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Abstract: We prove the existence of solutions of a coupled PDE-ODE system modeling the interaction of a large slow moving vehicle with the surrounding traffic flow. The model consists in a scalar conservation law with moving density constraint describing traffic evolution coupled with an ODE for the slow vehicle trajectory. The constraint location moves due to the surrounding traffic conditions, which in turn are affected by the presence of the slower vehicle, thus resulting in a strong non-trivial coupling.

Key-words: Scalar conservation laws with constraints; Traffic flow modeling; PDE-ODE coupling; Wave-front tracking approximations.

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Loi de conservation scalaire avec des contraintes mobiles dans la modélisation du trafic routier

Résumé : Le but de ce travail est d'étudier un modèle couplé de EDP-EDO et démontrer l'existence des solutions. Le modèle a été introduit pour décrire un goulot d'étranglement mobile. En particulier, il peut décrire le mouvement d'un poids lourd ou un bus, qui roule à une vitesse inférieure à celle des autres voitures, en réduisant la capacité de la route et générant ainsi un goulot d'étranglement.
On peut modéliser cette situation d'un point de vue macroscopique par un système couplant une EDP avec une EDO. L'EDP est une loi de conservation scalaire avec une contrainte mobile sur la densité et l'EDO décrive la trajectoire du véhicule plus lent.
Le bus se déplace avec une vitesse qui dépend du trafic routier, c'est-à-dire que le bus voyage à une vitesse constante tant qu'il n'est pas ralenti par les conditions de circulation en aval. Lorsque cela se produit, il adapte sa vitesse à la vitesse moyenne du trafic environnant. A son tour, la circulation est modifiée par la présence du véhicule plus lent. Il y a donc un couplage fort et non trivial.

Mots-clés : Lois de conservation scalaires avec contraintes; Modélisation du trafic routier; Couplage EDP-EDO; Méthode du suivi de fronts.

1 Introduction

A slow moving large vehicle, like a bus or a truck, reduces the road capacity and thus generates a moving bottleneck for the surrounding traffic flow. From the macroscopic point of view this can be modeled by a PDE-ODE coupled system consisting in a scalar conservation law with moving density constraint and an ODE describing the slower vehicle motion, i.e.,

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t,y(t)) \leq \alpha R, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases} \tag{1.1}$$

Above, $\rho = \rho(t,x) \in [0,R]$ is the scalar conserved quantity representing the mean traffic density, R is the maximal density allowed on the road and the flux function $f : [0,R] \to \mathbb{R}^+$ is a strictly concave function such that $f(0) = f(R) = 0$. It is given by the following flux-density relation

$$f(\rho) = \rho v(\rho),$$

where v is a smooth decreasing function denoting the mean traffic speed and here set to be $v(\rho) = V(1 - \frac{\rho}{R})$, V being the maximal velocity allowed on the road.
The time-dependent variable y denotes the slower vehicle position, that moves with a traffic density dependent speed of the form

$$\omega(\rho) = \begin{cases} V_b & \text{if } \rho \leq \rho^* \doteq R(1 - \frac{V_b}{V}), \\ v(\rho) & \text{otherwise,} \end{cases} \tag{1.2}$$

that is, it moves with constant speed $V_b < V$ as long as it is not slowed down by downstream traffic conditions. When this happens, it moves with the mean traffic speed.
Finally, the constant coefficient $\alpha \in]0,1[$ gives the reduction rate of the road capacity due to the presence of this large vehicle.

For our analytical purposes, it is not restrictive to assume that $R = V = 1$, so that the model becomes

$$\begin{cases} \partial_t \rho + \partial_x (\rho(1-\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t,y(t)) \leq \alpha, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases} \tag{1.3}$$

The above model was introduced in [10] to model the effect of urban transport systems, such as buses, in a road network. Other macroscopic models for moving bottlenecks in road traffic were recently proposed by [3, 12]. Compared to those approaches, the model described by (1.1) offers a more realistic definition of the slower vehicle speed and a description of its impact on traffic conditions which is simpler to handle both from the analytical and the numerical point of view.

From the analytical point of view, model (1.1) can be viewed as a generalization to moving constraints of the problem consisting in a scalar conservation law with a (fixed in space) constraint on the flux, introduced and studied in [1, 8]. In the present case, the constraint location moves due to the surrounding traffic conditions, which in turn are modified by the presence of the slower vehicle, thus resulting in a strong non-trivial coupling between the conservation equation and the trajectory of the vehicle.

The study of coupled PDE-ODE systems is not new in the conservation laws framework, we refer the reader to [2, 7, 9, 12]. Nevertheless, the problem posed here is slightly different. On one side, we deal with a strong coupling with the PDE and the ODE affecting each other, unlike [7, 9], where the PDE solution does not depend on the ODE. On the other side, even if the ODE has discontinuous right-hand side, the particular definition of the model allows us to consider classical Carathéodory solutions as in [2, 3, 4, 7] instead of the weaker Filippov's generalized solutions needed in [9, 12].

The present paper provides an existence result for solutions of (1.1) constructed by wave-front tracking approximations, as stated by Theorem 1 in Section 3. The continuous dependence on initial data is currently under investigation.

2 The Riemann problem with moving density constraint

Consider (1.3) with the particular choice

$$y_0 = 0 \quad \text{and} \quad \rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0. \end{cases} \tag{2.1}$$

We aim at defining a Riemann solver for the conservation law with moving density constraint. Therefore we consider the following Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, \\ \rho(0,x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0, \end{cases} \end{cases} \tag{2.2}$$

under the constraint

$$\rho(t, V_b t) \le \alpha. \tag{2.3}$$

Let $f_\alpha : [0,\alpha] \to \mathbb{R}^+$ be the function describing the constrained flow at $x = y(t)$, i.e.,

$$f_\alpha(\rho) = \rho \left(1 - \frac{\rho}{\alpha}\right),$$

and $\rho_\alpha \in \,]0, \alpha/2[$ such that $f'(\rho_\alpha) = V_b$, i.e.,

$$\rho_\alpha = \frac{\alpha}{2}\left(1 - V_b\right).$$

Problem (2.2), (2.3) can be recasted in the framework of conservation laws with flux constraint studied in [1, 8]. Rewriting the equations in the bus reference frame (setting $X = x - V_b t$), we get

$$\begin{cases} \partial_t \rho + \partial_X \left(f(\rho) - V_b \rho\right) = 0, \\ \rho(0,x) = \begin{cases} \rho_L & \text{if } X < 0, \\ \rho_R & \text{if } X > 0, \end{cases} \end{cases} \tag{2.4}$$

under the constraint

$$\rho(t, 0) \le \alpha. \tag{2.5}$$

Remark that solving problem (2.4), (2.5) is equivalent to solving (2.4) under the corresponding constraint on the flux

$$f(\rho(t,0)) - V_b \rho(t,0) \le f_\alpha(\rho_\alpha) - V_b \rho_\alpha \doteq F_\alpha.$$

We are now ready to define the Riemann solver for (1.3), (2.1) following [10, §V]. Denote by $\mathcal{R}$ the standard Riemann solver (i.e., without the constraint (2.3)) for (2.2), i.e., the (right continuous) map $(t,x) \mapsto \mathcal{R}(\rho_L, \rho_R)(\frac{x}{t})$ is the standard weak entropy solution to (2.2). Moreover, let $\check{\rho}_\alpha$ and $\hat{\rho}_\alpha$, with $\check{\rho}_\alpha \le \hat{\rho}_\alpha$, be the intersections of the flux function $f(\rho)$ and the line $f_\alpha(\rho_\alpha) + V_b(\rho - \rho_\alpha)$ (see Figure 1).

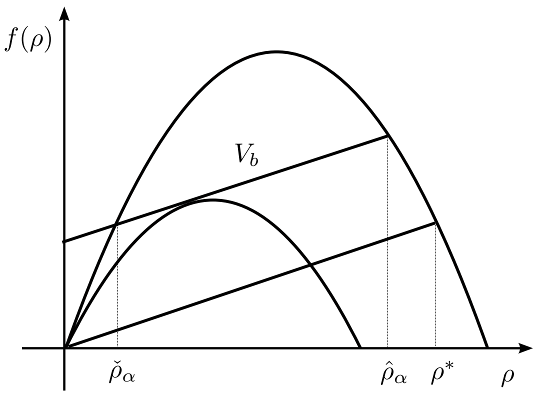

Figure 1: Flux function

Definition 1 *The constrained Riemann solver $\mathcal{R}^\alpha$ for* (1.3), (2.1) *is defined as follows.*

1. *If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) > F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R)(x) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho}_\alpha) & \text{if } x < V_b t, \\ \mathcal{R}(\check{\rho}_\alpha, \rho_R) & \text{if } x \ge V_b t, \end{cases} \qquad \text{and} \quad y(t) = V_b t.$$

2. *If $V_b \mathcal{R}(\rho_L, \rho_R)(V_b) \le f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \le F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = V_b t.$$

3. *If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = v(\rho_R) t.$$

Note that, when the constraint is enforced (point 1. in the above definition), a nonclassical shock arises, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition.

Remark 1 The above definition is well-posed even if the classical solution $\mathcal{R}(\rho_L, \rho_R)(x/t)$ displays a shock at $x = V_b t$. In fact, due to Rankine-Hugoniot equation, we have

$$f(\rho_L) = f(\rho_R) + V_b(\rho_L - \rho_R)$$

and hence

$$f(\rho_L) > f_\alpha(\rho_\alpha) + V_b(\rho_L - \rho_\alpha) \quad \Longleftrightarrow \quad f(\rho_R) > f_\alpha(\rho_\alpha) + V_b(\rho_R - \rho_\alpha).$$

Remark 2 The density constraint $\rho(t, y(t)) \leq \alpha$ does not appear explicitly in Definition 1, and in the following Definition 2. It is handled by the corresponding condition on the flux

$$f(\rho(t, y(t))) - \omega(\rho(t, y(t)))\rho(t, y(t)) \leq F_\alpha. \tag{2.6}$$

The corresponding density on the reduced roadway at $x = y(t)$ is found taking the solution to the equation

$$f(\rho_y) + \omega(\rho_y)(\rho - \rho_y) = \rho \left(1 - \frac{\rho}{\alpha}\right),$$

closer to $\rho_y \doteq \rho(t, y(t)))$.

3 The Cauchy problem: existence of solutions

The aim of this section is to study the existence of the solutions of problem (1.1), (1.3). A bus travels along a road modeled by

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, \\ \rho(0, x) = \rho_0(x), \\ \rho(t, y(t)) \leq \alpha. \end{cases} \tag{3.1}$$

The bus influences the traffic along the road but it is also influenced by it. The bus position $y = y(t)$ then solves

$$\begin{cases} \dot{y}(t) = \omega(\rho(t, y(t)+)), \\ y(0) = y_0. \end{cases} \tag{3.2}$$

Solutions to (3.2) will be intended in Carathéodory sense, i.e., as absolutely continuous functions which satisfy (3.2) for a.e. $t \geq 0$. In our setting, due to the strong PDE-ODE coupling, we will prove existence of both solutions to (3.1) and (3.2) at the same time. We start giving our definition of solution.

Definition 2 *A couple $(\rho, y) \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap BV(\mathbb{R}; [0, R])\right) \times \mathbf{W^{1,1}}(\mathbb{R}^+; \mathbb{R})$ is a solution to* (1.3) *if*

1. *ρ is a weak solution of the conservation law, i.e., for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R})$*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \; dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \; dx = 0 \; ; \tag{3.3a}$$

moreover, ρ satisfies the Kružhkov entropy conditions [11] on $(\mathbb{R}^+ \times \mathbb{R}) \backslash \{(t, y(t)) \colon t \in \mathbb{R}^+\}$, i.e., for every $k \in [0, 1]$ and for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^+)$ and $\varphi(t, y(t)) = 0$, $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}} & (|\rho - k| \partial_t \varphi + \operatorname{sgn}(\rho - k) \left(f(\rho) - f(k)\right) \partial_x \varphi) \, dx \; dt \\ & + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \; dx \geq 0 \; ; \end{aligned} \tag{3.3b}$$

2. *y is a Carathéodory solution of the ODE, i.e., for a.e. $t \in \mathbb{R}^+$*

$$y(t) = y_0 + \int_0^t \omega(\rho(s, y(s)+)) \; ds \; ; \tag{3.3c}$$

3. *the constraint is satisfied, in the sense that for a.e.* $t \in \mathbb{R}^+$

$$\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho\right)(t,x) \leq F_\alpha. \tag{3.3d}$$

Remark that the above traces exist because $\rho(t,\cdot) \in \mathrm{BV}(\mathbb{R})$ for all $t \in \mathbb{R}^+$.

Remark 3 Our choice of a Carathéodory solution of the ODE is motivated by the particular bus velocity defined by (1.2). With this choice it is not possible for a bus to end up trapped in a queue unless its speed is equal to V_b, in which case $\omega(\rho(t,y(t)+)) = \omega(\rho(t,y(t)-)) = V_b$. Therefore Carathéodory solutions are always well defined.

We are now ready to state the main result of the paper.

Theorem 1 *Let* $\rho_0 \in BV(\mathbb{R};[0,R])$*, then the problem* (1.1) *admits a solution in the sense of Definition 2.*

The rest of the section is devoted to the proof of Theorem 1. In particular, we will construct a sequence of approximate solutions via the wave-front tracking method, and prove its convergence. Finally, we will check that the limit functions satisfy conditions (3.3a)-(3.3d) of Definition 2.

3.1 Wave-front tracking

To construct piecewise constant approximate solutions, we adapt the standard wave-front tracking method, see for example [5, §6]. Fix a positive $n \in \mathbb{N}$, $n > 0$ and introduce in $[0,1]$ the mesh $\mathcal{M}_n = \{\rho_i^n\}_{i=0}^{2^n}$ defined by

$$\mathcal{M}_n = \left(2^{-n}\mathbb{N} \cap [0,1]\right) \cup \{\check{\rho}_\alpha, \hat{\rho}_\alpha\}.$$

In order to include the critical points $\check{\rho}_\alpha, \hat{\rho}_\alpha$, we modify the above mesh as follows:

- if $\min_i |\check{\rho}_\alpha - \rho_i^n| = 2^{-n-1}$, then we simply add the new point to the mesh:

$$\widetilde{\mathcal{M}}_n = \mathcal{M}_n \cup \{\check{\rho}_\alpha\};$$

- if $|\check{\rho}_\alpha - \rho_l^n| = \min_i |\check{\rho}_\alpha - \rho_i^n| < 2^{-n-1}$, then we replace ρ_l^n by $\check{\rho}_\alpha$:

$$\widetilde{\mathcal{M}}_n = \mathcal{M}_n \cup \{\check{\rho}_\alpha\} \setminus \{\rho_l^n\};$$

- we perform the same operations for $\hat{\rho}_\alpha$.

In this way, the distance between two points of the mesh $\widetilde{\mathcal{M}}_n = \{\tilde{\rho}_i^n\}_{i=0}^{2^n}$ satisfies the lower bound $\left|\tilde{\rho}_i^n - \tilde{\rho}_j^n\right| \geq 2^{-n-1}$.

Let f^n be the piecewise linear function which coincides with f on $\widetilde{\mathcal{M}}_n$, and let ρ_0^n be a piecewise constant function defined by

$$\rho_0^n = \sum_{j \in \mathbb{Z}} \rho_{0,j}^n \ \chi_{]x_{j-1},x_j]} \quad \text{with } \rho_{0,j}^n \in \widetilde{\mathcal{M}}_n,$$

which approximates ρ_0 in the sense of the strong $\mathbf{L^1}$ topology, that is

$$\lim_{n \to \infty} \|\rho_0^n - \rho_0\|_{\mathbf{L^1}(\mathbb{R})} = 0,$$

and such that $\mathrm{TV}(\rho_0^n) \leq \mathrm{TV}(\rho_0)$. Above, we set $x_0 = y_0$.

For small times $t > 0$, a piecewise approximate solution (ρ^n, y_n) to (1.3) is constructed piecing together the solutions to the Riemann problems

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \left(f^n(\rho)\right) = 0, \\ \rho(0,x) = \left\{\begin{array}{ll} \rho_0 & \text{if } x < y_0, \\ \rho_1 & \text{if } x > y_0, \end{array}\right. \\ \rho(t, y_n(t)) \leq \alpha, \end{array}\right. \qquad \left\{\begin{array}{l} \partial_t \rho + \partial_x \left(f^n(\rho)\right) = 0, \\ \rho(0,x) = \left\{\begin{array}{ll} \rho_j & \text{if } x < x_j, \\ \rho_{j+1} & \text{if } x > x_j, \end{array}\right. \\ j \neq 0, \end{array}\right. \tag{3.4}$$

where y_n satisfies

$$\left\{\begin{array}{l} \dot{y}_n(t) = \omega(\rho^n(t, y_n(t)+)), \\ y_n(0) = y_0. \end{array}\right. \tag{3.5}$$

Note that the solutions to the constrained Riemann problem in (3.4), left, coupled with (3.5), is constructed by means of $\mathcal{R}^\alpha$, see Definition 1.

The approximate solution ρ^n constructed above can be prolonged up to the first time $\bar{t} > 0$, where two discontinuities collide, or a discontinuity hits the bus trajectory. In both cases, a new Riemann problem arises and its solution, obtained in the former case with $\mathcal{R}$ and in the latter case with the constrained Riemann solver $\mathcal{R}^\alpha$, allows to extend ρ^n further in time.

3.2 Bounds on the total variation

Given an approximate solution $\rho^n = \rho^n(t, \cdot)$ constructed by the wave-front tracking method, we define the Glimm type functional

$$\Upsilon(t) = \Upsilon(\rho^n(t, \cdot)) = \mathrm{TV}(\rho^n) + \gamma = \sum_j \left|\rho_{j+1}^n - \rho_j^n\right| + \gamma, \tag{3.6}$$

where γ is given by

$$\gamma = \gamma(t) = \left\{\begin{array}{ll} 0 & \text{if } \rho^n(t, y_n(t)-) = \hat{\rho}_\alpha,\ \rho^n(t, y_n(t)+) = \check{\rho}_\alpha \\ 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| & \text{otherwise.} \end{array}\right. \tag{3.7}$$

The value of γ is chosen to have the following uniform bound on Υ.

Lemma 2 *For any $n \in \mathbb{N}$, the map $t \mapsto \Upsilon(t) = \Upsilon(\rho^n(t, \cdot))$ at any interaction either decreases by at least 2^{-n}, or remains constant and the number of waves does not increase.*

Lemma 2 in particular implies that the wave-front tracking procedure can be prolonged to any time $T > 0$.

Proof. In order to obtain a uniform bound on the total variation, we will consider the different types of interactions separately. In particular, we will assume that at any interaction time $t = \bar{t}$ either two waves interact or a single wave hits the bus trajectory.

(I1) We consider a classical collision between two waves (see Figure 2). In this case, either two shocks collide (which means that the number of waves diminishes) or a shock and a rarefaction cancel. However, in both cases, $\mathrm{TV}(\rho^n)$ is not increasing and γ is constant. Thus, we get $\Upsilon(\bar{t}+) \leq \Upsilon(\bar{t}-)$.

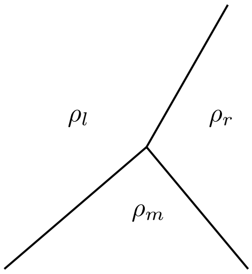

Figure 2: Interaction between two waves away from the bus trajectory

A particular case is when the bus trajectory coincides with one of the interacting waves. In this situation, it must be a classical shock between a left state belonging to $[0, \check{\rho}_\alpha]$ and a right state in $[\hat{\rho}_\alpha, \rho^*]$, and moving with speed equal to V_b. This interaction cannot generate a non-classical shock, it can therefore be treated as the general case above.

(I2) Assume that a wave between two states $\rho_l, \rho_r \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, 1]$ hits the bus trajectory (see Figure 3). In this case, the front crosses the bus trajectory and no new wave is created. Notice that this collision may eventually lead to a modification of the bus trajectory (for example, if $\rho_r > \rho^*$, after the collision the bus takes the velocity $v(\rho_r) \neq \omega(\rho_l)$). Nevertheless, $\mathrm{TV}(\rho^n)$, Υ and the number of waves remain constant.

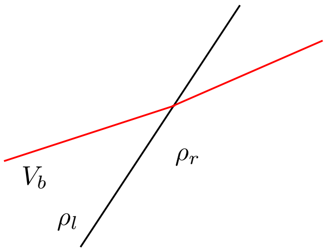

Figure 3: Interaction between a wave and the bus trajectory

(I3) Assume that we are in presence of the non-classical shock along the bus trajectory. Different types of interactions may occur.

(I3.1) Assume that the non-classical shock is present at $t < \bar{t}$, and a shock between $\rho_l \in [0, \check{\rho}_\alpha]$ and $\hat{\rho}_\alpha$ hits the bus trajectory on the left (Figure 4, left). After the collision, the number of discontinuities in ρ^n diminishes and the functional Υ remains constant:

$$\begin{aligned}\Delta\Upsilon(\bar{t}) &= \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-)\\ &= |\rho_l - \check{\rho}_\alpha| + 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \check{\rho}_\alpha|)\\ &= 0.\end{aligned}$$

Assume now a shock between $\check{\rho}_\alpha$ and $\rho_r \in [\hat{\rho}_\alpha, 1]$ hits the bus trajectory on the right (Figure 4, right). Then, after the collision, the bus takes the velocity $v(\rho_r)$ of the traffic mainstream, the number of discontinuities in ρ^n diminishes and the functional

Υ remains constant:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \Upsilon(\bar t+) - \Upsilon(\bar t-)\\ &= |\hat\rho_\alpha - \rho_r| + 2|\hat\rho_\alpha - \check\rho_\alpha| - (|\check\rho_\alpha - \rho_r| + |\hat\rho_\alpha - \check\rho_\alpha|)\\ &= 0.\end{aligned}$$

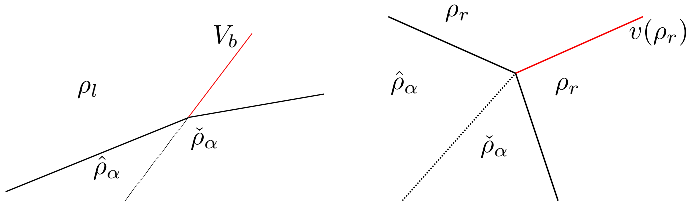

Figure 4: Interaction between a shock and the bus trajectory

(I3.2) Consider now the case of a non-classical shock arising at $t = \bar t$. We first analyze the case of a rarefaction front hitting the bus trajectory from the left (Figure 5, left). We have $\rho_r = \check\rho_\alpha < \rho_l \leq \hat\rho_\alpha$.

Figure 5: Interaction between a rarefaction and the bus trajectory

In this case, a new wave is created at $\bar t$ and the total variation is given by:

- $\mathrm{TV}(\bar t-) = |\check\rho_\alpha - \rho_l| \geq 2^{-n-1}$;
- $\mathrm{TV}(\bar t+) = |\hat\rho_\alpha - \check\rho_\alpha| + |\hat\rho_\alpha - \rho_l| \leq 2|\hat\rho_\alpha - \check\rho_\alpha|$,

where the second estimate is obtained by simple algebraic manipulation of the total variation $\mathrm{TV}(\bar t+)$. Then we are able to compute the changes in the functional as follows:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \Upsilon(\bar t+) - \Upsilon(\bar t-)\\ &= (|\hat\rho_\alpha - \check\rho_\alpha| + |\hat\rho_\alpha - \rho_l|) - (|\check\rho_\alpha - \rho_l| + 2|\hat\rho_\alpha - \check\rho_\alpha|)\\ &= 2(\check\rho_\alpha - \rho_l) \leq -2^{-n},\end{aligned}$$

hence the functional is strictly decreasing.
Let us consider now the case of a rarefaction front hitting the bus trajectory from the right (Figure 5, right). In this case, we have $\check\rho_\alpha \leq \rho_r < \rho_l = \hat\rho_\alpha$. A new wave is created at $\bar t$ and the total variation is given by:

- $\mathrm{TV}(\bar t-) = |\hat\rho_\alpha - \rho_r| \geq 2^{-n-1}$;
- $\mathrm{TV}(\bar t+) = |\hat\rho_\alpha - \check\rho_\alpha| + |\check\rho_\alpha - \rho_r| \leq 2|\hat\rho_\alpha - \check\rho_\alpha|$,

The functional changes as follows:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \Upsilon(\bar t+) - \Upsilon(\bar t-)\\ &= (|\hat\rho_\alpha - \check\rho_\alpha| + |\check\rho_\alpha - \rho_r|) - (|\hat\rho_\alpha - \rho_r| + 2|\hat\rho_\alpha - \check\rho_\alpha|)\\ &= 2(\hat\rho_\alpha - \rho_r) \leq -2^{-n},\end{aligned}$$

making the functional strictly decreasing.

□

3.3 Convergence of approximate solutions

In this section we prove that the limit of wave-front tracking approximations provides a solution (ρ, y) of the PDE-ODE model (1.1) in the sense of Definition 2.
We start showing the convergence of the wave-front tracking approximations.

Lemma 3 *Let ρ^n and y_n, $n \in \mathbb{N}$, be the wave-front tracking approximations to* (1.1) *constructed as detailed in Section 3.1, and assume $TV(\rho_0) \leq C$ be bounded, $0 \leq \rho_0 \leq 1$. Then, up to a subsequence, we have the following convergences*

$$\rho^n \to \rho \qquad \text{in } \mathbf{L^1_{loc}}(\mathbb{R}^+ \times \mathbb{R}); \tag{3.8a}$$

$$y_n(\cdot) \to y(\cdot) \qquad \text{in } \mathbf{L}^\infty([0,T]),\ \text{for all } T > 0; \tag{3.8b}$$

$$\dot y_n(\cdot) \to \dot y(\cdot) \qquad \text{in } \mathbf{L^1}([0,T]),\ \text{for all } T > 0; \tag{3.8c}$$

for some $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap BV(\mathbb{R})\right)$ and $y \in \mathbf{W^{1,1}}(\mathbb{R}^+)$.

Proof. Lemma 2 gives a uniform bound on the total variation of approximate solutions. Thus, we have $\mathrm{TV}(\rho^n(t,\cdot)) \leq \Upsilon(t) \leq \Upsilon(0)$. A standard procedure based on Helly's Theorem (see [5, Theorem 2.4]) ensures the existence of a subsequence converging to some function $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap \mathrm{BV}(\mathbb{R})\right)$, proving (3.8a).

Since $|\dot y_n(t)| \leq V_b$, the sequence $\{y_n\}$ is uniformly bounded and equicontinuous on any compact interval $[0,T]$. By Ascoli-Arzelà Theorem, there exists a subsequence converging uniformly, giving (3.8b).

In order to prove (3.8c), we have to show that $\mathrm{TV}\left(\dot y_n; [0,T]\right)$ is uniformly bounded. In fact, the analysis performed in Section 3.2 shows that $\dot y_n$ can change only at interactions with waves coming from its right. We can estimate the speed variation at interactions times $\bar t$ by the size of the interacting front:

$$|\dot y_n(\bar t+) - \dot y_n(\bar t-)| = |\omega(\rho_l) - \omega(\rho_r)| \leq |\rho_l - \rho_r|.$$

In particular, $\dot y_n$ is non-increasing at interactions with shock fronts and non-decreasing at interactions with rarefaction fronts, which must be originated at $t = 0$. In fact, the analysis performed in Section 3.2 shows that no new rarefaction front can arise at interactions. Therefore,

$$\mathrm{TV}\left(\dot y_n; [0,T]\right) \leq 2\,\mathrm{PV}\left(\dot y_n; [0,T]\right) + \|\dot y_n\|_{\mathbf{L}^\infty([0,T])} \leq 2\,\mathrm{TV}(\rho_0) + V_b$$

is uniformly bounded. Above, $\mathrm{PV}\left(\dot y_n; [0,T]\right)$ denotes the positive variation of $\dot y_n$, i.e., the total amount of positive jumps in the interval $[0,T]$. □

3.3.1 Proof of (3.3a) and (3.3b)

Since ρ^n converge strongly to ρ in $\mathbf{L^1_{loc}}(\mathbb{R}^+ \times \mathbb{R})$, it is straightforward to pass to the limit in the weak formulation of the conservation law, proving that the limit function ρ satisfies (3.3a). Kružhkov entropy condition (3.3b) can be recovered in the same way.

3.3.2 Proof of (3.3c)

We will prove that

$$\lim_{n\to\infty} \rho^n(t, y_n(t)+) = \rho^+(t) = \rho(t, y(t)+) \quad \text{for a.e. } t \in \mathbb{R}^+. \tag{3.9}$$

By pointwise convergence a.e. of ρ^n to ρ, there exists a sequence $z_n \geq y_n(t)$ such that $z_n \to y(t)$ and $\rho^n(t, z_n) \to \rho^+(t)$.

For a.e. $t > 0$, the point $(t, y(t))$ is for $\rho(t, \cdot)$ either a continuity point, or it belongs to a discontinuity curve (represented by $y(\cdot)$) that can be either a classical shock or a non-classical discontinuity between $\rho(t, y(t)-) = \hat{\rho}_\alpha$ and $\rho(t, y(t)+) = \check{\rho}_\alpha$.

Fix $\epsilon^* > 0$ and assume $\mathrm{TV}\left(\rho(t, \cdot);]y(t) - \delta, y(t) + \delta[\right) \leq \epsilon^*$, for some $\delta > 0$. Then by weak convergence of measures (see [6, Lemma 15]) we have $\mathrm{TV}\left(\rho^n(t, \cdot);]y(t) - \delta, y(t) + \delta[\right) \leq 2\epsilon^*$ for n large enough, and we can estimate

$$\left|\rho^n(t, y_n(t)+) - \rho^+(t)\right| \leq \left|\rho^n(t, y_n(t)+) - \rho^n(t, z_n)\right| + \left|\rho^n(t, z_n) - \rho^+(t)\right| \leq 3\epsilon^*$$

for n large enough.

If $\rho(t, \cdot)$ has a discontinuity of strength greater than ϵ^* at $y(t)$, then it holds also for the approximate solutions $|\rho^n(t, y_n(t)+) - \rho^n(t, y_n(t)-)| \geq \epsilon^*/2$ for n sufficiently large, and we proceed as in [6, Section 4]. That is, we set $\rho^{n,+} = \rho^n(t, y_n(t)+)$ and we show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all n large enough there holds

$$\left|\rho^n(s, x) - \rho^{n,+}\right| < \varepsilon \quad \text{for } |s - t| \leq \delta,\ |x - y(t)| \leq \delta,\ x > y_n(s). \tag{3.10}$$

In fact, if (3.10) does not hold, we could find $\varepsilon > 0$ and sequences $t_n \to t$, $\delta_n \to 0$ such that $\mathrm{TV}\left(\rho^n(t_n, \cdot);]y_n(t_n), y_n(t_n) + \delta_n[\right) \geq \varepsilon$. By strict concavity of the flux function f, there should be a uniformly positive amount of interactions in an arbitrarily small neighborhood of $(t, y(t))$, giving a contradiction. Therefore (3.10) holds and we get

$$\left|\rho^n(t, y_n(t)+) - \rho^+(t)\right| \leq \left|\rho^n(t, y_n(t)+) - \rho^n(t, z_n)\right| + \left|\rho^n(t, z_n) - \rho^+(t)\right| \leq 2\varepsilon$$

for n large enough, thus proving (3.9).

Combining (3.8c) and (3.9) we get $\dot{y}(t) = \omega(\rho(t, y(t)+))$ for a.e. $t > 0$.

3.3.3 Proof of (3.3d)

In order to verify that the limit solutions satisfy the constraint (3.3d), we can use directly the convergence result proved in the previous section or proceed as follows. Let us introduce the sets

$$\Omega^{\pm} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} \colon x \lesseqgtr y(t)\}$$

and

$$\Omega_n^{\pm} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} \colon x \lesseqgtr y_n(t)\}.$$

Let us now consider a test function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R})$, $\varphi \geq 0$, such that $\mathrm{supp}(\varphi) \cap \{(t, y(t)) \colon t > 0\} \neq 0$ and $\mathrm{supp}(\varphi) \cap \{(t, y_n(t)) \colon t > 0\} \neq 0$. Then by conservation on Ω_n^+ we have

$$\begin{aligned}
&\iint \chi_{\Omega_n^+} \big(\rho^n \partial_t \varphi + f^n(\rho^n) \partial_x \varphi\big) \, dx \, dt \\
&= \int_0^{+\infty} \int_{y_n(t)}^{+\infty} \big(\rho^n \partial_t \varphi + f^n(\rho^n) \partial_x \varphi\big) \, dx \, dt \\
&= \int_0^{+\infty} \big(f^n(\rho^n(t, y_n(t)+)) - \dot{y}_n(t) \rho^n(t, y_n(t)+)\big) \varphi(t, y_n(t)) \, dt \\
&= \int_0^{+\infty} \big(f^n(\rho^n(t, y_n(t)+)) - \omega(\rho^n(t, y_n(t)+)) \rho^n(t, y_n(t)+)\big) \varphi(t, y_n(t)) \, dt \\
&\leq \int_0^{T} F_\alpha \varphi(t, y_n(t)) \, dt,
\end{aligned} \tag{3.11}$$

where we have used the fact that wave-front tracking approximations satisfy the constraint (2.6) by construction. The same can be done for the limit solutions ρ and $y(t)$ that, by conservation on Ω, satisfy

$$\begin{aligned}
&\iint \chi_{\Omega^+} \big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi\big) \, dx \, dt \\
&= \int_0^{+\infty} \int_{y(t)}^{+\infty} \big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi\big) \, dx \, dt \\
&= \int_0^{+\infty} \big(f(\rho(t, y(t)+)) - \dot{y}(t) \rho(t, y(t)+)\big) \varphi(t, y(t)) \, dt \\
&= \int_0^{+\infty} \big(f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+)) \rho(t, y(t)+)\big) \varphi(t, y(t)) \, dt
\end{aligned} \tag{3.12}$$

By (3.8a) and (3.8b) we can pass to the limit in (3.11) and (3.12), which gives

$$\int_0^{+\infty} \big(f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+)) \rho(t, y(t)+) - F_\alpha\big) \varphi(t, y(t)) \, dt \leq 0.$$

Since the above inequality holds for every test function $\varphi \geq 0$, we have proved that

$$\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho\right)(t, x) \leq F_\alpha$$

for a. e. $t > 0$. The left-hand limit can be handled in the same way using the sets Ω_n^+ and Ω^+.

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