State estimation for linear switched systems with unstable invariant zeros and unknown inputs

Rios Hector, Davila Jorge, Leonid Fridman, Denis Efimov

State estimation for linear switched systems with unstable invariant zeros and unknown inputs

Héctor Ríos†, Jorge Davila‡, Leonid Fridman† and Denis Efimov§

Abstract— **In this paper the problem of continuous and discrete state estimation for a class of linear switched systems is studied. The class of systems under study can contain non-minimum phase zeros in some of their "operating modes". The conditions for exact reconstruction of the discrete state are given using structural properties of the switched system. The state-space is decomposed into the strongly observable part, the nonstrongly observable part and the unobservable part, to analyze the effect of the unknown inputs. A state observer based on high-order sliding-mode and Luenberger-like observers is proposed. For the case when the exact reconstruction of the state cannot be achieved, the ultimate bounds on the estimation errors are provided. The workability of the proposed method is illustrated by simulations.**

Index Terms— **High-order Sliding Modes, Non-minimum Phase, Linear Switched Systems, State Observers.**

I. Introduction

THE observation problem for switched systems (i.e. the estimation of the continuous and discrete state) is of great interest in many control areas. This problem has been studied by many authors using different kind of approaches. The main difference is related to the knowledge of the active discrete state: some approaches consider only continuous state uncertainty with known discrete state, while others assume that both of them are unknown.

In [1] and [2] a Luenberger observer approach and a high-order sliding-mode observer for linear systems are proposed for the known discrete state case. In [3] the problem of the simultaneous state and input estimation for hybrid systems subject to input disturbances is addressed by an algorithm based on the moving horizon estimation method. Considering that the continuous state is known, an algorithm for the discrete state reconstruction in nonlinear uncertain switched systems is presented in [4] based on sliding-mode control theory. For unknown discrete state, based on strong detectability and using a LMI approach, in [5] two state observers are designed for some classes of switched linear systems with unknown inputs. To Markovian jump singular systems, another class of switched systems, in [6] an integral sliding mode observer is designed to estimate the system states, and a sliding mode control scheme is synthesized for the reaching motion based on the state estimates.

The problem of observability in hybrid systems is also intensively studied in the literature. For instance, [7] and [8] analyze the observability of hybrid systems. For the detectability case, in [9] the detectability of linear switched systems is addressed that reduces to asymptotic stability of a suitable switched system with guards extracted from it (switching systems whose discrete state is triggered externally). In the nonlinear context, there are few works dealing with the observability of nonlinear switched systems with unknown inputs (see, for example [10] and [4]), which aim at designing unknown input observers for some classes of nonlinear switched systems. In the context of control, in [11] an integral sliding mode control of nonlinear singular stochastic systems with Markovian switching is proposed.

In this paper a solution of the problem of state estimation for linear switched systems with unknown inputs and unstable invariant zeros (non-minimum phase systems with respect to unknown inputs)[1] is presented. A state observer is proposed that is capable to estimate the continuous and discrete state in the presence of unknown inputs. The state observer is based on high-order sliding-mode (HOSM) and Luenberger-like observers.

The paper has the following structure. Section II deals with the problem statement. The preliminaries are recalled in section III. In section IV the system transformation is proposed. The observer design is presented in section V. The problem of discrete state estimation is presented in section VI. The simulation results is shown in section VII. Finally, some concluding remarks are given in section VIII.

II. Problem Statement

Consider the following linear switched system with unknown inputs:

$$\begin{aligned} \dot{x}(t) &= A_{j(x)}x(t) + B_{j(x)}u(t) + E_{j(x)}w(t), \\ y(t) &= C_{j(x)}x(t), \end{aligned} \tag{1}$$

where $x \in \mathscr{X} \subseteq \Re^n$ is the state vector, $u \in \mathscr{U} \subseteq \Re^p$ is the known input vector, $y \in \mathscr{Y} \subseteq \Re^m$ is the output, and $w \in \mathscr{W} \subseteq \Re^m$ is the unknown input term, which is bounded, i.e. $\|w(t)\| \leq w^+ < \infty$. The so-called "discrete state" $j(x) : \mathbb{R} \to \{1, \ldots, N\}$ determines the current system dynamics among the possible N "operating modes". The discrete state is generated

† Department of Control, Division of Electrical Engineering, Engineering Faculty, National Autonomous University of Mexico, C.P. 04510, Mexico, D.F. emails:hectorrios@comunidad.unam.mx, lfridman@unam.mx.

‡ Section of Graduate Studies and Research, ESIME-UPT, National Polytechnic Institute, C.P. 07340, Mexico D.F. email:jadavila@ieee.org.

§ The Non-A project at INRIA - LNE, Parc Scientifique de la Haute Borne, 40 avenue Halley, Bt.A Park Plaza, 59650 Villeneuve d'Ascq, France. email:denis.efimov@inria.fr.

[1] See, for example [12] for the observation problem for a class of non-minimum phase causal nonlinear systems.

by a scalar function of the system states (the switching signal) defined as

$$j(x) = \begin{cases} 1, & \forall x \mid \quad \rho_0 > Hx \geq \rho_1 \\ \vdots & \\ N, & \forall x \mid \quad \rho_{N-1} > Hx \geq \rho_N \end{cases} \tag{2}$$

where H, $\rho_0, \rho_2, ..., \rho_N \in \mathbb{R}$ are known matrix and constants, respectively. Notice that for every value of the continuous state $x(t)$ there is only one single value of the discrete state $j(x)$, i.e. the discrete state is distinguishable by definition.

The studied problems of this paper are the following:

- Estimation of the discrete state $j(x)$.
- Estimation of the continuous state $x(t)$.
- Estimation of bounds for the estimation error of the continuous state.

III. Preliminaries

Notation. The following notation is used. The pseudoinverse matrix of $F \in \Re^{n\times m}$ is defined as $F^+ \in \Re^{m\times n}$. Then, if $\text{rank}(F) = n \rightarrow FF^+ = I$, and if $\text{rank}(F) = m \rightarrow F^+F = I$. For a matrix $J \in \Re^{n\times m}$ with $n \geq m$ and $\text{rank}(J) = r$, $J^\perp \in \Re^{(n-r)\times n}$ is defined as a matrix such that $\text{rank}(J^\perp) = n-r$ and $J^\perp J = 0$. The matrix $J^{\perp\perp} \in \Re^{r\times n}$ corresponds to one of the matrices such that $\text{rank}(J^{\perp\perp}) = r$ and $J^\perp(J^{\perp\perp})^T = 0$. Its is clear that the matrices $J^\perp$ and $J^{\perp\perp}$ are not unique and that $\text{rank}\begin{bmatrix} (J^\perp)^T & (J^{\perp\perp})^T \end{bmatrix}^T = n$.

Some basic definitions for strong observability, strong detectability, invariant zeros and relative degree are recalled in this section (see, e.g. [13] and [14]).

Consider a linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \quad y = Cx(t) \tag{3}$$

where $x \in \Re^n$ is the state, $y \in \Re^m$ is the output, $w \in \Re^m$ is an unknown input and the known matrices A, C and E have corresponding dimensions. In this case, it can be assumed, without loss of generality, that the known input $u(t)$ is equal to zero.

Definition 1: *The system (3) is called strongly observable if for any initial condition $x(0)$ and for all unknown inputs $w(t)$, the identity $y(t) \equiv 0$ $\forall t \geq 0$ implies that also $x(t) \equiv 0$ $\forall t \geq 0$.*

Definition 2: *The system (3) is called strongly detectable if for any initial condition $x(0)$ and for all unknown inputs $w(t)$, the identity $y(t) \equiv 0$ $\forall t \geq 0$ implies that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Definition 3: *The complex number $s_0 \in \mathscr{C}$ is called an invariant zero of the triple (A,E,C) if $rank(R(s_0)) < n + rank(E)$, where $R(s)$ is the Rosenbrock matrix of system (1), i.e.*

$$R(s) = \begin{bmatrix} sI - A & -E \\ C & 0 \end{bmatrix}.$$

Fact 1: *The following statements are equivalent:*

i) The system (3) is strongly observable.

ii) The triple (A,E,C) has no invariant zeros.

Fact 2: *The following statements are equivalent:*

i) The system (3) is strongly detectable.

ii) The system is minimum-phase (i.e. the invariant zeros of the triple (A,E,C) satisfy $Re\{s\} < 0$).

Now, basing on the given statements, we can define the set of unstable invariant zeroes of the system (3) as the set of invariant zeros of the triple (A,E,C) satisfying $Re\{s\} \geq 0$. Moreover, notice that if there exist unstable invariant zeroes, then the system (3) is not strongly detectable.

In the case when $E = 0$, the notions of strong observability and strong detectability coincide with observability and detectability, respectively. Now, introduce the observability matrix

$$\text{O} = \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^{n-1})^T \end{bmatrix}^T.$$

Notice that system (3) is observable, independently of the unknown inputs, if and only if $\text{rank}(\text{O}) = n$. The unobservable subspace of the pair (A,C) is denoted by $\mathscr{N}$ and it is defined as $\mathscr{N} = \ker(\text{O})$.

Definition 4: *The scalar value $\lambda_0 \in \mathbb{C}$ is said to be an (A,C)-unobservable eigenvalue if*

$$rank\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n$$

Definition 5: *A point x_0 is called weakly unobservable if there exists an input $w(t)$, such that $y(t) \equiv 0$ $\forall t \geq 0$. The set of all weakly unobservable points of (3) is denoted by $\mathscr{V}^*$ and is called the weakly unobservable subspace of (3).*

The weakly unobservable subspace satisfies the following relations:

$$A\mathscr{V}^* \subset \mathscr{V}^* \oplus \mathscr{E}, \quad C\mathscr{V}^* = 0, \tag{4}$$

where $\mathscr{E}$ denotes the image of E. For any null-output $(A,E)-$invariant subspace, there exist a map $\bar{K} : \mathscr{X} \rightarrow \mathscr{W}$ such that

$$(A + E\bar{K})\mathscr{V}^* \subset \mathscr{V}^*, \quad C\mathscr{V}^* = 0. \tag{5}$$

Definition 6: *The output $y(t)$ is said to have a relative degree vector $(r_1, \ldots, r_m)$ with respect to the unknown input $w(t)$ if*

$$c_i A^k E = 0, \ \forall k < r_i - 1, \tag{6}$$

$$c_i A^{r_i - 1} E \neq 0, \ \forall i = 1, \ldots, m. \tag{7}$$

and

$$\det Q \neq 0, \quad Q = \begin{bmatrix} c_1 A^{r_1-1} E_1 & \cdots & c_1 A^{r_1-1} E_m \\ & \ddots & \\ c_m A^{r_m-1} E_1 & \cdots & c_m A^{r_m-1} E_m \end{bmatrix}, \tag{8}$$

where c_i is the $i-th$ row of matrix C, and E_j is the $j-th$ column of matrix E.

IV. System Transformation

Based on the previous definitions, the following assumption ensures the possibility for state estimation:

Assumption 1: *All the (A,C)-unobservable eigenvalues satisfy $Re\{\lambda\} < 0$.*

It is clear that, as a consequence of the Assumption 1, the system is detectable. Even more, the satisfaction of the above mentioned assumption ensures that the set of unstable invariant zeroes of (3) does not belong to the set of (A,C)-unobservable eigenvalues.

Now, a suitable transformation to decompose the system into the strongly observable part, the nonstrongly observable part and the unobservable part is applied to the system (1) (see, for example [15] and [14]).

For simplicity, there will be omitted the index of the discrete state. However, the transformation is applied for each of the dynamics generated by the discrete state.

Firstly, let us calculate a basis of the weakly unobservable subspace $\mathscr{V}^*$ by means of the computation of the matrices M_i defined by the following recursive algorithm[2]:

$$M_{i+1} = N_{i+1}^{\perp\perp} N_{i+1}, \quad M_1 = C,$$

$$N_{i+1} = G_i \begin{pmatrix} C \\ M_i A \end{pmatrix}, \quad G_i = \begin{pmatrix} 0_{p\times q} \\ M_i E \end{pmatrix}^{\perp}.$$

The algorithm ends when $\operatorname{rank}(M_{i+1}) = \operatorname{rank}(M_i)$. Therefore, it is possible to define $M_n = M_{n-1} = \ldots = M_i$. It was proven in [15] that $\mathscr{V}^* = \ker(M_n)$. Now, define $n_{\mathscr{V}} := \operatorname{rank}(M_n)$ with $M_n \in \Re^{n_{\mathscr{V}} \times n}$. Then, form the matrix $V \in \Re^{n\times(n-n_{\mathscr{V}})}$ whose columns form a basis of $\mathscr{V}^*$.

Secondly, assume that the following assumption is satisfied:

ASSUMPTION 2: *The output of the system (1) has a relative degree vector $(r_1,\ldots,r_m)$ such that $r_1+\cdots+r_m = n_{\mathscr{V}}$.*

According to Definition 6 and Assumption 2 it is possible to form the following matrix $U \in \Re^{n_{\mathscr{V}}\times n}$

$$U = \begin{bmatrix} c_1^T & (c_1A)^T & \cdots & (c_1A^{r_1-1})^T & \cdots \\ c_m^T & (c_mA)^T & \cdots & (c_mA^{r_m-1})^T \end{bmatrix}^T. \quad (9)$$

It is easy to see that $\operatorname{rank}(U) = n_{\mathscr{V}}$. Now, from the matrix U, form the following matrices $U_1 \in \Re^{n_{\mathscr{V}}-m\times n}$ and $U_2 \in \Re^{m\times n}$

$$U_1 = \begin{bmatrix} c_1^T & (c_1A)^T & \cdots & (c_1A^{r_1-2})^T & \cdots \\ c_m^T & (c_mA)^T & \cdots & (c_mA^{r_m-2})^T \end{bmatrix}^T, \quad (10)$$

$$U_2 = \begin{bmatrix} (c_1A^{r_1-1})^T & \cdots & (c_mA^{r_m-1})^T \end{bmatrix}^T. \quad (11)$$

Finally, form the matrix N whose columns form a basis of the unobservable subspace $\mathscr{N}$. It is clear by Definition 5 that $\mathscr{N} \subset \mathscr{V}^*$. Therefore, it is possible to select a full column rank matrix V forming a basis of $\mathscr{V}^*$ adapted to $\mathscr{N}$, i.e.

$$V = \begin{bmatrix} V_{\bar{\mathscr{N}}} & N \end{bmatrix}. \quad (12)$$

Define $n_{\mathscr{N}} = \dim(\mathscr{N})$. Then, $V_{\bar{\mathscr{N}}} \in \Re^{n\times(n-n_{\mathscr{V}}-n_{\mathscr{N}})}$ and $N \in \Re^{n\times n_{\mathscr{N}}}$. Moreover, matrix V satisfied the following equalities

$$AV + EK^* = VQ \quad \Leftrightarrow \quad (A+E\bar{K}^*)V = VQ, \quad (13)$$

$$CV = 0, \quad (14)$$

[2]The matrix $N_{i+1}^{\perp\perp}$ is introduced to excluded the linearly dependent terms of N_{i+1}. Therefore, M_{i+1} has full row rank (see [14]).

for some matrices $\bar{K}^* \in \Re^{m\times n}$, $K^* \in \Re^{m\times(n-n_{\mathscr{V}})}$ and $Q \in \Re^{(n-n_{\mathscr{V}})\times(n-n_{\mathscr{V}})}$. Notice that (13)-(14) are the matrix representations of the map (5), and that $V^+V = I$ implies $\bar{K}^* = K^*V^+$, satisfying (13).

The following nonsingular transformation matrix can be defined:

$$T := \begin{bmatrix} U_1^T & U_2^T & (V_{\bar{\mathscr{N}}}{}^+)^T & (N^+)^T \end{bmatrix}^T. \quad (15)$$

Then, the transformation $\bar{x}(t) = Tx(t)$, with matrix T designed according to (15), transforms the system (1) into the following form:

$$\begin{bmatrix} \dot{\bar{x}}_{11}(t) \\ \dot{\bar{x}}_{12}(t) \\ \dot{\bar{x}}_{21}(t) \\ \dot{\bar{x}}_{22}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \\ \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} \bar{w}(t), \quad (16)$$

$$y(t) = C_1 \begin{bmatrix} \bar{x}_{11}(t)^T & \bar{x}_{12}(t)^T \end{bmatrix}^T, \quad (17)$$

$$\bar{w}(t) = w(t) - K_1^* \bar{x}_{21}(t), \quad (18)$$

where $\bar{x}_{11}(t) \in \Re^{n_{\mathscr{V}}-m}$, $\bar{x}_{12}(t) \in \Re^m$, $\bar{x}_{21}(t) \in \Re^{n-n_{\mathscr{V}}-n_{\mathscr{N}}}$, $\bar{x}_{22}(t) \in \Re^{n_{\mathscr{N}}}$, $K_1^* \in \Re^{m\times(n-n_{\mathscr{V}}-n_{\mathscr{N}})}$ and

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} = T(A+E\bar{K}^*)T^{-1},$$

$$\begin{bmatrix} 0 \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} = TE, \quad C_1 = C \begin{bmatrix} (U_1^+)^T & (U_2^+)^T \end{bmatrix}^T,$$

$$K^* = \begin{bmatrix} K_1^* & 0 \end{bmatrix}, \ \begin{bmatrix} B_1^T & B_2^T & B_3^T & B_4^T \end{bmatrix}^T = TB.$$

For the system (16)-(18) it is possible to demonstrate that the set of invariant zeros that do not belong to unobservable subspace $\mathscr{N}$ is equal to the set of eigenvalues of matrix A_{33}, and, the set of invariant zeros that belong to unobservable subspace $\mathscr{N}$ is equal to the set of eigenvalues of matrix A_{44} (see, for example [16]).

Notice that, none assumption has been made on the eigenvalues of the matrix A_{33}. However, in this work we study the case when at least one of the eigenvalues of the matrix A_{33} satisfy $Re\{\lambda\} \geq 0$. Solution to the problem when all the eigenvalues of A_{33} are stable have been already reported, see e.g. [14], [16].

V. OBSERVER DESIGN

Consider the state estimation problem for a constant index $j(x) = j^* = const$. Let us describe the observer design for each partition of the state $\bar{x}(t)$. The proofs of all theorems and propositions are omitted for lack of space.

A. STATE OBSERVER FOR $\bar{x}_{11}(t)$ AND $\bar{x}_{12}(t)$

Consider the first two parts of the system (16)-(18) for $j(x) = j^*$ with the partition $\bar{x}_1(t) = \left[\begin{array}{cc} \bar{x}_{11}(t)^T & \bar{x}_{12}(t)^T \end{array}\right]^T$, i.e.

$$\begin{array}{l} \dot{\bar{x}}_1(t) = A_{1j^*}\bar{x}_1(t) + E_{1j^*}\bar{x}_1(t) + B_{12j^*}u(t), \\ y(t) = C_{1j^*}\bar{x}_1(t), \end{array} \tag{19}$$

where

$$A_{1j^*} = \left[\begin{array}{cc} A_{11j^*} & A_{12j^*} \\ A_{21j^*} & A_{22j^*} \end{array}\right], E_{1j^*} = \left[\begin{array}{c} 0 \\ E_{12j^*} \end{array}\right], B_{12j^*} = \left[\begin{array}{c} B_{1j^*} \\ B_{2j^*} \end{array}\right]$$

In accordance with the structure of the transformation, the states $\bar{x}_1(t)$ forms the strongly observable subspace. Then, the following observer for $\bar{x}_1(t)$ could be designed (see [14])

$$\hat{x}_{1_{j^*}}(t) = z_{1_{j^*}}(t) + P_{1j^*}^{-1}\nu_{j^*}(t) \tag{20}$$

$$\begin{aligned} \dot{z}_{1_{j^*}}(t) &= A_{1j^*}z_{1j^*}(t) + B_{12j^*}u(t) \\ &+ L_{1j^*}\left(y(t) - C_{1j^*}z_{1j^*}(t)\right), \end{aligned} \tag{21}$$

$$\dot{\nu}_{j^*}(t) = W_{j^*}(y(t) - C_{1j^*}z_{1t^*}(t), \nu_{j^*}(t)), \tag{22}$$

where $z_{1_{j^*}}(t)$,$\hat{x}_{1_{j^*}}(t) \in \Re^{n_{\mathscr{V}}}$ and the matrix $L_{1j^*} \in \Re^{n_{\mathscr{V}} \times m}$ is selected such that matrix $\left(A_{1j^*} - L_{1j^*}C_{1j^*}\right) = \tilde{A}_{1_{j^*}}$ is Hurwitz[3]. The distribution matrix P_{1j^*} takes the following structure

$$\begin{aligned} P_{1j^*} = \Big[\; & c_{1j^*}^T \quad (c_{1j^*}\tilde{A}_{1_{j^*}})^T \quad \cdots \quad (c_{1j^*}\tilde{A}_{1_{j^*}}^{r_1-1})^T \quad \cdots \\ & c_{mj^*}^T \quad (c_{mj^*}\tilde{A}_{1_{j^*}})^T \quad \cdots \quad (c_{mj^*}\tilde{A}_{1_{j^*}}^{r_m-1})^T \;\Big]^T. \end{aligned} \tag{23}$$

According to Assumption 2 the condition $\text{rank}(P_{1j^*}) = n_{\mathscr{V}}$ is satisfied. The vector $\nu_{j^*}(t) = \left(\nu_{1_{j^*}}, \ldots, \nu_{n_{\mathscr{V}}j^*}\right)$ and the nonlinear function W_{j^*} are chosen using the HOSM differentiator (for more details, see [17])

$$\begin{aligned} &\dot{\nu}_{1_{j^*}} = \nu_{2_{j^*}} \\ &\quad -\alpha_{1_{j^*}}M_{j^*}^{\frac{1}{n_{\mathscr{V}}+1}}(t)\left|\nu_{1_{j^*}} - e_{y_{j^*}}\right|^{\frac{n_{\mathscr{V}}}{n_{\mathscr{V}}+1}}\text{sign}\left(\nu_{1_{j^*}} - e_{y_{j^*}}\right), \\ &\dot{\nu}_{2_{j^*}} = \nu_{3_{j^*}} \\ &\quad -\alpha_{2_{j^*}}M_{j^*}^{\frac{1}{n_{\mathscr{V}}}}(t)\left|\nu_{2_{j^*}} - \dot{\nu}_{1_{j^*}}\right|^{\frac{n_{\mathscr{V}}-1}{n_{\mathscr{V}}}}\text{sign}\left(\nu_{2_{j^*}} - \dot{\nu}_{1_{j^*}}\right), \\ &\quad\quad \vdots \\ &\dot{\nu}_{(n_{\mathscr{V}}-1)_{j^*}} = \nu_{n_{\mathscr{V}}j^*} \\ &\quad -\alpha_{(n_{\mathscr{V}}-1)_{j^*}}M_{j^*}^{\frac{1}{2}}(t)\left|\nu_{(n_{\mathscr{V}}-1)_{j^*}} - \dot{\nu}_{(n_{\mathscr{V}}-2)_{j^*}}\right|^{\frac{1}{2}} \\ &\quad \times\text{sign}\left(\nu_{(n_{\mathscr{V}}-1)_{j^*}} - \dot{\nu}_{(n_{\mathscr{V}}-2)_{j^*}}\right), \\ &\dot{\nu}_{n_{\mathscr{V}}j^*} = -\alpha_{n_{\mathscr{V}}j^*}M_{j^*}(t)\text{sign}\left(\nu_{n_{\mathscr{V}}j^*} - \dot{\nu}_{(n_{\mathscr{V}}-1)_{j^*}}\right), \end{aligned} \tag{24}$$

where $e_{y_{j^*}}(t) = y(t) - C_1z_{1j^*}(t)$ and the constants $\alpha_{k_{j^*}}$ are chosen recursively and sufficiently large. In particular, according to [17], one possible choice is $\alpha_{6_{j^*}} = 1.1$, $\alpha_{5_{j^*}} = 1.5$, $\alpha_{4_{j^*}} = 2$, $\alpha_{3_{j^*}} = 3$, $\alpha_{2_{j^*}} = 5$, $\alpha_{1_{j^*}} = 8$, which is enough for the case that $n_{\mathscr{V}} \leq 6$.

The continuous function $M_{j^*}(t)$ is a known locally Lipschitz constant, at time t, for $\dot{\bar{x}}_{12}(t)$, and it can be computed in the following way.

[3]Due to Assumption 2 and Definition 6 such matrix L_{1j^*} always exist.

ASSUMPTION 3: *The set of eigenvalues of matrix $A_{22_{j^*}}$ are stable.*

PROPOSITION 1: *Let Assumption 3 be satisfied. Then, there exist a time $\tilde{t}$ and known positive constants β_{1,j^*}, β_{2,j^*}, β_{3,j^*}, β_{4,j^*} and λ_{j^*} such that*

$$\left\|\dot{\bar{x}}_{12}(t)\right\| \leq M_{j^*}(t), \quad \forall t > \tilde{t}, \tag{25}$$

with

$$\begin{aligned} M_{j^*}(t) &= \beta_{1,j^*} + \left\|A_{21j^*}\right\|\bar{x}_{11M}(t) + \left\|B_{2j^*}\right\|u_M(t) \\ &+ \left\|E_{12j^*}\right\|\bar{w}_M(t) + \beta_{2,j^*}\int_0^t \exp(-\lambda_{j^*}(t-\tau))\bar{x}_{11M}(\tau)d\tau \\ &+ \beta_{3,j^*}\int_0^t \exp(-\lambda_{j^*}(t-\tau))u_M(\tau)d\tau \\ &+ \beta_{4,j^*}\int_0^t \exp(-\lambda_{j^*}(t-\tau))\bar{w}_M(\tau)d\tau, \end{aligned} \tag{26}$$

where $\bar{x}_{11M}(t) = \max\|\bar{x}_{11}(t)\|$, $u_M(t) = \max\|u(t)\|$ *and* $\bar{w}_M(t) = \max\|\bar{w}(t)\|$ *and* $\exp(A_{22_{j^*}})(t - t_0)) \leq k_{1_{j^*}}\exp(-\lambda_{j^*}(t-t_0))$.

Taking into account the previous explanations, the following theorem can be stated.

THEOREM 1: *Let $j(x) = j^* = const$, and the observer (20)-(22) with the correction terms (24) be applied to the system (19). Let Assumptions 1 - 3 be satisfied. Then, provided that constants $\alpha_{k_{j^*}}$ are chosen properly and M_{j^*} is selected as in Proposition 1, the state estimation error for $\bar{x}_1(t)$ converges to zero exactly and in a finite time, i.e. $e_{1_{j^*}}(t) = \bar{x}_1(t) - \hat{x}_{1_{j^*}}(t) = 0$ $\forall t \in [t_{i^*}, t_1)$*[4].

B. STATE OBSERVER FOR $\bar{x}_{21}(t)$

Let $\hat{x}_{21_{j^*}}(t)$ be the state observer for $\bar{x}_{21}(t)$ defined by

$$\hat{x}_{21_{j^*}}(t) = z_{2j^*}(t) + L_{2j^*}\hat{x}_{12_{j^*}}(t), \tag{27}$$

$$\begin{aligned} \dot{z}_{2_{j^*}}(t) &= \bar{A}_{1_{j^*}}\hat{x}_{11_{j^*}}(t) + \bar{A}_{2_{j^*}}\hat{x}_{12_{j^*}}(t) \\ &\quad + A_{L_2j^*}\hat{x}_{21_{j^*}}(t) + \bar{B}_{2_{j^*}}u(t), \end{aligned} \tag{28}$$

where the estimations of $\hat{x}_{11_{j^*}}(t)$ and $\hat{x}_{12_{j^*}}(t)$ are provided by observer (20)-(22). The matrices in (28) are defined by

$$\bar{A}_{1_{j^*}} = A_{31j^*} - L_{2j^*}A_{21j^*}, \tag{29}$$

$$\bar{A}_{2_{j^*}} = A_{32j^*} - L_{2j^*}A_{22j^*}, \tag{30}$$

$$A_{L_2j^*} = A_{33j^*} - K_{1\,j^*}^*E_{21j^*} + L_{2j^*}E_{12j^*}K_{1\,j^*}^*, \tag{31}$$

$$\bar{B}_{2_{j^*}} = B_{3j^*} - L_{2j^*}B_{2j^*}. \tag{32}$$

Then, the following theorem can be stated.

THEOREM 2: *Let $j(x) = j^* = const$ be satisfied. Then, provided that the matrix L_{2j^*} is selected such that the matrix $A_{L_2j^*}$ is Hurwitz*[5]*, the state estimation error for $\bar{x}_{21}(t)$ is ultimately bounded by a constant $\gamma_{j^*}w^+$, i.e. $\left\|e_{21_{j^*}}(t)\right\| = \left\|\bar{x}_{21}(t) - \hat{x}_{21_{j^*}}(t)\right\| \leq \gamma_{j^*}w^+$ as $t \to \infty$.*

[4]t_{i^*} is the time when the observer (20)-(22) has converged to zero, and t_1 is the first switching time.

[5]Such matrix L_{2j^*} always exist if the pair $\left(A_{33j^*} - K_{1\,j^*}^*E_{21j^*}, E_{12j^*}K_{1\,j^*}^*\right)$ is detectable and it is possible to compute it using robust pole assignment or LMI solvers.

C. State Observer for $\bar{x}_{22}(t)$

Without loss of generality it is assumed that

$$\mathrm{rank}(E_{2\,j^*}) = \mathrm{rank}\begin{bmatrix} E_{12\,j^*} \\ E_{21\,j^*} \end{bmatrix} = m.$$

Let $\hat{x}_{22_{j^*}}(t)$ be the state observer for $\bar{x}_{22}(t)$ defined by

$$\hat{x}_{22_{j^*}}(t) = z_{3\,j^*}(t) + E_{22\,j^*}E_{2\,j^*}^{+}\begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{21}(t) \end{bmatrix}, \qquad (33)$$

$$\begin{aligned}\dot{z}_{3_{j^*}}(t) &= A_{41_{j^*}}\hat{x}_{11_{j^*}}(t) + A_{42_{j^*}}\hat{x}_{12_{j^*}}(t) + A_{43_{j^*}}\hat{x}_{21_{j^*}}(t) \\ &\quad + A_{44_{j^*}}\hat{x}_{22_{j^*}} + B_{4\,j^*}u_{j^*}(t) \\ &\quad - E_{22\,j^*}E_{2\,j^*}^{+}\begin{bmatrix} \Big(A_{21_{j^*}}\hat{x}_{11_{j^*}}(t) + A_{22_{j^*}}\hat{x}_{12_{j^*}}(t) \\ +B_{2_{j^*}}u(t)\Big) \\ \Big(A_{31_{j^*}}\hat{x}_{11_{j^*}}(t) + A_{32_{j^*}}\hat{x}_{12_{j^*}}(t) \\ +A_{33_{j^*}}\hat{x}_{21_{j^*}}(t) + B_{3_{j^*}}u(t)\Big) \end{bmatrix}, \end{aligned} \qquad (34)$$

where the estimations of $\hat{x}_{11_{j^*}}(t) - \hat{x}_{12_{j^*}}(t)$ and $\hat{x}_{21_{j^*}}(t)$ are provided by observers (20)-(22) and (27)-(28), respectively.

Then, the following theorem can be stated.

Theorem 3: *Let $j(x) = j^* = const$ and $rank(E_{2\,j^*}) = m$ be satisfied. Then, the state estimation error for $\bar{x}_{22}(t)$ is ultimately bounded by a constant $\delta_{j^*}w^+$, i.e. $\left\| e_{22_{j^*}}(t)\right\| = \left\|\bar{x}_{22}(t) - \hat{x}_{22_{j^*}}(t)\right\| \leq \delta_{j^*}w^+$ as $t \to \infty$.*

To solve the continuous state estimation problem for the non-minimum phase system (1), the following bank of observers is proposed

$$\tilde{x}_{1_\lambda}(t) = z_{1\lambda}(t) + P_{1\,\lambda}^{-1}\nu_\lambda(t), \qquad (35)$$

$$\tilde{x}_{21_\lambda}(t) = z_{2\lambda}(t) + L_{2\lambda}\tilde{x}_{12_\lambda}(t), \qquad (36)$$

$$\tilde{x}_{22_\lambda}(t) = z_{3\lambda}(t) + E_{22\lambda}E_{2\,\lambda}^{+}\begin{bmatrix} \tilde{x}_{12}(t) \\ \tilde{x}_{21}(t) \end{bmatrix}, \qquad (37)$$

$$\tilde{y}(t) = C_{1\lambda}\tilde{x}_1(t),\ \forall\lambda = 1,\ldots,N, \qquad (38)$$

where $\tilde{x}_{1_\lambda}(t)$, $\tilde{x}_{21_\lambda}(t)$, $\tilde{x}_{22_\lambda}(t)$ and theirs components are designed according to Section V. Now, the following assumptions are stated.

Assumption 4: *Assume that there exists a number $\tau_d > 0$ such that the switchings times $t_1, t_2, \ldots$ satisfy the inequality $t_{j+1} - t_j \geq \tau_d$ for all j.*

Assumption 5: *Assume that the initial discrete state is known.*

Now, Theorems 1, 2 and 3 establish that the j^* observer reconstructs the continuous state correctly, and according to Assumption 5 we know the j^* observer has made it on the interval $[0, t_1)$. According to [18], to know when the j^* observer has converged it is sufficient to verify that the following inequality is satisfied

$$\left| e_{\tilde{y}_{j^*}}(t)\right| \leq \xi_{j^*}M_{j^*}^{+}h^{n_{\mathscr{V}}},\ \forall t \in \left[t_1 - \xi_{j^*_t}h, t_1\right), \qquad (39)$$

where $e_{\tilde{y}_{j^*}} = y - \tilde{y}_{j^*}$, ξ_{j^*} and $\xi_{j^*_t}$ are positive constants, h is the sampling time and $M_{j^*}^{+} = \max(M_{j^*}(t))$. It is natural to estimate the constants ξ_{i^*} and $\xi_{i^*_t}$ through simulation. Thus, in this way it is possible to determine when the $j^* - th$ observer has converged to the correct continuous state during the time interval $t \in [0, t_1)$.

Then, the real estimated state $\hat{x}$ is defined as follows:

$$\hat{x} = \tilde{x}_{j^*},\quad \forall t \in [t_{j^*}, t_1).^6 \qquad (40)$$

Due to the transformation $\bar{x}(t) = Tx(t)$, the bank of observers for the original state vector has to be designed as:

$$\hat{x}(t) = T_\lambda^{-1}\begin{bmatrix} \tilde{x}_{11_\lambda}^T(t) & \tilde{x}_{12_\lambda}^T(t) & \tilde{x}_{21_\lambda}^T(t) & \tilde{x}_{22_\lambda}^T(t) \end{bmatrix}^T. \qquad (41)$$

Theorem 4: *The original state estimation error generated by the observer (41) and system (1) is ultimately bounded by a positive constant $\rho_\lambda w^+$, i.e.*

$$\hat{x}(t) - x(t) \underset{t\to\infty}{\to} \rho_\lambda w^+. \qquad (42)$$

VI. Discrete State Estimation and State Estimation On Switching Times

Once the continuous state is estimated, the following discrete state observer is proposed:

$$\hat{\lambda}(\hat{x}) = \begin{cases} 1, & \forall\hat{x} \mid \quad \rho_0 > H\hat{x} \geq \rho_1 \\ \vdots & \\ N, & \forall\hat{x} \mid \quad \rho_{N-1} > H\hat{x} \geq \rho_N \end{cases}, \qquad (43)$$

For each discrete state j define the observability matrix of the discrete state as

$$Q_j = \begin{bmatrix} H^T & (HA_j)^T & \cdots & (HA_j^{n-1})^T \end{bmatrix}^T.$$

The exactly and finite time discrete state estimation is described by the following theorem.

Theorem 5: *The discrete state $j(x)$ is estimated exactly and in finite time if the following condition is satisfied*

$$\mathscr{V}_j^* \subset ker(Q_j),\ \forall j = 1,\ldots,N. \qquad (44)$$

Notice that an exact estimation of the discrete stated is achieved only if the scalar function $j(x)$ can be represented as a combination of the strongly observable states, i.e. $\bar{x}_{11}$ and $\bar{x}_{12}$.

Let $t_i{}^+$ be the time instants after the switching times t_i. In order to maintain the state estimation on the switching times the following proposition is done.

Proposition 2: *The state estimation of system is maintained in spite of the switchings if the following reset equations are implemented in the bank of observers (35)-(38) for all $\lambda \neq j^*$*

$$\nu_\lambda(t_i{}^+) \quad = \quad 0,\quad z_{1_\lambda}(t_i{}^+) = \tilde{x}_{1_{j^*}}(t_i), \qquad (45)$$

$$z_{2_\lambda}(t_i{}^+) \quad = \quad \tilde{x}_{21_{j^*}}(t_i) - L_{2_{j^*}}\tilde{x}_{12_{j^*}}(t_i), \qquad (46)$$

$$z_{3_\lambda}(t_i{}^+) \quad = \quad \tilde{x}_{22_{j^*}}(t_i) - E_{22\,j^*}E_{2\,j^*}^{+}\begin{bmatrix} \tilde{x}_{12}(t_i) \\ \tilde{x}_{21}(t_i) \end{bmatrix}. \qquad (47)$$

[6]Notice that it is always possible to design the gain $M_{j^*}(t)$ in such a way that inequality $t_{j^*} < t_1$ is satisfied.

VII. Simulation Results

Consider the following switched linear system to illustrate the proposed approach. The discrete state is given by

$$j(x)=\begin{cases} 1, & \forall x \mid 20000 > Hx > -50 \\ 2, & \forall x \mid -50 > Hx > -10000 \end{cases},$$

with $Hx=-2x_1(t)+5x_2(t)$. The following matrices correspond to the dynamics equations (1)

$$A_1=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}, A_2=\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & -1 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, B_2=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, E_1=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, E_2=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, C_2=\begin{bmatrix} 1.5 & 0 & 0 & 0 \end{bmatrix}.$$

The output, known and unknown inputs are given by $y=C_{j(x)}x_1(t)$, $u(t)=5\sin t$ and $w(t)=\cos(2t)+0.5$, respectively. The system initial conditions are set to $x(0)=\begin{bmatrix} 2 & 3 & 2 & 1 \end{bmatrix}^T$ and $i(x(0))=1$. Simulations have been done in the MATLAB Simulink environment, with the Euler discretization method and sampling time $h=0.0001[sec]$. It is possible to show that all assumptions stated along the paper are satisfied. The values of the designed matrices are the following

$$T_1=I_{4\times 4},\ T_2=\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$K_{1\,1}^*=K_{1\,2}^*=\begin{bmatrix} 1 & 0 \end{bmatrix},\ \bar{K}_1^*=\bar{K}_2^*=\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

$$L_{11}=\begin{bmatrix} 33 & 267 \end{bmatrix}^T, L_{12}=\begin{bmatrix} 22 & -178 \end{bmatrix}^T,$$

$$P_{11}=\begin{bmatrix} 1 & 0 \\ -34 & 1 \end{bmatrix}, P_{12}=\begin{bmatrix} 1.5 & 0 \\ -51 & -1.5 \end{bmatrix},$$

$$L_{21}=-5,\ L_{22}=-10.$$

The HOSM differentiators in (24) are designed for $n_{\mathscr{V}}=2$ with M_1 and M_2 according to Proposition 1. The reset equations in Proposition 2 are implemented. The continuos state estimation results are depicted in Fig. 1.

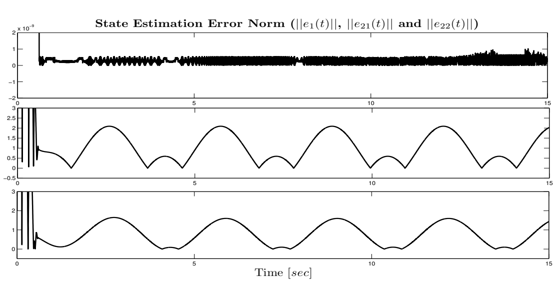

Fig. 1. State Estimation Error, Euclidian Norm

VIII. Conclusions

A state observer is proposed that is capable to estimate the continuous and discrete state in the presence of unknown inputs for linear switched systems. The system under study may contain unstable invariant zeros. The ultimate bounds on the estimation errors are provided for the case when the exact reconstruction of the state cannot be achieved. Simulation results support the developed estimation approach.

IX. Acknowledgments

, The authors gratefully acknowledge the financial support from CONACyT grants 132125, 151855 and CVU-270504, PAPIIT-UNAM grant 111211, SIP-IPN grant 20120218 and PAEP-IPN.

References

[1] A. Alessandri and P. Coletta, "Design of observers for switched discretetime linear systems," in *Proceedings of the 2003 American Control Conference*, Denver, CO, USA, 2003, pp. 2785–2790.

[2] F. Bejarano, A. Pisano, and E. Usai, "Finite-time converging jump observer for switched linear systems with unknown inputs," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 174–188, 2011.

[3] L. Pina and M. Botto, "Simultaneous state and input estimation of hybrid systems with unknown inputs," *Automatica*, vol. 42, pp. 1755–762, 2006.

[4] N. Orani, A. Pisano, M. Franceschelli, A. Giua, and E. Usai, "Robust reconstruction of the discrete state for a class of nonlinear uncertain switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 220–232, 2011.

[5] F. Bejarano and A. Pisano, "Switched observers for switched linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 681–686, 2011.

[6] L. Wu, P. Shi, and H. Gao, "State estimation and sliding-mode control of markovian jump singular systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1213–1219, 2010.

[7] J. Barbot, H. Saadaoui, M. Djemai, and N. Manamanni, "Nonlinear observer for autonomous switching systems with jumps," *Nonlinear Analysis: Hybrid Systems*, vol. 1, no. 4, pp. 537–547, 2007.

[8] M. Fliess, C. Join, and W. Perruquetti, "Real-time estimation of the switching signal for perturbed switched linear systems," in *Proceedings of the 3rd IFAC Conference on Analysis and Design of Hybrid Systems*, Zaragoza, Spain, 2009, pp. 409–414.

[9] E. De-Santis, M. Di-Benedetto, and G. Pola, "A structural approach to detectability for a class of hybrid systems," *Automatica*, vol. 45, pp. 1202–1206, 2009.

[10] S. Chaib, D. Boutat, A. Benali, and F. Kratz, "Failure detection and reconstruction in switched nonlinear systems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, pp. 225–238, 2009.

[11] L. Wu and D. Ho, "Sliding mode control of singular stochastic hybrid systems," *Automatica*, vol. 46, pp. 779–783, 2010.

[12] Y. B. Shtessel, S. Baev, C. Edwards, and S. Spurgeon, "Hosm observer for a class of non-minimum phase causal nonlinear mimo systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 543–548, February 2010.

[13] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control theory for linear systems*. London, Great Britain: Springer-Verlag, 2001.

[14] L. Fridman, J. Davila, and A. Levant, "High-order sliding-mode observation for linear systems with unknown inputs," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 189–205, 2011.

[15] B. P. Molinari, "A strong controllability and observability in linear multivariable control," *IEEE Transactions on Automatic Control*, vol. 21, no. 5, pp. 761–764, 1976.

[16] F. Bejarano, "Partial unknown input reconstruction for linear systems," *Automatica*, vol. 47, no. 8, pp. 1751–1756, 2011.

[17] A. Levant, "High-order sliding modes: differentiation and output-feedback control," *International Journal of Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[18] M. Angulo, L. Fridman, and A. Levant, "Robust exact finite-time output based control using high-order sliding modes," *International Journal of System Science*, vol. 42, no. 11, pp. 1847–1857, 2011.