Finite-time output stabilization of the double integrator

Emmanuel Bernuau, Wilfrid Perruquetti, Denis Efimov, Emmanuel Moulay

Finite-time output stabilization of the double integrator

Emmanuel Bernuau, Wilfrid Perruquetti, Denis Efimov and Emmanuel Moulay

***Abstract*— The problem of finite-time output stabilization of the double integrator is addressed applying the homogeneity approach. A homogeneous controller and a homogeneous observer are designed (for different degree of homogeneity) ensuring the finite-time stabilization. Their combination under mild conditions is shown to stay homogeneous and finite-time stable as well. The efficiency of the obtained solution is demonstrated in computer simulations.**

In many applications the nominal models have the double integrator form (mechanical planar systems, for instance). Despite of its simplicity this model is rather important in the control theory since frequently a design method developed for the double integrator can be extended to a more general case (via backstepping, for example). Most of the current techniques for nonlinear feedback stabilization provide an asymptotic stability: the obtained closed-loop dynamics is Lipschitzian and the system trajectories settle at the origin when the time is approaching infinity. Such a rate of convergence is not admissible in many applications, this is why the Finite-Time Stability (FTS) notion is quickly developing during the last decades: solutions of a FTS system reach the equilibrium point in a finite time. For example, the solutions $x(t, x_0)$ of

$$\dot{x} = -\text{sign}(x)|x|^{\alpha}, \quad x \in \mathbb{R},\ \alpha \in (0,1), \tag{1}$$

starting from $x_0 \in \mathbb{R}$ at $t_0 = 0$ are for $\varsigma = 1 - \alpha$:

$$x(t, x_0) = \begin{cases} \text{sign}(x_0)[|x_0|^{\varsigma} - \varsigma t]^{\frac{1}{\varsigma}} & \text{if} \quad 0 \le t \le \frac{|x_0|^{\varsigma}}{\varsigma}, \\ 0 & \text{if} \quad t > \frac{|x_0|^{\varsigma}}{\varsigma}. \end{cases} \tag{2}$$

Let us note that the right hand side of the above differential equation is not Lipschitz. In fact, finite-time convergence implies non-uniqueness of solutions (in reverse time) which is not possible in the presence of Lipschitz-continuous dynamics, where different maximal trajectories never cross.

Engineers are interested in the FTS because one can manage the time for solutions to reach the equilibrium which is called the *settling time*. An important issue is the settling time function regularity at the origin. The problem of finite-time stability has been developed for continuous systems giving sufficient and necessary condition (see [1], [2]). In addition, necessary and sufficient conditions appear for discontinuous systems involving uniqueness of solutions in forward time under continuity of the settling time function at the origin (see [3], [4]). It was observed in many papers that FTS can be achieved if the system is locally asymptotically stable and *homogeneous* with negative degree [5]. This is why the homogeneity plays a central role in the FTS system design. The reader may found additional properties and results on homogeneity in [6], [7], [8], [9], [10]. The homogeneity property was used many times to design FTS state controls [11], [12], [13], [14], [15], [16], FTS observers [17], [18] and FTS output feedback [19], [20].

Our goal is to develop the techniques of a FTS output feedback controller design for the double integrator. Since the double integrator is controllable, open-loop control strategies can be used to drive the state to the origin in a finite time (see [21], [22], [23] for a minimum time optimal control). Based on homogeneity, Bhat and Bernstein in their paper [11] provide a homogeneous FTS state controller for the double integrator. Our objective in this work is to relax the applicability conditions for the control obtained in [24], and to develop that solution to the FTS output control.

The outline of this work is as follows. Notations, and introduction of the FTS and the homogeneity concepts are given in Section 2. The precise problem formulation is presented in Section 3. The output FTS controller is designed in Section 4. The results of computer simulations of the proposed control algorithm are presented in Section 5.

E. Bernuau and W. Perruquetti are with LAGIS UMR 8219, Université Lille Nord de France, Ecole Centrale de Lille, Avenue Paul Langevin, BP 48, 59651 Villeneuve d'Ascq, France, {emmanuel.bernuau; wilfrid.perruquetti}@ec-lille.fr. W. Perruquetti and D. Efimov are with the Non-A project at INRIA Lille - Nord Europe, Parc Scientifique de la Haute Borne, 40 avenue Halley, Bât.A Park Plaza, 59650 Villeneuve d'Ascq, France,{denis.efimov; wilfrid.perruquetti}@inria.fr. E. Moulay is with Systèmes de Communications, Département XLIM-SIC, UMR CNRS 7252, Bât. SP2MI, Téléport 2, Bvd Marie et Pierre Curie, BP 30179, 86962 Futuroscope Chasseneuil Cedex, France, emmanuel.moulay@univ-poitiers.fr.

I. Preliminaries

A. Notations

Through the paper the following notations will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : \ x > 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R} : \ x < 0\}$, where $\mathbb{R}$ is the set of real numbers.
- If Q is a symmetric positive definite matrix and $x \in \mathbb{R}^n$, we denote $\|x\|_Q = x^T Q x$. When $Q = I_n$ (Euclidean norm), it will be simply denoted by $\|.\|$.
- For any real number $\alpha \geq 0$ and for all real x with $\alpha > 0$ and $x \in \mathbb{R}$ define$\lceil x \rfloor^{\alpha} = \text{sign}(x)|x|^{\alpha}$, then we have

$$\frac{d\lfloor x \rceil^{\alpha}}{dx} = \alpha\, |x|^{\alpha-1}, \qquad \frac{d\,|x|^{\alpha}}{dx} = \alpha\, \lfloor x \rceil^{\alpha-1}. \tag{3}$$

- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing; the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.

B. Finite-time stabilization

Let us consider

$$\dot{x} = G(x, u),$$

and the following closed loop system

$$\dot{x} = F(x), \tag{4}$$

where $F(x) := G(x, u(x))$ for a given feedback $u(x)$. In the following, $\Psi^t(x)$ denotes a solution of the system (4) starting from x at time zero. When F is not Lipschitz but for example continuous it may happen that any solution goes to zero in a finite time as it is the case for $\dot{x} = -\lfloor x \rceil^{\alpha}$, $\alpha \in (0,1)$, see (1)–(2). The system (4) is said to have unique solutions in forward time on a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ if for any $x \in \mathcal{U}$ and two right maximally defined solutions of (4), $\Psi_1^t(x) : [0, T_1) \to \mathbb{R}^n$ and $\Psi_2^t(x) : [0, T_2) \to \mathbb{R}^n$, there exists $0 < T \leq \min\{T_1, T_2\}$ such that $\Psi_1^t(x) = \Psi_2^t(x)$ for all $t \in [0, T)$. It can be assumed that for each $x \in \mathcal{U}$, T is chosen to be the largest in $\mathbb{R}_+ \cup \{+\infty\}$. Various sufficient conditions for forward uniqueness can be found in [25].

Let us consider the system (4) where F is continuous on $\mathbb{R}^n$ and where F has unique solutions in forward time. We recall the definition of finite-time stability [6].

Definition 1: The origin of the system (4) is *finite-time stable* (FTS) iff there exists a neighborhood of the origin $\mathcal{V}$ such that:

1) There exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_+$ where for all $x \in \mathcal{V} \setminus \{0\}$, $\Psi^t(x)$ is defined (and unique) on $[0, T(x))$, $\Psi^t(x) \in \mathcal{V} \setminus \{0\}$ for all $t \in [0, T(x))$ and $\lim\limits_{t \to T(x)} \Psi^t(x) = 0$. T is called the *settling-time function* of the system (4).
2) For all neighborhoods of the origin $\mathcal{U}_1$, there exists a neighborhood of the origin $\mathcal{U}_2 \subset \mathcal{V}$ such that for every $x \in \mathcal{U}_2$, $\Psi^t(x) \in \mathcal{U}_1$ for all $t \in [0, T(x))$.

The following result gives a sufficient condition for the system (4) to be FTS (see [26], [27] for ordinary differential equations, and [28] for differential inclusions):

Theorem 1: [26], [27] Let the origin be an equilibrium point for the system (4), and let F be continuous on an open neighborhood $\mathcal{V}$ of the origin. If there exists a Lyapunov function[1] $V : \mathcal{V} \to \mathbb{R}_+$ and a function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\frac{dV}{dt}(x) \leq -r(V(x)), \tag{5}$$

along the solutions of (4) and $\varepsilon > 0$ such that

$$\int_0^{\varepsilon} \frac{dz}{r(z)} < +\infty, \tag{6}$$

then the origin of the system (4) is finite-time stable.

In particular, assuming forward uniqueness of the solution and the continuity of the settling time function, Bhat and Bernstein (see [3, Definition 2.2]) showed that FTS of the origin is equivalent to the existence of a Lyapunov function satisfying (5) where $r(x) = cx^a$, with $a \in (0,1)$, $c > 0$. In order to circumvent the classical Lyapunov function art of design, one can use homogeneity conditions recalled below.

[1] V is a continuously differentiable function defined on $\mathcal{V}$ such that V is positive definite.

C. Homogeneity

Let $r = (r_1, \ldots, r_n)$ be a $n-$uplet of positive real numbers, then for any positive number λ

$$\Lambda_r x = (\ldots, \lambda^{r_i} x_i, \ldots) \tag{7}$$

represents a mapping $x \mapsto \Lambda_r x$ usually called a dilation (see [8]).

Definition 2: A function h defined on $\mathbb{R}^n$ is homogeneous with degree $k \in \mathbb{R}$ with respect to dilation Λ_r if for all $x \in \mathbb{R}^n$ we have [8]:

$$h(\Lambda_r x) = \lambda^k h(x). \tag{8}$$

When such a property holds, we write: $\deg(h) = k$.

Definition 3: A vector field F defined on $\mathbb{R}^n$ is homogeneous with degree m with respect to dilation Λ_r if for all $x \in \mathbb{R}^n$, we have (see [8]): $\deg(F_i) = m + r_i, \forall i \in \{1, \ldots, n\}$, which could be written as: $\Lambda_r^{-1} F(\Lambda_r x) = \lambda^m F(x)$. When such a property holds the corresponding nonlinear ODE (4) is said to be homogeneous with degree m with respect to dilation Λ_r.

Theorem 2: [5] Let F be a continuous vector field on $\mathbb{R}^n$ homogeneous with degree $m < 0$ (with respect to dilation Λ_r); if the origin is LAS (locally asymptotically stable) then it is globally FTS.

Let us recall a useful theorem from [24].

Theorem 3: Suppose the vector field F is homogeneous with respect to the dilation Δ_r. If K is a compact subset of $\mathbb{R}^n$ such that $\Psi^t(K) \subseteq K$ for all $t > 0$ (K is said to be strictly positively invariant or SPI), then $0 \in K$ and 0 is globally asymptotically stable.

By adding the two previous results, we get:

Corollary 1: If F is a homogeneous vector field with respect to the dilation Δ_r, of negative degree, and if there exists a SPI compact subset of $\mathbb{R}^n$, then 0 is globally FTS.

II. Problem formulation

Our contribution aims at designing a FTS output feedback based on homogeneity for the following double-integrator system:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u(x_1, x_2), \\ y & = & x_1, \end{array} \right. \tag{9}$$

where x_1 and x_2 are the states of the system, u is the input and y is the output. We will proceed in three steps:

1) Design a homogeneous state feedback control ensuring FTS for the double integrator:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u. \end{array} \right. \tag{10}$$

2) Design a homogeneous observer:

$$\left\{ \begin{array}{lcl} \dot{\hat{x}}_1 & = & \hat{x}_2 - \chi_1(y - \hat{x}_1), \\ \dot{\hat{x}}_2 & = & -\chi_2(y - \hat{x}_1), \end{array} \right. \tag{11}$$

where χ_1 and χ_2 are functions to be designed such that the origin is FTS for the error $e = x - \hat{x}$ equation:

$$\left\{ \begin{array}{rcl} \dot{e}_1 & = & e_2 + \chi_1(e_1), \\ \dot{e}_2 & = & \chi_2(e_1). \end{array} \right. \tag{12}$$

3) Show a separation principle such that the obtained observer-based closed loop system is FTS.

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u(y, \hat{x}_2), \\ y & = & x_1, \end{array} \right. \tag{13}$$

where $\hat{x}_2$ is obtained from (11).

III. Finite-time output feedback based on homogeneity

A. Finite-time control

Let us consider the double integrator (10). It is homogeneous with degree m w.r.t. to dilation Λ_r with weight $r = (r_1, r_2)$ as soon as u is homogeneous of degree r_u and the following relations hold:

$$\begin{array}{rcl} r_1 + m & = & r_2, \\ r_2 + m & = & r_u. \end{array}$$

Thus fixing $r_u = 1$ (without loss of generality) a necessary and sufficient condition for (10) to be homogeneous is

$$r_1 = 2r_2 - 1, \tag{14}$$

$$m = 1 - r_2. \tag{15}$$

To have FTS it is necessary and sufficient that (10) is LAS and that $m < 0$. Let us find conditions for which the following feedback leads to LAS of the origin of the system (10):

$$u = k_1 \lfloor x_1 \rceil^{\alpha_1} + k_2 \lfloor x_2 \rceil^{\alpha_2}, \tag{16}$$

and $m < 0$. The feedback (16) is homogeneous of degree r_u iff $r_u = 1 = r_i \alpha_i$. From (14) and (15), setting $\alpha := \alpha_2$, we get: $m = 1 - \frac{1}{\alpha}$, $r_1 = \frac{2-\alpha}{\alpha}$, $r_2 = \frac{1}{\alpha}$ and $\alpha_1 = \frac{\alpha}{2-\alpha}$. The condition $m < 0$ is equivalent to $\alpha \in (0, 1)$, which in turn implies that $\alpha_1 \in (0, 1)$. In all the sequel, we assume $\alpha \in (0, 1)$.

The system (10) with the feedback (16) takes the form

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 \rceil^{\alpha}. \end{array} \right. \tag{17}$$

We would like to find the conditions on the coefficients k_i providing LAS for the system (17) (that due to homogeneity implies FTS). In the work [24] these conditions have been obtained for α sufficiently close to one (here we consider $\alpha \in (0, 1)$).

Consider the following functions:

$$\varphi : x = (x_1, x_2)^T \mapsto \varphi(x) = (x_1, \lfloor x_2 \rceil^{2-\alpha})^T,$$

$$V : x \mapsto \varphi(x)^T P \varphi(x), \tag{18}$$

where P is a symmetric positive definite matrix such that $PA + A^T P = -I$ with $A = \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix}$:

$$P = \begin{pmatrix} \frac{k_2^2 + k_1^2 - k_1}{2k_1 k_2} & \frac{-1}{2k_1} \\ \frac{-1}{2k_1} & \frac{1-k_1}{2k_1 k_2} \end{pmatrix}. \tag{19}$$

The functions φ and V are homogeneous of degree $1/\alpha_1 > 1$ and $2/\alpha_1$ with respect to Λ_r, then

$$\dot{V} = 2\varphi(x)^T P \begin{pmatrix} x_2 \\ (2-\alpha)|x_2|^{1-\alpha} u \end{pmatrix}$$

and a direct computation leads to:

$$\begin{array}{rcl} \dot{V} & = & |x_2|^{1-\alpha} \Bigg[(\alpha - 2)|x_1|^{\frac{2}{2-\alpha}} + (\alpha - 2 + \dfrac{1-\alpha}{k_1}) x_2^2 \\ & & + \dfrac{k_1^2 - k_1 + (\alpha - 1)k_2^2}{k_1 k_2} x_1 \lfloor x_2 \rceil^{\alpha} \\ & & - (2-\alpha) \dfrac{k_1 - 1}{k_2} \lfloor x_1 \rceil^{\frac{2}{2-\alpha}} \lfloor x_2 \rceil^{2-\alpha} \Bigg]. \end{array}$$

Let us denote $y = \lfloor x_1 \rceil^{\frac{1}{2-\alpha}}$ and $z = x_2$, then we obtain:

$$\begin{array}{rcl} \dot{V} & = & |z|^{1-\alpha} \Bigg[(\alpha - 2)(y^2 + z^2) + (1-\alpha) \dfrac{z^2}{k_1} \\ & & + \dfrac{k_1^2 - k_1 + (\alpha - 1)k_2^2}{k_1 k_2} \lfloor y \rceil^{2-\alpha} \lfloor z \rceil^{\alpha} \\ & & - (2-\alpha) \dfrac{k_1 - 1}{k_2} \lfloor z \rceil^{2-\alpha} \lfloor y \rceil^{\alpha} \Bigg]. \end{array}$$

Lemma 1: Set $f(y, z) = \lfloor y \rceil^{2-\alpha} \lfloor z \rceil^{\alpha}$ and

$$M = \frac{(2-\alpha)^{1-\alpha/2} \alpha^{\alpha/2}}{2},$$

then $-M \le f \le M$ on the circle $\mathbb{S} = \{y^2 + z^2 = 1\}$.

The proofs of lemmas 1–4 and Theorem 4 are excluded due to space limitations.

Remark 1: Direct computations show that for all $\alpha \in (0, 1)$ we have $M^2(1-\alpha) < 1$.

We will be interested in the following condition on k_2:

$$k_2 < \frac{M}{M^2(1-\alpha) - 1}. \tag{C.1}$$

Remark 2: The condition (C.1) implies $k_2 < 0$ for all $\alpha \in (0, 1)$.

Lemma 2: Set $\Delta_1 = 1 + 4(1-\alpha)k_2^2$ and $\Delta_2 = ((3-\alpha) + \frac{(2-\alpha)k_2}{M})^2 + 4(1-\alpha)(3-\alpha)k_2^2$. Under the condition (C.1) we have the following inequalities:

$$\frac{1}{2} + \frac{(2-\alpha)k_2}{2M(3-\alpha)} - \frac{\sqrt{\Delta_2}}{2(3-\alpha)} < \frac{1 - \sqrt{\Delta_1}}{2} < M(1-\alpha)k_2.$$

Theorem 4: If k_2 is chosen in accordance with the condition (C.1) and k_1 belongs to the following interval

$$k_1 \in \left[\frac{1}{2} + \frac{(2-\alpha)k_2}{2M(3-\alpha)} - \frac{\sqrt{\Delta_2}}{2(3-\alpha)}, M(1-\alpha)k_2 \right], \tag{C.2}$$

then the system (17) is FTS.

Remark 3: The theorem 4 proves that the origin of (10) with the control (16) is FTS for $\alpha \in (0, 1)$ under the conditions (C.1) and (C.2). Notice that when $\alpha \to 1$, the proposition (8.1) of [24] ensures the FTS for any k_1 and k_2.

B. Finite-time observer design

A finite-time observer for a canonical observable form was constructed for the first time in [17]. The proof of finite-time stability is based on homogeneity property. The class of considered systems is:

$$\begin{cases} \dot{x} = \widetilde{A}(a_1, \ldots, a_n)x + f(y, u, \dot{u}, \ldots, u^{(r)}) \\ y = Cx \end{cases}, \tag{20}$$

where $x \in \mathbb{R}^n$ and r is a positive integer, and:

$$\begin{array}{rcl} \widetilde{A}(a_1, \ldots, a_n) & = & \left(\begin{array}{ccccc} a_1 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & 0 & 1 \\ a_n & 0 & 0 & 0 & 0 \end{array} \right), \\ C & = & (1\ 0\ \ldots\ 0), \end{array}$$

where $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$. The proposed observer is:

$$\left(\begin{array}{c} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \dot{\hat{x}}_n \end{array} \right) = \widetilde{A} \left(\begin{array}{c} y \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{array} \right) + f(.) - \left(\begin{array}{c} l_1 \lceil y - \hat{x}_1 \rfloor^{\beta_1} \\ l_2 \lceil y - \hat{x}_1 \rfloor^{\beta_2} \\ \vdots \\ l_n \lceil y - \hat{x}_1 \rfloor^{\beta_n} \end{array} \right),$$

where $f(.) = f(y, u, \dot{u}, \ldots, u^{(r)})$. The powers β_i are defined such that the error dynamics can be written as follows:

$$\begin{cases} \dot{e}_1 = e_2 + l_1 \lceil e_1 \rfloor^{\beta_1}, \\ \vdots \\ \dot{e}_n = l_n \lceil e_1 \rfloor^{\beta_n}, \end{cases}$$

where $e = x - \hat{x}$ and the right hand side is homogeneous with a negative degree with respect to the weights $\rho = (\rho_1, \ldots, \rho_n)$. It is homogeneous with degree m w.r.t. to dilation Λ_ρ as soon as the following relations hold

$$\begin{array}{rcl} \rho_i + m & = & \rho_{i+1} = \rho_1 \beta_i, i \in \{1, \ldots, n-1\}, \\ \rho_n + m & = & \rho_1 \beta_n. \end{array}$$

Thus a necessary and sufficient condition for (11) to be homogeneous is

$$\begin{array}{rcl} \rho_i & = & \rho_1 + (i-1)\, m, \\ m & = & (\beta_1 - 1)\rho_1. \end{array}$$

Therefore, setting $\beta = \beta_1$, the relation $\rho_{i+1} = \rho_1 \beta_i$ gives

$$\begin{array}{rcl} \rho_i & = & \rho_1[(i-1)\beta + 2 - i], \\ \beta_i & = & i\beta + (1-i), \end{array}$$

with $\beta \in (1 - \frac{1}{n}, 1)$. The gains l_i, $i = 1, \ldots, n$, are defined such that the matrix $\widetilde{A}(l_1, \ldots, l_n)$ is Hurwitz.

However, in [17] FTS was proved for $\beta \in (1 - \varepsilon, 1)$ for a sufficiently small $\varepsilon > 0$. Here we concentrate on the case $n = 2$ and show that the system is FTS for all $\beta \in (\frac{1}{2}, 1)$ and all $\rho_1 > 0$. From the previous relations we get: $\beta_1 = \beta \in (\frac{1}{2}, 1)$, $\beta_2 = 2\beta - 1$, $\rho_1 = 1$, $\rho_2 = \rho_1 \beta$ and $m = \rho_1(\beta - 1)$. The system becomes:

$$\left\{ \begin{array}{rcl} \dot{e}_1 & = & e_2 + l_1 \lfloor e_1 \rceil^{\beta}, \\ \dot{e}_2 & = & l_2 \lfloor e_1 \rceil^{2\beta - 1}. \end{array} \right. \tag{21}$$

Let us denote

$$A_\beta = \left(\begin{array}{cc} \beta l_1 & \beta \\ l_2 & 0 \end{array} \right).$$

Let P and Q be symmetric positive definite matrices such that $PA_\beta + A_\beta^T P = -Q$. This equation has a solution if and only if A_β is Hurwitz. But the characteristic polynomial of A_β is $X^2 - \beta l_1 X - \beta l_2$, and this polynomial is Hurwitz since $l_1 < 0$ and $l_2 < 0$. Consider the following function:

$$V(e) = \left(\begin{array}{c} \lfloor e_1 \rceil^{\beta} \\ e_2 \end{array} \right)^T P \left(\begin{array}{c} \lfloor e_1 \rceil^{\beta} \\ e_2 \end{array} \right).$$

The function V is positive definite, homogeneous of degree $2\rho_1\beta$, continuous everywhere and differentiable on the open set $U = \{e_1 \neq 0\}$. Furthermore, on U we have:

$$\begin{array}{rcl} \dot{V} & = & 2 \left(\begin{array}{c} \lfloor e_1 \rceil^{\alpha} \\ e_2 \end{array} \right)^T P \left(\begin{array}{c} \beta |e_1|^{\beta - 1}(e_2 + l_1 \lfloor e_1 \rceil^{\beta}) \\ l_2 \lfloor e_1 \rceil^{2\beta - 1} \end{array} \right) \\ & = & |e_1|^{\beta - 1} \left(\begin{array}{c} \lfloor e_1 \rceil^{\beta} \\ e_2 \end{array} \right)^T (PA_\beta + A_\beta^T P) \left(\begin{array}{c} \lfloor e_1 \rceil^{\beta} \\ e_2 \end{array} \right) \\ & = & -|e_1|^{\beta - 1} \left\| \left(\begin{array}{c} \lfloor e_1 \rceil^{\beta} \\ e_2 \end{array} \right) \right\|_Q^2 \\ & < & 0. \end{array}$$

Since $\dot{V}$ is strictly negative on U, for all $e \in U$, the function $t \mapsto V(\Psi^t(e))$ is strictly decreasing as long as $\Psi^t(e)$ belongs to U, where Ψ denotes the semi-flow of the vector field F given in the right hand side of (21).

Now, let $e = (0, e_2) \neq 0$. We have for $t > 0 : \Psi^t(e) = e + tF(e) + o(t)$, thus $\Psi^t(e) = (te_2, e_2) + o(t)$, where $o(t)$ is the Landau notation. Therefore we get that for all $e \in \mathbb{R}^2 \backslash \{0\}$, there exists $T_e > 0$ such that for all $t \in (0, T_e)$, $\Psi^t(e) \in U$.

Set $e \neq 0$. If $e \in U$, for all $s < t \in [0, T_e)$, we have $V(\Psi^s(e)) > V(\Psi^t(e))$. If $e \notin U$, for all $t \in (0, T_e)$, we have $\Psi^t(e) \in U$, thus for all $s < t \in (0, T_e)$ we have $V(\Psi^s(e)) > V(\Psi^t(e))$. By continuity of $t \mapsto V(\Psi^t(e))$, we have $V(e) \geq V(\Psi^t(e))$ for all $t \in [0, T_e)$. Assume there exists $t \in (0, T_e)$ such that $V(e) = V(\Psi^t(e))$. Then for $0 < s < t$ we have $V(e) \geq V(\Psi^s(e)) > V(\Psi^t(e)) = V(e)$ and this is a contradiction.

A similar proof leads to $V(\Psi^s(e)) > V(\Psi^t(e))$ for all $0 \leq s < t \leq T_e$. Finally, for all $e \neq 0$, the function $t \mapsto V(\Psi^t(e))$ is strictly decreasing.

Consider now the compact $K = \{V \leq 1\}$. Since $t \mapsto V(\Psi^t(e))$ is strictly decreasing, K is strictly positively invariant, and thus, by corollary (1), the system is globally FTS and we have proven the following theorem.

Theorem 5: The observer (11) with $\chi_1(e_1) = \lfloor e_1 \rceil^{\beta}$, $\chi_2(e_1) = \lfloor e_1 \rceil^{2\beta - 1}$ for any $\beta \in (\frac{1}{2}, 1)$ is globally FTS in the coordinates (e_1, e_2).

Thus the observer (11) ensures observation of the state of the system (9) in a finite time for any initial conditions (without loosing generality we always may assume that $e_1(0) = 0$).

C. Finite-time stable observer based control

To construct our observer-based control, we will introduce some restrictions on the observer parameters. We choose $\beta = \frac{1}{2-\alpha}$ and $\rho_1 = \frac{2-\alpha}{\alpha}$, then it follows that $\beta_2 = \frac{\alpha}{2-\alpha}$. Let us rewrite the system (13) for the designed FTS control (16) and the FTS observer (21) (in the estimation error coordinates):

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & k_1\lfloor x_1\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor x_2 - e_2\rceil^{\alpha}, \\ \dot{e}_1 & = & e_2 + l_1\lfloor e_1\rceil^{\frac{1}{2-\alpha}}, \\ \dot{e}_2 & = & l_2\lfloor e_1\rceil^{\frac{\alpha}{2-\alpha}}. \end{array}\right. \tag{22}$$

Note that $x_2 - e_2 = \hat{x}_2$, thus the control depends on the measured output x_1 only. To prove the FTS property of this system we need three auxiliary lemmas.

Lemma 3: With this choice of β and ρ_1, the system (22) is homogeneous w.r.t. the dilation given by:

$$(x_1, x_2, e_1, e_2)^T \mapsto (\lambda^{\frac{2-\alpha}{\alpha}} x_1, \lambda^{\frac{1}{\alpha}} x_2, \lambda^{\frac{2-\alpha}{\alpha}} e_1, \lambda^{\frac{1}{\alpha}} e_2)^T.$$

Let us denote $\Delta = \lfloor x_2 - e_2\rceil^{\alpha_2} - \lfloor x_2\rceil^{\alpha_2}$.

Lemma 4: For all $e_2 \in \mathbb{R}$, and all $x_2 \in \mathbb{R}$ we have $|\Delta| \leq 2|e_2|^{\alpha_2}$.

Consider now the function V defined in (18). Denoting by u the control defined in (16), we have:

$$\begin{array}{rcl} \dot{V} & = & dV\begin{pmatrix} x_2 \\ u + k_2\Delta \end{pmatrix} \\ & = & dV\begin{pmatrix} x_2 \\ u \end{pmatrix} + dV\begin{pmatrix} 0 \\ k_2\Delta \end{pmatrix} \\ & \leq & 2\varphi^T P\begin{pmatrix} 0 \\ k_2\Delta(2-\alpha)|x_2|^{1-\alpha} \end{pmatrix} \\ & \leq & \varphi^T P\varphi + (2-\alpha)^2|x_2|^{2-2\alpha}\Delta^2 P_{22} \\ & \leq & V + 4(2-\alpha)^2 P_{22}|e_2|^{2\alpha_2}|x_2|^{2-2\alpha}. \end{array}$$

Let us denote $B(e_2) = 4(2-\alpha)^2 P_{22}|e_2|^{2\alpha_2}$. Since the observer is stable, we easily show that there exists a function $\sigma \in \mathcal{K}_\infty$ s.t. $B(e_2) \leq \sigma(||\mathbf{e}_0||)$, where $\mathbf{e}_0 = (e_1(0), e_2(0))^T$. Noting that $|x_2|^{2-2\alpha} \leq 1 + |x_2|^{4-2\alpha}$, we get:

$$\begin{array}{rcl} \dot{V} & \leq & V + B(e_2)(1 + |x_2|^{4-2\alpha}) \\ & \leq & V + \sigma(||\mathbf{e}_0||)(1 + x_1^2 + |x_2|^{4-2\alpha}). \end{array}$$

Denoting $\mu > 0$ the smallest eigenvalue of the matrix P defined in (19), we have $\mu(x_1^2 + x_2^{4-2\alpha}) = \mu\|\varphi(x)\|^2 \leq V(x)$, e.g. $x_1^2 + x_2^{4-2\alpha} \leq \frac{V}{\mu}$. Finally we have:

$$\dot{V} \;\leq\; \left(1 + \frac{\sigma(||\mathbf{e}_0||)}{\mu}\right)V + \sigma(||\mathbf{e}_0||),$$

and it follows that for all $t \geq 0$:

$$V \leq \iota(t, V(x(0)), ||\mathbf{e}_0||),$$

$$\iota(t, r, s) = \left(r + \frac{\sigma(s)}{1 + \sigma(s)\mu^{-1}}\right)\exp[(1 + \sigma(s)\mu^{-1})t].$$

Lemma 5: The solutions of the system (22) are defined for all $t \geq 0$.

Now we are in position to formulate the main result.

Theorem 6: The system (22) is globally FTS for any $\alpha \in (0, 1)$ and $\beta = \frac{1}{2-\alpha} \in (1/2, 1)$ provided that k_1, k_2 are chosen in accordance with (C.1), (C.2) and for any $l_1 < 0$, $l_2 < 0$.

Proof: Let us denote by $T_1(\text{resp. } T_2) : \mathbb{R}^2 \to \mathbb{R}_+$ the settling time function of the system (17) (resp. the system (11)). By lemma 5, the solutions of the system (22) exist for all $t > 0$. For $t \geq T_2(\mathbf{e}_0)$, we have $\Delta = 0$ and the system (22) becomes equivalent to the system (17). Thus the system (22) converges to the origin in a finite-time, namely $T_2(\mathbf{e}_0) + T_1(x_1(T_2(\mathbf{e}_0)), x_2(T_2(\mathbf{e}_0)))$. Therefore the origin is a global finite-time equilibrium. Since the system (22) is homogeneous and attractive, by [24, proposition 6.1] the system is globally asymptotically stable, and thus globally FTS. ■

Remark 4: If the function T_2 is locally bounded, the stability can be proved without the homogeneity of the system (22).

Remark 5: It is worth to stress that the system (22) is FTS in coordinates (e_1, e_2) (see Theorem 5) and it is FTS in coordinates (x_1, x_2, e_1, e_2) (Theorem 6). However, it is not FTS in the isolated coordinates (x_1, x_2) since the time of convergence in these coordinates depends on the convergence of the observer.

IV. Simulations

Let us consider two cases $\alpha = 0.3$ and $\alpha = 0.6$. Let $l_1 = -1$ and $l_2 = -2$. The straightforward calculation shows that the choice $k_1 = -1$ and $k_2 = -2$ also admits the conditions (C.1), (C.2) for both values of α:

- $\alpha = 0.3$:

$$M = 0.655,\ k_2 \leq -0.937,\ -1.619 \leq k_1 \leq -0.917;$$

- $\alpha = 0.6$:

$$M = 0.543,\ k_2 \leq -0.615,\ -1.624 \leq k_1 \leq -0.434.$$

The results of the system simulation are presented in figures 1, 2. In figures 1.a, 2.a and 1.b, 2.b the examples of transients in time are given for the system state (x_1, x_2) and the estimation error (e_1, e_2) respectively. As we can conclude from these figures, the system is converging to zero in a finite time for both pairs of variables, but for (x_1, x_2) the convergence is not monotone (that justifies the theoretical results obtained above). The fact that the system is FTS in coordinates (e_1, e_2) and (x_1, x_2, e_1, e_2) (but not in (x_1, x_2)) becomes more evident from analysis of figures 1.c, 2.c and 1.d, 2.d, where the trajectories are shown for different initial conditions for (x_1, x_2) and (e_1, e_2) respectively. In addition, as we can observe from these figures, the behavior (the rate of convergence) of the system trajectories is changed when the trajectories approach the line $x_2 = 0$, that was also detected in the theoretical part, where the Lyapunov function becomes identical zero close to this line (see Theorem 4).

V. Conclusion

The problems of finite-time control and estimation for the double integrator are studied. An extension of applicability conditions of the homogeneous control algorithm from [11]

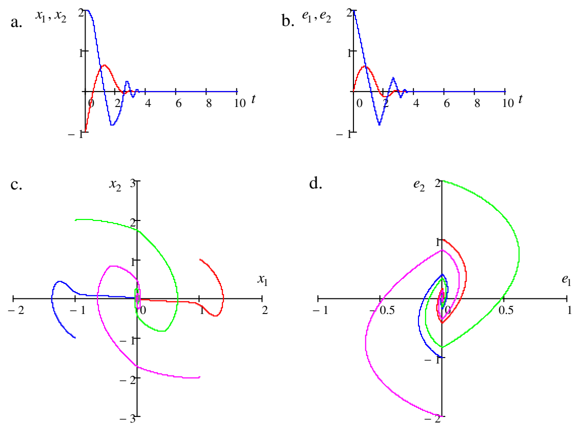

Fig. 1. The results of simulation for $\alpha = 0.3$

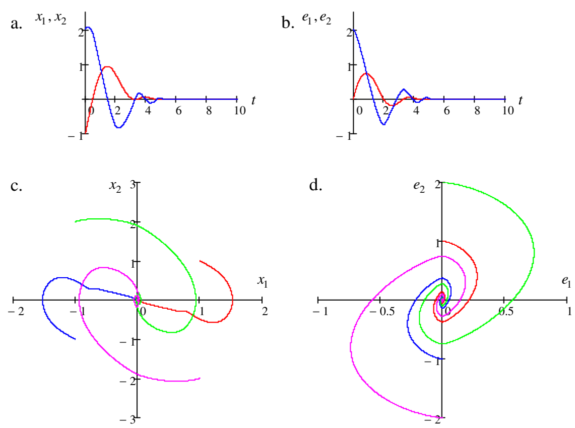

Fig. 2. The results of simulation for $\alpha = 0.6$

is obtained. A finite-time output control is designed. The efficiency of the obtained solution is demonstrated by computer simulations.

Development of the approach to the case of n^{th}-dimensional differentiator, analysis of robustness with respect to external disturbances and measurement noise, evaluation of the settling time function are the possible future directions of the research.

References

[1] V. Haimo, "Finite time controllers," *SIAM J. Control Optim.*, vol. 24, no. 4, pp. 760–770, 1986.

[2] E. Moulay and W. Perruquetti, "Finite time stability of non linear systems," in *IEEE Conference on Decision and Control*, Hawaii, USA, 2003, pp. 3641–3646.

[3] S. Bhat and D. Bernstein, "Finite time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[4] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM J. Control Optim.*, vol. 43, no. 4, pp. 1253–1271, 2005.

[5] L. Rosier, "Etude de quelques problèmes de stabilisation," Ph.D. dissertation, Ecole Normale Supérieure de Cachan, 1993.

[6] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*, 2nd ed., B. Springer, Ed. Springer, 2005.

[7] S. Bhat and D. Bernstein, "Finite-time stability of homogeneous systems," in *Proceedinds of the American Control Conference*, Albuquerque, USA, 1997, pp. 2513–2514.

[8] H. Hermes, *Homogeneous coordinates and continuous asymptotically stabilizing feedback controls*, ser. Differential Equations: Stability and Control. Marcel Dekker, 1991, vol. 109, pp. 249–260.

[9] M. Kawski, "Geometric homogeneity and stabilization," in *Proc. IFAC Nonlinear Control Symposium*, A. Krener and D. Mayne, Eds., Lake Tahoe, CA, 1995, pp. 164–169.

[10] Y. Orlov, "Finite time stability of homogeneous switched systems," in *IEEE Conference on Decision and Control*, Hawaii, USA, 2003, pp. 4271–4276.

[11] S. P. Bhat and D. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrator," *IEEE Trans. Automat. Control*, vol. 43, no. 5, pp. 678–682, may 1998.

[12] T. Floquet, J. P. Barbot, and W. Perruquetti, "Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems," *Automatica*, vol. 39, no. 6, pp. 1077–1083, 2003.

[13] Y. Hong, "Finite-time stabilization and stabilizability of a class of controllable systems," *Systems & Control Letters*, vol. 46, no. 4, pp. 231–236, 2002.

[14] E. Moulay and W. Perruquetti, "Finite time stability and stabilization of a class of continuous systems," *Journal of Mathematical Analysis and Application*, vol. 323, no. 2, pp. 1430–1443, 2006.

[15] C. Qian and W. Lin, "Non-lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization," *Systems & Control Letters*, vol. 42, no. 3, pp. 185 – 200, 2001.

[16] L. Praly, "Generalized weighted homogeneity and state dependent time scale for linear controllable systems," in *Proc. IEEE CDC 97*. San Diego, California USA: IEEE, 1997, pp. 4342–4347.

[17] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: application to secure communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 356–360, 2008.

[18] Y. Shen and X. Xia, "Semi-global finite-time observers for nonlinear systems," *Automatica*, vol. 44, no. 12, pp. 3152–3156, 2008.

[19] Y. Hong, J. Huang, and Y. Xu, "On an output feedback finite-time stabilization problem," *IEEE Trans. Automat. Control*, vol. 46, no. 2, pp. 305–309, 2001.

[20] J. Li and C. Qian, "Global finite-time stabilization by dynamic output feedback for a class of continuous nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 879–884, 2006.

[21] M. Athans and P. L. Falb, *Optimal control : an introduction to the theory and its applications*, ser. Lincoln Laboratory publications. McGraw-Hill, 1966.

[22] E. Ryan, "Singular optimal controls for second-order saturating systems," *Internat. J. Control*, vol. 3, no. 4, pp. 549–564, 1979.

[23] W. Wonham, *Linear multivariable control : a geometric approach*, 3rd ed. Springer New York, 1985.

[24] S. Bhat and D. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[25] M. Kawski, "Stabilization of nonlinear systems in the plane," *Systems & Control Letters*, vol. 12, pp. 169–175, 1989.

[26] E. Moulay and W. Perruquetti, "Finite-time stability conditions for non-autonomous continuous systems," *Internat. J. Control*, vol. 81(5), pp. 797–803, 2008.

[27] W. Perruquetti and S. Drakunov, "Finite time stability and stabilisation," in *IEEE Conference on Decision and Control*, no. 2, Sydney, Australia, 2000, pp. 1894–1899.

[28] E. Moulay and W. Perruquetti, "Finite time stability of differential inclusions," *IMA J. Math. Control Inform*, 2005.