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► **To cite this version:**

Laurent Busé, Jean-Pierre Jouanolou. On the discriminant scheme of homogeneous polynomials. *Mathematics in Computer Science*, Springer, 2014, Special Issue in Computational Algebraic Geometry, 8 (2), pp.175-234. <10.1007/s11786-014-0188-7>. <hal-00747930v2>

**HAL Id: hal-00747930**

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Submitted on 8 Jul 2014

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# On the discriminant scheme of homogeneous polynomials

Laurent Busé and Jean-Pierre Jouanolou

**Keywords.** Elimination theory; discriminant of homogeneous polynomials; resultant of homogeneous polynomials; inertia forms.

**Abstract.** In this paper, the discriminant scheme of homogeneous polynomials is studied in two particular cases: the case of a single homogeneous polynomial and the case of a collection of  $n - 1$  homogeneous polynomials in  $n \geq 2$  variables. In both situations, a normalized discriminant polynomial is defined over an arbitrary commutative ring of coefficients by means of the resultant theory. An extensive formalism for this discriminant is then developed, including many new properties and computational rules. Finally, it is shown that this discriminant polynomial is faithful to the geometry: it is a defining equation of the discriminant scheme over a general coefficient ring  $k$ , typically a domain, if  $2 \neq 0$  in  $k$ . The case where  $2 = 0$  in  $k$  is also analyzed in detail.

## 1. Introduction

The discriminant of a collection of polynomials gives information about the nature of the common roots of these polynomials. Following the example of the very classical discriminant of a single univariate polynomial, it is a fundamental tool in algebraic geometry which is very useful and has many applications. Let us recall briefly the usual definition of the discriminant (see [GKZ94]): given integers  $1 \leq c \leq n$  and  $1 \leq d_1, \dots, d_c$ , denote by  $S$  the set of all  $c$ -uples of homogeneous polynomials  $f_1, \dots, f_c$  in the polynomial ring  $\mathbb{C}[X_1, \dots, X_n]$  ( $n \geq 2$ ) of degrees  $d_1, \dots, d_c$ , respectively. The subset  $D$  of  $S$  corresponding to the  $c$ -uples  $f_1, \dots, f_c$  such that  $\{f_1 = f_2 = \dots = f_c = 0\}$  is not smooth and of codimension  $c$  is called the discriminant locus. It is well-known that  $D$  is an irreducible algebraic variety of codimension one providing  $d_i \geq 2$  for some  $i \in \{1, \dots, c\}$  or  $c = n$ . The discriminant is then defined as an equation of  $D$  (and set to be 1 if  $D$  is not of codimension one).

There are drawbacks to this definition of the discriminant. First, it is not stable under change of basis. In other words, the discriminant is a polynomial in the coefficients of the polynomials  $f_1, \dots, f_c$  and a given specialization of these coefficients does not always commute with this construction of the discriminant. Such a property is however a natural request. Notice that it is actually well satisfied when defining the discriminant of a single univariate polynomial  $f$  as the determinant of the Sylvester matrix associated to  $f$  and its first derivative (see [EH00, §V.3] for more details). Second, the discriminant is defined up to multiplication by a nonzero constant. This is not satisfactory when the value of the discriminant is important and not only its vanishing, as this is for instance the case for some applications in the fields of arithmetic geometry and number theory (see for instance the recent paper [Sai12]). Another point is that this definition is only valid under the hypothesis that the ground ring is a field, often assumed to be algebraically closed and of characteristic zero. But

for many applications, it is very useful to understand the behavior of the discriminant under general ground rings. These drawbacks are important obstructions that prevent the discriminant from having a well developed formalism, in particular some properties and formulas that allow to handle it as a computational tool. In many situations such a formalism is actually more important than the value of the discriminant itself which is very often unreachable by direct computations. Moreover, the discriminant gives more insights if it is defined without ambiguity (in particular not up to a nonzero constant multiplicative factor) over a general coefficient ring (see e.g. [BM09]).

The goal of this paper is to develop the formalism of the discriminant over an arbitrary commutative ring  $k$ . To work in this setting, it is necessary to give a scheme structure to the above-mentioned set  $D \subset S$ . For all  $i = 1, \dots, c$ , set

$$f_i(X_1, \dots, X_n) := \sum_{|\alpha|=d_i \geq 1} U_{i,\alpha} X^\alpha \in {}_k A[X_1, \dots, X_n]_{d_i}$$

where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  is a multi-index,  $|\alpha| := \sum_{i=1}^n \alpha_i$ ,  $X^\alpha := X_1^{\alpha_1} \dots X_n^{\alpha_n}$  and where  ${}_k A := k[U_{i,\alpha} \mid |\alpha| = d_i, i = 1, \dots, n]$  is the universal ring of coefficients. Now, let  $\mathcal{D}$  be the ideal of  ${}_k A[X_1, \dots, X_n]$  generated by  $f_1, \dots, f_c$  and all the  $c$ -minors of their Jacobian matrix. Then, the *discriminant scheme* of  $f_1, \dots, f_c$  can be defined as the scheme-theoretic image of the canonical projection of the incidence scheme  $\text{Proj}({}_k A[X_1, \dots, X_n]/\mathcal{D})$  onto  $\text{Spec}({}_k A)$ . It is a closed subscheme of  $\text{Spec}({}_k A)$  whose defining ideal is the inertia forms ideal

$$\{a \in {}_k A \mid \exists \nu \in \mathbb{N} \, \mathfrak{m}^\nu a \subset \mathcal{D}\} \subset {}_k A$$

where  $\mathfrak{m}$  stands for the ideal generated by the variables  $X_1, \dots, X_n$ . In this paper, we will study this discriminant scheme in the two cases  $c = n - 1$  and  $c = 1$ , in Section 3 and Section 4 respectively. In geometric terms, the case  $c = n - 1$  corresponds to the discriminant of a finite set of points in complete intersection in a projective space, whereas the case  $c = 1$  corresponds to the study of the discriminant of a hypersurface in a projective space.

Our strategy is the same for both cases  $c = n - 1$  and  $c = 1$ . We first define a *universal discriminant polynomial*  ${}_Z \text{Disc} \in {}_Z A$  by means of certain resultants. Taking advantage of the existing formalism of the resultant that we will briefly overview in Section 2, we then develop a rich formalism for this discriminant polynomial: computation of the partial degrees, multiplicativity, reduction properties, change of basis formulas, etc. As far as we know, such a formalism did not appear under this form in the existing literature (see the related works below). Most of the properties we give are new and for the others, our contribution consists in providing rigorous proofs in our general framework. As a typical byproduct of this formalism, we mention the following interesting result: whereas it is classical that the discriminant of a univariate polynomial  $f$  belongs to the ideal generated by  $f$  itself and its first derivative  $\partial f$ , we prove that it actually belongs to the ideal generated by  $f$  and  $(\partial f)^2$  (see Theorem 3.25 and Corollary 4.30 for more general statements).

The second step of our strategy is to show that the canonical specialization of the discriminant polynomial  ${}_Z \text{Disc}$  in  ${}_k A$ , denoted  ${}_k \text{Disc}$ , is a defining equation of the discriminant scheme under some suitable assumptions. Our definition of the discriminant polynomial is hence faithful to the geometry. To be more precise, in the case  $c = n - 1$  we prove that  ${}_k \text{Disc}$  is a defining equation of the discriminant scheme if  $2 \neq 0$  in  $k$ , and that it is moreover prime if  $k$  is a domain (see Theorem 3.23). If  $2 = 0$  in  $k$  and  $k$  is a domain,  ${}_k \text{Disc}$  is shown to be the square of a prime polynomial which is a defining equation of the discriminant scheme (see Theorem 3.24). In the case  $c = 1$ , assuming that  $k$  is a domain we show that  ${}_k \text{Disc}$  is a prime polynomial defining the discriminant scheme if  $2 \neq 0$  or  $n$  is odd, and that it is the square of a prime polynomial defining the discriminant scheme otherwise, i.e. if  $2 = 0$  in  $k$  and  $n$  is even (see Theorem 4.26). At the end of Section 4 we also provide a detailed study of the birationality of the canonical projection of the incidence scheme onto the discriminant scheme for the case  $c = 1$  (see Theorem 4.36). This is one of the delicate properties of the discriminant that is not always treated rigorously in the existing literature.

The paper ends with an appendix where we give rigorous proofs of two remarkable formulas that are due to F. Mertens [Mer92, Mer86]. We will use these formulas at some point in text, but they are definitely interesting on their own.

**Related works.** As far as we know, the literature on the theory of the discriminant goes back to an outstanding paper by Sylvester [Syl64b, Syl64a] where among others, an explicit formula for the degree of the discriminant is given for any integer  $1 \leq c \leq n$ . Then, one finds the works by Mertens [Mer86, Mer92], where the concept of inertia forms is already used, and some other works by König [Kön03], by Kronecker [Kro82], by Ostrowski [Ost19] and also by Henrici [Hen68]. There is also an important contribution by Krull [Kru39, Kru42] who studied Jacobian ideals and some properties of the discriminant, especially in the case  $c = n - 1$ . The case  $c = 1$  is by far the most classical case of study of the discriminant in the literature, mostly because of the so-called “Cayley trick” (see [GKZ94]). A detailed study can be found in a Bourbaki manuscript by Demazure [Dem69] that was unfortunately left unpublished until very recently [Dem12]. Another treatment is proposed in [GKZ94, chapter 12.B] and more recently another one in [Sai12]. For the past twenty years, one can observe a regain of interest, in particular regarding properties with respect to the shape (total degree, partial degrees, Newton polyhedron, etc) of the discriminant. Unlike the previously mentioned works, the techniques are here more advanced and use homological methods. The book by Gelfand, Kapranov and Zelevinsky [GKZ94] was definitely a turning point in this modern approach. One can also mention the paper by Scheja and Storch [SS08] and the more recent one by Esterov [Est10] that deals with more general grading of the polynomials (they correspond to anisotropic projective spaces and more general toric varieties respectively). It is as well worth mentioning the recent paper by Benoist [Ben12] where the degree of the discriminant is carefully studied (see also [Syl64b, Syl64a]).

In the sequel, all rings are assumed to be commutative and with unity.

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## 2. Preliminaries

We recall here the basic definitions and properties of inertia forms and the resultant that we will use in the rest of this paper to study the discriminant of homogeneous polynomials. Our main source is the monograph [Jou91] where a detailed exposition can be found.

Suppose given  $r \geq 1$  homogeneous polynomials of positive degrees  $d_1, \dots, d_r$ , respectively, in the variables  $X_1, \dots, X_n$ , all assumed to have weight 1,

$$f_i(X_1, \dots, X_n) = \sum_{|\alpha|=d_i} U_{i,\alpha} X^\alpha, \quad i = 1, \dots, r.$$

Let  $k$  be a commutative ring and set  ${}_kA := k[U_{i,\alpha} \mid i = 1, \dots, r, |\alpha| = d_i]$  the universal coefficient ring over  $k$ . Then  $f_i \in {}_kC := {}_kA[X_1, \dots, X_n]$  for all  $i = 1, \dots, r$ . We define the ideal  $I := (f_1, \dots, f_r) \subset {}_kC$  and the graded quotient ring  ${}_kB := {}_kC/I$ . The main purpose of elimination theory is the study of the image of the canonical projection

$$\text{Proj}({}_kB) \rightarrow \text{Spec}({}_kA)$$

which corresponds to the elimination of the variables  $X_1, \dots, X_n$  in the polynomial system  $f_1 = \dots = f_r = 0$ . It turns out that this image is closed (the latter projection is a projective morphism) and its defining ideal, that we will denote by  ${}_k\mathfrak{A}$  and which is usually called the *resultant* (or *eliminant*) *ideal*, consists of the elements of  ${}_kA$  which are contained in  $I$  after multiplication by some power of the maximal ideal  $\mathfrak{m} := (X_1, \dots, X_n) \subset {}_kC$ . In other words,  ${}_k\mathfrak{A}$  is the degree 0 part of the 0th local cohomology module of  ${}_kB$  with respect to  $\mathfrak{m}$ , i.e.  ${}_k\mathfrak{A} = H_{\mathfrak{m}}^0({}_kB)_0$ .

### 2.1. Inertia forms

First introduced by Hurwitz, *inertia forms* reveal a powerful tool to study the resultant ideals, notably in the case  $r = n$  corresponding to the theory of the resultant, and more generally elimination theory.

**Definition 2.1.** *The ideal of inertia forms of the ideal  $I$  with respect to the ideal  $\mathfrak{m}$  is the ideal of  ${}_kC$*

$${}_k\text{TF}_{\mathfrak{m}}(I) := \pi^{-1}(H_{\mathfrak{m}}^0({}_kB)) = \{f \in {}_kC : \exists \nu \in \mathbb{N} \mathfrak{m}^\nu f \subset I\} \subset {}_kC$$

where  $\pi$  denotes the canonical projection  ${}_kC \rightarrow {}_kB = {}_kC/I$ .

Observe that the inertia forms ideal is naturally graded and that  ${}_k\mathfrak{A} = {}_k\text{TF}_{\mathfrak{m}}(I)_0$ . We recall two useful other descriptions of this ideal.

Let us distinguish, for all  $i = 1, \dots, r$ , the particular coefficient  $\mathcal{E}_i := U_{i,(0,\dots,0,d_i)}$  of the polynomial  $f_i$  which can be rewritten in  ${}_kC[X_n^{-1}]$

$$f_i = X_n^{d_i} (\mathcal{E}_i + \sum_{\alpha \neq (0,\dots,0,d_i)} U_{i,\alpha} X^\alpha X_n^{-d_i}).$$

Then we get the isomorphism of  $k$ -algebras

$$\begin{aligned} {}_kB_{X_n} &\xrightarrow{\sim} k[U_{j,\alpha} : U_{j,\alpha} \neq \mathcal{E}_i][X_1, \dots, X_n][X_n^{-1}] \\ \mathcal{E}_i &\mapsto \mathcal{E}_i - \frac{f_i}{X_n^{d_i}} = - \sum_{\alpha \neq (0,\dots,0,d_i)} U_{i,\alpha} X^\alpha X_n^{-d_i} \end{aligned} \quad (2.1.1)$$

and of course similar isomorphisms for all the  ${}_kB_{X_i}$ 's. They show that  $X_i$  is a nonzero divisor in  ${}_kB_{X_j}$  for all pair  $(i, j) \in \{1, \dots, n\}^2$ , and by the way that, for all  $i \in \{1, \dots, n\}$ ,

$${}_k\text{TF}_{\mathfrak{m}}(I) = \{f \in {}_kC : \exists \nu \in \mathbb{N} X_i^\nu f \subset I\} = \text{Ker}({}_kC \rightarrow {}_kB_{X_i}). \quad (2.1.2)$$

In particular, if the commutative ring  $k$  is a domain, it follows that the  ${}_kB_{X_i}$ 's are also domains and thus that  ${}_k\text{TF}_{\mathfrak{m}}(I)$  is a *prime* ideal of  ${}_kC$ , as well as  ${}_k\mathfrak{A}$ . Note also that, as a simple consequence, we obtain the equality

$${}_k\mathfrak{A} = {}_k\text{TF}_{\mathfrak{m}}(I)_0 = {}_kA \cap (\tilde{f}_1, \dots, \tilde{f}_r) \quad (2.1.3)$$

where  $\tilde{f}_i(X_1, \dots, X_{n-1}) = f_i(X_1, \dots, X_{n-1}, 1) \in {}_kA[X_1, \dots, X_{n-1}]$ .

The combination of (2.1.2) and (2.1.1) also gives another interesting description of  ${}_k\text{TF}_m(I)$ . Indeed, similarly to (2.1.1), we define the morphism

$$\tau : {}_kC \rightarrow k[U_{i,\alpha} \mid U_{i,\alpha} \neq \mathcal{E}_i][X_1, \dots, X_n][X_n^{-1}] : \mathcal{E}_i \mapsto \mathcal{E}_i - \frac{f_i}{X_n^{d_i}}$$

which is sometimes called the Kronecker substitution. Then, it follows that

$${}_k\text{TF}_m(I) = \{f \in {}_kC : \tau(f) = 0\}. \quad (2.1.4)$$

In other words, an inertia form is a polynomial in  ${}_kC$  that vanishes after the substitution of each  $\mathcal{E}_i$  by  $\mathcal{E}_i - f_i/X_n^{d_i}$  for all  $i = 1, \dots, r$ . This property yields in particular the following refinement of (2.1.3):

$${}_k\mathfrak{A} = {}_k\text{TF}_m(I)_0 = {}_kA \cap \sum_{i=1}^r \tilde{f}_i \cdot {}_kA[\tilde{f}_1, \dots, \tilde{f}_r]. \quad (2.1.5)$$

## 2.2. The resultant

We now turn to the particular case  $r = n$ , usually called the *principal case of elimination*. As we are going to recall, in this situation the resultant ideal  ${}_k\mathfrak{A}$  is principal and the resultant is one of its generator. We will need the

**Notation 2.2.** *Let  $k$  be a commutative ring. Suppose given a  $k$ -algebra  $R$  and, for all integers  $i \in \{1, \dots, n\}$ , a homogeneous polynomial of degree  $d_i$  in the variables  $X_1, \dots, X_n$*

$$g_i = \sum_{|\alpha|=d_i} u_{i,\alpha} X^\alpha \in R[X_1, \dots, X_n]_{d_i}.$$

We denote by  $\theta$  the  $k$ -algebra morphism  $\theta : {}_kA \rightarrow R : U_{j,\alpha} \mapsto u_{j,\alpha}$  corresponding to the specialization of the polynomials  $f_i$  to the polynomials  $g_i$ . Then, for any element  $a \in {}_kA$  we set  $a(g_1, \dots, g_n) := \theta(a)$ . In particular, if  $R = {}_kA$  and  $\theta$  is the identity (i.e.  $g_i = f_i$  for all  $i$ ), then  $a = a(f_1, \dots, f_n)$ .

**Proposition 2.3** ([Jou91, §2]). *The ideal  ${}_Z\mathfrak{A}$  of  ${}_ZA$  is principal and has a unique generator, denoted  ${}_Z\text{Res}$ , which satisfies*

$${}_Z\text{Res}(X_1^{d_1}, \dots, X_n^{d_n}) = 1. \quad (2.2.1)$$

Moreover, for any commutative ring  $k$ , the ideal  ${}_k\mathfrak{A}$  of  ${}_kA$  is also principal and generated by  ${}_k\text{Res} := \lambda({}_Z\text{Res})$ , where  $\lambda$  denotes the canonical morphism

$$\lambda : {}_ZA := \mathbb{Z}[U_{i,\alpha}] \rightarrow {}_kA = k[U_{i,\alpha}] : U_{j,\alpha} \mapsto U_{j,\alpha}.$$

In addition,  ${}_k\text{Res}$  is a nonzero divisor in  ${}_kA$ .

In view of Notation 2.2, we have defined the resultant of any set of homogeneous polynomials of positive degrees  $f_1, \dots, f_n \in k[X_1, \dots, X_n]$ , where  $k$  denotes any commutative ring; we will denote it by  $\text{Res}(f_1, \dots, f_n)$  without any possible confusion. Indeed, this resultant is by definition obtained as a specialization of the corresponding resultant in the generic case over  $\mathbb{Z}$ , that is to say  ${}_Z\text{Res}$  (with the corresponding choice of degrees for the input polynomials). Therefore, the resultant has the property to be stable under specialization whereas this is not the case of the inertia forms ideal in general. Nevertheless, we have the following property.

**Proposition 2.4.** *The ideal of inertia forms is stable under specialization up to radical. More precisely, let  $R$  be a commutative ring and  $\rho : {}_ZA \rightarrow R$  be a specialization morphism. Then, the ideals  $\rho({}_Z\text{TF}_m(I)_0) \cdot R = (\rho({}_Z\text{Res}))$  and  $\text{TF}_m(\rho(I) \cdot R)_0$  are two ideals in  $R$  that have the same radical.*

*Proof.* This result corresponds to a general property of proper morphisms under change of basis. As we already said, the canonical projection  $\text{Proj}(\mathbb{Z}B) \rightarrow \text{Spec}(\mathbb{Z}A)$  is a projective, hence proper, morphism whose image is closed and defined by the ideal  ${}_{\mathbb{Z}}\text{TF}_m(I)_0 \subset \mathbb{Z}A$ . The specialization  $\rho$  corresponds to a change of basis from  $\text{Spec}(R)$  to  $\text{Spec}(\mathbb{Z}A)$ . Since the support of the closed image of a proper morphism is stable under change of basis, we deduce that, as claimed, the support of the inverse image of the closed image of  $\text{Proj}(\mathbb{Z}B) \rightarrow \text{Spec}(\mathbb{Z}A)$  is equal to the support of the closed image of

$$\text{Proj}(\mathbb{Z}B) \times_{\text{Spec}(\mathbb{Z}A)} \text{Spec}(R) \rightarrow \text{Spec}(R).$$

We can give another proof, somehow more elementary, of this proposition. Indeed, by specialization it is clear that

$$\rho({}_{\mathbb{Z}}\text{TF}_m(I)_0.R) = (\rho({}_{\mathbb{Z}}\text{Res})) = (\text{Res}(\rho(f_1), \dots, \rho(f_n))) \subset \text{TF}_m(\rho(I).R)_0.$$

Let  $a \in \text{TF}_m(\rho(I).R)_0$ , so that there exists an integer  $N$  such that for all  $i = 1, \dots, n$

$$X_i^N a \in (\rho(f_1), \dots, \rho(f_n)) \subset R[X_1, \dots, X_n].$$

It follows that

$$(X_1^N a, X_2^N a, \dots, X_n^N a) \subset (\rho(f_1), \dots, \rho(f_n)) \subset R[X_1, \dots, X_n]$$

and hence that  $\text{Res}(\rho(f_1), \dots, \rho(f_n))$  divides  $\text{Res}(X_1^N a, \dots, X_n^N a)$  in  $R$  by [Jou91, §5.6]. Now, using [Jou91, Proposition 2.3(ii)], we obtain that

$$\text{Res}(X_1^N a, \dots, X_n^N a) = a^{nN^{n-1}} \text{Res}(X_1^N, \dots, X_n^N) = a^{nN^{n-1}} \in R.$$

Therefore,  $\text{Res}(\rho(f_1), \dots, \rho(f_n))$  divides  $a^{nN^{n-1}}$  in  $R$  and hence  $\text{TF}_m(\rho(I).R)_0$  is contained in the radical of the ideal  $(\text{Res}(\rho(f_1), \dots, \rho(f_n))) \subset R$ .  $\square$

The resultant have a lot of interesting properties that we are going to use all along this paper; we refer the reader to [Jou91, §5] and each time we will need one of these properties we will quote a precise reference from this source (as we have just done in the proof of the previous proposition).

We end this paragraph by recalling the old-fashion way, still very useful in some cases, to define the resultant (see for instance [Zar37]). To do this, let us introduce  $n$  new indeterminates  $T_1, \dots, T_n$ . From (2.1.5) we deduce easily that

$${}_k\text{TF}_m((f_1 - T_1 X_n^{d_1}, \dots, f_n - T_n X_n^{d_n}))_0 = \{P(T_1, \dots, T_n) \in {}_kA[T_1, \dots, T_n] : P(\tilde{f}_1, \dots, \tilde{f}_n) = 0\},$$

equality which can be rephrased by saying that *the kernel of the map*

$$\phi : {}_kA[T_1, \dots, T_n] \rightarrow {}_kA[X_1, \dots, X_{n-1}] : T_i \mapsto \tilde{f}_i$$

is a principal ideal generated by  $\text{Res}(f_1 - T_1 X_n^{d_1}, \dots, f_n - T_n X_n^{d_n})$ . Thus, we obtain an explicit formulation of (2.1.5) under the form

$$\text{Res}(f_1 - \tilde{f}_1 X_n^{d_1}, \dots, f_n - \tilde{f}_n X_n^{d_n}) = 0. \quad (2.2.2)$$

### 2.3. A generalized weight property

When dealing with the discriminant of  $n - 1$  homogeneous polynomials in  $n$  variables, we will need a property of homogeneity for the resultant that is due to Mertens [Mer86] and that has been generalized by Zariski about fifty years later [Zar37, Theorem 6]. For the convenience of the reader, we provide a proof of this result.

Suppose given  $n$  integers  $\mu_1, \dots, \mu_n$  such that for all  $i = 1, \dots, n$  we have  $0 \leq \mu_i \leq d_i$  and set  $f_i = X_n^{\mu_i} g_i + h_i$  where all the monomials having a nonzero coefficient in the polynomial  $h_i$

is not divisible by  $X_n^{\mu_i}$ , i.e. is such that  $\alpha_n < \mu_i$ . Now, define the weight of each coefficient  $U_{i,\alpha}$ ,  $i = 1, \dots, n$ ,  $|\alpha| = d_i$  by

$$\text{weight}(U_{i,\alpha}) := \begin{cases} 0 & \text{if } \alpha_n < \mu_i \\ \alpha_n - \mu_i & \text{if } \alpha_n \geq \mu_i \end{cases} \quad (2.3.1)$$

(we will refer to this grading as the Zariski grading) and set

$$\text{Res}(f_1, \dots, f_n) = H(f_1, \dots, f_n) + N(f_1, \dots, f_n) \in {}_k A$$

where  $H$  is the homogeneous part of minimum degree of the resultant, using the above weights definition.

**Proposition 2.5.** *With the above notation, there exists an element*

$$H_1(f_1, \dots, f_n) \in {}_k A$$

which is of degree zero and that satisfies

$$H(f_1, \dots, f_n) = \text{Res}(g_1, \dots, g_n)H_1(f_1, \dots, f_n) \in {}_k A.$$

In particular, the degree of  $H$  is equal to  $\prod_{i=1}^n (d_i - \mu_i)$ .

Here is an immediate corollary that is the form under which we will use this property later on.

**Corollary 2.6.** *For all  $i = 1, \dots, n$ , define the polynomials  $h_i$  and rename some coefficients  $U_{i,\alpha}$  of  $f_i$  so that  $f_i = X_n^{d_i-1}(\sum_{j=1}^n V_{i,j}X_j) + h_i$ . Then, we have*

$$\text{Res}(f_1, \dots, f_n) - \det((V_{i,j})_{i,j=1,\dots,n})H_1 \in (V_{1,n}, \dots, V_{n,n})^2 \subset {}_k A.$$

*Proof.* Let  $\phi \in \text{TF}_m(f_1, \dots, f_n) \cap A$ . There exists an integer  $N$  such that  $X_n^N \phi \in (f_1, \dots, f_n)$ , and define  $\phi_0 \in A$  as the homogeneous part of minimum degree of  $\phi$  with respect to the weights given in (2.3.1). We begin by showing that  $\phi_0 \in \text{TF}_m(g_1, \dots, g_n) \cap A$ .

In addition of the weights (2.3.1), we set  $\text{weight}(X_i) = 1$  for all  $i = 1, \dots, n-1$  and  $\text{weight}(X_n) = 0$ . In this way, for all  $i = 1, \dots, n$  the terms in the decomposition  $f_i = X_n^{\mu_i}g_i + h_i$  are such that  $X_n^{\mu_i}g_i$  is homogeneous of degree  $d_i - \mu_i$  whereas  $h_i$  contains monomials that are homogeneous of degree strictly bigger than  $d_i - \mu_i$ . To emphasize this property, introduce a new indeterminate  $t$  and consider the linear transformation

$$\begin{cases} X_i \mapsto tX_i, & i = 1, \dots, n-1 \\ X_n \mapsto X_n \\ U_{i,\alpha} \mapsto t^{\text{weight}(U_{i,\alpha})}U_{i,\alpha}, & i = 1, \dots, n, |\alpha| = d_i. \end{cases}$$

Denoting by  $\nu$  the degree of  $\phi_0$  and applying the above transformation, we deduce that

$$X_n^N t^\nu (\phi_0 + t\omega_0) \in (t^{d_1-\mu_1}(X_n^{\mu_1}g_1 + t\omega_1), t^{d_2-\mu_2}(X_n^{\mu_2}g_2 + t\omega_2), \dots, t^{d_n-\mu_n}(X_n^{\mu_n}g_n + t\omega_n)) \quad (2.3.2)$$

where  $\omega_i \in A[X_1, \dots, X_n, t]$  for all  $i = 0, \dots, n$ . Having in mind to use the characterization (2.1.4) of inertia forms, for all  $i = 1, \dots, n$  we set  $g_i = \eta_i X_n^{d_i-\mu_i} + \varphi_i$ ,  $\tilde{g}_i = g_i(X_1, \dots, X_{n-1}, 1)$  and  $\tilde{\varphi}_i = \varphi_i(X_1, \dots, X_{n-1}, 1)$ . Now, the specialization of  $X_n$  to 1 in (2.3.2) yields

$$t^\nu (\phi_0 + t\omega_0) \in (\tilde{g}_1 + t\omega_1, \tilde{g}_2 + t\omega_2, \dots, \tilde{g}_n + t\omega_n)$$

and then the specializations of  $\eta_i$  to  $-\tilde{\varphi}_i - t\omega_i$  for all  $i = 1, \dots, n$  give

$$t^\nu (\phi_0(-\tilde{\varphi}_1 - t\omega_1, \dots, -\tilde{\varphi}_n - t\omega_n) + t\omega_0(-\tilde{\varphi}_1 - t\omega_1, \dots, -\tilde{\varphi}_n - t\omega_n)) = 0 \quad (2.3.3)$$

in  $A[X_1, \dots, X_n, t]$ , where the quoted arguments of  $\phi_0$  and  $\phi_1$  are those corresponding to the coefficients  $\eta_1, \dots, \eta_n$  respectively. But since  $t$  is a nonzero divisor, we can simplify (2.3.3) by  $t^\nu$ . Then, by specializing  $t$  to 0 we deduce that  $\phi_0(-\tilde{\varphi}_1, \dots, -\tilde{\varphi}_n) = 0$  and hence that  $\phi_0 \in \text{TF}_m(g_1, \dots, g_n)$ .



Now, applying the above property to  $\phi = \text{Res}(f_1, \dots, f_n)$  we deduce that there exists  $H_1 \in A$  such that  $H = \text{Res}(g_1, \dots, g_n)H_1$ . However, to conclude the proof it remains to show that  $H_1$  is of degree zero, or equivalently that  $H$  and  $\text{Res}(g_1, \dots, g_n)$  have the same degree with respect to the weights (2.3.1). Notice that we already know that  $\text{Res}(g_1, \dots, g_n)$  has degree  $\prod_{i=1}^n (d_i - \mu_i)$  by the property [Jou91, §5.13.2] and hence, the degree of  $H$  is greater or equal to  $\prod_{i=1}^n (d_i - \mu_i)$ . In order to show that it is actually an equality, we consider the following specialization

$$\begin{cases} f_1 &= X_1^{d_1 - \mu_1} X_n^{\mu_1} \\ f_2 &= X_1^{d_2} + X_2^{d_2 - \mu_2} X_n^{\mu_2} \\ f_3 &= X_2^{d_3} + X_3^{d_3 - \mu_3} X_n^{\mu_3} \\ &\vdots \\ f_{n-1} &= X_{n-2}^{d_{n-1}} + X_{n-1}^{d_{n-1} - \mu_{n-1}} X_n^{\mu_{n-1}} \\ f_n &= X_{n-1}^{d_{n-1}} + t^{d_n - \mu_n} X_n^{d_n} \end{cases}$$

where, for all  $i = 1, \dots, n$ , the coefficient  $U_{i,\alpha}$  of each monomial  $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ ,  $|\alpha| = d_i$ , of  $f_i$  that appears in this specialization has been also specialized to  $t^{\text{weight}(U_{i,\alpha})}$ . Let us compute the resultant of  $f_1, \dots, f_n$ . Applying the multiplicativity property of resultants [Jou91, §5.7], we get

$$\begin{aligned} \text{Res}(f_1, \dots, f_n) &= \text{Res}(X_1^{d_1 - \mu_1}, f_2, \dots, f_n) \text{Res}(X_n^{\mu_1}, f_2, \dots, f_n) \\ &= \text{Res}(X_1, f_2, \dots, f_n)^{d_1 - \mu_1} \text{Res}(X_n, X_1, X_2, \dots, X_{n-1})^{\mu_1 d_2 d_3 \dots d_{n-1}} \\ &= (-1)^{(n-1)\mu_1 d_2 d_3 \dots d_{n-1}} \text{Res}(X_1, f_2, \dots, f_n)^{d_1 - \mu_1}, \end{aligned}$$

then

$$\begin{aligned} &\text{Res}(X_1, f_2, \dots, f_n) \\ &= \text{Res}(X_1, X_2^{d_2 - \mu_2}, f_3, \dots, f_n) \text{Res}(X_1, X_n^{\mu_2}, f_3, \dots, f_n) \\ &= \text{Res}(X_1, X_2, f_3, \dots, f_n)^{d_2 - \mu_2} \text{Res}(X_1, X_n, X_2, X_3, \dots, X_{n-1})^{\mu_2 d_3 \dots d_{n-1}} \\ &= (-1)^{(n-2)\mu_2 d_3 \dots d_{n-1}} \text{Res}(X_1, X_2, f_3, \dots, f_n)^{d_2 - \mu_2} \end{aligned}$$

and continuing this way we arrive at the equality

$$\text{Res}(f_1, \dots, f_n) = \pm \text{Res}(X_1, \dots, X_{n-1}, f_n)^{(d_1 - \mu_1) \dots (d_{n-1} - \mu_{n-1})}.$$

But since  $f_n$  is specialized to  $X_{n-1}^{d_{n-1}} + t^{d_n - \mu_n} X_n^{d_n}$ , we deduce that

$$\begin{aligned} &\text{Res}(f_1, \dots, f_n) \\ &= \pm \text{Res}(X_1, \dots, X_{n-1}, t^{d_n - \mu_n} X_n^{d_n})^{(d_1 - \mu_1) \dots (d_{n-1} - \mu_{n-1})} \\ &= \pm t^{(d_1 - \mu_1) \dots (d_{n-1} - \mu_{n-1})(d_n - \mu_n)} \text{Res}(X_1, \dots, X_{n-1}, X_n)^{(d_1 - \mu_1) \dots (d_{n-1} - \mu_{n-1})d_n} \\ &= \pm t^{\prod_{i=1}^n (d_i - \mu_i)}. \end{aligned}$$

Therefore, for this particular specialization, we get that  $\text{Res}(f_1, \dots, f_n)$  is of degree  $\prod_{i=1}^n (d_i - \mu_i)$ , and hence that, in the generic context, the degree of  $H$  can not be greater than  $\prod_{i=1}^n (d_i - \mu_i)$  which concludes the proof.  $\square$

We mention that from an historical point of view, the above result is the beginning of the theory of the *reduced resultant*. Indeed, Zariski proved [Zar37] that the factor  $H_1$  is a generator of a principal ideal whose geometric interpretation is that the polynomials  $h_1, \dots, h_n$  have a common root in addition of the root  $X_1 = \dots = X_{n-1} = 0$  that they already have in common. It is called the *reduced resultant*. We refer the interested reader to [Zar37] and [OM88] for more details.

## 2.4. The Dedekind-Mertens Lemma

We end this section of preliminaries by recalling the Dedekind-Mertens Lemma and give an important corollary that we will use several times in this text (sometimes even implicitly).

Let  $A$  be a commutative ring and  $\underline{X} := (X_1, \dots, X_n)$  be a sequence of  $n \geq 1$  indeterminates. Given a  $A$ -module  $M$  and an element

$$m = \sum_{\alpha} c_{\alpha} X^{\alpha} \in M[\underline{X}] := M[X_1, \dots, X_n]$$

we define the *support* of  $m$  as

$$\text{supp}(m) = \{\alpha \in \mathbb{N}^n : c_{\alpha} \neq 0\}$$

and the *length* of  $m$ , denoted  $l(m)$ , as the cardinal of  $\text{supp}(m)$ . Observe that  $l(m) = 0$  if and only if  $m = 0$ . Moreover, for any subring  $R$  of  $A$ , we define the  $R$ -content of  $m$  as the  $R$ -submodule of  $M$ :

$$C_R(m) := \sum_{\alpha \in \text{supp}(m)} c_{\alpha} R.$$

**Lemma 2.7 (Dedekind-Mertens).** *Let  $M$  be a  $A$ -module,  $f$  be a polynomial in  $A[\underline{X}]$  and  $m$  a polynomial in  $M[\underline{X}]$ . Then, for all subrings  $R$  of  $A$  we have*

$$C_R(f)^{l(m)} C_R(m) = C_R(f)^{l(m)-1} C_R(fm)$$

where we set, by convention,  $C_R(f)^{-1} = R$ .

**Corollary 2.8.** *Let  $M$  be a  $A$ -module and  $f \in A[\underline{X}]$  a polynomial. Then, the following are equivalent:*

- (i) *The polynomial  $f$  is a nonzero divisor in the  $A[\underline{X}]$ -module  $M[\underline{X}]$ .*
- (ii) *The ideal  $C_A(f)$  does not divide zero in  $M$  (there does not exist  $m \in M$  such that  $m \neq 0$  and  $C_A(f)m = 0$ ).*

*Proof.* Assume that (i) holds and that there exists  $m \in M$  such that  $C_A(f)m = 0$ . Then we have  $(m \cdot 1_{A[\underline{X}]})f = 0$  in  $M[\underline{X}]$  and hence  $m = 0$ , which proves that (i) implies (ii).

Now, assume that (ii) holds and that there exists  $m \in M[\underline{X}]$  such that  $mf = 0$ . Then, by the Dedekind-Mertens lemma, we deduce that  $C_A(f)^{l(m)} C_A(m) = 0$  and from (ii) that  $C_A(m) = 0$ . It follows that  $m = 0$  and the corollary is proved.  $\square$

Finally, recall that a polynomial  $f \in A[\underline{X}]$  is said to be *primitive* if  $C_A(f) = A$ .

## 3. The discriminant of a finite set of points

### 3.1. Definition and first properties

In this section, we give the definition of the discriminant of  $n - 1$  homogeneous polynomials in  $n$  variables. We begin the section with some properties on Jacobian determinants. Then, we provide computational rules for handling this discriminant and we show that its definition has the expected geometric property: its vanishing corresponds to the detection of a singular locus.

Hereafter, we suppose given  $n - 1$ , with  $n \geq 2$ , homogeneous polynomials  $f_1, \dots, f_{n-1}$  of positive degree  $d_1, \dots, d_{n-1}$ , respectively,

$$f_i(X_1, \dots, X_n) = \sum_{|\alpha|=d_i} U_{i,\alpha} X^{\alpha}, \quad i = 1, \dots, n - 1.$$

We denote by  $k$  an arbitrary commutative ring and set  ${}_k A := k[U_{i,\alpha}]$  the universal coefficient ring over  $k$ . Thus,  $f_i \in {}_k A[X_1, \dots, X_n]_{d_i}$  for all  $i = 1, \dots, n - 1$ .

**3.1.1. Jacobian determinants.** For all  $i = 1, \dots, n$ , consider the Jacobian determinant

$$J_i(f_1, \dots, f_{n-1}) := (-1)^{n-i} \begin{vmatrix} \partial_{X_1} f_1 & \cdots & \partial_{X_{i-1}} f_1 & \partial_{X_{i+1}} f_1 & \cdots & \partial_{X_n} f_1 \\ \partial_{X_1} f_2 & \cdots & \partial_{X_{i-1}} f_2 & \partial_{X_{i+1}} f_2 & \cdots & \partial_{X_n} f_2 \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{X_1} f_{n-1} & \cdots & \partial_{X_{i-1}} f_{n-1} & \partial_{X_{i+1}} f_{n-1} & \cdots & \partial_{X_n} f_{n-1} \end{vmatrix} \quad (3.1.1)$$

that is obviously a homogeneous polynomial in the variables  $X_1, \dots, X_n$  of degree  $\deg(J_i) = \sum_{j=1}^{n-1} (d_j - 1)$ . Notice that this degree is independent of  $i \in \{1, \dots, n\}$ .

**Lemma 3.1.** *For all integers  $i \in \{1, \dots, n\}$ , we have:*

- i) *the Jacobian determinant  $J_i := J_i(f_1, \dots, f_{n-1})$  is irreducible in  $\mathbb{Z}A[X_1, \dots, X_n]$ ,*
- ii) *the polynomial  $J_i(X_1, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)$  is primitive, hence a nonzero divisor, in  ${}_k A[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$ ,*
- iii) *if  $k$  is a domain then  $J_i(X_1, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)$  is prime in the polynomial ring  ${}_k A[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$ .*

*Proof.* It is sufficient to prove this result for  $J_n := J_n(f_1, \dots, f_{n-1})$ . Observe first that  $J_n$  is homogeneous of degree 1 in each set of variables  $(U_{i,\alpha})_{|\alpha|=d_i}$  with  $i \in \{1, \dots, n-1\}$ . Now, consider the specialization  $\rho$  that sends each polynomial  $f_i$ ,  $i = 1, \dots, n-1$ , to

$$f_i \mapsto U_{i,1} X_1 X_n^{d_i-1} + U_{i,2} X_2 X_n^{d_i-1} + \cdots + U_{i,n-1} X_{n-1} X_n^{d_i-1}.$$

We have

$$\rho(J_n) = X_n^{\deg(J_n)} \begin{vmatrix} U_{1,1} & \cdots & U_{1,n-1} \\ \vdots & & \vdots \\ U_{n-1,1} & \cdots & U_{n-1,n-1} \end{vmatrix}. \quad (3.1.2)$$

Let us assume first that  $k$  is a UFD. Then the determinant in (3.1.2) is known to be irreducible in  $k[U_{i,j} | i, j = 1, \dots, n-1]$ . Since  $\rho$  preserves the homogeneity with respect to each set of variables  $(U_{i,\alpha})_{|\alpha|=d_i}$ ,  $i \in \{1, \dots, n-1\}$ , we deduce that iii) holds (under the assumption that  $k$  is a UFD).

Moreover, assuming that  $k = \mathbb{Z}$ , (3.1.2) implies that  $J_n$  decomposes as a product  $P \cdot Q$  where  $P$  is irreducible and depends on the  $U_{i,\alpha}$ 's,  $Q$  does not depend on the  $U_{i,\alpha}$ 's. Moreover  $Q \in \mathbb{Z}[X_1, \dots, X_n]$  so that it must divide  $X_n^{\deg(J_n)}$ . Now, if we specialize each polynomial  $f_i$  to  $X_i^{d_i}$ , then  $J_n$  specializes to  $\prod_{i=1}^{n-1} d_i X_i^{d_i-1}$ . It follows that  $Q$  must also divide this latter polynomial and we deduce that  $Q$  is equal to  $\pm 1 \in \mathbb{Z}$ . This proves i).

Now, we prove that iii) holds under the weaker assumption that  $k$  is a domain. For that purpose, consider the quotient ring

$${}_k Q := {}_k A[X_1, \dots, X_{n-1}] / (J_n(X_1, \dots, X_{n-1}, 1))$$

and set  $Q := {}_{\mathbb{Z}} Q$  for simplicity in the notation. We have already proved that  ${}_k Q$  is a domain as soon as  $k$  is a UFD. In particular  $Q$  is a domain. Since  $Q$  contains  $\mathbb{Z}$ ,  $Q$  is a torsion-free abelian group and hence it is flat. It follows that the canonical inclusion of rings  $k \subset K := \text{Frac}(k)$  gives rise to an injective map

$${}_k Q = k \otimes_{\mathbb{Z}} Q \rightarrow K \otimes_{\mathbb{Z}} Q = {}_K Q.$$

But we have proved that  ${}_K Q$  is a domain, so we deduce that  ${}_k Q$  is also a domain and hence that  $J_n(X_1, \dots, X_{n-1}, 1)$  is a prime element in  ${}_k A[X_1, \dots, X_{n-1}]$  as claimed.

Finally, from i) we deduce that  $J_n(X_1, \dots, X_{n-1}, 1)$  is a primitive polynomial in the polynomial ring  ${}_k A[X_1, \dots, X_{n-1}]$ . It follows that it is also primitive over any commutative ring  $k$ , hence a nonzero-divisor by the Dedekind-Mertens Lemma.  $\square$

**Remark 3.2.** Notice that the Jacobian determinant  $J_i \in {}_kA[X_1, \dots, X_n]$  is not irreducible in general. Indeed, take for instance  $n = 2$  and set  $f_1(X_1, X_2) = \sum_{i=0}^d U_i X_1^i X_2^{d-i}$ . Then

$$J_2 = \frac{\partial f_1}{\partial X} = dU_d X_1^{d-1} + (d-1)U_{d-1} X_1^{d-2} X_2 + \dots + U_1 X_2^{d-1}$$

and hence if  $d = 0$  in  $k$  then  $X_2$  divides  $J_2$ .

Similarly, the Jacobian determinant of  $n$  homogeneous polynomials in  $n$  homogeneous variables is not irreducible in general. For instance, the Jacobian of the polynomials

$$f_1(X_1, X_2) = aX_1^2 + bX_1X_2 + cX_2^2, \quad f_2(X_1, X_2) = uX_1^2 + vX_1X_2 + wX_2^2$$

is equal to the determinant

$$\begin{vmatrix} bX_2 & bX_1 \\ vX_2 & vX_1 \end{vmatrix}$$

which is identically to zero in  $k[a, b, c, u, v, w][X_1, X_2]$  if  $2 = 0$  in  $k$ .

Now, introduce the generic homogeneous polynomial of degree  $d \geq 1$  in the set of variables  $X_1, \dots, X_n$

$$F(X_1, \dots, X_n) := \sum_{|\alpha|=d} U_\alpha X^\alpha$$

and set  ${}_kA' := {}_kA[U_\alpha : |\alpha| = d]$ . The Jacobian determinant

$$J(f_1, \dots, f_{n-1}, F) := \begin{vmatrix} \partial_{X_1} f_1 & \partial_{X_2} f_1 & \cdots & \partial_{X_n} f_1 \\ \partial_{X_1} f_2 & \partial_{X_2} f_2 & \cdots & \partial_{X_n} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{X_1} f_{n-1} & \partial_{X_2} f_{n-1} & \cdots & \partial_{X_n} f_{n-1} \\ \partial_{X_1} F & \partial_{X_2} F & \cdots & \partial_{X_n} F \end{vmatrix} \quad (3.1.3)$$

is a homogeneous polynomial of degree  $\deg(J) = (d-1) + \sum_{i=1}^{n-1} (d_i - 1)$  in the set of variables  $X_1, \dots, X_n$ . By developing the determinant (3.1.3) with respect to its last row, we obtain the equality

$$J(f_1, \dots, f_{n-1}, F) = \sum_{i=1}^n \frac{\partial F}{\partial X_i} J_i(f_1, \dots, f_{n-1})$$

that holds in the ring  ${}_kA'[X_1, \dots, X_n]$ .

**Lemma 3.3.** With the above notation, we have:

i) for all integers  $i \in \{1, \dots, n\}$

$$X_i J(f_1, \dots, f_{n-1}, F) - dF J_i(f_1, \dots, f_{n-1}) \in (d_1 f_1, \dots, d_{n-1} f_{n-1}) \subset {}_kA'[X_1, \dots, X_n].$$

ii) for all pairs  $(i, j)$  of distinct integers in  $\{1, \dots, n\}$

$$X_i J_j(f_1, \dots, f_{n-1}) - X_j J_i(f_1, \dots, f_{n-1}) \in (d_1 f_1, \dots, d_{n-1} f_{n-1}) \subset {}_kA'[X_1, \dots, X_n].$$

*Proof.* These properties follow straightforwardly by using Euler's identities

$$\sum_{j=1}^n X_j \frac{\partial f_i}{\partial X_j} = d_i f_i, \quad i = 1, \dots, n$$

in the determinants (3.1.1) and (3.1.3). □

**3.1.2. Definition of the discriminant.** The definition of the discriminant of the homogeneous polynomials  $f_1, \dots, f_{n-1}$  is based on the

**Proposition 3.4.** *With the previous notation,*

$$d^{d_1 \dots d_{n-1}} \text{Res}(f_1, \dots, f_{n-1}, F) \text{ divides } \text{Res}(f_1, \dots, f_{n-1}, J(f_1, \dots, f_{n-1}, F))$$

in  ${}_k A'$ . Moreover, for all  $i \in \{1, \dots, n\}$ , we have the equality

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, J(f_1, \dots, f_{n-1}, F)) \text{Res}(f_1, \dots, f_{n-1}, X_i) = \\ d^{d_1 \dots d_{n-1}} \text{Res}(f_1, \dots, f_{n-1}, F) \text{Res}(f_1, \dots, f_{n-1}, J_i) \end{aligned}$$

*Proof.* By specialization, it is sufficient to prove this proposition over the integers, that is to say by assuming that  $k = \mathbb{Z}$ .

By Lemma 3.3 we know that  $X_i J(f_1, \dots, f_{n-1}, F)$  and  $dJ_i(f_1, \dots, f_{n-1})F$  are homogeneous polynomials of the same degree in the variables  $X_1, \dots, X_n$  that are equal modulo the ideal  $(f_1, \dots, f_{n-1})$ . It follows that, in  ${}_k A'$ ,

$$\text{Res}(f_1, \dots, f_{n-1}, X_i J(f_1, \dots, f_{n-1}, F)) = \text{Res}(f_1, \dots, f_{n-1}, dJ_i(f_1, \dots, f_{n-1})F).$$

The result then follows from standard properties of resultants [Jou91, §5].  $\square$

We are now ready to state the definition of the discriminant of the polynomials  $f_1, \dots, f_{n-1}$ .

**Definition 3.5.** *If  $\sum_{i=1}^{n-1} (d_i - 1) \geq 1$  then the discriminant of the polynomials  $f_1, \dots, f_{n-1}$ , denoted  $\text{Disc}(f_1, \dots, f_{n-1})$ , is defined as the unique non-zero element in  $\mathbb{Z}A$  such that*

$$\text{Disc}(f_1, \dots, f_{n-1}) \text{Res}(f_1, \dots, f_{n-1}, X_i) = \text{Res}(f_1, \dots, f_{n-1}, J_i) \quad (3.1.4)$$

for all  $i \in \{1, \dots, n\}$ . If  $\sum_{i=1}^{n-1} (d_i - 1) = 0$ , or equivalently if  $d_1 = \dots = d_{n-1} = 1$ , we set  $\text{Disc}(f_1, \dots, f_{n-1}) = 1 \in \mathbb{Z}A$ .

Let  $R$  be a commutative ring and suppose given  $n - 1$  homogeneous polynomials

$$g_i = \sum_{|\alpha|=d_i} u_{i,\alpha} X^\alpha \in R[X_1, \dots, X_n], \quad i = 1, \dots, n - 1,$$

of degree  $d_1, \dots, d_{n-1}$  respectively. As in §2.2, denote by  $\theta$  the ring morphism  $\theta : \mathbb{Z}A \rightarrow R : U_{j,\alpha} \mapsto u_{j,\alpha}$  corresponding to the specialization of the polynomial  $f_i$  to the polynomial  $g_i$  for all  $i = 1, \dots, n - 1$ . Then, the discriminant of  $g_1, \dots, g_{n-1}$  is defined as

$$\text{Disc}(g_1, \dots, g_{n-1}) := \theta(\text{Disc}(f_1, \dots, f_{n-1})) \in R.$$

**Remark 3.6.** *We recall that, for all integers  $i \in \{1, \dots, n\}$ ,*

$$\text{Res}(f_1, \dots, f_{n-1}, X_i) = \text{Res}(f_1^{(i)}, \dots, f_{n-1}^{(i)}) \in {}_k A$$

where  $f_1^{(i)}, \dots, f_{n-1}^{(i)}$  are the polynomials obtained from  $f_1, \dots, f_{n-1}$ , respectively, by substituting  $X_i$  for 0 (see [Jou91, Lemma 4.8.9]). It is a nonzero divisor in  ${}_k A$  (see Proposition 2.3).

A direct consequence of the definition of the discriminant is the following. From Proposition 3.4, it follows immediately that, in  ${}_k A'$ ,

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, J(f_1, \dots, f_{n-1}, F)) = \\ d^{d_1 \dots d_{n-1}} \text{Disc}(f_1, \dots, f_{n-1}) \text{Res}(f_1, \dots, f_{n-1}, F). \end{aligned} \quad (3.1.5)$$

Moreover, if  $\deg(F) = d = 1$  then  $J(f_1, \dots, f_{n-1}, F)$  can be replaced by  $F(J_1, \dots, J_n)$  in this formula and we get

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, U_1 J_1 + \dots + U_n J_n) = \\ \text{Disc}(f_1, \dots, f_{n-1}) \text{Res}(f_1, \dots, f_{n-1}, U_1 X_1 + \dots + U_n X_n). \end{aligned}$$

More generally, we have the

**Proposition 3.7.** *For all  $d \geq 1$  the following equality holds in  ${}_k A'$ :*

$$\text{Res}(f_1, \dots, f_{n-1}, F(J_1, \dots, J_n)) = \text{Disc}(f_1, \dots, f_{n-1})^d \text{Res}(f_1, \dots, f_{n-1}, F).$$

*Proof.* Indeed, Lemma 3.3 shows that both polynomials  $X_i^d F(J_1, \dots, J_n)$  and  $J_i^d F(X_1, \dots, X_n)$  are homogeneous of the same degree in the variables  $X_1, \dots, X_n$  and equal up to an element in the ideal  $(f_1, \dots, f_{n-1})$ . It follows that

$$\text{Res}(f_1, \dots, f_{n-1}, X_i^d F(J_1, \dots, J_n)) = \text{Res}(f_1, \dots, f_{n-1}, J_i^d F(X_1, \dots, X_n))$$

and the claimed formula is obtained using the multiplicativity property of the resultants [Jou91, §5.7].  $\square$

An important property of the *generic* discriminant is that, similarly to the generic resultant, it is universally a nonzero divisor.

**Proposition 3.8.** *The discriminant  $\text{Disc}(f_1, \dots, f_{n-1}) \in {}_k A$  is a nonzero divisor.*

*Proof.* By specializing each polynomial  $f_i$  to a product of generic linear form, the discriminant specializes to a primitive polynomial (the ideal generated by its coefficients is equal to  $k$ ) by Corollary 3.17. It follows that  $\text{Disc}(f_1, \dots, f_{n-1}) \in {}_k A$  itself a primitive polynomial in  ${}_k A$ . Therefore, the claimed result follows by Dedekind-Mertens Lemma.  $\square$

**3.1.3. The degree of the discriminant.** As inheritance of the resultant, the discriminant is multi-homogeneous : it is homogeneous with respect to the coefficients of *each* polynomial  $f_1, \dots, f_{n-1}$ . The following result gives the precise multi-degree of the discriminant.

**Proposition 3.9.** *With the notation of §3.1.2,  $\text{Disc}(f_1, \dots, f_{n-1})$  is a homogeneous polynomial in  ${}_k A$  of total degree*

$$(n-1) \prod_{i=1}^{n-1} d_i + (d_1 + \dots + d_{n-1} - n) \left( \sum_{i=1}^{n-1} \frac{d_1 \cdots d_{n-1}}{d_i} \right).$$

Moreover, for all  $i \in \{1, \dots, n-1\}$  it is homogeneous with respect to the coefficients of the polynomial  $f_i$  of degree

$$\frac{d_1 \cdots d_{n-1}}{d_i} \left( (d_i - 1) + \sum_{j=1}^{n-1} (d_j - 1) \right). \quad (3.1.6)$$

*Proof.* Let us fix an integer  $i \in \{1, \dots, n-1\}$  and introduce a new variable  $t$ . We know that the Jacobian polynomial  $J_n$  is homogeneous in the variables  $X_1, \dots, X_n$  of degree  $\sum_{i=1}^{n-1} (d_i - 1)$ . It also obviously satisfies

$$J_n(f_1, \dots, t f_i, \dots, f_{n-1}) = t J_n(f_1, \dots, f_i, \dots, f_{n-1}). \quad (3.1.7)$$

Therefore, by multi-homogeneity property of the resultant [Jou91, 2.3(ii)], we deduce that

$$\begin{aligned} & \text{Res}(f_1, \dots, t f_i, \dots, f_{n-1}, J_n(f_1, \dots, t f_i, \dots, f_{n-1})) \\ &= t^{\frac{d_1 \cdots d_{n-1}}{d_i} \sum_{j=1}^{n-1} (d_j - 1)} \text{Res}(f_1, \dots, f_{n-1}, J_n(f_1, \dots, t f_i, \dots, f_{n-1})) \\ &= t^{\frac{d_1 \cdots d_{n-1}}{d_i} \sum_{j=1}^{n-1} (d_j - 1)} \text{Res}(f_1, \dots, f_{n-1}, t J_n(f_1, \dots, f_n)) \\ &= t^{\frac{d_1 \cdots d_{n-1}}{d_i} \sum_{j=1}^{n-1} (d_j - 1) + \prod_{i=1}^{n-1} d_i} \text{Res}(f_1, \dots, f_{n-1}, J_n(f_1, \dots, f_n)) \end{aligned}$$

and

$$\text{Res}(f_1, \dots, t f_i, \dots, f_{n-1}, X_n) = t^{\frac{d_1 \cdots d_{n-1}}{d_i}} \text{Res}(f_1, \dots, f_i, \dots, f_{n-1}, X_n).$$

From Definition 3.5 of the discriminant, it follows that

$$\text{Disc}(f_1, \dots, t f_i, \dots, f_{n-1}) = t^{\frac{d_1 \cdots d_{n-1}}{d_i} ((d_i - 1) + \sum_{j=1}^{n-1} (d_j - 1))} \text{Disc}(f_1, \dots, f_{n-1})$$

as claimed. The total degree is obtained by adding all these partial degrees.  $\square$

**Remark 3.10.** *Observe that the integers (3.1.6) are always even. This is expected because, as we will see later on, in characteristic 2 it turns out that the discriminant is the square of an irreducible polynomial.*

**3.1.4. The classical case  $n = 2$ .** Let us show that our definition of the discriminant coincides with the classical case  $n = 2$ .

Let  $f$  be a polynomial homogeneous in the variable  $X, Y$  of degree  $d \geq 2$

$$f := V_d X^d + V_{d-1} X^{d-1} Y + V_{d-2} X^{d-2} Y^2 + \cdots + V_1 X Y^{d-1} + V_0 Y^d.$$

According to (3.1.4) we have

$$\text{Res}(f, J_2(f)) = \text{Disc}(f) \text{Res}(f, Y) \in k[V_0, \dots, V_d].$$

But it is easy to see that  $\text{Res}(f, Y) = V_d$  and that  $J_2(f) = \frac{\partial f}{\partial X}$ . Therefore we recover the usual definition  $V_d \text{Disc}(f) = \text{Res}(f, \frac{\partial f}{\partial X})$ . Moreover, from Proposition 3.9 we also obtain that it is a homogeneous polynomial in the coefficients of  $f$ , i.e.  $V_0, \dots, V_d$ , of degree  $2d - 2$ .

A lot of properties are known for this discriminant (see e.g. [AJ06] or [GKZ94, chapter 12.B]) and we will generalize most of them to the case of  $n - 1$  homogeneous polynomials in  $n$  variables in the sequel.

**3.1.5. Vanishing of the discriminant.** Assume that  $k$  is an algebraically closed field and let  $f_1, \dots, f_{n-1}$  be  $n - 1$  homogeneous polynomials in  $k[X_1, \dots, X_n]$  such that the variety

$$Y := V(f_1, \dots, f_{n-1}) \subset \mathbb{P}_k^{n-1}$$

is finite. The following proposition says that the discriminant of  $f_1, \dots, f_{n-1}$  vanishes if and only if the polynomial system  $f_1 = \cdots = f_{n-1} = 0$  has a multiple root.

**Proposition 3.11.** *With the above notation,  $\text{Disc}(f_1, \dots, f_{n-1}) = 0$  if and only if there exists a point  $\xi \in Y$  such that  $Y$  is singular at  $\xi$ .*

*Proof.* First, without loss of generality we can assume  $Y \cap V(X_n) = \emptyset$ , so that the resultant  $\text{Res}(f_1, \dots, f_{n-1}, X_n)$  is not equal to zero in  $k$  and

$$\text{Disc}(f_1, \dots, f_{n-1}) = \frac{\text{Res}(f_1, \dots, f_{n-1}, J_n)}{\text{Res}(f_1, \dots, f_{n-1}, X_n)} \in k.$$

By the Poisson's formula [Jou91, Proposition 2.7], we have the equality

$$\frac{\text{Res}(f_1, \dots, f_{n-1}, J_n)}{\text{Res}(f_1, \dots, f_{n-1}, X_n)^{\deg(J_n)}} = \prod_{\xi \in Y} J_n(\xi)^{\mu_\xi}$$

where  $\mu_\xi$  denotes the multiplicity of  $\xi \in Y$ . It follows that  $\text{Disc}(f_1, \dots, f_{n-1}) = 0$  if and only if there exists a point  $\xi \in Y$  such that  $J_n(\xi) = 0$ .

Now, a classical necessary and sufficient condition for  $\xi \in Y$  to be a singular point of  $Y$  is that  $J_i(\xi) = 0$  for all  $i = 1, \dots, n$  (see e.g. [Har77, Chapter I, Theorem 5.1]). But from Lemma 3.3, ii), we have  $J_i(\xi) = \xi_i J_n(\xi)$  for all  $\xi \in Y$  and all  $i = 1, \dots, n - 1$ , where  $\xi = (\xi_1 : \xi_2 : \cdots : \xi_{n-1} : 1) \in Y \subset \mathbb{P}_k^{n-1}$ . Therefore, we deduce that  $\xi \in Y$  is a singular point of  $Y$  if and only if  $J_n(\xi) = 0$ .  $\square$

This proposition gives a geometric interpretation of the discriminant of  $n - 1$  homogeneous polynomials in  $n$  variables. We will give a more precise description of its geometry in Section 3.3.

### 3.2. Formulas and formal properties

In this section we give some properties of the discriminant. Thanks to the definition we gave of the discriminant in terms of the resultant, it turns out that most of these properties can be derived from the known ones of the resultant.

Hereafter  $R$  will denote an arbitrary commutative ring.

**3.2.1. Elementary transformations.** The discriminant of  $n - 1$  homogeneous polynomials  $f_1, \dots, f_{n-1}$  is invariant under a permutation of the  $f_i$ 's. It is also invariant if one adds to one of the  $f_i$ 's an element in the ideal generated by the others.

**Proposition 3.12.** *for all  $j = 1, \dots, n - 1$ , let  $f_j$  be a homogeneous polynomial of degree  $d_j \geq 1$  in  $R[X_1, \dots, X_n]$ . Then,*

i) *for any permutation  $\sigma$  of the set  $\{1, \dots, n - 1\}$  we have*

$$\text{Disc}(f_{\sigma(1)}, \dots, f_{\sigma(n-1)}) = \text{Disc}(f_1, \dots, f_{n-1}) \text{ in } R.$$

ii) *for all  $i \in \{1, \dots, n - 1\}$  we have*

$$\text{Disc}(f_1, \dots, f_i + \sum_{j \neq i} h_{i,j} f_j, \dots, f_{n-1}) = \text{Disc}(f_1, \dots, f_{n-1}) \text{ in } R,$$

*where the  $h_{i,j}$ 's are arbitrary homogeneous polynomials in  $R[X_1, \dots, X_n]$  of respective degrees  $d_i - d_j$  (therefore  $h_{i,j} = 0$  if  $d_i < d_j$ ).*

*Proof.* Of course, it is sufficient to prove these properties in the generic case. The property ii) is an immediate consequence of [Jou91, §5.9].

To prove i), we first remark that

$$J_n^\sigma := J_n(f_{\sigma(1)}, \dots, f_{\sigma(n-1)}) = \epsilon(\sigma) J_n(f_1, \dots, f_{n-1}).$$

Then, using [Jou91, §5.8] we deduce that

$$\begin{aligned} \text{Res}(f_{\sigma(1)}, \dots, f_{\sigma(n-1)}, J_n^\sigma) &= \epsilon(\sigma)^{d_1 \dots d_{n-1}} \text{Res}(f_{\sigma(1)}, \dots, f_{\sigma(n-1)}, J_n) \\ &= \epsilon(\sigma)^{d_1 \dots d_{n-1}} \epsilon(\sigma)^{d_1 \dots d_{n-1} \deg(J_n)} \text{Res}(f_1, \dots, f_n, J_n), \end{aligned}$$

and

$$\text{Res}(f_{\sigma(1)}, \dots, f_{\sigma(n-1)}, X_n) = \epsilon(\sigma)^{d_1 \dots d_{n-1}} \text{Res}(f_1, \dots, f_{n-1}, X_n).$$

From here the claimed result follows from (3.1.4) (with  $i = n$ ) and the fact that

$$d_1 \dots d_{n-1} \deg(J_n) = d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)$$

is always an even integer. □

**3.2.2. Reduction on the variables.** Hereafter, for any polynomial  $f \in R[X_1, \dots, X_n]$  we denote by  $f^{(j)}$  the polynomial obtained by substituting  $X_j$  with 0 in  $f$ . Notice that

$$f^{(j)} \in R[X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n].$$

**Proposition 3.13** ( $n \geq 3$ ). *For all  $i = 1, \dots, n - 2$ , let  $f_i$  be a homogeneous polynomial of degree  $d_i \geq 1$  in  $R[X_1, \dots, X_n]$ . The following equality holds in  $R$ :*

$$\text{Disc}(f_1, \dots, f_{n-2}, X_n) = (-1)^{d_1 \dots d_{n-2}} \text{Disc}(f_1^{(n)}, \dots, f_{n-2}^{(n)}).$$



*Proof.* It is sufficient to prove this formula in the generic context. From the definition of the discriminant we thus have the equality

$$\operatorname{Res}(f_1, \dots, f_{n-2}, X_n, J_{n-1}(f_1, \dots, f_{n-2}, X_n)) = \operatorname{Disc}(f_1, \dots, f_{n-2}, X_n) \operatorname{Res}(f_1, \dots, f_{n-2}, X_n, X_{n-1}).$$

But, from (3.1.1) we deduce that

$$J_{n-1}(f_1, \dots, f_{n-2}, X_n) = - \left| \frac{\partial(f_1, \dots, f_{n-2}, X_n)}{\partial(X_1, \dots, X_{n-2}, X_n)} \right| = (-1)^n \left| \frac{\partial(f_1, \dots, f_{n-2})}{\partial(X_1, \dots, X_{n-2})} \right|.$$

And since

$$\operatorname{Res}(f_1, \dots, f_{n-2}, X_n, J_{n-1}(f_1, \dots, f_{n-2}, X_n)) = (-1)^{d_1 \dots d_{n-2} \sum_{i=1}^{n-2} (d_i - 1)} \operatorname{Res}(f_1, \dots, f_{n-2}, J_{n-1}(f_1, \dots, f_{n-2}, X_n), X_n)$$

where  $d_1 \dots d_{n-2} \sum_{i=1}^{n-2} (d_i - 1)$  is even, it comes

$$\operatorname{Res}(f_1, \dots, f_{n-2}, X_n, J_{n-1}(f_1, \dots, f_{n-2}, X_n)) = \operatorname{Res}(f_1^{(n)}, \dots, f_{n-2}^{(n)}, J_{n-1}(f_1^{(n)}, \dots, f_{n-2}^{(n)})). \quad (3.2.1)$$

Moreover, we also have

$$\begin{aligned} \operatorname{Res}(f_1, \dots, f_{n-2}, X_n, X_{n-1}) &= (-1)^{d_1 \dots d_{n-2}} \operatorname{Res}(f_1, \dots, f_{n-2}, X_{n-1}, X_n) \\ &= (-1)^{d_1 \dots d_{n-2}} \operatorname{Res}(f_1^{(n)}, \dots, f_{n-2}^{(n)}, X_{n-1}). \end{aligned}$$

Now taking the ratio of both previous quantities we obtain,

$$(-1)^{d_1 \dots d_{n-2}} \operatorname{Res}(f_1^{(n)}, \dots, f_{n-2}^{(n)}, J_{n-1}(f_1^{(n)}, \dots, f_{n-2}^{(n)})) = \operatorname{Disc}(f_1, \dots, f_{n-2}, X_n) \operatorname{Res}(f_1^{(n)}, \dots, f_{n-2}^{(n)}, X_{n-1})$$

so that, as claimed,

$$\operatorname{Disc}(f_1, \dots, f_{n-2}, X_n) = (-1)^{d_1 \dots d_{n-2}} \operatorname{Disc}(f_1^{(n)}, \dots, f_{n-2}^{(n)}). \quad \square$$

The following proposition and corollary give reductions of the discriminant in cases where certain polynomials  $f_1, \dots, f_{n-1}$  do not depend on all the variables  $X_1, \dots, X_n$ .

**Proposition 3.14** ( $n \geq 3$ ). *Let  $k \in \{2, \dots, n-1\}$  and for all  $i = 1, \dots, n-1$  let  $f_i$  be a homogeneous polynomial of degree  $d_i \geq 1$  in  $R[X_1, \dots, X_n]$  such that  $\sum_{i=1}^{k-1} (d_i - 1) \geq 1$ . Assume moreover that  $f_1, \dots, f_{k-1}$  only depend on the variables  $X_1, \dots, X_k$ . Then, denoting for all integers  $i = k, \dots, n-1$*

$$\hat{f}_i = f_i(0, \dots, 0, X_{k+1}, \dots, X_n) \in R[X_{k+1}, \dots, X_n],$$

*we have the equality*

$$\begin{aligned} \operatorname{Disc}(f_1, \dots, f_{n-1}) &= (-1)^{(n-k) \prod_{i=1}^{n-1} d_i} \operatorname{Disc}(f_1, \dots, f_{k-1}) \prod_{i=k}^{n-1} d_i \\ &\operatorname{Res}(\hat{f}_k, \dots, \hat{f}_{n-1})^{\left(\prod_{i=1}^{k-1} d_i\right) \left(\sum_{i=1}^{k-1} d_i - k\right)} \operatorname{Res}\left(f_1, \dots, f_{n-1}, \left| \frac{\partial(f_k, \dots, f_{n-1})}{\partial(X_{k+1}, \dots, X_n)} \right| \right). \end{aligned}$$

*Proof.* As always, it is sufficient to prove this formula in the generic case. By definition we have

$$\operatorname{Res}(f_1, \dots, f_{n-1}, X_1) \operatorname{Disc}(f_1, \dots, f_{n-1}) = \operatorname{Res}(f_1, \dots, f_{n-1}, J_1(f_1, \dots, f_{n-1})).$$

From the hypothesis, the Jacobian determinant involved in this formula decomposes into four square blocks and one of them is identically zero. More precisely, one has

$$J_1(f_1, \dots, f_{n-1}) = \left| \frac{\partial(f_1, \dots, f_{k-1})}{\partial(X_2, \dots, X_k)} \right| \left| \frac{\partial(f_k, \dots, f_{n-1})}{\partial(X_{k+1}, \dots, X_n)} \right|$$

and by multiplicativity of the resultant [Jou91, §5.7] we deduce

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, J_1(f_1, \dots, f_{n-1})) &= \text{Res}\left(f_1, \dots, f_{n-1}, \left| \frac{\partial(f_1, \dots, f_{k-1})}{\partial(X_2, \dots, X_k)} \right| \right) \\ &\quad \times \text{Res}\left(f_1, \dots, f_{n-1}, \left| \frac{\partial(f_k, \dots, f_{n-1})}{\partial(X_{k+1}, \dots, X_n)} \right| \right). \end{aligned}$$

Now, permuting polynomials in the resultant [Jou91, §5.8],

$$\begin{aligned} \text{Res}\left(f_1, \dots, f_{n-1}, \left| \frac{\partial(f_1, \dots, f_{k-1})}{\partial(X_2, \dots, X_k)} \right| \right) &= \\ (-1)^\nu \text{Res}\left(f_1, \dots, f_{k-1}, \left| \frac{\partial(f_1, \dots, f_{k-1})}{\partial(X_2, \dots, X_k)} \right|, f_k, \dots, f_{n-1}\right) & \quad (3.2.2) \end{aligned}$$

where  $\nu := (n-k)(\prod_{i=1}^{n-1} d_i)(\sum_{i=1}^{k-1} (d_i - 1)) \geq 1$  and is even, and using Laplace's formula [Jou91, §5.10] this latter resultant is equal to

$$\text{Res}\left(f_1, \dots, f_{k-1}, \left| \frac{\partial(f_1, \dots, f_{k-1})}{\partial(X_2, \dots, X_k)} \right| \right)^{\prod_{i=k}^{n-1} d_i} \text{Res}(\hat{f}_k, \dots, \hat{f}_{n-1})^{\left(\prod_{i=1}^{k-1} d_i\right) \sum_{i=1}^{k-1} (d_i - 1)}.$$

Similarly, we have

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, X_1) &= \\ (-1)^{(n-k)(\prod_{i=1}^{n-1} d_i)} \text{Res}(f_1, \dots, f_{k-1}, X_1)^{\prod_{i=k}^{n-1} d_i} \text{Res}(\hat{f}_k, \dots, \hat{f}_{n-1})^{\left(\prod_{i=1}^{k-1} d_i\right)} & \quad (3.2.3) \end{aligned}$$

and the claimed formula follows easily by gathering these computations.  $\square$

**Corollary 3.15.** *Let  $k \in \{1, \dots, n-1\}$  and for all  $i = 1, \dots, n-1$ , let  $f_i$  be a homogeneous polynomial of degree  $d_i \geq 1$  in  $R[X_1, \dots, X_n]$ . Assume moreover that  $d_1 \geq 2$ . If the polynomials  $f_1, \dots, f_k$  only depend on the variables  $X_1, \dots, X_k$  then  $\text{Disc}(f_1, \dots, f_{n-1}) = 0$ .*

*Proof.* First assume that  $k \geq 2$ ; since  $d_1 \geq 2$  we have  $\sum_{i=1}^{n-1} (d_i - 1) \geq 1$ . Since  $f_k$  only depends on the variables  $X_1, \dots, X_k$  we deduce that, according to the notation of the previous proposition,  $\hat{f}_k = 0$ . Consequently, using the formula of this proposition we immediately get that  $\text{Disc}(f_1, \dots, f_{n-1}) = 0$ .

Now assume that  $k = 1$ ; thus  $f_1 = U_1 X_1^{d_1}$ . One may also assume that the polynomials  $f_2, \dots, f_{n-1}$  are generic in all the variables  $X_1, \dots, X_n$ . It follows that  $\text{Res}(f_1, \dots, f_{n-1}, X_n)$  is nonzero and we know that

$$\text{Res}(f_1, \dots, f_{n-1}, X_n) \text{Disc}(f_1, \dots, f_{n-1}) = \text{Res}(f_1, \dots, f_{n-1}, J_n(f_1, \dots, f_{n-1})).$$

But since  $f_1 = U_1 X_1^{d_1}$  we deduce that  $X_1^{d_1-1}$  divides  $J_n$  and consequently that the resultant  $\text{Res}(f_1, \dots, f_{n-1}, J_n)$  vanishes.  $\square$

**3.2.3. Multiplicativity.** We now describe the multiplicativity property of the discriminant, property that was already known to Sylvester [Syl64b]. Recall that the discriminant of  $n - 1$  homogeneous polynomials of degree 1 equals 1 (the unit of the ground ring) by convention.

**Proposition 3.16.** *Let  $f'_1, f''_1, f_2, \dots, f_{n-1}$  be  $n$  homogeneous polynomials in  $R[X_1, \dots, X_n]$  of positive degree  $d'_1, d''_1, d_2, \dots, d_{n-1} \geq 1$ , respectively. Then,*

$$\text{Disc}(f'_1 f''_1, f_2, \dots, f_{n-1}) = (-1)^s \text{Disc}(f'_1, f_2, \dots, f_{n-1}) \text{Disc}(f''_1, f_2, \dots, f_{n-1}) \text{Res}(f'_1, f''_1, f_2, \dots, f_{n-1})^2,$$

where  $s := d'_1 d''_1 d_2 \dots d_{n-1}$ .

*Proof.* It is sufficient to prove this result in the generic case, so let us assume that  $f'_1, f''_1, f_2, \dots, f_n$  are generic polynomials. It is easy to see that

$$J_n(f'_1 f''_1, f_2, \dots, f_n) = f'_1 J_n(f''_1, f_2, \dots, f_n) + f''_1 J_n(f'_1, f_2, \dots, f_n) =: f'_1 J''_n + f''_1 J'_n.$$

Assume first that  $\deg(J''_n) \geq 1$  and  $\deg(J'_n) \geq 1$ . Using [Jou91, §5.7 & §5.8] we obtain

$$\begin{aligned} & \text{Res}(f'_1 f''_1, f_2, \dots, f_{n-1}, J_n) \\ &= \text{Res}(f'_1, f_2, \dots, f_{n-1}, f''_1 J'_n) \text{Res}(f''_1, f_2, \dots, f_{n-1}, f'_1 J''_n) \\ &= (-1)^s \text{Res}(f'_1, f''_1, f_2, \dots, f_{n-1})^2 \text{Res}(f'_1, f_2, \dots, f_{n-1}, J'_n) \text{Res}(f''_1, f_2, \dots, f_{n-1}, J''_n) \end{aligned}$$

where  $s := d'_1 d''_1 d_2 \dots d_{n-1}$ . And since

$$\text{Res}(f'_1 f''_1, f_2, \dots, f_{n-1}, X_n) = \text{Res}(f'_1, f_2, \dots, f_{n-1}, X_n) \text{Res}(f''_1, f_2, \dots, f_{n-1}, X_n),$$

we deduce the expected formula by applying (3.1.4).

Assume now that  $\deg(J'_n) = 0$  and  $\deg(J''_n) \geq 1$ . Then, in the previous computations, the resultant  $\text{Res}(f'_1, f_2, \dots, f_{n-1}, J'_n)$  must be replaced by  $J'_n$  (under our hypothesis  $d'_1 d_2 \dots d_{n-1} = 1$ ). But it turns out that, always since  $\deg(J'_n) = 0$ ,  $J'_n = \text{Res}(f'_1, f_2, \dots, f_{n-1}, X_n)$  and consequently the whole formula remains exact. A similar argument shows that this formula is also exact if  $\deg(J'_n) = 1$  and  $\deg(J''_n) \geq 0$ , and if  $\deg(J'_n) = \deg(J''_n) \geq 0$ .  $\square$

**Corollary 3.17.** *Let  $d_1, \dots, d_{n-1}$  be  $n-1$  integers greater or equal to 2 and let  $l_{i,j}$ , for  $1 \leq i \leq n-1$  and  $1 \leq j \leq d_i$ , be linear forms in  $R[X_1, \dots, X_n]$ . Then*

$$\text{Disc} \left( \prod_{j=1}^{d_1} l_{1,j}, \dots, \prod_{j=1}^{d_{n-1}} l_{n-1,j} \right) = (-1)^s \prod_I \det(l_{1,j_1}, l_{2,j_2}, \dots, l_{n-1,j_{n-1}}, l_{i,j})^2$$

where  $s := \frac{1}{2} \prod_{i=1}^{n-1} d_i \sum_{i=1}^{n-1} (d_i - 1)$  and the product runs over the set

$$I := \{(j_1, \dots, j_{n-1}, i, j) \mid 1 \leq j_1 \leq d_1, 1 \leq j_2 \leq d_2, \dots, 1 \leq j_{n-1} \leq d_{n-1}, \\ 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq d_i \text{ such that } j \neq j_i\}.$$

**3.2.4. Covariance.** Assume that  $n \geq 2$  and suppose given a sequence of  $n - 1$  positive integers  $d_1, \dots, d_{n-1}$  such that  $\sum_{i=1}^{n-1} (d_i - 1) \geq 1$ . For all  $d \in \mathbb{N}$  set  $I_d := \{i \in \{1, \dots, n\} \mid d_i = d\}$  and define  $L := \{d \in \mathbb{N} \mid I_d \neq \emptyset\}$ . In this way, the set  $\{1, \dots, n\}$  is the disjoint union of  $I_d$  with  $d \in L$ .

Let  $\varphi$  be a square matrix of size  $n - 1$  with coefficients in  $R$

$$\varphi = \begin{bmatrix} u_{1,1} & \cdots & u_{1,n-1} \\ \vdots & & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} \end{bmatrix}.$$

We will say that  $\varphi$  is adapted to the sequence  $d_1, \dots, d_n$  if and only if

$$u_{i,j} \neq 0 \Rightarrow d_i = d_j.$$

Equivalently,  $\varphi$  is adapted to the sequence  $d_1, \dots, d_n$  if and only if  $\varphi$  can be transformed by row and column permutations into a block diagonal matrix whose diagonal blocs are given by  $\varphi_d := \varphi|_{I_d \times I_d}$  for all  $d \in L$ ; in particular  $\det(\varphi) = \prod_{d \in L} \det(\varphi_d) \in R$ .

**Proposition 3.18.** *Assume that  $n \geq 2$  and suppose given a sequence of  $n - 1$  positive integers  $d_1, \dots, d_{n-1}$  such that  $\sum_{i=1}^{n-1} (d_i - 1) \geq 1$  and a sequence of  $n - 1$  homogeneous polynomials  $f_1, \dots, f_{n-1}$  in  $R[X_1, \dots, X_n]$  of degree  $d_1, \dots, d_{n-1}$  respectively. Then, for all  $i = 1, \dots, n - 1$  and all matrix  $\varphi = (u_{i,j})_{1 \leq i, j \leq n-1}$  with coefficients in  $R$  adapted to  $d_1, \dots, d_{n-1}$ , the polynomial  $\sum_{j=1}^{n-1} u_{i,j} f_j \in R$  is homogeneous of degree  $d_i$  and we have*

$$\text{Disc} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) = \left( \prod_{d \in L} \det(\varphi_d)^{\frac{d_1 \dots d_{n-1} ((d-1) + \sum_{i=1}^{n-1} (d_i - 1))}{d}} \right) \text{Disc}(f_1, \dots, f_{n-1}).$$

*Proof.* By specialization, we can assume that the coefficients of the polynomials  $f_1, \dots, f_{n-1}$  and all the  $u_{i,j}$  are distinct indeterminates so that  $R$  is the polynomial ring of these indeterminates over the integers.

By definition of the discriminant we have

$$\text{Res}(f_1, \dots, f_{n-1}, J_n(f_1, \dots, f_{n-1})) = \text{Disc}(f_1, \dots, f_{n-1}) \text{Res}(f_1, \dots, f_{n-1}, X_n) \quad (3.2.4)$$

and

$$\begin{aligned} & \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, J_n \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) \right) \\ &= \text{Disc} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, X_n \right). \end{aligned} \quad (3.2.5)$$

Now, it is not hard to check that

$$J_n \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) = \det(\varphi) J_n(f_1, \dots, f_{n-1})$$

so that

$$\begin{aligned} & \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, J_n \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) \right) \\ &= \det(\varphi)^{d_1 \dots d_{n-1}} \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, J_n(f_1, \dots, f_{n-1}) \right). \end{aligned}$$

But since  $J_n(f_1, \dots, f_{n-1})$  is a polynomial of degree  $\sum_{i=1}^{n-1} (d_i - 1) \geq 1$ , the covariance property of the resultant [Jou91, §5.11] yields

$$\begin{aligned} \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, J_n(f_1, \dots, f_{n-1}) \right) \\ = \left( \prod_{d \in L} \det(\varphi_d)^{\frac{d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)}{d}} \right) \text{Res}(f_1, \dots, f_n, J_n(f_1, \dots, f_{n-1})) \end{aligned}$$

and we deduce that

$$\begin{aligned} \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, J_n \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j \right) \right) \\ = \det(\varphi)^{d_1 \dots d_{n-1}} \left( \prod_{d \in L} \det(\varphi_d)^{\frac{d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)}{d}} \right) \text{Res}(f_1, \dots, f_n, J_n). \quad (3.2.6) \end{aligned}$$

Again by the covariance formula for resultants, we have

$$\begin{aligned} \text{Res} \left( \sum_{j=1}^{n-1} u_{1,j} f_j, \dots, \sum_{j=1}^{n-1} u_{n-1,j} f_j, X_n \right) = \\ \left( \prod_{d \in L} \det(\varphi_d)^{\frac{d_1 \dots d_{n-1}}{d}} \right) \text{Res}(f_1, \dots, f_{n-1}, X_n) \quad (3.2.7) \end{aligned}$$

and therefore, since  $\det(\varphi) = \prod_{d \in L} \det(\varphi_d)$ , the comparison of (3.2.4), (3.2.5), (3.2.6) and (3.2.7) gives the claimed formula.  $\square$

**3.2.5. Reduction modulo  $\delta$ .** Recall from Lemma 3.3 that, for all  $1 \leq i, j \leq n$  we have

$$X_i J_j(f_1, \dots, f_{n-1}) - X_j J_i(f_1, \dots, f_{n-1}) \in \delta \cdot (f_1, \dots, f_{n-1}) \subset \delta \cdot A[X_1, \dots, X_n] \quad (3.2.8)$$

where  $\delta := \gcd(d_1, \dots, d_{n-1})$ . Considering the (cohomological) Koszul complex associated to the sequence  $X_1, \dots, X_n$  in the ring  $A/\delta \cdot A[X_1, \dots, X_n]$

$$0 \rightarrow \frac{A}{\delta \cdot A}[X_1, \dots, X_n] \xrightarrow{\mathbf{d}_1 = {}^t[X_1, \dots, X_n]} \bigoplus_{i=1}^n \frac{A}{\delta \cdot A}[X_1, \dots, X_n] \xrightarrow{\mathbf{d}_2} \dots,$$

we notice that since  $n \geq 2$ , its cohomology groups  $H^0$  and  $H^1$  are both equal to 0. In addition, the equations (3.2.8) imply that  $(J_1, \dots, J_n)$  belongs to the kernel of  $\mathbf{d}_2$ . Therefore, we deduce that there exists a polynomial  $\Delta \in A[X_1, \dots, X_n]$  whose residue class in  $A/\delta \cdot A[X_1, \dots, X_n]$  is unique and such that

$$J_i(f_1, \dots, f_{n-1}) = X_i \Delta \pmod{\delta \cdot A[X_1, \dots, X_n]}, \quad 1 \leq i \leq n. \quad (3.2.9)$$

From here, we get the following property.

**Proposition 3.19.** *With the above notation, we have the following equality in  ${}_k A$ :*

$$\text{Disc}(f_1, \dots, f_{n-1}) = \text{Res}(f_1, \dots, f_{n-1}, \Delta) \pmod{\delta}.$$

*Proof.* From (3.2.9) and the multiplicativity of the resultant, we obtain that

$$\text{Res}(f_1, \dots, f_{n-1}, J_n) = \text{Res}(f_1, \dots, f_{n-1}, X_n) \text{Res}(f_1, \dots, f_{n-1}, \Delta) \pmod{\delta}.$$

By definition of the discriminant, it follows that

$$\begin{aligned} \text{Res}(f_1, \dots, f_{n-1}, X_n) \text{Disc}(f_1, \dots, f_{n-1}) = \\ \text{Res}(f_1, \dots, f_{n-1}, X_n) \text{Res}(f_1, \dots, f_{n-1}, \Delta) \pmod{\delta} \end{aligned}$$

from we deduce the claimed equality since  $\text{Res}(f_1, \dots, f_{n-1}, X_n)$  is a nonzero divisor in the ring  $A/\delta \cdot A[X_1, \dots, X_n]$  by Proposition 2.3.  $\square$

Obviously, this result is useless if  $\delta = 1$ , but as soon as  $\delta > 1$  it allows to explicit the discriminant as a single resultant modulo  $\delta$ . For instance, suppose given the two quadrics

$$\begin{aligned} f_1 &:= a_0 X_1^2 + a_1 X_1 X_2 + a_2 X_1 X_3 + a_3 X_2^2 + a_4 X_2 X_3 + a_5 X_3^2, \\ f_2 &:= b_0 X_1^2 + b_1 X_1 X_2 + b_2 X_1 X_3 + b_3 X_2^2 + b_4 X_2 X_3 + b_5 X_3^2. \end{aligned}$$

We have  $\delta = 2$  and it is not hard to see that  $J_i = X_i \Delta \pmod{2}$ ,  $i = 1, 2, 3$ , where

$$\Delta = X_1 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + X_2 \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} + X_3 \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}.$$

It follows that

$$\text{Disc}(f_1, f_2) = \text{Res}(f_1, f_2, \Delta) \pmod{2 \cdot \mathbb{Z}[a_0, \dots, a_5, b_0, \dots, b_5]}.$$

### 3.3. Inertia forms and the discriminant

The discriminant was originally introduced to give a condition for the existence of a singular root in a polynomial system. The aim of this section is to show that the definition we gave of the discriminant of  $n - 1$  homogeneous polynomials in  $n$  variables (i.e. Definition 3.5) fits this goal.

Hereafter we take again the notation of Section 3.1:  $k$  is a commutative ring and for all  $i = 1, \dots, n - 1$ ,  $n \geq 2$ , we set

$$f_i(X_1, \dots, X_n) := \sum_{|\alpha|=d_i \geq 1} U_{i,\alpha} X^\alpha \in {}_k A[X_1, \dots, X_n]_{d_i}$$

where  ${}_k A := k[U_{i,\alpha} \mid |\alpha| = d_i, i = 1, \dots, n - 1]$ . Notice that we will often omit the subscript  $k$  to not overload the notation, but we will print it whenever there is a confusion or a need to emphasis it.

Now, we define the ideals of  $C = A[X_1, \dots, X_n]$

$$\mathcal{D} = (f_1, \dots, f_{n-1}, J_1, \dots, J_n), \quad \mathfrak{m} = (X_1, \dots, X_n)$$

and set  $B := C/\mathcal{D}$ . The ring  $B$  is graded (setting  $\text{weight}(X_i) = 1$ ) and we can thus consider the projective scheme  $\text{Proj}(B) \subset \mathbb{P}_A^{n-1}$  that corresponds set-theoretically to the points  $((u_{i,\alpha})_{i,\alpha}, x) \in \text{Spec}(A) \times \mathbb{P}_k^{n-1}$  such that the  $f_i$ 's and the  $J_i$ 's vanish simultaneously. The scheme-theoretic image of the projection

$$\text{Proj}(B) \rightarrow \text{Spec}(A)$$

is a closed subscheme of  $\text{Spec}(A)$  whose defining ideal is exactly

$$\mathfrak{P} := H_{\mathfrak{m}}^0(B)_0 = \text{TF}_{\mathfrak{m}}(\mathcal{D})_0$$

where we recall that

$$\text{TF}_{\mathfrak{m}}(\mathcal{D}) = \ker \left( C \rightarrow \prod_{i=1}^n B_{X_i} \right). \quad (3.3.1)$$

**Proposition 3.20.** *If  $k$  is a domain then for all  $i = 1, \dots, n$  the ring  $B_{X_i}$  is a domain.*

*Proof.* For simplicity, we prove the claim for  $i = n$ ; the other cases can be treated exactly in the same way.

Let  $h_1, h_2$  be two elements in  $C$  such that their product  $h_1 h_2$  vanishes in  $B_{X_n}$  (recall that we have the canonical projection  $C \rightarrow B = C/\mathcal{D}$ ). This means that, up to multiplication by some power of  $X_n$ , this product is in the ideal  $\mathcal{D}$ . Thus, using Lemma 3.3, ii), we deduce that there exists  $\nu \in \mathbb{N}$  such that

$$X_n^\nu h_1 h_2 \in (f_1, \dots, f_{n-1}, J_n).$$

Now, taking the additional notation of the subsection 2.1, we substitute each  $\mathcal{E}_i$  by  $\mathcal{E}_i - \tilde{f}_i$  and obtain that  $h_1 h_2 (\mathcal{E}_i - \tilde{f}_i) \in (\tilde{J}_n)$  in  $A[X_1, \dots, X_{n-1}]$  (since  $f_i(\mathcal{E}_i - \tilde{f}_i) = 0$ ). But by Lemma 3.1  $\tilde{J}_n$  is prime in  $A[X_1, \dots, X_{n-1}]$  and it follows that it divides  $h_1 (\mathcal{E}_i - \tilde{f}_i)$  or  $h_2 (\mathcal{E}_i - \tilde{f}_i)$ , say  $h_1 (\mathcal{E}_i - \tilde{f}_i)$ . Therefore there exists  $\mu \in \mathbb{N}$  such that

$$X_n^\mu h_1 \in (f_1, \dots, f_{n-1}, J_n) \subset \mathcal{D},$$

that is to say  $h_1$  equals 0 in  $B_{X_n}$ , and the claim is proved.  $\square$

**Corollary 3.21.** *Moreover, for all  $i = 1, \dots, n$  we have*

$$\mathrm{TF}_m(\mathcal{D}) = \mathrm{TF}_{(X_i)}(\mathcal{D}) = \ker(C \rightarrow B_{X_i}), \quad H_m^0(B) = H_{(X_i)}^0(B).$$

*In particular,*

$$\mathfrak{P} = A \cap (\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n) \subset A[X_1, \dots, X_{n-1}].$$

*As a consequence, if  $k$  is a domain then  $\mathrm{TF}_m(\mathcal{D})$  and  $\mathfrak{P}$  are prime ideals of  ${}_k C$  and  ${}_k A$  respectively.*

*Proof.* The only thing to prove in that for all pairs of integers  $(i, j) \in \{1, \dots, n\}^2$  the variable  $X_i$  is a nonzero divisor in the ring  $B_{X_j}$ . Indeed, this property implies immediately the equalities given in this corollary (similarly to (2.1.2) and (2.1.3) for the case of the resultant). From here, assuming moreover that  $k$  is a domain we deduce that  $\mathrm{TF}_m(\mathcal{D})$  and  $\mathfrak{P}$  are prime ideals by Proposition 3.20.

So let us fix a pair of integers  $(i, j) \in \{1, \dots, n\}^2$  and prove that  $X_i$  is a nonzero divisor in  ${}_k B_{X_j}$  (for any commutative ring  $k$ ). By Proposition 3.20, this property holds if  $k$  is a domain. On the one hand, this implies that  ${}_Z B_{X_j}$  is a torsion-free abelian group, hence flat (as a  $\mathbb{Z}$ -module). On the other hand, this implies that the multiplications by  $X_i$  in  ${}_Z B_{X_j}$  and  ${}_Z/pZ B_{X_j}$ ,  $p$  a prime integer, are all injective maps. Denoting by  ${}_Z Q$  the quotient abelian group of the multiplication by  $X_i$  in  ${}_Z B_{X_j}$ , we deduce that  ${}_Z Q$  is a torsion-free, hence flat, abelian group. Indeed, the exact sequence of abelian groups

$$0 \rightarrow {}_Z B_{X_j} \xrightarrow{\times X_i} {}_Z B_{X_j} \rightarrow {}_Z Q \rightarrow 0 \tag{3.3.2}$$

is a flat resolution of  ${}_Z Q$  and it remains exact after tensorization by  $\mathbb{Z}/p\mathbb{Z}$  over  $\mathbb{Z}$  for all prime integers  $p$ . Therefore  $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, {}_Z Q) = 0$  and hence  ${}_Z Q$  is torsion-free, hence flat. As a consequence, for any commutative ring  $k$  we have  $\mathrm{Tor}_1^{\mathbb{Z}}({}_Z Q, k) = 0$  and therefore the multiplication by  $X_i$  in  ${}_k B_{X_j}$  is an injective map, i.e.  $X_i$  is a nonzero divisor in  ${}_k B_{X_j}$ .  $\square$

**Lemma 3.22.**  *$\mathrm{Disc}(f_1, \dots, f_{n-1})$  belongs to the ideal  $\mathfrak{P} \subset {}_k A$ .*

*Proof.* By specialization, it is sufficient to prove this property under the assumption that  $k = \mathbb{Z}$ . Denote  $\rho := \mathrm{Res}(f_1, \dots, f_{n-1}, X_n)$ . From Definition 3.5 and Lemma 3.3, ii) we deduce that there exists  $\nu$  such that

$$X_n^\nu \rho \mathrm{Disc}(f_1, \dots, f_{n-1}) \in (f_1, \dots, f_{n-1}, J_n).$$

Now, taking again the notation of subsection 2.1 and substituting each  $\mathcal{E}_i$  by  $\mathcal{E}_i - \tilde{f}_i$  we deduce that  $\rho \mathrm{Disc}(f_1, \dots, f_{n-1}) (\mathcal{E}_i - \tilde{f}_i) \in (\tilde{J}_n)$  in  $A[X_1, \dots, X_{n-1}]$ . But  $\tilde{J}_n$  is prime in  $A[X_1, \dots, X_{n-1}]$  by Lemma 3.1, and it is coprime with  $\rho$  since  $\rho$  does not depend on the variables  $X_1, \dots, X_n$  and

is also prime. Therefore  $\tilde{J}_n$  must divide  $\text{Disc}(f_1, \dots, f_{n-1})(\mathcal{E}_i - \tilde{f}_i)$  and we obtain that there exists  $\mu \in \mathbb{N}$  such that

$$X_n^\mu \text{Disc}(f_1, \dots, f_{n-1}) \in (f_1, \dots, f_{n-1}, J_n) \subset \mathcal{D}. \quad (3.3.3)$$

In other words,  $\text{Disc}(f_1, \dots, f_{n-1}) \in \text{TF}_{(X_n)}(\mathcal{D}) = \text{TF}_m(\mathcal{D})$ .  $\square$

**Theorem 3.23.** *If 2 is a nonzero divisor in  $k$  then  $\mathfrak{P}$  is generated by  $\text{Disc}(f_1, \dots, f_{n-1})$ . In particular, if  $k$  is moreover assumed to be a domain then  $\text{Disc}(f_1, \dots, f_{n-1})$  is a prime polynomial in  ${}_k A$ .*

*Proof.* We first prove this theorem under the assumption that  $k$  is a UFD. So assume that  $k$  is a UFD and let  $a \in \mathfrak{P} = \text{TF}_m(\mathcal{D}) \cap A$ . Then there exists  $\nu \in \mathbb{N}$  such that  $X_n^\nu a \in (f_1, \dots, f_{n-1}, J_n)$ . Therefore we have the inclusion

$$(f_1, \dots, f_{n-1}, X_n^\nu a) \subset (f_1, \dots, f_{n-1}, J_n)$$

from we deduce, using the divisibility property of the resultant [Jou91, §5.6], that

$$\text{Res}(f_1, \dots, f_{n-1}, J_n) \text{ divides } \text{Res}(f_1, \dots, f_{n-1}, X_n^\nu a).$$

Let us denote by  $\rho := \text{Res}(f_1, \dots, f_{n-1}, X_n) = \text{Res}(f_1^{(n)}, \dots, f_{n-1}^{(n)})$  (see Remark 3.6). From Definition 3.5 and the multiplicativity property of the resultant [Jou91, §5.7] we obtain that

$$\text{Disc}(f_1, \dots, f_{n-1}) \text{ divides } a^{d_1 \dots d_{n-1}} \rho^{\nu-1} \quad (3.3.4)$$

for all  $a \in \mathfrak{P}$ . But it turns out that  $\text{Disc}(f_1, \dots, f_{n-1})$  and  $\rho$  are coprime in  $A$ . Indeed, since  $\rho$  is irreducible, if  $D := \text{Disc}(f_1, \dots, f_{n-1})$  and  $\rho$  are not coprime then  $\rho$  must divide  $D$ . Consider the specialization where each polynomial  $f_i$  is specialized to a product of generic linear forms. Then,  $\rho$  specializes to a product of determinants where each determinant is a prime polynomial (see for instance [BV88, Theorem 2.10]) in the coefficients of these linear forms except the ones of the variables  $X_n$ . On the other hand,  $D$  specializes to a product of square of determinants (see Corollary 3.17), where each determinant is a prime polynomial in the coefficients of the generic linear form and does depend on the ones of the variable  $X_n$ . We thus obtain a contradiction and deduce that  $\rho$  and  $D$  are coprime. [BV88, Theorem 2.10] Therefore, from (3.3.4) and the fact that  $\rho$  is prime in  $A$  we deduce that for all  $a \in \mathfrak{P}$  the discriminant  $D$  divides  $a^{d_1 \dots d_{n-1}}$  and hence that

$$\mathfrak{P}^{d_1 \dots d_{n-1}} \subset (D) \subset \mathfrak{P}.$$

Since  $\mathfrak{P}$  is prime, we deduce that  $D = c.P^p$  where  $c$  is an invertible element in  $k$ ,  $p$  is a positive integer and  $P$  is an irreducible element in  $A$  such that  $\mathfrak{P}$  is a principal ideal generated by  $P$ .

Now, always under the assumption that  $k$  is UFD, we will prove that  $p = 1$  if  $2 \neq 0$  in  $k$ . Notice that we can assume  $d_1 \geq 2$  because if  $d_1 = \dots = d_{n-1} = 1$  then  $\mathfrak{P} = (D) = A$  and we can permute polynomials by Proposition 3.12, i). To begin with, consider the specialization of the polynomial  $f_1$  to a product of a generic linear form  $l$  and a generic polynomial  $f'_1$  of degree  $d_1 - 1$ . By Proposition 3.16,  $D$  specializes, up to sign, to the product

$$\text{Disc}(l, f_2, \dots, f_{n-1}) \text{Disc}(f'_1, f_2, \dots, f_{n-1}) \text{Res}(l, f'_1, f_2, \dots, f_{n-1})^2. \quad (3.3.5)$$

Since all the polynomials  $l, f'_1, f_2, \dots, f_{n-1}$  are generic of positive degree, this product is nonzero. Moreover, the factor  $\text{Res}(l, f'_1, f_2, \dots, f_{n-1})$  is irreducible and is clearly coprime with the two discriminants appearing in (3.3.5). It follows that necessarily  $p \leq 2$ , i.e.  $p = 1$  or  $p = 2$ .

To prove that  $p = 1$ , equivalently that  $D$  is irreducible, we proceed by induction on the integer  $r := \sum_{i=1}^{n-1} d_i$ . The intricate point is actually the initialization step. Indeed, assume that  $D$  is irreducible for  $r = n$  (observe that  $D = 1$  if  $r = n - 1$ ). Then, using the specialization (3.3.5), we deduce immediately by induction that both discriminants in (3.3.5) are irreducible and



coprime, and consequently that  $D$  is also irreducible. Therefore, we have to show that if  $d_1 = 2$  and  $d_2 = \dots = d_{n-1} = 1$  then  $D$  is irreducible. For that purpose, we consider the specialization

$$\begin{cases} f_1 &= U_{1,1}X_1^2 + U_{1,2}X_1X_2 + U_{2,2}X_2^2 + \sum_{i=3}^n U_{i,i}X_i^2 \\ f_2 &= X_3 - V_3X_1 \\ &\vdots \\ f_{n-1} &= X_n - V_nX_1 \end{cases}$$

and the matrix

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ -V_3 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -V_n & 0 & \cdots & 0 & 1 \end{bmatrix}$$

that corresponds to a linear change of coordinates such that  $f_i = X_{i+1} \circ \varphi$  for all  $i = 2, \dots, n-1$ . Applying Proposition (3.27) then Proposition 3.13, we get

$$\begin{aligned} \text{Disc}(f_1, f_2, \dots, f_{n-1}) &= \text{Disc}(f_1 \circ \varphi^{-1}, X_3, \dots, X_n) \\ &= \text{Disc}\left(U_{1,1}X_1^2 + U_{1,2}X_1X_2 + U_{2,2}X_2^2 + \sum_{i=3}^n U_{i,i}V_i^2X_1^2\right) \\ &= \text{Disc}\left(\left(U_{1,1} + \sum_{i=3}^n U_{i,i}V_i^2\right)X_1^2 + U_{1,2}X_1X_2 + U_{2,2}X_2^2\right) \\ &= U_{1,2}^2 - 4U_{2,2}\left(U_{1,1} + \sum_{i=3}^n U_{i,i}V_i^2\right). \end{aligned}$$

Since  $2 \neq 0$  in  $k$ , this is an irreducible polynomial. Therefore, we deduce that necessarily  $p = 1$ , i.e. that  $D = c.P$ . Since  $c.P$  also generates  $\mathfrak{P} = (P)$ , we conclude that  $D$  is an irreducible polynomial that generates  $\mathfrak{P}$ . This concludes the proof of the theorem under the assumptions that  $k$  is a UFD and  $2 \neq 0$  in  $k$ .

It remains to show that this theorem holds with the single assumption that 2 is a nonzero divisor in  $k$ ,  $k$  being an arbitrary commutative ring. For that purpose, consider the exact sequence of abelian groups

$$0 \rightarrow {}_{\mathbb{Z}}A \xrightarrow{\times D} {}_{\mathbb{Z}}A \rightarrow {}_{\mathbb{Z}}B_{X_n} \rightarrow E \rightarrow 0 \quad (3.3.6)$$

where the map on the left is the multiplication by  $D$ , the map on the middle is the canonical one and where  $E$  is the cokernel of this latter. By what we have just proved above, this sequence is exact and remains exact after tensorization by  $\mathbb{Z}/p\mathbb{Z}$  over  $\mathbb{Z}$  for all prime integers  $p \neq 2$  (they are all UFD). Since  ${}_{\mathbb{Z}}A$  and  ${}_{\mathbb{Z}}B_{X_n}$  are torsion-free, the exact sequence (3.3.6) is a flat resolution of  $E$  and therefore for all integers  $i \geq 2$  the abelian group  $\text{Tor}_i^{\mathbb{Z}}(-, E)$  is supported on  $V((2))$ . Now, let  $M$  be an abelian group without 2-torsion. The abelian group  $M_{(2)}$  is a flat  $\mathbb{Z}_{(2)}$ -module and hence for all  $i \geq 1$  we have

$$\text{Tor}_i^{\mathbb{Z}}(M, E)_{(2)} \simeq \text{Tor}_i^{\mathbb{Z}_{(2)}}(\mathbb{Z}_{(2)} \otimes M, \mathbb{Z}_{(2)} \otimes E) \simeq \text{Tor}_i^{\mathbb{Z}_{(2)}}(M_{(2)}, \mathbb{Z}_{(2)} \otimes E) = 0.$$

It follows that  $\text{Tor}_i^{\mathbb{Z}}(M, E)_{(p)} = 0$  for all  $i \geq 2$  and all prime integers  $p$ , so that  $\text{Tor}_i^{\mathbb{Z}}(M, E) = 0$  for all  $i \geq 2$ . Consequently, since 2 is a nonzero divisor in  $k$ ,  $k$  has no 2-torsion and we deduce that the sequence obtained by tensorization of (3.3.6) by  $k$  over  $\mathbb{Z}$

$$0 \rightarrow {}_kA \xrightarrow{\times D} {}_kA \rightarrow {}_kB_{X_n} \rightarrow k \otimes E \rightarrow 0$$

is exact and the theorem is proved.  $\square$

It is reasonable to ask what happens if 2 is a zero divisor in  $k$ . As shown in [AJ06, §8.5.2], one can not expect in this case that the discriminant generates  $\mathfrak{P}$ , nor even that  $\mathfrak{P}$  is a principal ideal. Indeed, in *loc. cit.* the authors exhibit an example where  $\mathfrak{P}$  is not a principal ideal with the settings  $n = 2$ ,  $d_1 = 2$  and  $k = \mathbb{Z}/2^r\mathbb{Z}$  with  $r \geq 2$ . Nevertheless, we will show in the following theorem that the situation is not so bad if  $k$  is assumed to be a domain.

**Theorem 3.24.** *Assume that  $k$  is a domain and that  $2 = 0$  in  $k$ . Then*

$$\text{Disc}(f_1, \dots, f_{n-1}) = P^2$$

where  $P$  is a prime polynomial that generates  $\mathfrak{P}$ .

*Proof.* We first prove this theorem under the stronger assumption that  $k$  is a UFD such that  $2 = 0$  in  $k$ . To begin with, recall that in the proof of Theorem 3.23 it is shown that there exists a prime element  $P \in A$  and an integer  $p \leq 2$  such that the discriminant  $D := \text{Disc}(f_1, \dots, f_{n-1})$  satisfies  $D = c.P^p$ ,  $P$  being a generator of the prime and principal ideal  $\mathfrak{P}$ . We will show that  $p = 2$  under our assumptions. Our strategy is based on the use of a Mertens' formula that allows to rely on a discriminant of a unique bivariate and homogeneous polynomial. Indeed, in this case (i.e.  $n = 2$ ) it is known that the claimed result holds [AJ06, Proposition 60] (see also Theorem 4.26 in the case  $n = 2$  for a self-contained reference).

Introduce some notation related to the Mertens' formulae given in the appendix at the end of this paper. Let  $U_1, \dots, U_n$  be new indeterminates and define

$$\theta(U_1, \dots, U_n) := \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n U_i X_i) \in A[U_1, \dots, U_n]$$

and  $\theta_i(U_1, \dots, U_n) := \partial\theta/\partial U_i \in A[U_1, \dots, U_n]$  for all  $i = 1, \dots, n$ . In addition, let  $V_1, \dots, V_n, W_1, \dots, W_n, X, Y$  be a collection of some other new indeterminates and consider the ring morphisms

$$\begin{aligned} \rho : A[U_1, \dots, U_n] &\rightarrow A[V_1, \dots, V_n, W_1, \dots, W_n][X_1, \dots, X_n] \\ U_i &\mapsto V_i \left( \sum_{j=1}^n W_j X_j \right) - W_i \left( \sum_{j=1}^n V_j X_j \right) \end{aligned}$$

and

$$\begin{aligned} \bar{\rho} : A[U_1, \dots, U_n] &\rightarrow A[V_1, \dots, V_n, W_1, \dots, W_n][X, Y] \\ U_i &\mapsto V_i X + W_i Y. \end{aligned}$$

To not overload the notation, we will sometimes denote a collection of variables with its corresponding letter underlined. For instance,  $V_1, \dots, V_n$  will be shortcut by  $\underline{V}$ .

Our aim is to show that the multivariate discriminant  $\text{Disc}(f_1, \dots, f_n) \in A$  divides the bivariate discriminant  $\text{Disc}_{X,Y}(\bar{\rho}) \in A[\underline{V}, \underline{W}]$ . To begin with, introduce two collections of new indeterminates  $t_1, \dots, t_n$  and  $Z_1, \dots, Z_n$ , and define the matrix

$$\varphi := \begin{bmatrix} t_n & 0 & \cdots & 0 & Z_1 \\ 0 & t_n & \vdots & \vdots & Z_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & t_n & Z_{n-1} \\ -t_1 & -t_2 & \cdots & -t_{n-1} & Z_n \end{bmatrix}.$$

Applying the base change formula for the resultant [Jou91, §5.12], we get

$$\begin{aligned} \theta_Z &:= \text{Res}(f_1 \circ \varphi, \dots, f_{n-1} \circ \varphi, \left( \sum_{i=1}^n U_i X_i \right) \circ \varphi) \\ &= \det(\varphi)^{d_1 \dots d_{n-1}} \theta(\underline{U}) = t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \theta(\underline{U}) \end{aligned} \quad (3.3.7)$$

in the extended ring  $A[\underline{U}, t, \underline{Z}]$ . Now, set  $f_n := \sum_{i=1}^n U_i X_i$ . Having in mind to use Corollary 2.6, we need to identify for all  $i, j = 1, \dots, n$  the coefficient, say  $V_{i,j}$ , of the monomial  $X_j X_n^{d_i-1}$  in the polynomial  $f_i \circ \varphi$ . The coefficients  $V_{i,n}$  are easily seen to be equal to  $f_i(Z_1, \dots, Z_n)$  since one only has to evaluate  $f_i \circ \varphi$  at  $X_1 = \dots = X_{n-1} = 0$  and  $X_n = 1$ . Then, to get the coefficients  $V_{i,j}$  with  $j \neq n$ , we have to differentiate  $f_i \circ \varphi$  with respect to  $X_j$  and finally evaluate the result at  $X_1 = \dots = X_{n-1} = 0$  and  $X_n = 1$ ; we find

$$V_{i,j} = t_n \frac{\partial f_i}{\partial X_j}(Z_1, \dots, Z_n) - t_j \frac{\partial f_i}{\partial X_n}(Z_1, \dots, Z_n), \quad j \neq n.$$

We claim that

$$\mathcal{D} := \det(V_{i,j})_{i,j=1,\dots,n} = \left( \sum_{i=1}^n U_i Z_i \right) t_n^{n-2} \Delta_t \bmod (f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z})) \quad (3.3.8)$$

in  $A[\underline{U}, t, \underline{Z}]$ , where  $\Delta_t$  stands for the Jacobian matrix

$$\Delta_t := \frac{\partial(f_1, \dots, f_{n-1}, \sum_{i=1}^n t_i X_i)}{\partial(X_1, \dots, X_n)}(Z_1, \dots, Z_n) \in A[t, \underline{Z}].$$

Indeed, from the definition, it is easy to see that

$$\mathcal{D} = \left( \sum_{i=1}^n U_i Z_i \right) \begin{vmatrix} t_n \frac{\partial f_1}{\partial X_1}(\underline{Z}) - t_1 \frac{\partial f_1}{\partial X_n}(\underline{Z}) & \cdots & t_n \frac{\partial f_1}{\partial X_{n-1}}(\underline{Z}) - t_{n-1} \frac{\partial f_1}{\partial X_n}(\underline{Z}) \\ \vdots & & \vdots \\ t_n \frac{\partial f_{n-1}}{\partial X_1}(\underline{Z}) - t_1 \frac{\partial f_{n-1}}{\partial X_n}(\underline{Z}) & \cdots & t_n \frac{\partial f_{n-1}}{\partial X_{n-1}}(\underline{Z}) - t_{n-1} \frac{\partial f_{n-1}}{\partial X_n}(\underline{Z}) \end{vmatrix} \bmod (f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z})).$$

Denote by  $M$  the determinant appearing in this equality. Then, it is clear that

$$t_n M = \begin{vmatrix} t_n \frac{\partial f_1}{\partial X_1}(\underline{Z}) & \cdots & t_n \frac{\partial f_1}{\partial X_{n-1}}(\underline{Z}) & \frac{\partial f_1}{\partial X_n}(\underline{Z}) \\ \vdots & & \vdots & \vdots \\ t_n \frac{\partial f_{n-1}}{\partial X_1}(\underline{Z}) & \cdots & t_n \frac{\partial f_{n-1}}{\partial X_{n-1}}(\underline{Z}) & \frac{\partial f_{n-1}}{\partial X_n}(\underline{Z}) \\ t_n t_1 & \cdots & t_n t_{n-1} & t_n \end{vmatrix} = t^{n-1} \Delta_t$$

and (3.3.8) is proved. Therefore, by Corollary 2.6 there exists  $H_1 \in A[\underline{U}, t, \underline{Z}]$  such that

$$\theta_Z - \mathcal{D} H_1 \in (f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z}), \sum_{i=1}^n U_i Z_i)^2$$

and hence, using (3.3.7) and (3.3.8), we obtain that

$$\begin{aligned} t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \theta(\underline{U}) \in \\ \left( f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z}), \left( \sum_{i=1}^n U_i Z_i \right) t_n^{n-2} \Delta_t, \left( \sum_{i=1}^n U_i Z_i \right)^2 \right). \end{aligned}$$

Applying the operator  $\sum_{i=1}^n t_i \partial(-)/\partial U_i$ , we get

$$t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \sum_{i=1}^n t_i \theta_i(\underline{U}) \in \left( f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z}), \left( \sum_{i=1}^n t_i Z_i \right) t_n^{n-2} \Delta_t, \left( \sum_{i=1}^n U_i Z_i \right) \right).$$

Now, we send this relation through the morphism  $\rho$  and substitute  $\underline{X}$  to  $\underline{Z}$ . It turns out that  $\sum_{i=1}^n U_i Z_i$  is sent to zero and hence we obtain that

$$t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \sum_{i=1}^n t_i \rho(\theta_i)(\underline{Z}) \in (f_1(\underline{Z}), \dots, f_{n-1}(\underline{Z}), \Delta_t).$$

By the divisibility property of the resultant [Jou91, §5.6], we deduce that  $\text{Res}(f_1, \dots, f_{n-1}, \Delta_t)$  divides

$$\text{Res} \left( f_1, \dots, f_{n-1}, t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \sum_{i=1}^n t_i \rho(\theta_i)(\underline{Z}) \right).$$

But by definition,

$$\text{Res}(f_1, \dots, f_{n-1}, \Delta_t) = \text{Disc}(f_1, \dots, f_{n-1}) \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n t_i X_i)$$

and by the second Mertens' formula and the multiplicativity property of the resultant we have

$$\begin{aligned} \text{Res} \left( f_1, \dots, f_{n-1}, t_n^{(n-2)d_1 \dots d_{n-1}} \left( \sum_{i=1}^n t_i Z_i \right)^{d_1 \dots d_{n-1}} \sum_{i=1}^n t_i \rho(\theta_i)(\underline{Z}) \right) = \\ (-1)^{d_1 \dots d_{n-1}} t_n^{(n-2)d_1^2 \dots d_{n-1}^2} \text{Disc}_{X,Y}(\bar{\rho}(\theta)) \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n t_i X_i)^{d_1 \dots d_{n-1} + 1}. \end{aligned}$$

Since  $\text{Disc}(f_1, \dots, f_{n-1})$  and  $\text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n t_i X_i)$  are coprime (the latter is irreducible and depends on  $\underline{t}$  which is not the case of the discriminant) we deduce that there exists  $H \in A[\underline{V}, \underline{W}]$  such that

$$\text{Disc}_{X,Y}(\bar{\rho}(\theta)) = H \text{Disc}(f_1, \dots, f_n).$$

To finish the proof, we will show that  $H$  and  $\text{Disc}(f_1, \dots, f_n)$  are coprime, so that  $p$  must be equal to 2 since  $\text{Disc}(\bar{\rho}(\theta))$  is a square, as a specialization of a square. For that purpose, we proceed as in the proof of Lemma A (in the appendix): we specialize each polynomial  $f_i$ ,  $i = 1, \dots, n-1$  to the product of  $d_i$  generic linear forms

$$l_{i,j} := U_{i,j,1} X_1 + U_{i,j,2} X_2 + \dots + U_{i,j,d_i} X_{d_i} = \sum_{r=1}^{d_i} U_{i,j,r} X_r, \quad i = 1, \dots, n, \quad j = 1, \dots, d_i.$$

After this specialization, we get (see the proof of Lemma A)

$$\begin{aligned} \text{Disc}(\bar{\rho}(\theta)) = \\ \pm \prod_{\lambda < \mu} (\Delta_\lambda(V_1, \dots, V_n) \Delta_\mu(W_1, \dots, W_n) - \Delta_\lambda(W_1, \dots, W_n) \Delta_\mu(V_1, \dots, V_n))^2. \end{aligned}$$

On the other hand, Proposition 3.17 yields

$$\text{Disc} \left( \prod_{j=1}^{d_1} l_{1,j}, \dots, \prod_{j=1}^{d_{n-1}} l_{n-1,j} \right) = \pm \prod_I \det(l_{1,j_1}, l_{2,j_2}, \dots, l_{n-1,j_{n-1}}, l_{i,j})^2.$$

Moreover, if  $\lambda = (j_1, \dots, j_{n-2}, j'_{n-1})$  and  $\mu = (j_1, \dots, j_{n-2}, j''_{n-1})$  then we have the equality

$$\Delta_\lambda(V_1, \dots, V_n) \Delta_\mu(W_1, \dots, W_n) - \Delta_\lambda(W_1, \dots, W_n) \Delta_\mu(V_1, \dots, V_n) = \\ \det(l_{1,j_1}, l_{2,j_2}, \dots, l_{n-1,j'_{n-1}}, l_{n-1,j''_{n-1}}) \times \det(l_{1,j_1}, l_{2,j_2}, \dots, V, W)$$

(it is easy to check this formula in the case  $n = 2$ ; then the general case can be deduced from this by developing each determinant in this equality with respect to their two last columns). Therefore,  $H$  and  $\text{Disc}(f_1, \dots, f_n)$  are coprime. So we have proved that  $D = c.P^2$  under the assumptions  $k$  is a UFD and  $2 = 0$  in  $k$ .

Now, assume that  $k$  is a domain such that  $2 = 0$ , and set  $F := \mathbb{Z}/2\mathbb{Z}$  for simplicity. The injective map  $F \hookrightarrow k$  is flat for  $k$  is a torsion-free  $F$ -module ( $k$  is not the trivial ring). Therefore, the canonical exact sequence (see Corollary 3.21)

$$0 \rightarrow {}_F\text{TF}_m(\mathcal{D}) \rightarrow {}_F C \rightarrow {}_F B_{X_n}$$

remains exact after tensorization by  $k$  over  $F$ . Since  ${}_F C \otimes_F k \simeq {}_k C$  and  ${}_F B_{X_n} \otimes_F k \simeq {}_k B_{X_n}$  we deduce that

$${}_k \text{TF}_m(\mathcal{D}) \simeq {}_F \text{TF}_m(\mathcal{D}) \otimes_F k$$

and hence that  ${}_k \mathfrak{P} \simeq {}_F \mathfrak{P} \otimes_F k$ . Moreover,  $F$  is a UFD and hence we have proved that  ${}_F D = P^2$  where  $P$  is a prime element that generates  ${}_F \mathfrak{P}$  (observe that the unit  $c$  is necessarily equal to 1 in  $F$ ). Considering the specialization  $\rho : {}_F A \rightarrow {}_k A$ , it follows that  $\rho(P)$  generates  ${}_k \mathfrak{P}$  and  ${}_k D = \rho({}_F D) = \rho(P)^2$  (by definition of the discriminant) and this concludes the proof of this theorem.  $\square$

Before closing this section, we give a refined relationship for the discriminant. Let  $R$  be a commutative ring and suppose given  $f_1, \dots, f_{n-1}$  homogeneous polynomials in  $R[X_1, \dots, X_n]$  of respective positive degree  $d_1, \dots, d_{n-1}$ . Recall the notation

$$\tilde{f}_i(X_1, \dots, X_{n-1}) := f_i(X_1, \dots, X_{n-1}, 1) \in R[X_1, \dots, X_{n-1}]$$

(and similarly for  $\tilde{J}_n$ ). An immediate consequence of the proof of Lemma 3.22 (see (3.3.3)) is that

$$\text{Disc}(f_1, \dots, f_{n-1}) \in (\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n) \subset A[X_1, \dots, X_{n-1}].$$

The following theorem, which appears in [AJ06] for the case  $n = 2$ , improves this result.

**Theorem 3.25.** *With the above notation we have*

$$\text{Disc}(f_1, \dots, f_{n-1}) \in R \cap \left( \tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n^2 \right) \subset R[X_1, \dots, X_{n-1}].$$

*Proof.* As always, it is sufficient to prove this theorem in the generic case of Section 3.1, that is to say  $f_1, \dots, f_{n-1}$  are supposed to be homogeneous polynomials in  $A[X_1, \dots, X_n]$ , where  $A$  is the universal coefficient ring, of respective positive degree  $d_1, \dots, d_{n-1}$ . We recall that  $J_n$  denotes the Jacobian determinant  $|\frac{\partial(f_1, \dots, f_{n-1})}{\partial(X_1, \dots, X_{n-1})}|$  and that for any polynomial  $P$  in  $X_1, \dots, X_n$  we denote by  $\tilde{P}$  (resp.  $\bar{P}$ ) the polynomial in  $X_1, \dots, X_{n-1}$  obtained by substituting  $X_n$  by 1 (resp. 0) in  $P$ .

Let us introduce the new indeterminates  $T_1, \dots, T_n$ . Setting  $\delta := \deg(J_n) = \sum_{i=1}^{n-1} (d_i - 1)$ , we consider both resultants

$$\rho := \text{Res}(f_1, \dots, f_{n-1}, X_n) = \text{Res}(\tilde{f}_1, \dots, \tilde{f}_{n-1}) \in A,$$

$$\mathcal{R} := \text{Res}(f_1 - T_1 X_n^{d_1}, \dots, f_{n-1} - T_{n-1} X_n^{d_{n-1}}, J_n - T_n X_n^\delta) \in A[T_1, \dots, T_n].$$

Since the  $f_i$ 's are generic polynomials, we know that  $\rho$  is an irreducible element in  $A$  generating the inertia forms ideal

$$\mathcal{T} := \text{TF}_{(X_1, \dots, X_{n-1})}(\bar{f}_1, \dots, \bar{f}_{n-1})_0 = \text{TF}_{(X_1, \dots, X_n)}(f_1, \dots, f_{n-1}, X_n)_0 \subset A.$$

From Lemma 3.3, ii) (take  $i = 1, \dots, n-1$  and  $j = n$ ), we deduce that  $J_n \in \mathcal{T}$ . Consequently, polynomials  $f_1 - T_1 X_n^{d_1}, \dots, f_{n-1} - T_{n-1} X_n^{d_{n-1}}$  and  $J_n - T_n X_n^\delta$  are in  $\mathcal{T} \otimes_A A[T_1, \dots, T_n]$  and it follows that  $\mathcal{R}$  itself is in  $\mathcal{T} \otimes_A A[T_1, \dots, T_n]$ . This implies that  $\rho$  divides  $\mathcal{R}$ : there exists  $H(T_1, \dots, T_n) \in A[T_1, \dots, T_n]$  such that

$$\mathcal{R} = \rho H(T_1, \dots, T_n) \in A[T_1, \dots, T_n].$$

This polynomial  $H$  have the two following important properties:

- $H(0, \dots, 0) = \text{Disc}(f_1, \dots, f_{n-1}) \in A$  (by (3.1.4)),
- $H(\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n) = 0 \in A[X_1, \dots, X_{n-1}]$  (by (2.2.2)).

Therefore  $H(T_1, \dots, T_n)$  gives (similarly to (2.2.2) for the resultant) an *explicit expression* of the discriminant of  $f_1, \dots, f_{n-1}$  as a polynomial in  $\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n$  with coefficients in  $A$  and without constant term, i.e. as an element in  $\tilde{J}_n A[\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n] + \sum_{i=1}^{n-1} \tilde{f}_i A[\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n]$ . We claim that the coefficient of  $H$  (seen as a polynomial in the  $T_i$ 's) of the monomial  $T_n$  is zero, and this implies our theorem.

To prove this claim, it is sufficient to prove the same claim for  $\mathcal{R} \in A[T_1, \dots, T_n]$ , and even, by performing the specialization (which leaves  $J_n$  invariant)

$$f_i \mapsto f_i + T_i X_n^{d_i} \text{ for all } i = 1, \dots, n-1,$$

for the resultant

$$\text{Res}(f_1, \dots, f_{n-1}, J_n - T_n X_n^\delta) \in A[T_n].$$

Let  $K$  be the quotient field of  $A$  and  $\bar{K}$  its algebraic closure. Then the  $f_i$ 's have  $d_1 \dots d_{n-1}$  simple roots, none at infinity, in  $\mathbb{P}_{\bar{K}}^{n-1}$ . As in the proof of Proposition 3.11, the Poisson's formula gives

$$\frac{\text{Res}(f_1, \dots, f_{n-1}, J_n - T_n X_n^\delta)}{\text{Res}(\bar{f}_1, \dots, \bar{f}_{n-1})^\delta} = \prod_{\xi \in I} (\tilde{J}_n(\xi) - T_n),$$

where  $I := \{\xi \in \mathbb{A}_{\bar{K}}^{n-1} : f_1(\xi) = \dots = f_{n-1}(\xi) = 0\}$ . But the coefficient of  $T_n$ , up to a nonzero multiplicative constant, equals

$$\left( \prod_{\xi \in I} \tilde{J}_n(\xi) \right) \cdot \left( \sum_{\xi \in I} \frac{1}{\tilde{J}_n(\xi)} \right).$$

This latter quantity vanishes since its second factor is zero by the well known Jacobi formula.  $\square$

**Remark 3.26.** Observe that we actually proved that

$$\text{Disc}(f_1, \dots, f_{n-1}) \in \tilde{J}_n^2 A[\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n] + \sum_{i=1}^{n-1} \tilde{f}_i A[\tilde{f}_1, \dots, \tilde{f}_{n-1}, \tilde{J}_n].$$

### 3.4. The base change formula

In this section, we investigate the behavior of the discriminant of  $n-1$  homogeneous polynomials in  $n$  variables under polynomial compositions. Although the situation is much more involved compared to the case of the resultant [Jou91, §5.12], we provide a detailed base change formula. We begin with the case of a linear change of coordinates.

**Proposition 3.27.** *Let  $R$  be a commutative ring and  $f_i$  ( $i = 1, \dots, n-1$ ) be a homogeneous polynomial of degree  $d_i \geq 1$  in  $R[X_1, \dots, X_n]$ . Given a matrix  $\varphi = [c_{i,j}]_{1 \leq i, j \leq n}$  with entries in  $R$  and denoting, for all  $f \in R[X_1, \dots, X_n]$ ,*

$$f \circ \varphi(X_1, \dots, X_n) := f \left( c_{1,1}X_1 + \dots + c_{1,n}X_n, \dots, \sum_{j=1}^n c_{i,j}X_j, \dots, \sum_{j=1}^n c_{n,j}X_n \right),$$

we have

$$\text{Disc}(f_1 \circ \varphi, \dots, f_{n-1} \circ \varphi) = \det(\varphi)^{d_1 \dots d_{n-1} (\sum_{i=1}^{n-1} (d_i - 1))} \text{Disc}(f_1, \dots, f_{n-1}).$$

*Proof.* We prove this proposition in the generic case. By Definition 3.5, we have

$$\begin{aligned} \text{Res}(f_1 \circ \varphi, \dots, f_{n-1} \circ \varphi, X_n \circ \varphi) \text{Disc}(f_1 \circ \varphi, \dots, f_{n-1} \circ \varphi) \\ = \text{Res}(f_1 \circ \varphi, \dots, f_{n-1} \circ \varphi, J_n(f \circ \varphi)). \end{aligned}$$

Now, since  $J_n(f \circ \varphi) = J_n(f_1, \dots, f_{n-1}) \circ [\varphi] \cdot \det(\varphi)$  (the classical formula for changing variables), we deduce from [Jou91, §5.12] and the homogeneity of the resultant that the numerator of the previous display is equal to

$$\det(\varphi)^{d_1 \dots d_{n-1}} \text{Res}(f_1, \dots, f_{n-1}, J_n) \det(\varphi)^{d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)},$$

and the denominator is equal to

$$\text{Res}(f_1, \dots, f_{n-1}, X_n) \det(\varphi)^{d_1, \dots, d_{n-1}}.$$

The result follows by simplifying  $\det(\varphi)^{d_1, \dots, d_{n-1}}$  in both previous equalities.  $\square$

**Corollary 3.28.** *Take again the notation of §3.1.2. Let  $m$  be a fixed integer in  $\{1, \dots, n\}$  and define a grading on the ring  ${}_k A = k[U_{i,\alpha} \mid |\alpha| = d_i]$  by*

$$\text{weight}(U_{i,\alpha_1, \dots, \alpha_n}) := \alpha_m.$$

Then  $\text{Disc}(f_1, \dots, f_{n-1}) \in {}_k A$  is homogeneous of total weight

$$d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1).$$

*Proof.* It is an immediate corollary of Proposition 3.27 by taking the diagonal matrix  $\varphi = [c_{i,j}]$  where  $c_{m,m} = t$ , where  $t$  be a new indeterminate, and  $c_{i,i} = 1$  if  $i \neq m$ .  $\square$

We now turn to the general situation.

**Proposition 3.29.** *For all  $i = 1, \dots, n-1$ , let  $f_i$  be a homogeneous polynomial of degree  $d_i \geq 1$  in  $R[X_1, \dots, X_n]$ , where  $R$  is a commutative ring. If  $g_1, \dots, g_n$  are  $n$  homogeneous polynomials of the same degree  $d \geq 2$  in  $R[X_1, \dots, X_n]$  then, denoting  $f_i \circ g := f_i(g_1, \dots, g_n)$  for all  $i = 1, \dots, n-1$ , we have*

$$\begin{aligned} d^{d^{n-1} \prod_{i=1}^{n-1} d_i} \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g) &= \text{Disc}(f_1, \dots, f_{n-1})^{d^{n-1}} \\ \text{Res}(g_1, \dots, g_n)^{d_1 \dots d_{n-1} ((\sum_{i=1}^{n-1} (d_i - 1)) - 1)} &\text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, J(g_1, \dots, g_n)). \end{aligned}$$

*Proof.* As always, we assume that we are in the generic situation over the integers, which is sufficient to prove this formula. Let us introduce the polynomials  $F := U_1 g_1 + \dots + U_n g_n$  which is homogeneous of degree  $d$  in the variables  $X_1, \dots, X_n$ . Then by (3.1.5) we get

$$d^{d^{n-1} \prod_{i=1}^{n-1} d_i} \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g) = \frac{\text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, J(f \circ g, F))}{\text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, F)}.$$

But

$$J(f_1 \circ g, \dots, f_{n-1} \circ g, F) = J(f_1, \dots, f_{n-1}, \sum_{i=1}^n U_i X_i) \circ g \times J(g_1, \dots, g_n)$$

and  $\deg(J(g_1, \dots, g_n)) = n(d-1) \geq 1$ . By the base change formula for the resultant [Jou91, §5.12] we deduce that, denoting  $l := \sum_{i=1}^n U_i X_i$  and using obvious notation,

$$\begin{aligned} \text{Res}(f \circ g, J(f \circ g, F)) &= \text{Res}(f \circ g, J(f_1, \dots, f_{n-1}, l) \circ g) \text{Res}(f \circ g, J(g)) = \\ &= \text{Res}(f, J(f_1, \dots, f_{n-1}, l))^{d^{n-1}} \text{Res}(g_1, \dots, g_n)^{d_1 \dots d_{n-1} (\sum_{i=1}^{n-1} (d_i - 1))} \text{Res}(f \circ g, J(g)) \end{aligned}$$

and

$$\text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, F) = \text{Res}(f_1, \dots, f_{n-1}, l)^{d^{n-1}} \text{Res}(g_1, \dots, g_n)^{d_1 \dots d_{n-1}}.$$

Therefore the claimed formula follows.  $\square$

This first base change formula is not completely factorized. Indeed, it is not hard to see that  $\text{Res}(g_1, \dots, g_n)^{d_1 \dots d_{n-1}}$  divides  $\text{Res}(f \circ g, J(g_1, \dots, g_n))$  and this latter must contain other factors by degree evidence. Let us state this property more precisely.

**Lemma 3.30.** *There exists a polynomial in the coefficients of the  $f_i$ 's and the  $g_i$ 's, denoted  $K(f, g)$ , such that*

$$\text{Res}(f \circ g, J(g_1, \dots, g_n)) = d^{d^{n-1} \prod_{i=1}^n d_i} \text{Res}(g_1, \dots, g_n)^{\prod_{i=1}^n d_i} K(f, g).$$

*Proof.* As always, we assume that we are in the generic situation over the integers, which is sufficient to prove this formula. For all  $i = 1, \dots, n-1$ , it is clear that  $f_i \circ g \in (g_1, \dots, g_n)^{d_i}$ . Moreover, we also have that  $X_n J(g_1, \dots, g_n) \in (g_1, \dots, g_n)$ . Therefore, applying the general divisibility lemma for the resultant [Jou91, Proposition 6.2.1], we deduce that  $\text{Res}(g_1, \dots, g_n)^{\prod_{i=1}^n d_i}$  divides

$$\text{Res}(f \circ g, X_n J(g_1, \dots, g_n)) = \text{Res}(f \circ g, J(g_1, \dots, g_n)) \text{Res}(f \circ g, X_n).$$

Now, we claim that  $\text{Res}(g_1, \dots, g_n)$  and  $\text{Res}(f \circ g, X_n)$  are relatively prime, which concludes the proof. Indeed,  $\text{Res}(g_1, \dots, g_n)$  being irreducible, if it divides  $\text{Res}(f \circ g, X_n)$ , then it must divide any specialization of this latter resultant where the  $g_i$ 's are left generic. So, if we specialize each polynomial  $f_i$  to  $X_i^{d_i}$  then this resultant specialize to  $\text{Res}(g_1, \dots, g_{n-1}, X_n)$  which is irreducible and independent of the polynomial  $g_n$ . Therefore, we obtain a contradiction.  $\square$

By gathering Proposition 3.29 and Lemma 3.30, we are ready to give a base change formula which is completely factorized.

**Theorem 3.31.** *With the notation of Proposition 3.29 and Lemma 3.30, we have*

$$\begin{aligned} \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g) &= \text{Disc}(f_1, \dots, f_{n-1})^{d^{n-1}} \\ &\quad \text{Res}(g_1, \dots, g_n)^{d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)} K(f_1, \dots, f_{n-1}, g_1, \dots, g_n). \end{aligned}$$

*The polynomial  $K(f, g)$  is homogeneous with respect to the coefficients of the polynomials  $g_1, \dots, g_n$  of degree*

$$n(n-1)(d-1)d^{n-2} \prod_{i=1}^n d_i$$

*and, for all  $i = 1, \dots, n-1$ , it is homogeneous with respect to the coefficients of the polynomial  $f_i$  of degree*

$$n(d-1)d^{n-2} \left( \frac{d_1 \dots d_n}{d_i} \right).$$

*Moreover, if  $k$  is a domain then  $K(f_1, \dots, f_{n-1}, g_1, \dots, g_n) \in {}_k A$  satisfies to the following properties:*



- i)  $K(f, g)$  is irreducible if  $2 \neq 0$  in  $k$ ,
- ii)  $K(f, g)$  is the square of an irreducible polynomial if  $2 = 0$  in  $k$ .

*Proof.* The first equality follows directly from Proposition 3.29 and Lemma 3.30. The computations of the degrees of  $K$  can be deduced from this formula and the degrees for the discriminant and the resultant. Indeed, since for all  $i = 1, \dots, n-1$  the polynomial  $f_i \circ g$  is homogeneous of degree  $dd_i$  in the  $X_i$ 's, by Proposition 3.9 we deduce that  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$  is homogeneous of degree

$$D_i := d^{n-2} \frac{\prod_{j=1}^{n-1} d_j}{d_i} \left( (dd_i - 1) + \sum_{j=1}^{n-1} (dd_j - 1) \right)$$

with respect to the coefficients of the polynomial  $f_i$  and of degree

$$D := \sum_{i=1}^{n-1} d_i D_i = nd^{n-2} \left( \prod_{j=1}^{n-1} d_j \right) \sum_{i=1}^{n-1} (dd_i - 1)$$

with respect to the coefficients of the polynomials  $g_1, \dots, g_n$ . Therefore, it follows that  $K$  is homogeneous with respect to the coefficients of the polynomial  $f_i$  of degree

$$D_i - d^{n-1} \frac{\prod_{j=1}^{n-1} d_j}{d_i} \left( (d_i - 1) + \sum_{j=1}^{n-1} (d_j - 1) \right) = n(d-1)d^{n-2} \left( \frac{d_1 \cdots d_n}{d_i} \right)$$

and is homogeneous with respect to the coefficients of the polynomials  $g_1, \dots, g_n$  of degree

$$D - nd^{n-1} \left( \prod_{i=1}^{n-1} d_i \right) \sum_{i=1}^{n-1} (d_i - 1) = n(n-1)(d-1)d^{n-2} \prod_{i=1}^n d_i$$

since  $\text{Res}(g_1, \dots, g_n)$  is homogeneous of degree  $nd^{n-1}$  with respect to the coefficients of the polynomials  $g_1, \dots, g_n$ .

Now, we turn to the proof of the irreducibility of  $K$ . First we observe that it is sufficient to prove the claimed properties in  $\text{Frac}(k)$  so that we will always work in a UFD. We begin with the case where  $2 \neq 0$  in  $k$ . We will proceed by induction on the integer  $r = d_1 + d_2 + \cdots + d_{n-1}$ . The difficult point is actually to prove this irreducibility property for  $r = n-1$ , that is to say for the case  $d_1 = \cdots = d_{n-1} = 1$ . Indeed, let us assume this for a moment and suppose that  $r > n-1$ . Then, at least one of the degree  $d_i$  is greater or equal to 2 and we can assume without loss of generality that it is  $d_1$  by permuting the  $f_i$ 's if necessary. Consider the specialization that sends  $f_1$  to the product of a generic form  $l$  and a generic polynomial  $f'_1$  of degree  $d_1 - 1$ . Lemma 3.30 implies that  $K$  has a multiplicativity property with respect to the polynomial  $f_1, \dots, f_n$ , so that this specialization sends  $K(f_1, \dots, f_n)$  (we omit the  $g_i$ 's in the notation for simplicity) to the product  $K(l, f_2, \dots, f_n)K(f'_1, f_2, \dots, f_n)$ . Now, if  $K$  is reducible then all its irreducible factors depending on the polynomial  $f_1$  must depend on  $l$  and  $f'_1$  after the above specialization. Therefore, since  $K(l, f_2, \dots, f_n)$  and  $K(f'_1, f_2, \dots, f_n)$  are both irreducible by our inductive hypothesis and distinct, we deduce that  $K(f_1, \dots, f_n)$  is also irreducible.

So, it remains to prove that  $K$  is irreducible in the case  $d_1 = \cdots = d_{n-1} = 1$ . Set  $f_i = \sum_{j=1}^n U_{i,j} X_j$  for all  $i = 1, \dots, n-1$ , introduce new indeterminates  $W_1, \dots, W_n$  and define the determinant

$$\Lambda := \begin{vmatrix} U_{1,1} & U_{1,2} & \cdots & U_{1,n} \\ U_{2,1} & U_{2,2} & \cdots & U_{2,n} \\ \vdots & \vdots & & \vdots \\ U_{n-1,1} & U_{n-1,2} & \cdots & U_{n-1,n} \\ W_1 & W_2 & \cdots & W_n \end{vmatrix}.$$

By (3.1.5) and the covariance property of resultants [Jou91, §5.11], we have

$$\begin{aligned} & \text{Res} \left( f_1 \circ g, \dots, f_{n-1} \circ g, J \left( f_1 \circ g, \dots, f_{n-1} \circ g, \sum_{i=1}^n W_i g_i \right) \right) \\ &= d^{d^{n-1}} \text{Res} \left( f_1 \circ g, \dots, f_{n-1} \circ g, \sum_{i=1}^n W_i g_i \right) \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g) \\ &= d^{d^{n-1}} \Lambda^{d^{n-1}} \text{Res}(g_1, \dots, g_n) \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g). \end{aligned}$$

On the other hand, since  $\Lambda \cdot J(g_1, \dots, g_n) = J(f_1 \circ g, \dots, f_{n-1} \circ g, \sum_{i=1}^n W_i g_i)$  we obtain that

$$\begin{aligned} & \text{Res} \left( f_1 \circ g, \dots, f_{n-1} \circ g, J \left( f_1 \circ g, \dots, f_{n-1} \circ g, \sum_{i=1}^n W_i g_i \right) \right) \\ &= \text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, \Lambda \cdot J(g_1, \dots, g_n)) \\ &= \Lambda^{d^{n-1}} \text{Res}(f_1 \circ g, \dots, f_{n-1} \circ g, J(g_1, \dots, g_n)) \\ &= \Lambda^{d^{n-1}} d^{d^{n-1}} \text{Res}(g_1, \dots, g_n) K(f_1, \dots, f_n, g_1, \dots, g_n). \end{aligned}$$

where the last equality follows from Lemma 3.30. Therefore, by comparison of these two computations (in the generic case over the integers and then by specialization) we deduce that

$$K(f_1, \dots, f_n, g_1, \dots, g_n) = \text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g) \quad (3.4.1)$$

under our assumption  $d_1 = \dots = d_{n-1} = 1$ . In order to show that this discriminant is irreducible, we will compare several specializations.

We begin with the specialization of the polynomials  $f_1, \dots, f_{n-1}$  to  $X_1, \dots, X_{n-1}$  respectively. Under this specialization, the polynomial  $f_i \circ g$  is sent to  $g_i$  for all  $i = 1, \dots, n-1$  and hence  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$  is sent to  $\text{Disc}(g_1, \dots, g_{n-1})$  which is known to be an irreducible polynomial in the coefficients of the polynomials  $g_1, \dots, g_{n-1}$  by Theorem 3.23. It follows that if  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$  is reducible, then necessarily there exists a non constant and irreducible polynomial  $P(U_{i,j})$  which is independent of the coefficients of the polynomials  $g_1, \dots, g_n$  and that divides  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$ .

Now, consider the specialization that sends the polynomial  $g_n$  to 0. Then, the discriminant  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$  is sent to

$$\begin{aligned} & \text{Disc} \left( \sum_{i=1}^{n-1} U_{1,i} g_i, \dots, \sum_{i=1}^{n-1} U_{n-1,i} g_i \right) = \\ & \left| \begin{array}{ccc} U_{1,1} & \cdots & U_{1,n-1} \\ \vdots & & \vdots \\ U_{n-1,1} & \cdots & U_{n-1,n-1} \end{array} \right|^{n(d-1)d^{n-2}} \times \text{Disc}(g_1, \dots, g_{n-1}) \end{aligned}$$

where the equality holds by the covariance property given in Proposition 3.18. We deduce that  $P(U_{i,j})$  is equal to the determinant of the matrix  $(U_{i,j})_{1 \leq i, j \leq n-1}$  up to multiplication by an invertible element in  $k$ . But if we consider the specialization that sends the polynomial  $g_1$  to 0, then

by a similar argument we get that  $\text{Disc}(f_1 \circ g, \dots, f_{n-1} \circ g)$  is sent to

$$\text{Disc} \left( \sum_{i=2}^n U_{1,j} g_j, \dots, \sum_{i=2}^n U_{n-1,j} g_j \right) = \begin{vmatrix} U_{1,2} & \cdots & U_{1,n} \\ \vdots & & \vdots \\ U_{n-1,2} & \cdots & U_{n-1,n} \end{vmatrix}^{n(d-1)d^{n-2}} \times \text{Disc}(g_2, \dots, g_n).$$

Therefore, we deduce that up to multiplication by an invertible element in  $k$ ,  $P(U_{i,j})$  should also be equal to the determinant of the matrix  $(U_{i,j})_{1 \leq i, j \leq n-1}$ . Hence, we get a contradiction. This concludes the proof of the irreducibility of  $K$  when  $2 \neq 0$  in  $k$ .

Now, we turn to the proof that  $K$  is the square of an irreducible polynomial under the assumption  $2 = 0$  in  $k$ . By Theorem 3.24, the discriminant is the square of a polynomial, irreducible in the generic case, that we will denote by  $\Delta$ . Now, define the polynomial  $\chi$  by the equality

$$\Delta(f_1 \circ g, \dots, f_{n-1} \circ g) = \Delta(f_1, \dots, f_{n-1})^{d^{n-1}} \text{Res}(g_1, \dots, g_n)^{\frac{1}{2}d_1 \dots d_{n-1} \sum_{i=1}^{n-1} (d_i - 1)} \chi(f_1, \dots, f_{n-1}, g_1, \dots, g_n)$$

so that  $K(f_1, \dots, f_{n-1}, g_1, \dots, g_n) = \chi(f_1, \dots, f_{n-1}, g_1, \dots, g_n)^2$ . To prove that  $\chi$  is an irreducible polynomial we can proceed similarly to the case where  $2 \neq 0$  in  $k$ : we proceed by induction on the integer  $r = d_1 + \dots + d_{n-1} \geq n - 1$ . Assuming for a moment that the statement holds for  $r = n - 1$ , then the reasoning is exactly the same:  $\chi$  inherits of a multiplicative property from  $K$  and hence by specializing one polynomial of degree  $\geq 2$ , say  $f_1$ , to the product of a linear form  $l$  and a polynomial  $f'_1$  of degree  $d_1 - 1$  then we conclude that  $\chi$  is irreducible.

To prove that  $\chi$  is indeed irreducible when  $d_1 = \dots = d_{n-1} = 1$ , we also proceed similarly to the case where  $2 \neq 0$ . Using (3.4.1) that holds in the generic case other the integers, we deduce that there exists  $\epsilon \in k$  such that  $\epsilon^2 = 1$  and

$$\chi(f_1, \dots, f_{n-1}, g_1, \dots, g_n) = \epsilon \Delta(f_1 \circ g, \dots, f_{n-1} \circ g)$$

in  $k$  under the assumption  $d_1 = \dots = d_{n-1} = 1$ . From here, we conclude that  $\chi$  is irreducible by exploiting, as in the case  $2 \neq 0$ , the three specializations  $f_i \mapsto X_i$  for all  $i = 1, \dots, n - 1$ , then  $g_n \mapsto 0$  and finally  $g_1 \mapsto 0$ , the argumentation being the same.  $\square$

## 4. The discriminant of a hypersurface

In this section we study the discriminant of a single homogeneous polynomial in several variables.

Let  $k$  be a commutative ring and  $f$  be a homogeneous polynomial of degree  $d \geq 2$  in the polynomial ring  $k[X_1, \dots, X_n]$  ( $n \geq 1$ ). We will denote by  $\partial_i f$  the partial derivative of  $f$  with respect to the variable  $X_i$ . Recall the classical Euler identity

$$df = \sum_{i=1}^n X_i \partial_i f.$$

We will also often denote by  $\bar{f}$ , respectively  $\tilde{f}$ , the polynomial  $f(X_1, \dots, X_{n-1}, 0)$ , respectively  $f(X_1, \dots, X_{n-1}, 1)$ , in  $k[X_1, \dots, X_{n-1}]$ .

We aim to study the quotient ring

$$k[X_1, \dots, X_n] / (f, \partial_1 f, \dots, \partial_n f)$$

and its associated inertia forms of degree 0 with respect to  $\mathfrak{m} := (X_1, \dots, X_n)$ . The geometric interpretation of the *generic case* over the commutative ring  $k$  is the following. Let  $d$  be an integer greater or equal to 2. We suppose that

$$f(X_1, \dots, X_n) = \sum_{|\alpha|=d} U_\alpha X^\alpha$$

and denote  ${}_k A := k[U_\alpha \mid |\alpha| = d]$ ,  ${}_k C := {}_k A[X_1, \dots, X_n]$  and

$${}_k B := {}_k C / (f, \partial_1 f, \dots, \partial_n f)$$

The closed image of the canonical projection  $\pi$  of  $\text{Proj}({}_k B)$  to  $\text{Spec}({}_k A)$  is defined by the ideal  $H_{\mathfrak{m}}^0({}_k B)_0$ ; roughly speaking, it parameterizes all the homogeneous forms of degree  $d$  with coefficients in  $k$  whose zero locus has a singular point.

#### 4.1. Regularity of certain sequences

We suppose that we are in the generic case over the commutative ring  $k$ . We begin with two technical results. Given a sequence of elements  $r_1, \dots, r_s$  in a ring  $R$ , we will denote by  $H_i(r_1, \dots, r_s; R)$  the  $i^{\text{th}}$  homology group of the Koszul complex associated to this sequence.

**Lemma 4.1.** *For all  $i \geq 2$  we have  $H_i(f, \partial_1 f, \dots, \partial_n f; {}_k C) = 0$ .*

*Proof.* Let us emphasize some coefficients of  $f$  by rewriting it as

$$f(X_1, \dots, X_n) = g(X_1, \dots, X_n) + \sum_{i=1}^n \mathcal{E}_i X_i X_n^{d-1}$$

where  $g \in {}_k C$ . Then, it appears that the sequence  $(\partial_1 f, \dots, \partial_{n-1} f, f)$  is, in this order, regular in the ring  ${}_k C_{X_n}$ . Indeed, the quotient by  $\partial_1 f$  amounts to express  $\mathcal{E}_1$  in the polynomial ring

$$k[U_\alpha \mid U_\alpha \neq \mathcal{E}_1, \dots, \mathcal{E}_n][X_1, \dots, X_n]_{X_n}.$$

Then the quotient by  $\partial_2 f$  amounts to express  $\mathcal{E}_2$  and so on. We end with the quotient by  $f$  that amounts to express  $\mathcal{E}_n$ . From this property and the well known properties of the Koszul complex, it follows that

$$H_i(f, \partial_1 f, \dots, \partial_n f; {}_k C)_{X_n} = 0 \text{ for all } i \geq 2.$$

But we can argue similarly by choosing another variable  $X_j$  instead of  $X_n$  and therefore we actually deduce that

$$H_i(f, \partial_1 f, \dots, \partial_n f; {}_k C)_{X_j} = 0 \text{ for all } i \geq 2 \text{ and } j = 1, \dots, n. \quad (4.1.1)$$

Now, the consideration of the two spectral sequences

$$'E_1^{p,q} = H_{\mathfrak{m}}^q(K^\bullet(f, \partial_1 f, \dots, \partial_n f; {}_k C)) \implies E^n = H_{\mathfrak{m}}^n(K^\bullet(f, \partial_1 f, \dots, \partial_n f; {}_k C))$$

$$''E_2^{p,q} = H_{\mathfrak{m}}^p(H^q(f, \partial_1 f, \dots, \partial_n f; {}_k C)) \implies E^n = H_{\mathfrak{m}}^n(K^\bullet(f, \partial_1 f, \dots, \partial_n f; {}_k C))$$

shows that for all  $i \geq 2$  we have  $H_i(f, \partial_1 f, \dots, \partial_n f; {}_k C) = 0$ , as claimed.  $\square$

**Proposition 4.2.** *The two following statements hold:*

- (i) *For all  $i \in \{1, \dots, n\}$  the sequence  $(f, \partial_1 f, \dots, \widehat{\partial_i f}, \dots, \partial_n f)$  is regular in the ring  ${}_k C$ .*
- (ii) *If  $d$  is a nonzero divisor in  $k$  then the sequence  $(\partial_1 f, \dots, \partial_n f)$  is regular in the ring  ${}_k C$ .*

*Proof.* We prove (i) in the case  $i = n$  to not overload the notation; the other cases can be treated similarly. For simplicity, we set

$$K_\bullet := K_\bullet(f, \partial_1 f, \dots, \partial_n f; {}_k C), \quad L_\bullet := K(f, \partial_1 f, \dots, \partial_{n-1} f; {}_k C).$$

Since  $K_\bullet = L_\bullet \otimes_{kC} K_\bullet(\partial_n f; kC)$ , we deduce, using the two spectral sequences associated to the two filtrations of a double complex having only two rows, that we have an exact sequence

$$0 \rightarrow H_0(\partial_n f; H_2(L_\bullet)) \rightarrow H_2(K_\bullet) \rightarrow H_1(\partial_n f; H_1(L_\bullet)) \rightarrow 0.$$

But by Lemma 4.1, we know that  $H_2(K_\bullet) = 0$ ; it follows that  $\partial_n f$  is a nonzero divisor in  $H_1(L)$ . The homology of  $L_\bullet$  is annihilated by the ideal generated by  $(f, \partial_1 f, \dots, \partial_{n-1} f)$ . So, by the Euler identity we deduce that  $X_n \partial_n f$  annihilates  $H_1(L)$ . But since we have just proved that  $\partial_n f$  is a nonzero divisor in  $H_1(L)$  we obtain  $X_n H_1(L) = 0$ .

Denoting  $\bar{f}(X_1, \dots, X_{n-1}) := f(X_1, \dots, X_{n-1}, 0) \in kA[X_1, \dots, X_{n-1}]$ , we have the exact sequence of complexes

$$0 \rightarrow L_\bullet \xrightarrow{\times X_n} L_\bullet \rightarrow L_\bullet / X_n L_\bullet \rightarrow 0$$

where the complex  $L_\bullet / X_n L_\bullet$  is nothing but the Koszul complex

$$L_\bullet / X_n L_\bullet = K_\bullet(\bar{f}, \partial_1 \bar{f}, \dots, \partial_{n-1} \bar{f}; kA[X_1, \dots, X_{n-1}]).$$

It follows that  $H_2(L_\bullet / X_n L_\bullet) = 0$  and hence the long exact sequence of homology

$$\dots \rightarrow H_2(L_\bullet / X_n L_\bullet) \rightarrow H_1(L_\bullet) \xrightarrow{\times X_n} H_1(L_\bullet) \rightarrow H_1(L_\bullet / X_n L_\bullet) \rightarrow \dots$$

shows that  $X_n$  is a nonzero divisor in  $H_1(L)$ . This, with the equality  $X_n H_1(L_\bullet) = 0$  obtained above, implies that  $H_1(L_\bullet) = 0$  which means that  $(f, \partial_1 f, \dots, \partial_{n-1} f)$  is a regular sequence in  $kC$ .

Setting  $M_\bullet := K_\bullet(\partial_1 f, \dots, \partial_n f; kC)$ , we will prove the point (ii) by showing that  $H_1(M_\bullet) = 0$ . Since  $K_\bullet = M_\bullet \otimes_{kC} K_\bullet(f; kC)$  we have the two exact sequences

$$0 \rightarrow H_0(f; H_2(M_\bullet)) \rightarrow H_2(K_\bullet) \rightarrow H_1(f; H_1(M_\bullet)) \rightarrow 0 \quad (4.1.2)$$

$$0 \rightarrow H_0(f; H_1(M_\bullet)) \rightarrow H_1(K_\bullet) \rightarrow H_1(f; H_1(M_\bullet)) \rightarrow 0. \quad (4.1.3)$$

First, by (4.1.1) we know that  $H_2(K_\bullet) = 0$  and hence the exact sequence (4.1.2) shows that  $H_1(f, H_1(M)) = 0$ , that is to say that  $f$  is a nonzero divisor in  $H_1(M)$ . But the Euler identity implies that  $df$  annihilates  $H_1(M_\bullet)$ , so  $dH_1(M) = 0$ . Second, from the exact sequence of complexes

$$0 \rightarrow K_\bullet \xrightarrow{\times d} K_\bullet \rightarrow K_\bullet / dK_\bullet \rightarrow 0$$

we get the long exact sequence

$$\dots \rightarrow H_2(K_\bullet / dK_\bullet) \rightarrow H_1(K_\bullet) \xrightarrow{\times d} H_1(K_\bullet) \rightarrow \dots$$

which shows, since  $H_2(K_\bullet / dK_\bullet) = 0$ , that  $d$  is a nonzero divisor in  $H_1(K_\bullet)$ .

Finally, the exact sequence (4.1.3) combined with the two facts  $dH_1(M) = 0$  and  $d$  is a nonzero divisor in  $H_1(K_\bullet)$ , implies that  $H_0(f; H_1(M_\bullet)) = 0$ , that is to say that the multiplication map  $\times f : H_1(M_\bullet) \rightarrow H_1(M_\bullet)$  is surjective. It follows that, by composition, for any integer  $m \geq 1$  the multiplication map  $\times f^m : H_1(M_\bullet) \rightarrow H_1(M_\bullet)$  is also surjective. But  $H_1(M)$  is a  $\mathbb{Z}$ -graded module and  $f$  has degree  $d$  for this graduation, so we have, for any  $\nu \in \mathbb{Z}$  and  $m \in \mathbb{N}^*$ , a surjective map

$$H_1(M_\bullet)_{\nu-dm} \xrightarrow{\times f^m} H_1(M)_\nu.$$

As  $H_1(M)_\mu = 0$  for  $\mu \ll 0$  we finally get, by choosing  $m \gg 0$ , that  $H_1(M)_\nu = 0$  for all  $\nu \in \mathbb{Z}$ .  $\square$

**Corollary 4.3.** *For all  $i \in \{1, \dots, n\}$ , the resultant*

$$\text{Res}(\partial_1 f, \dots, \widehat{\partial_i f}, \dots, \partial_{n-1} f, f) \in kA$$

*is a primitive polynomial, hence nonzero divisor, in  $kA$ .*

*Proof.* This result is a consequence of Proposition 4.2 and [Jou92, Proposition 3.12.4.2]. The last claim is obtained by observing that this resultant is a nonzero divisor in  $\mathbb{Z}/p\mathbb{Z}A$  for all integers  $p$ , which implies that it is a primitive polynomial in  $\mathbb{Z}A$ , hence in  $kA$ .  $\square$

#### 4.2. Definition of the discriminant

**Lemma 4.4.** *Let  $k$  be a commutative ring and  $f \in k[X_1, \dots, X_n]$  be a homogeneous polynomial of degree  $d \geq 2$ . Then, we have the following equality in  $k$ :*

$$d^{(d-1)^{n-1}} \operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, f) = \operatorname{Res}(\partial_1 f, \dots, \partial_n f) \operatorname{Res}(\partial_1 \bar{f}, \dots, \partial_{n-1} \bar{f}).$$

*Proof.* On the one hand we have, using the homogeneity of the resultant,

$$\operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, df) = d^{(d-1)^{n-1}} \operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, f),$$

and on the other hand we have, using successively [Jou91, §5.9], [Jou91, §5.7] and [Jou91, Lemma 4.8.9],

$$\begin{aligned} \operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, df) &= \operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, X_n \partial_n f) \\ &= \operatorname{Res}(\partial_1 f, \dots, \partial_n f) \operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, X_n) \\ &= \operatorname{Res}(\partial_1 f, \dots, \partial_n f) \operatorname{Res}(\partial_1 \bar{f}, \dots, \partial_{n-1} \bar{f}). \end{aligned}$$

Comparing these two computations we deduce the claimed equality.  $\square$

**Proposition 4.5.** *Let  $f(X_1, \dots, X_n) = \sum_{|\alpha|=d} U_\alpha X^\alpha$  be the generic homogeneous polynomial of degree  $d \geq 2$  over the integers. Then the resultant  $\operatorname{Res}(\partial_1 f, \dots, \partial_n f)$  is divisible by  $d^{a(n,d)}$  in the ring  $\mathbb{Z}A$  where*

$$a(n, d) := \frac{(d-1)^n - (-1)^n}{d} \in \mathbb{Z}.$$

*Proof.* By Corollary 4.3, we know that  $\operatorname{Res}(\partial_1 f, \dots, \partial_{n-1} f, f)$  is a primitive polynomial in  $\mathbb{Z}A$ . Denoting by  $c(n, d)$  the content of  $\operatorname{Res}(\partial_1 f, \dots, \partial_n f)$  for all  $n, d \geq 2$ , Lemma 4.4 implies that

$$c(n, d)c(n-1, d) = d^{(d-1)^{n-1}} \quad \text{for all } n \geq 3 \text{ and } d \geq 2$$

and also that  $c(2, d) = d^{d-2} = d^{a(2,d)}$  for all  $d \geq 2$  (just remark that we have  $\operatorname{Res}(dU X_1^{d-1}, dU) = dU$ ). Therefore, we can proceed by induction on  $n$  to prove the claimed result: assume that  $c(n-1, d) = d^{a(n-1,d)}$ , which is true for  $n = 3$ , then

$$c(n, d) = d^{(d-1)^{n-1} - a(n-1,d)} = d^{a(n,d)}$$

since it is immediate to check that  $a(n-1, d) + a(n, d) = (d-1)^{n-1}$ .  $\square$

We are now ready to define the discriminant of a homogeneous polynomial of degree  $d \geq 2$ .

**Definition 4.6.** *Let  $f(X_1, \dots, X_n) = \sum_{|\alpha|=d} U_\alpha X^\alpha \in \mathbb{Z}A$  be the generic homogeneous polynomial of degree  $d \geq 2$ . The discriminant of  $f$ , that will be denoted  $\operatorname{Disc}(f)$ , is the unique element in  $\mathbb{Z}A$  such that*

$$d^{a(n,d)} \operatorname{Disc}(f) = \operatorname{Res}(\partial_1 f, \dots, \partial_n f). \quad (4.2.1)$$

*Let  $R$  be a commutative ring and  $g = \sum_{|\alpha|=d} u_\alpha X^\alpha$  be a homogeneous polynomial of degree  $d \geq 2$  in  $R[X_1, \dots, X_n]$ . Then we define the discriminant of  $g$  as  $\operatorname{Disc}(g) := \lambda(\operatorname{Disc}(f))$  where  $\lambda$  is the canonical (specialization) morphism  $\lambda : \mathbb{Z}A \rightarrow R : U_\alpha \mapsto u_\alpha$ .*

#### 4.3. Formal properties

Up to a nonzero integer constant factor, the discriminant of a homogeneous polynomial corresponds to a resultant. Consequently, most of its properties follow from the properties of the resultant.

**Proposition 4.7.** *Let  $k$  be a commutative ring and  $f$  be a homogeneous polynomial in  $k[X_1, \dots, X_n]$  of degree  $d \geq 2$ .*

- (i) *For all  $t \in k$ , we have  $\operatorname{Disc}(tf) = t^{n(d-1)^{n-1}} \operatorname{Disc}(f)$ .*

(ii) For all  $n \geq 2$ , we have the equality in  $k$

$$\text{Disc}(f)\text{Disc}(\bar{f}) = \text{Res}(\partial_1 f, \dots, \partial_{n-1} f, f).$$

*Proof.* To prove (i), we use the homogeneity of the resultant: one obtains

$$\text{Res}(t\partial_1 f, \dots, t\partial_n f) = t^{\sum_{i=1}^n (d-1)^{n-1}} \text{Res}(\partial_1 f, \dots, \partial_n f).$$

To prove (ii), we first assume that we are in the generic case, that is to say that  $f = \sum_{|\alpha|=d} U_\alpha X^\alpha$  and  $k = {}_{\mathbb{Z}}A := \mathbb{Z}[U_\alpha \mid |\alpha| = d]$ . Using the notation of Proposition 4.5, we have  $a(n, d) + a(n-1, d) = d^{(d-1)^{n-1}}$  for all  $n \geq 3$  and  $d \geq 2$ . Moreover, from Definition 4.6, we deduce that

$$\text{Res}(\partial_1 f, \dots, \partial_n f)\text{Res}(\partial_1 \bar{f}, \dots, \partial_{n-1} \bar{f}) = d^{(d-1)^{n-1}} \text{Disc}(f)\text{Disc}(\bar{f}).$$

Now, comparing with Lemma 4.4, we get the claimed formula in  ${}_{\mathbb{Z}}A$  and then over any commutative ring  $k$  by specialization.  $\square$

**Remark 4.8.** In the case where  $n = 2$  and  $d$  is a nonzero divisor of  $k$  (equivalently  $\text{char}(k)$  does not divide  $d$ ), the point (ii) recovers a well known formula: set  $f := U_0 X_1^d + U_1 X_1^{d-1} X_2 + \dots + U_d X_2^d$  for simplicity, then

$$U_0 \text{Disc}(f) = \text{Res}(\partial_1 f, f) = \text{Res}(f, \partial_1 f).$$

This follows from Definition 4.6 since we have

$$d^{a(1,d)} \text{Disc}(\bar{f}) = d \text{Disc}(U_0 X_1^d) = \text{Res}(dU_0 X_1^{d-1}) = dU_0$$

in  ${}_k A$ .

**Corollary 4.9.** Let  $f(X_1, \dots, X_n) = \sum_{|\alpha|=d} U_\alpha X^\alpha \in {}_k A$  be the generic homogeneous polynomial of degree  $d \geq 2$  over the commutative ring  $k$ . Then  $\text{Disc}(f)$  is a primitive polynomial, hence nonzero divisor, in  ${}_k A$ .

*Proof.* The first claim is a combination of both Corollary 4.3 and Proposition 4.7, (ii). To prove the second claim we can argue as in the proof of Corollary 4.3.  $\square$

We continue with some particular examples.

**Example 4.10.** Let  $h(X_1, \dots, X_{n-1}) = \sum_{|\alpha|=d} V_\alpha X^\alpha$  be the generic homogeneous polynomial of degree  $d \geq 2$  in the variables  $X_1, \dots, X_{n-1}$  over the commutative ring  $k$  and consider the homogeneous polynomial

$$g(X_1, \dots, X_n) = U X_n^d + h(X_1, \dots, X_{n-1}) \in k[U, V_\alpha \mid |\alpha| = d][X_1, \dots, X_n].$$

Then, we have

$$\text{Disc}(g) = d^{(d-1)^{n-1} + (-1)^n} U^{(d-1)^{n-1}} \text{Disc}(h)^{d-1}.$$

*Proof.* Notice that without loss of generality, it is enough to prove this formula in the case  $k = \mathbb{Z}$ . Now, since  $\partial_n g = dU X_n^{d-1}$  and  $\partial_i g = \partial_i h$  for  $i = 1, \dots, n-1$ , we deduce that

$$\text{Res}(\partial_1 g, \dots, \partial_n g) = (dU)^{(d-1)^{n-1}} \text{Res}(\partial_1 h, \dots, \partial_{n-1} h)^{d-1}.$$

Therefore, from the definition of the discriminant we get

$$d^{a(n,d)} \text{Disc}(g) = (dU)^{(d-1)^{n-1}} d^{(d-1)a(n-1,d)} \text{Disc}(h)^{d-1}$$

and the claimed formula follows from a straightforward computation.  $\square$

**Example 4.11** ([Dem69]). Consider the homogeneous polynomial of degree  $d \geq 2$

$$g(X_1, \dots, X_n) = A_1 X_1^d + \dots + A_n X_n^d \in \mathbb{Z}[A_1, \dots, A_n][X_1, \dots, X_n].$$

Then, its discriminant consists of only one monomial; more precisely,

$$\text{Disc}(g) = d^{n(d-1)^{n-1} - a(n,d)} (A_1 A_2 \dots A_n)^{(d-1)^{n-1}} \in \mathbb{Z}[A_1, \dots, A_n].$$

*Proof.* Indeed, since  $\partial_i g = dA_i X_i^{d-1}$  for all  $i = 1, \dots, n$ , from the classical properties of the resultant we get

$$\begin{aligned} \text{Res}(\partial_1 g, \dots, \partial_n g) &= d^{n(d-1)^{n-1}} \text{Res}(A_1 X_1^{d-1}, \dots, A_n X_n^{d-1}) \\ &= d^{n(d-1)^{n-1}} (A_1 A_2 \dots A_n)^{(d-1)^{n-1}}. \end{aligned}$$

The claimed result follows by comparing this equality with (4.2.1).  $\square$

**Example 4.12** ([Dem69]). *Consider the homogeneous polynomial of degree  $d \geq 2$*

$$g(X_1, \dots, X_n) = X_1^d + U X_1 X_2^{d-1} + X_2 X_3^{d-1} + \dots + X_{n-1} X_n^{d-1} \in \mathbb{Z}[U][X_1, \dots, X_n].$$

*Then, its discriminant contains only one monomial modulo  $d$ . More precisely,*

$$\text{Disc}(g) = U^{(d-1)^{n-1} + (-1)^n} \pmod{(d)} \in \mathbb{Z}[U].$$

*Proof.* To prove this formula, we proceed by induction on the number  $n$  of variables. So, assume first that  $n = 2$ . We have  $g = X_1 + U X_1 X_2^{d-1}$  and we easily compute in the ring  $\mathbb{Z}[U]$

$$\begin{aligned} \text{Res}(\partial_1 g, \partial_2 g) &= \text{Res}(dX_1^{d-1} + U X_2^{d-1}, (d-1)U X_1 X_2^{d-2}) \\ &= (d-1)^{d-1} U^{d-1} \text{Res}(U X_2^{d-1}, X_1) \text{Res}(dX_1^{d-1}, X_2^{d-2}) \\ &= (-1)^{d-1} (d-1)^{d-1} d^{d-2} U^d. \end{aligned} \tag{4.3.1}$$

From (4.2.1) and since  $a(2, d) = d - 2$ , we deduce that

$$\text{Disc}(g) = (-1)^{d-1} (d-1)^{d-1} U^d = U^d \pmod{(d)}.$$

Now, fix the integer  $n > 2$  and suppose that the claimed formula is proved at the step  $n - 1$ . Again, an easy computation of resultants in  $\mathbb{Z}[U]$  yields

$$\begin{aligned} \text{Res}(\partial_1 g, \dots, \partial_{n-1} g, g) & \tag{4.3.2} \\ &= \text{Res}(U X_2^{d-1}, -U X_1 X_2^{d-2} + X_3^{d-1}, \dots, -X_{n-2} X_{n-1}^{d-2} + X_n^{d-1}, g) \pmod{(d)} \\ &= U^{d(d-1)^{n-2}} \text{Res}(X_2^{d-1}, X_3^{d-1}, \dots, X_n^{d-1}, X_1^d) \pmod{(d)} \\ &= U^{d(d-1)^{n-2}} \pmod{(d)}. \end{aligned}$$

By Proposition 4.7, (ii), it follows that, in  $\mathbb{Z}[U]$ ,

$$\begin{aligned} U^{d(d-1)^{n-2}} &= \text{Disc}(\bar{g}) \text{Disc}(g) \pmod{(d)} \\ &= U^{(d-1)^{n-2} + (-1)^{n-2}} \text{Disc}(g) \pmod{(d)}. \end{aligned}$$

We deduce that

$$\text{Disc}(g) = U^{d(d-1)^{n-2} - (d-1)^{n-2} - (-1)^{n-2}} = U^{(d-1)^{n-1} + (-1)^{n-1}} \pmod{(d)} \in \mathbb{Z}[U].$$

$\square$

Next, we provide two formulas that encapsulate the behavior of the discriminant under a linear change of coordinates and under a general base change formula.

**Proposition 4.13.** *Let  $k$  be a commutative ring and  $f$  be a homogeneous polynomial of degree  $d \geq 2$  in  $k[X_1, \dots, X_n]$ . Being given a matrix  $\varphi = [c_{i,j}]_{1 \leq i, j \leq n}$  with entries in  $k$  and denoting*

$$f \circ \varphi(X_1, \dots, X_n) := f \left( c_{1,1} X_1 + \dots + c_{1,n} X_n, \dots, \sum_{j=1}^n c_{i,j} X_j, \dots, \sum_{j=1}^n c_{n,j} X_n \right),$$

*we have*

$$\text{Disc}(f \circ \varphi) = \det(\varphi)^{d(d-1)^{n-1}} \text{Disc}(f).$$



*Proof.* By specialization, it is sufficient to prove this formula in the generic setting, that is to say with  $f = \sum_{|\alpha|=d} U_\alpha X^\alpha \in \mathbb{Z}A$ . Since  $f \circ \varphi$  and  $f$  have the same degree  $d$  as polynomials in the  $X_i$ 's, it is equivalent to prove that

$$\text{Res}(\partial_1(f \circ \varphi), \dots, \partial_n(f \circ \varphi)) = \det(\varphi)^{d(d-1)^{n-1}} \text{Res}(\partial_1 f, \dots, \partial_n f).$$

To do this, we remark, by basic differential calculus, that

$$[\partial_1(f \circ \varphi), \dots, \partial_n(f \circ \varphi)] = [\partial_1(f) \circ \varphi, \dots, \partial_n(f) \circ \varphi] \cdot \det(\varphi),$$

as matrices. Therefore, the covariance formula of the resultant [Jou91, §5.11.2] shows that

$$\text{Res}(\partial_1(f \circ \varphi), \dots, \partial_n(f \circ \varphi)) = \det(\varphi)^{(d-1)^{n-1}} \text{Res}(\partial_1(f) \circ \varphi, \dots, \partial_n(f) \circ \varphi).$$

Moreover, the formula for linear change of coordinates for the resultant [Jou91, §5.13.1] gives

$$\text{Res}(\partial_1(f) \circ \varphi, \dots, \partial_n(f) \circ \varphi) = \det(\varphi)^{(d-1)^n} \text{Res}(\partial_1 f, \dots, \partial_n f),$$

and we conclude the proof by observing that  $(d-1)^n + (d-1)^{n-1} = d(d-1)^{n-1}$ .  $\square$

One consequence of this invariance property is the following generalization of the formula defining the discriminant given in Proposition 4.7, (ii).

**Proposition 4.14.** *Let  $k$  be a commutative ring, let  $f$  be a homogeneous polynomial of degree  $d \geq 2$  in  $k[X_1, \dots, X_n]$  and let  $\varphi = [c_{i,j}]_{1 \leq i \leq n, 1 \leq j \leq n-1}$  be a  $n \times (n-1)$ -matrix with coefficients in  $k$ . Then, we have*

$$\text{Disc}(f) \text{Disc}(f([X_1, \dots, X_{n-1}] \circ {}^t \varphi)) = \text{Res}(f, [\partial_1 f, \dots, \partial_n f] \circ \varphi).$$

*Proof.* By specialization, it is sufficient to prove this equality for  $f$  the generic homogeneous polynomial of degree  $d$  over the integers and for  $\varphi := [V_{i,j}]_{1 \leq i \leq n, 1 \leq j \leq n-1}$  a matrix of indeterminates. Adding another column of indeterminates to  $\varphi$ , we introduce the matrix  $\psi := [V_{i,j}]_{1 \leq i, j \leq n}$ .

Now, consider the following resultant

$$\begin{aligned} \Omega := \text{Res}(f([X_1, \dots, X_n] \circ {}^t \psi), \\ [\partial_1 f([X_1, \dots, X_n] \circ {}^t \psi), \dots, \partial_n f([X_1, \dots, X_n] \circ {}^t \psi)] \circ \varphi). \end{aligned}$$

On the one hand, by the invariance property of the resultant [Jou91, §5.13] we have

$$\Omega = \det(\psi)^{d(d-1)^{n-1}} \text{Res}(f, [\partial_1 f, \dots, \partial_n f] \circ \varphi). \quad (4.3.3)$$

On the other hand, since

$$\begin{aligned} [\partial_1 f([X_1, \dots, X_n] \circ {}^t \psi), \dots, \partial_n f([X_1, \dots, X_n] \circ {}^t \psi)] \circ \varphi = \\ \left[ \frac{\partial}{\partial X_1} (f([X_1, \dots, X_n] \circ {}^t \psi)), \dots, \frac{\partial}{\partial X_n} (f([X_1, \dots, X_n] \circ {}^t \psi)) \right] \end{aligned}$$

by the composition rule of the derivatives, we get from Proposition 4.7, (ii) that

$$\begin{aligned} \Omega &= \text{Disc}(f([X_1, \dots, X_n] \circ {}^t \psi)) \text{Disc}(f([X_1, \dots, X_{n-1}], 0] \circ {}^t \psi)) \\ &= \text{Disc}(f([X_1, \dots, X_n] \circ {}^t \psi)) \text{Disc}(f([X_1, \dots, X_{n-1}] \circ {}^t \varphi)) \\ &= \det(\psi)^{d(d-1)^{n-1}} \text{Disc}(f) \text{Disc}(f([X_1, \dots, X_{n-1}] \circ {}^t \varphi)) \end{aligned} \quad (4.3.4)$$

where the last equality holds by invariance of the discriminant; see Proposition 4.13. Finally, the claimed formula follows by comparing (4.3.3) and (4.3.4), taking into account the fact that  $\det(\psi)$  is a nonzero divisor in our generic setting.  $\square$

Now, we turn to the more general problem of the behavior of the discriminant under a general change of basis.

**Proposition 4.15.** *Let  $k$  be a commutative ring,  $f$  be a homogeneous polynomial of degree  $m \geq 2$  and  $g_1, \dots, g_n$  be homogeneous polynomials of degree  $d \geq 1$ . There exists a polynomial  $K$  that depends on the coefficients of the polynomials  $f, g_1, \dots, g_n$  such that*

$$\text{Disc}(f(g_1, \dots, g_n)) = \text{Disc}(f)^{d^{n-1}} \text{Res}(g_1, \dots, g_n)^{m(m-1)^{n-1}} K(f, g_1, \dots, g_n).$$

*Proof.* We prove the existence of  $K$  in the universal setting over the integers so that the claimed result follows by specialization. From the equality of matrices

$$\begin{bmatrix} \partial_{X_1}(f(\underline{g})) & \cdots & \partial_{X_n}(f(\underline{g})) \end{bmatrix} = \begin{bmatrix} \partial_{X_1}f(\underline{g}) & \cdots & \partial_{X_n}f(\underline{g}) \end{bmatrix} \begin{bmatrix} \partial_{X_1}g_1 & \cdots & \partial_{X_n}g_1 \\ \vdots & & \vdots \\ \partial_{X_1}g_n & \cdots & \partial_{X_n}g_n \end{bmatrix}$$

we deduce that for all  $i = 1, \dots, n$

$$\partial_{X_i}(f(\underline{g})) \in (\partial_{X_1}f(\underline{g}), \dots, \partial_{X_n}f(\underline{g})). \quad (4.3.5)$$

Therefore, applying the divisibility property of the resultant [Jou91, §5.6], we obtain that

$$\text{Res}(\partial_{X_1}f(\underline{g}), \dots, \partial_{X_n}f(\underline{g})) \text{ divides } \text{Res}(\partial_{X_1}(f(\underline{g})), \dots, \partial_{X_n}(f(\underline{g}))).$$

On the one hand, using the base change formula of the resultant [Jou91, §5.12], we have

$$\begin{aligned} \text{Res}(\partial_{X_1}f(\underline{g}), \dots, \partial_{X_n}f(\underline{g})) &= \text{Res}(g_1, \dots, g_n)^{(m-1)^n} \text{Res}(\partial_{X_1}f, \dots, \partial_{X_n}f) \\ &= m^{a(m,d)d^{n-1}} \text{Disc}(f)^{d^{n-1}} \text{Res}(g_1, \dots, g_n)^{(m-1)^n} \end{aligned}$$

and on the other hand

$$\text{Res}(\partial_{X_1}(f(\underline{g})), \dots, \partial_{X_n}(f(\underline{g}))) = (md)^{a(n,md)} \text{Disc}(f(g_1, \dots, g_n)).$$

Therefore, since  $\text{Disc}(f)$  is a primitive polynomial, we deduce that  $\text{Disc}(f)^{d^{n-1}}$  divides  $\text{Disc}(f(\underline{g}))$ .

Now, notice that we have  $f(g_1, \dots, g_n) \in (g_1, \dots, g_n)^m$  and that for all  $i = 1, \dots, n$  we have  $\partial_{X_i}(f(\underline{g})) \in (g_1, \dots, g_n)^{m-1}$  by using from (4.3.5). Using the generalized divisibility property of the resultant [Jou91, §6.2], it follows that

$$\text{Res}(g_1, \dots, g_n)^{m(m-1)^{n-1}} \text{ divides } \text{Res}(\partial_{X_1}(f(\underline{g})), \dots, \partial_{X_{n-1}}(f(\underline{g})), f(\underline{g})).$$

But

$$\text{Res}(\partial_{X_1}(f(\underline{g})), \dots, \partial_{X_{n-1}}(f(\underline{g})), f(\underline{g})) = \text{Disc}(f(\underline{g})) \text{Disc}(\overline{f(\underline{g})})$$

and  $\text{Res}(g_1, \dots, g_n)$  is an irreducible polynomial that depends on all the coefficients of all the polynomials  $g_1, \dots, g_n$ . We deduce that

$$\text{Res}(g_1, \dots, g_n)^{m(m-1)^{n-1}} \text{ divides } \text{Disc}(f(g_1, \dots, g_n)).$$

Finally, since  $\text{Res}(g_1, \dots, g_n)$  and  $\text{Disc}(f)$  are obviously coprime, the existence of the polynomial  $K$  is proved.  $\square$

#### 4.4. Inertia forms and the discriminant

By definition, it is clear that the discriminant of a homogeneous polynomial  $f \in R[X_1, \dots, X_n]$ , where  $R$  is a field, of degree  $d \geq 2$  vanishes if and only if  $\partial_1 f, \dots, \partial_n f$  (and hence  $f$  if  $\text{char}(k)$  does not divide  $d$ ) have a non trivial common root in an algebraic extension of  $R$ . The purpose of this section is to study the behavior of the discriminant when  $R$ , the coefficient ring of the homogeneous polynomial  $f$ , is not assumed to be a field.

Let  $d \geq 2$  be a fixed integer and consider the polynomial

$$f(X_1, \dots, X_n) := \sum_{|\alpha|=d} U_\alpha X^\alpha.$$

Let  $k$  be a commutative ring and denote by  ${}_kA := k[U_\alpha \mid |\alpha| = d]$  the coefficient ring of  $f$  over  $k$ . Then  $f \in {}_kA[X_1, \dots, X_n]$ ; it is the generic homogeneous polynomial of degree  $d$  over  $k$ . Defining the ideals of  ${}_kC := {}_kA[X_1, \dots, X_n]$

$$\mathcal{D} := (f, \partial_1 f, \dots, \partial_n f), \quad \mathfrak{m} := (X_1, \dots, X_n),$$

we recall that  $\mathfrak{P} := \mathrm{TF}_m(\mathcal{D})_0 = H_m^0({}_kB)_0$  where  ${}_kB$  is the quotient ring  ${}_kC/\mathcal{D}$ . This latter ideal is nothing but the defining ideal of the closed subscheme of  $\mathrm{Spec}({}_kA)$  obtained as the image of the canonical projective morphism

$$\mathrm{Proj}({}_kB) \rightarrow \mathrm{Spec}({}_kA).$$

In the sequel, our aim is to relate the discriminant of  $f$  as defined in Definition 4.6 with this ideal of inertia forms  $\mathfrak{P} \subset {}_kA$ .

**Proposition 4.16.** *For  $j \in \{1, \dots, n\}$  we have an isomorphism of  $k[X_1, \dots, X_n]$ -algebras*

$${}_kB_{X_j} \xrightarrow{\sim} k[U_\alpha \mid |\alpha| = d, \alpha_j < d-1][X_1, \dots, X_n][X_j^{-1}]. \quad (4.4.1)$$

*In particular, for all  $j \in \{1, \dots, n\}$  the ring  ${}_kB_{X_j}$  is a domain if  $k$  is a domain.*

*Proof.* Let  $i$  be a fixed integer in  $\{1, \dots, n\}$ . The Euler equality  $df = \sum_{j=1}^n X_j \partial_j f$  shows that, after localization by the variable  $X_j$ , we have

$$\mathcal{D}_{X_j} = (\partial_1 f, \dots, \partial_{j-1} f, \partial_{j+1} f, \dots, \partial_n f, f) \subset {}_kC_{X_j}.$$

In order to emphasize some particular coefficients of the polynomial  $f$ , let us rewrite it as

$$f(X_1, \dots, X_n) = \sum_{i=1}^n \mathcal{E}_i X_i X_j^{d-1} + \sum_{|\alpha|=d, \alpha_j < d-1} U_\alpha X^\alpha.$$

Then, denoting by  $Q(X_1, \dots, X_n)$  the second term of the right side of this equality, for all integers  $i \in \{1, \dots, n\}$  such that  $i \neq j$  we have

$$\partial_i f(X_1, \dots, X_n) = \mathcal{E}_i X_j^{d-1} + \partial_i Q(X_1, \dots, X_n).$$

It follows that the following  $k[X_1, \dots, X_n]$ -algebras morphism

$$\begin{aligned} {}_kC_{X_j} &\longrightarrow k[U_\alpha \mid |\alpha| = d, \alpha_j < d-1][X_1, \dots, X_n][X_j^{-1}] \\ \mathcal{E}_i \ (i \neq j) &\mapsto -X_j^{-d+1} \partial_i Q \\ \mathcal{E}_j &\mapsto -X_j^{-d} Q + \sum_{i \neq j, i=1}^n X_i X_j^{-d} \partial_i Q = -X_j^{-d} ((1-d)Q + X_j \partial_j Q) \end{aligned}$$

has kernel  $\mathcal{D}_{X_j}$  and therefore induces an isomorphism of  $k[X_1, \dots, X_n]$ -algebras

$${}_kB_{X_j} \xrightarrow{\sim} k[U_\alpha \mid |\alpha| = d, \alpha_j < d-1][X_1, \dots, X_n][X_j^{-1}].$$

□

**Corollary 4.17.** *For all  $i = 1, \dots, n$  we have*

$$\mathrm{TF}_m(\mathcal{D}) = \ker({}_kC \rightarrow {}_kB_{X_i})$$

where  ${}_kC \rightarrow {}_kB_{X_i}$  is the canonical map, so that

$$\mathrm{TF}_m(\mathcal{D})_0 = \ker({}_kA \rightarrow {}_kB_{X_i}) = H_{(X_i)}^0({}_kB)_0.$$

*In particular, if  $k$  is a domain then  $\mathrm{TF}_m(\mathcal{D})$  and  $\mathfrak{P}$  are prime ideals.*

*Proof.* Observe first that by definition we have

$$\mathrm{TF}_m(\mathcal{D}) = \ker(kC \rightarrow \prod_{i=1}^n kB_{X_i}).$$

The isomorphisms (4.4.1) show that for any pair of integers  $(i, j) \in \{1, \dots, n\}^2$  the variable  $X_i$  is a nonzero divisor in  $kB_{X_j}$  and hence that the canonical map  $kB_{X_i} \rightarrow kB_{X_i X_j}$  is injective. By considering the commutative diagrams, for all pairs  $(i, j) \in \{1, \dots, n\}^2$ ,

$$\begin{array}{ccc} C & \longrightarrow & B_{X_i} \\ \downarrow & & \downarrow \\ B_{X_j} & \longrightarrow & B_{X_i X_j} \end{array}$$

we obtain that  $\mathrm{TF}_m(\mathcal{D}) = \ker(kC \rightarrow kB_{X_i})$  for all  $i = 1, \dots, n$ . From here, assuming that  $k$  is domain we deduce easily that  $\mathrm{TF}_m(\mathcal{D})$  is a prime ideal of  $kC$  and that  $\mathfrak{P} = \mathrm{TF}_m(\mathcal{D})_0$  is a prime ideal of  $kA$ .  $\square$

We now turn to the relation between the ideal of inertia forms  $\mathrm{TF}_m(\mathcal{D})$  and the discriminant of  $f$ .

**Theorem 4.18.** *Let  $R$  be a commutative ring and  $f$  a homogeneous polynomial in  $R[X_1, \dots, X_n]_d$  with  $d \geq 2$ . Then, we have the following inclusions of ideals in  $R$ :*

$$(\mathrm{Disc}(f)) \subset \mathrm{TF}_m((f, \partial_1 f, \dots, \partial_n f)) \cap R \subset \sqrt{(\mathrm{Disc}(f))}.$$

*Proof.* We first prove these inclusions in the generic case over the integers, that is to say with  $f = \sum_{|\alpha|=d} U_\alpha X^\alpha$  and  $R = {}_{\mathbb{Z}}A = \mathbb{Z}[U_\alpha \mid |\alpha| = d]$ .

By definition of the discriminant, we have

$$d^{a(n,d)} \mathrm{Disc}(f) = \mathrm{Res}(\partial_1 f, \dots, \partial_n f) \text{ in } {}_{\mathbb{Z}}A.$$

But since  $\mathrm{Res}(\partial_1 f, \dots, \partial_n f)$  is an inertia form of the ideal  $(\partial_1 f, \dots, \partial_n f)$  with respect to  $\mathfrak{m}$ , we deduce that

$$d^{a(n,d)} \mathrm{Disc}(f) \in \mathrm{TF}_m(\mathcal{D})_0$$

which is a prime ideal ( $\mathbb{Z}$  is a domain). Moreover, we claim that  $d^{a(n,d)} \notin \mathrm{TF}_m(\mathcal{D})_0$  because

$$\mathrm{TF}_m(\mathcal{D})_0 \cap \mathbb{Z} = (0).$$

Indeed, this equality can be checked using any particular specialization of the coefficients  $U_\alpha$ ; for instance, if we specialize  $f$  to  $X_1^d$ , then  $\mathcal{D}$  specializes to the ideal  $(X_1^d, dX_1^{d-1})$  in  $\mathbb{Z}[X_1, \dots, X_n]$  and clearly  $\mathrm{TF}_m((X_1^d, dX_1^{d-1}))_0 = (0) \subset \mathbb{Z}$ . Finally, we deduce that  $\mathrm{Disc}(f) \in \mathrm{TF}_m(\mathcal{D})_0$ .

We turn to the proof of the second inclusion, always in the generic case over the integers. Suppose given  $a \in H_m^0({}_{\mathbb{Z}}B)_0$  and denote by  ${}_{\mathbb{Z}}B'$  the quotient ring  ${}_{\mathbb{Z}}C/(\partial_1 f, \dots, \partial_n f)$ . By the Euler identity,  $da \in H_m^0({}_{\mathbb{Z}}B')_0$ . Since both ideals  $H_m^0({}_{\mathbb{Z}}B')_0$  and  $(\mathrm{Res}(\partial_1 f, \dots, \partial_n f))$  of  ${}_{\mathbb{Z}}A$  have the same radical, we deduce that there exists an integer  $N$  such that  $\mathrm{Res}(\partial_1 f, \dots, \partial_n f)$  divides  $(da)^N$ . Using (3.1.4), there exists  $a' \in {}_{\mathbb{Z}}A$  such that

$$d^N a^N = d^{a(n,d)} a' \mathrm{Disc}(f) \text{ in } {}_{\mathbb{Z}}A.$$

Taking the contents in the above equality, we deduce that

$$a^N = \frac{a'}{C_k(a')} C_k(a)^N \mathrm{Disc}(f) \text{ in } {}_{\mathbb{Z}}A$$

and this proves that  $\mathrm{TF}_m(\mathcal{D})_0 \subset \sqrt{(\mathrm{Disc}(f))}$ .

To conclude the proof, we first remark that the inclusion

$$(\text{Disc}(f)) \subset \text{TF}_m((f, \partial_1 f, \dots, \partial_n f)) \cap R$$

is, by specialization, an immediate consequence of the same inclusion in the generic case over the integers. The rest of the proof is a consequence of a base change property, exactly as in the proof of Proposition 2.4.  $\square$

**Corollary 4.19.** *Let  $k$  be a domain and  $f = \sum_{|\alpha|=d} U_\alpha X^\alpha$  be the generic homogeneous polynomial of degree  $d \geq 2$  over  $k$ . Then,  $\text{Disc}(f) = c.P^r$  where  $c$  is an invertible element in  $k$ ,  $r$  is a positive integer and  $P$  is a prime polynomial that generates the ideal  $\mathfrak{P} \subset {}_k A$ .*

*Proof.* Let us first assume that  $k$  is a UFD. Theorem 4.18 implies that both ideals  $\mathfrak{P} = \text{TF}_m(\mathcal{D})_0$  and  $(\text{Disc}(f))$  of  ${}_k A$  have the same radical and Corollary 4.17 shows that  $\mathfrak{P}$  is a prime ideal. Therefore, we deduce immediately that  $\text{Disc}(f) = c.P^r$  as claimed.

Now, assume that  $k$  is a domain. Depending on its characteristic, it contains either  $\mathbb{Z}$  or  $\mathbb{Z}/p\mathbb{Z}$ ,  $p$  a prime integer, that we will denote by  $F$  in the sequel. Thus, we have an injective map  $F \hookrightarrow k$  which is moreover flat (for  $k$  is a torsion-free  $F$ -module). Therefore, the canonical exact sequence (see Corollary 4.17)

$${}_F \text{TF}_m(\mathcal{D}) \rightarrow {}_F C \rightarrow {}_F B_{X_n}$$

remains exact after tensorization by  $k$  over  $F$ . Since  ${}_F C \otimes_F k = {}_k C$  and  ${}_F B_{X_n} \otimes_F k = {}_k B_{X_n}$ , this latter being an immediate consequence of (4.4.1), we deduce that

$${}_k \text{TF}_m(\mathcal{D}) = {}_F \text{TF}_m(\mathcal{D}) \otimes_F k. \quad (4.4.2)$$

Since  $F$  is a UFD, we know that  ${}_F \text{Disc}(f) = c.P^r$  where  $c$  is an invertible element in  $F$ ,  $r$  is a positive integer and  $P$  is a prime polynomial in  ${}_F A$  that generates  ${}_F \mathfrak{P}$ . Now, considering the canonical specialization  $\rho : {}_F A \rightarrow {}_k A$ , we get

$${}_k \text{Disc}(f) = \rho({}_F \text{Disc}(f)) = \rho(c) \cdot \rho(P)^r,$$

where the first equality follows from the definition of the discriminant. But by (4.4.2),  $\rho(P)$  generates  ${}_k \mathfrak{P}$  and since  ${}_k \mathfrak{P}$  is a prime ideal by Corollary 4.17, we deduce that  $\rho(P)$  is a prime polynomial in  ${}_k A$ . To conclude, observe that  $\rho(c)$  is clearly an invertible element in  $k$  because  $F$  is contained in  $k$ .  $\square$

**Remark 4.20.** *From the proof of the above corollary we see that the only dependence of  $r$  on  $k$  is the characteristic of  $k$ , for  $F$  only depends on this characteristic.*

With this property, we can explore the behavior of the discriminant in some particular cases. Here are two such examples.

**Proposition 4.21.** *The universal discriminant over the integers is a prime polynomial in  ${}_{\mathbb{Z}} A$  that generates the ideal  ${}_{\mathbb{Z}} \mathfrak{P}$ .*

*Proof.* By Corollary 4.19, there exists an irreducible polynomial  $P \in {}_{\mathbb{Z}} A$  that generates  ${}_{\mathbb{Z}} \mathfrak{P}$  and an integer  $r \geq 1$  such that  ${}_{\mathbb{Z}} \text{Disc}(f) = \pm P^r$ . In order to prove that  $r = 1$  we will use two specializations.

First, consider the specialization that sends  $f$  to  $UX_n^d + f(X_1, \dots, X_{n-1}, 0)$  where  $U$  denotes, for simplicity, the coefficient of  $X_n^d$  of  $f$ . By Example 4.10, we get that  ${}_{\mathbb{Z}} \text{Disc}(f)$  specializes to

$$d^{(d-1)^{n-1} + (-1)^n} U^{(d-1)^{n-1}} \text{Disc}(f(X_1, \dots, X_{n-1}, 0))^{d-1} \in {}_{\mathbb{Z}} A.$$

Since  $U$  is an irreducible polynomial in  ${}_{\mathbb{Z}} A$  and  $U$  does not divide  $\text{Disc}(f(X_1, \dots, X_{n-1}, 0))$  (this latter discriminant actually does not depend on  $U$ ), we deduce that  $r$  divides  $(d-1)^{n-1}$ .

Second, consider the specialization that sends  $f$  to the polynomial

$$g \in \mathbb{Z}/d\mathbb{Z}[U][X_1, \dots, X_n]$$

given in Example 4.12. We have seen that  $\text{Disc}(f)$  specializes to  $U^{(d-1)^{n-1}+(-1)^n}$ . It follows that  $r$  divides  $(d-1)^{n-1}+(-1)^n$ .

Finally, we have shown that  $r$  divides two consecutive and positive integers, namely  $(d-1)^{n-1}$  and  $(d-1)^{n-1}+(-1)^n$ . Therefore,  $r$  must be equal to 1.  $\square$

**Proposition 4.22.** *Let  $k$  be a domain and  $f = \sum_{|\alpha|=2} U_\alpha X^\alpha$  be the generic homogeneous polynomial of degree 2 over  $k$ . If  $\text{char}(k) \neq 2$  or  $n$  is odd, then  $\text{Disc}(f)$  is a prime polynomial in  ${}_k A$  that generates  $\mathfrak{P}$ . Otherwise, if  $\text{char}(k) = 2$  and  $n$  is even, then  $\text{Disc}(f) = P^2$  where  $P$  is a prime polynomial that generates  $\mathfrak{P}$ .*

*Proof.* As explained in the proof of Corollary 4.19, it is enough to prove this proposition under the assumption that  $k$  is a UFD. So let us assume hereafter that this is the case.

By Corollary 4.19, there exists an irreducible polynomial  $P \in {}_k A$  that generates  ${}_k \mathfrak{P}$ , an integer  $r \geq 1$  and  $c$  an invertible element in  $k$  such that  ${}_k \text{Disc}(f) = c.P^r$ . Depending on the characteristic of  $k$  and the parity of  $n$  we will prove that  $r$  is equal to 1 or 2.

Rewriting  $f(X_1, \dots, X_n)$  as  $f = \sum_{0 \leq i < j \leq n} A_{i,j} X_i X_j$  (so that  ${}_k A$  is now the polynomial ring  $k[A_{i,j}, 0 \leq i < j \leq n]$ ), for all  $i \in \{1, \dots, n\}$  we have

$$\partial_i f = A_{1,i} X_1 + \dots + A_{i-1,i} X_{i-1} + 2A_{i,i} X_i + A_{i,i+1} X_{i+1} + \dots + A_{i,n} X_n$$

in  ${}_k A[X_1, \dots, X_n]$ . Then, Definition 4.6 implies that

$$\begin{vmatrix} 2A_{1,1} & A_{1,2} & \cdots & A_{1,n-1} & A_{1,n} \\ A_{1,2} & 2A_{2,2} & & & A_{2,n} \\ \vdots & & \ddots & & \vdots \\ A_{1,n-1} & & & 2A_{n-1,n-1} & A_{n-1,n} \\ A_{1,n} & A_{2,n} & \cdots & A_{n-1,n} & 2A_{n,n} \end{vmatrix} = \begin{cases} 2 \text{Disc}(f) = 2c.P^r & \text{if } n \text{ is odd} \\ \text{Disc}(f) = c.P^r & \text{if } n \text{ is even} \end{cases} \quad (4.4.3)$$

in the polynomial ring  ${}_k A$ .

Let us first assume that  $\text{char}(k) \neq 2$ . Denote by  $\rho$  the specialization that leaves invariant  $A_{i,i}$  for all  $i$  and sends  $A_{i,j}$  to 0 for all  $i \neq j$ . The specialization of (4.4.3) by  $\rho$  yields

$$2^n A_{1,1} A_{2,2} \dots A_{n,n} = \begin{cases} 2c.\rho(P)^r & \text{if } n \text{ is odd} \\ c.\rho(P)^r & \text{if } n \text{ is even} \end{cases}$$

and from here we deduce that  $r$  must be equal to 1.

Now, assume that  $\text{char}(k) = 2$  and that  $n$  is even. Since  $\text{char}(k) = 2$ , the determinant in (4.4.3) can be seen as the determinant of a skew-symmetric matrix, and since  $n$  is even it is known that it is equal to the square of its pfaffian. Therefore, (4.4.3) implies that  $r \geq 2$ .

Consider the specialization  $\varphi$  that leaves invariant  $A_{1+2k,2+2k}$  for all integers  $k = 0, 1, \dots, (n-2)/2$  and that sends all the other variables  $A_{i,j}$  to 0. The matrix in (4.4.3) then specializes by  $\varphi$  to the block diagonal matrix

$$\text{diag} \left( \begin{bmatrix} 0 & A_{1,2} \\ A_{1,2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{3,4} \\ A_{3,4} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & A_{n-1,n} \\ A_{n-1,n} & 0 \end{bmatrix} \right)$$

and therefore (4.4.3) yields

$$\prod_{k=0}^{(n-2)/2} A_{1+2k,2+2k}^2 = c.\varphi(P)^r$$

This implies that  $r \leq 2$  and hence we conclude that  $r = 2$  if  $\text{char}(k) = 2$  and  $n$  is even. Then, to conclude observe that  ${}_{\mathbb{Z}/2\mathbb{Z}} \text{Disc}(f)$  is a square (necessarily  $c = 1$  in this case), so that we deduce that  $c$  is actually a square in  $k$  via the canonical specialization from  $\mathbb{Z}/2\mathbb{Z}$  to  $k$ . It follows that

${}_k\text{Disc}(f) = (uP)^2$  where  $u^2 = c$  and  $u$  is an invertible element in  $k$ , and the claimed result follows as  $uP$  is an irreducible element that generates  $\mathfrak{P}$ .

Let us turn to the last case:  $\text{char}(k) = 2$  and  $n$  is odd. Consider the specialization  $\phi$  that leaves invariant  $A_{n-2,n}$ ,  $A_{n-1,n}$  and  $A_{1+2k,2+2k}$  for all  $k = 0, 1, \dots, (n-3)/2$ , and that sends all the other variables  $A_{i,j}$  to 0. In order to determine the image of  ${}_k\text{Disc}(f)$  by this specialization, we remark that we have the following commutative diagram of specializations

$$\begin{array}{ccc} \mathbb{Z}A & \xrightarrow{\phi} & \mathbb{Z}[A_{n-2,n}, A_{n-1,n}, A_{1+2k,2+2k} \mid k = 0, 1, \dots, (n-3)/2] \\ \downarrow & & \downarrow \\ {}_kA & \xrightarrow{\phi} & k[A_{n-2,n}, A_{n-1,n}, A_{1+2k,2+2k} \mid k = 0, 1, \dots, (n-3)/2] \end{array}$$

where the vertical arrows are induced by the ring morphism  $\mathbb{Z} \rightarrow k$ . So, we can first perform the specialization  $\phi$  over the integers and then specialize to  $k$ .

The matrix in (4.4.3) specializes by  $\phi$  to the block diagonal matrix

$$\text{diag} \left( \begin{bmatrix} 0 & A_{1,2} \\ A_{1,2} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & A_{n-4,n-3} \\ A_{n-4,n-3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{n-2,n-1} & A_{n-2,n} \\ A_{n-2,n-1} & 0 & A_{n-1,n} \\ A_{n-2,n} & A_{n-1,n} & 0 \end{bmatrix} \right).$$

Therefore, the specialization of (4.4.3) by  $\phi$  over the integers yields the equality

$$2A_{n-2,n-1}A_{n-2,n}A_{n-1,n} \prod_{k=0}^{(n-5)/2} -(A_{1+2k,2+2k})^2 = 2\phi({}_\mathbb{Z}\text{Disc}(f))$$

so that,

$$A_{n-2,n-1}A_{n-2,n}A_{n-1,n} \prod_{k=0}^{(n-5)/2} -(A_{1+2k,2+2k})^2 = \phi({}_\mathbb{Z}\text{Disc}(f)).$$

Now, we specialize this equality to  $k$  and we obtain

$$A_{n-2,n-1}A_{n-2,n}A_{n-1,n} \prod_{k=0}^{(n-5)/2} A_{1+2k,2+2k}^2 = \phi({}_k\text{Disc}(f)) = c \cdot \phi(P)^r.$$

From here, we deduce that  $r$  must be equal to 1. □

Our next step is to prove that the conclusion of this proposition holds without restriction on the degree  $d$ . This is Theorem 4.26. Notice that in the case  $n = 2$  we already know that such a result is valid by Theorem 3.23 and Theorem 3.24 (see also [AJ06, §8.5]).

#### 4.5. Zariski weight of the discriminant

Let  $k$  be a commutative ring and consider the generic homogeneous polynomial in the variables  $X_1, \dots, X_n$  of degree  $d \geq 2$

$$f := \sum_{|\alpha|=d} U_\alpha X^\alpha \in C := A[X_1, \dots, X_n]$$

where  $A := k[U_\alpha \mid |\alpha| = d]$ . Define also the ideals  $\mathfrak{m} := (X_1, \dots, X_n)$  and  $\mathfrak{n} := (X_1, \dots, X_{n-1})$  of  $C$  and rewrite the polynomial  $f$  as  $f = \sum_{t=0}^d f_{d-t} X_n^t$  where  $f_l$  is the generic homogeneous polynomial of degree  $l$  in  $A[X_1, \dots, X_{n-1}]$  for all  $l = 0, \dots, d$ .

Now, fix an integer  $\mu$  such that  $0 \leq \mu \leq d$  and define the polynomials

$$h := \sum_{t=0}^{\mu} f_{d-t} X_n^t \in C_d \text{ and } g := \sum_{t=\mu}^d f_{d-t} X_n^{t-\mu} \in C_{d-\mu}.$$

**Proposition 4.23.** *For all integers  $0 \leq \mu \leq d$  the sequence  $h, \partial_1 h, \dots, \partial_{n-2} h$  is  $C$ -regular. Moreover, for all integers  $1 \leq \mu \leq d$ , the sequence  $h, \partial_1 h, \dots, \partial_{n-1} h$  is  $C$ -regular outside  $V(\mathfrak{n})$ .*

*Proof.* By Proposition 4.2, the sequence  $f_d, \partial_1 f_d, \dots, \partial_{n-2} f_d$  is  $C$ -regular. It follows that the sequence  $X_n, h, \partial_1 h, \dots, \partial_{n-2} h$  is also  $C$ -regular. Since all the elements of this sequence are homogeneous of positive degree, this sequence remains  $C$ -regular under any permutation of its elements. Therefore,  $h, \partial_1 h, \dots, \partial_{n-2} h, X_n$  is  $C$ -regular, in particular  $h, \partial_1 h, \dots, \partial_{n-2} h$  is  $C$ -regular.

To prove the second assertion, we have to prove that the sequence  $h, \partial_1 h, \dots, \partial_{n-1} h$  is  $C_{X_j}$ -regular for all  $1 \leq j \leq n-1$ . Up to a permutation of the variables  $X_1, \dots, X_{n-1}$ , one can assume that  $j = n-1$ .

For the sake of simplicity in the notation, we rename by  $V_i$  the coefficient of the monomial  $X_i X_{n-1}^{d-1}$  in  $f_d$  for all  $i = 1, \dots, n-1$  so that

$$f_d = V_1 X_1 X_{n-1}^{d-1} + V_2 X_2 X_{n-1}^{d-1} + \dots + V_{n-2} X_{n-2} X_{n-1}^{d-1} + V_{n-1} X_{n-1}^d + \dots.$$

We also define the polynomial  $v$  by the equality

$$h = v + V_1 X_1 X_{n-1}^{d-1} + V_2 X_2 X_{n-1}^{d-1} + \dots + V_{n-2} X_{n-2} X_{n-1}^{d-1} + V_{n-1} X_{n-1}^d.$$

Now, perform the following successive specializations:

$$\begin{aligned} V_{n-1} &\mapsto \frac{-1}{X_{n-1}^d} (v + V_1 X_1 X_{n-1}^{d-1} + V_2 X_2 X_{n-1}^{d-1} + \dots + V_{n-2} X_{n-2} X_{n-1}^{d-1}), \quad (4.5.1) \\ V_i &\mapsto \frac{-1}{X_{n-1}^{d-1}} \partial_i v, \quad 1 \leq i \leq n-2. \end{aligned}$$

They successively annihilate  $h, \partial_1 h, \dots, \partial_{n-1} h$  and we recover that  $h, \partial_1 h, \dots, \partial_{n-2} h$  is a regular sequence (outside  $V(\mathfrak{n})$ ). In addition, (4.5.1) yields an isomorphism

$$C_{X_{n-1}} / (h, \partial_1 h, \dots, \partial_{n-2} h) \longrightarrow A'[X_1, \dots, X_n][X_{n-1}^{-1}]$$

where  $A' := k[U_\alpha \mid |\alpha| = d, U_\alpha \neq V_i \forall i \in \{1, \dots, n-1\}]$ . Therefore, it remains to prove that the image of  $\partial_{n-1} h$  by the specializations (4.5.1) is a nonzero divisor in  $A'[X_1, \dots, X_n][X_{n-1}^{-1}]$ . For that purpose, we observe that the Euler identity implies that

$$X_1 \partial_1 h + \dots + X_{n-1} \partial_{n-1} h = \sum_{t=0}^{\mu} (d-t) f_{d-t} X_n^t = dh - \sum_{t=1}^{\mu} t f_{d-t} X_n^t.$$

But the polynomials  $f_{d-t}$  for  $1 \leq t \leq \mu$  do not depend on the variables  $V_1, V_2, \dots, V_{n-1}$ , so we deduce that  $X_{n-1} \partial_{n-1} h$  is specialized to  $-\sum_{t=1}^{\mu} t f_{d-t} X_n^t$  by (4.5.1). Assuming  $\mu \geq 1$ , the  $k$ -content of this polynomial contains the  $k$ -content of  $f_{d-1}$  which is a primitive polynomial over  $k$ , and we conclude the proof by the Dedekind-Mertens Lemma.  $\square$

By definition, the polynomial  $h \in C$  is homogeneous of degree  $d$  with respect to the variables  $X_1, \dots, X_n$  and of valuation  $d - \mu$  with respect to the variables  $X_1, \dots, X_{n-1}$ . Therefore, for all  $i = 1, \dots, n-1$ , the polynomial  $\partial_i h$  is of degree  $d-1$  with respect to the variables  $X_1, \dots, X_n$  and of valuation  $d-1-\mu$  with respect to the variables  $X_1, \dots, X_{n-1}$ . We will denote by  $\text{Red}(h, \partial_1 h, \partial_2 h, \dots, \partial_{n-1} h)$  the reduced resultant of  $h, \partial_1 h, \partial_2 h, \dots, \partial_{n-1} h$  with respect to these degrees and weights. It is well defined for all  $\mu$  such that  $1 \leq \mu \leq d-2$  ([Zar37, OM88]).



**Proposition 4.24.** *For all  $1 \leq \mu \leq d - 2$  the reduced resultant*

$$\text{Red}(h, \partial_1 h, \partial_2 h, \dots, \partial_{n-1} h)$$

*is a primitive polynomial, hence a nonzero divisor, in  $A$ .*

*Proof.* The reduced resultant is a nonzero divisor by Proposition 4.23 and the Poisson formula ([Zar37, Theorem 5.1 and Theorem 5.2], [OM88, Chapter IV]). Then, we deduce that it is primitive over the integers, hence over  $k$ , by applying the previous property with  $k = \mathbb{Z}$  and  $k = \mathbb{Z}/p\mathbb{Z}$  for all prime integers  $p$ .  $\square$

**Theorem 4.25.** *Assume that the ring  $A = k[U_\alpha \mid |\alpha| = d]$  is graded by the Zariski weight, i.e. by setting  $\text{weight}(c) := 0$  for all  $c \in k$  and  $\text{weight}(U_\alpha) := \max(\alpha_n - \mu, 0)$ . Then, the discriminant  $\text{Disc}(f) \in A$  is of valuation  $(d - \mu)(d - 1 - \mu)^{n-1}$ . Moreover, its isobaric part  $H$  of weight  $(d - \mu)(d - 1 - \mu)^{n-1}$  satisfies the equality*

$$\text{Disc}(g)\text{Disc}(\bar{g})\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) = H.\text{Disc}(\bar{f}) \in A$$

*where  $\text{Disc}(\bar{g}) = \text{Disc}(f_{d-\mu})$ ,  $\text{Disc}(\bar{f}) = \text{Disc}(f_d)$  and  $\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h)$  are all isobaric polynomials of zero weight.*

*Proof.* Let  $f_0 := \sum_{|\alpha|=d} V_{0,\alpha} X^\alpha$  and  $f_i := \sum_{|\alpha|=d-1} V_{i,\alpha} X^\alpha$  for  $i = 1, \dots, n-1$  be generic homogeneous polynomials of degree  $d, d-1, \dots, d-1$  respectively and let  $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$  be their generic specialization of degree  $d, d-1, \dots, d-1$  and of valuation  $d-\mu, d-\mu-1, \dots, d-\mu-1$  respectively. Notice that we consider here the canonical grading of  $k[V_{i,\alpha} \forall i, \alpha]$ , so that

$$f_0 = \sum_{|\alpha|=d} V_{0,\alpha} X^\alpha, \quad f_i = \sum_{|\alpha|=d-1} V_{i,\alpha} X^\alpha, \quad \varphi_0 = \sum_{\substack{|\alpha|=d \\ \alpha_n \leq \mu}} V_{0,\alpha} X^\alpha, \quad \varphi_i = \sum_{\substack{|\alpha|=d-1 \\ \alpha_n \leq \mu}} V_{i,\alpha} X^\alpha$$

for all  $i = 1, \dots, n-1$ . Moreover, we also define the polynomials

$$g_0 := \sum_{\substack{|\alpha|=d \\ \alpha_n \geq \mu}} V_{0,\alpha} X^\alpha / X_n^\mu, \quad g_i := \sum_{\substack{|\alpha|=d-1 \\ \alpha_n \geq \mu}} V_{i,\alpha} X^\alpha / X_n^\mu \in k[V_{i,\alpha}, \forall i, \alpha][X_1, \dots, X_n]$$

for all  $i = 1, \dots, n-1$ .

Now, consider the grading of  $k[V_{i,\alpha} \forall i, \alpha]$  defined in this theorem, namely

$$\text{weight}(V_{i,\alpha}) := \max(\alpha_n - \mu, 0) \text{ for all } i = 0, \dots, n-1.$$

Then, by definition of the reduced resultant of  $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$ , we have the equality

$$\begin{aligned} \text{Res}(f_0, \dots, f_{n-1}) &= \text{Res}(g_0, \dots, g_n)\text{Red}(\varphi_0, \dots, \varphi_n) + \\ &\quad \text{terms of weight } > (d - \mu)(d - 1 - \mu)^{n-1} \end{aligned} \quad (4.5.2)$$

Denote by  $\rho$  the specialization from  $k[V_{i,\alpha} \forall i, \alpha]$  to  $A = k[U_\alpha \mid |\alpha| = d]$  (and also, by abusing notation, its canonical extension to polynomial rings) which is such that  $\rho(f_0) = f$  and  $\rho(f_i) = \partial_i f$  for all  $i = 1, \dots, n-1$ . It is easy to check that  $\rho(g_0) = g$ ,  $\rho(\varphi_0) = h$  and that  $\rho(g_i) = \partial_i g$ ,  $\rho(\varphi_i) = \partial_i h$  for all  $i = 1, \dots, n-1$ . Moreover,  $\rho$  is isobaric with respect to the Zariski grading of  $k[V_{i,\alpha} \forall i, \alpha]$  and  $A$  because each variable  $U_\alpha$  has the same Zariski weight in  $f$  and  $\partial_1 f, \dots, \partial_{n-1} f$ . Therefore, the specialization of (4.5.2) yields the equality

$$\begin{aligned} \text{Res}(f, \partial_1 f, \dots, \partial_{n-1} f) &= \text{Res}(g, \partial_1 g, \dots, \partial_{n-1} g)\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) \\ &\quad + \text{terms of weight } > (d - \mu)(d - 1 - \mu)^{n-1}. \end{aligned}$$

By Proposition 4.7, we deduce that

$$\text{Disc}(f)\text{Disc}(\bar{f}) = \text{Disc}(g)\text{Disc}(\bar{g})\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) + \text{terms of weight } > (d - \mu)(d - 1 - \mu)^{n-1}.$$

But  $\text{Disc}(g) \neq 0$ ,  $\text{Disc}(\bar{g}) \neq 0$  and by Proposition 4.24  $\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) \neq 0$ . Since  $\text{Disc}(\bar{f})$ ,  $\text{Disc}(\bar{g})$  and  $\text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) \neq 0$  have null Zariski weight and  $\text{Disc}(g)$  is isobaric of Zariski weight  $(d - \mu)(d - 1 - \mu)^{n-1}$ , we deduce that  $\text{Disc}(f) \in A$  is of valuation  $(d - \mu)(d - 1 - \mu)^{n-1}$  with respect to Zariski weight as claimed.

Pushing further the computations, we see that

$$\text{Disc}(\bar{f}) = \text{Disc}(\bar{h}) \text{ divides } \text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h)$$

and hence we deduce the formula of the theorem. To see this property, notice that the reduced resultant is a reduced inertia form, that is to say that there exists an integer  $N$  such that

$$(X_1, \dots, X_{n-1})^N \text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) \subset (h, \partial_1 h, \dots, \partial_{n-1} h).$$

Specializing  $X_n$  to 0, we get

$$(X_1, \dots, X_{n-1})^N \text{Red}(h, \partial_1 h, \dots, \partial_{n-1} h) \subset (\bar{h}, \partial_1 \bar{h}, \dots, \partial_{n-1} \bar{h}) \subset A[X_1, \dots, X_{n-1}]$$

from we deduce the claimed property by Proposition 4.21.  $\square$

We are now ready to extend Proposition 4.22 to the generic homogeneous polynomial of arbitrary degree  $d \geq 2$ .

**Theorem 4.26.** *Let  $k$  be a domain and  $f = \sum_{|\alpha|=d} U_\alpha X^\alpha$  be the generic homogeneous polynomial of degree  $d \geq 2$  over  $k$ . If  $\text{char}(k) \neq 2$  or  $n$  is odd, then  $\text{Disc}(f)$  is a prime polynomial in  ${}_k A$  that generates  $\mathfrak{P}$ . Otherwise, if  $\text{char}(k) = 2$  and  $n$  is even, then  $\text{Disc}(f) = P^2$  where  $P$  is a prime polynomial that generates  $\mathfrak{P}$ .*

*Proof.* By Corollary 4.19, there exists an invertible element  $c$  in  $k$ , a prime polynomial  $P$  that generates  $\mathfrak{P}$  and an integer  $r$  such that  $\text{Disc}(f) = c.P^r$ .

Now, grading  $A$  with the Zariski weight, for all integers  $1 \leq \mu \leq d - 2$  Theorem 4.25 shows that

$$\text{Disc}(f) = Q_\mu(f).\text{Disc}(g) + \text{terms of weight } > (d - \mu)(d - \mu - 1)^{n-1}$$

where  $Q_\mu(f)$  has weight zero and  $\text{Disc}(g)$  is isobaric of weight  $(d - \mu)(d - \mu - 1)^{n-1}$ . Let  $P_s$  be the isobaric part of smallest weight  $s$  of  $P$ . Then, we deduce that for all integers  $1 \leq \mu \leq d - 2$

$$Q_\mu(f).\text{Disc}(g) = c.(P_s)^r.$$

In particular, if  $\mu = d - 2$  then  $g$  is the generic homogeneous polynomial in  $X_1, \dots, X_n$  of degree 2. But by Proposition 4.22 we know that  $\text{Disc}(g)$  is prime if  $n$  is odd or  $2 \neq 0$  in  $k$ , and that it is equal to the square of a prime polynomial otherwise. We deduce that  $r = 1$  in the first case and that necessarily  $r \leq 2$  in the second case.

Assume now that  $2 = 0$  in  $k$  and  $n$  is even. We have just seen that  $r \in \{1, 2\}$ . We claim that in this case, the canonical projection  $\text{Proj}(B) \rightarrow \text{Spec}(A)$  is not birational onto its image  $\text{Spec}(A/\mathfrak{P})$ . This implies that  $r$  cannot be equal to 1, so  $r = 2$  and  $\text{Disc}(f) = c.P^2$ . Then, to conclude observe that  ${}_{\mathbb{Z}/2\mathbb{Z}}\text{Disc}(f)$  is a square (necessarily  $c = 1$  in this case), so that we deduce that  $c$  is actually a square in  $k$  via the canonical specialization from  $\mathbb{Z}/2\mathbb{Z}$  to  $k$ . It follows that  ${}_k \text{Disc}(f) = (uP)^2$  where  $u^2 = c$  and  $u$  is an invertible element in  $k$ , and the claimed result follows as  $uP$  is an irreducible element that generates  $\mathfrak{P}$ .

To prove that  $\text{Proj}(B) \rightarrow \text{Spec}(A)$  is not birational, we examine the module of relative differentials  $\Omega_{B(X_n)/A}$ . In the following section, we will prove in Lemma 4.27 that it is isomorphic to the cokernel of a Hessian matrix. Moreover, under the assumptions that  $2 = 0$  in  $k$  and  $n$  is even it

turns out that the determinant of this Hessian matrix is equal to zero (see the beginning of Section 4.6 below). Consequently, the projection  $\text{Proj}(B) \rightarrow \text{Spec}(A)$  cannot be birational.  $\square$

#### 4.6. Inertia forms and the Hessian

Let  $k$  be a commutative ring. Given a polynomial  $f \in k[X_1, \dots, X_n]$ , we will denote by  $\text{Hess}(f)$ , and call it the Hessian of  $f$ , the determinant of the (symmetric) matrix

$$\mathbb{H}(f) := \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n}.$$

When  $2 = 0$  in  $k$ , the elements on the diagonal of  $\mathbb{H}(f)$  all vanish and  $\mathbb{H}(f)$  is then a skew-symmetric matrix. Consequently,  $\text{Hess}(f) = 0$  if  $n$  is odd and  $\text{Hess}(f)$  is the square of a polynomial (its Pfaffian) if  $n$  is even. Regarding this behavior, the case where  $f$  is a generic polynomial of degree 2 is particularly instructive.

**Lemma 4.27.** *Set  $A := k[U_{i,j} \mid 1 \leq i < j \leq n]$  and let*

$$f := \sum_{1 \leq i < j \leq n} U_{i,j} X_i X_j \in A[X_1, \dots, X_n]$$

*be the generic homogeneous polynomial of degree 2 over the ring  $k$ . If  $n$  is even or if 2 is a nonzero divisor in  $k$  then  $\text{Hess}(f)$  is a nonzero divisor in  $A$ .*

*Proof.* If  $n$  is even, the monomial  $U_{1,2}^2 U_{3,4}^2 \dots U_{n-1,n}^2$  appears in  $\text{Hess}(f)$  with a coefficient  $\pm 1$  (to see it, one can for instance specialize all the other variables to zero). We deduce that the  $k$ -content of  $\text{Hess}(f)$  is equal to  $k$  and therefore that  $\text{Hess}(f)$  is a nonzero divisor in  $A$  by Dedekind-Mertens Lemma.

Now, assume that  $n$  is odd and that 2 is a nonzero divisor in  $k$ . By specializing  $U_{1,j}$  to 0 for all  $1 < j \leq n$ ,  $\text{Hess}(f)$  specializes to  $2U_{1,1}\text{Hess}(g)$  where  $g = \sum_{2 \leq i < j \leq n} U_{i,j} X_i X_j$ . But since  $n - 1$  is even,  $\text{Hess}(g)$  is a nonzero divisor in  $A$  and it follows that  $\text{Hess}(f)$  is also a nonzero divisor.  $\square$

**Proposition 4.28.** *Set  $A := k[U_\alpha \mid |\alpha| = d]$  and let*

$$f := \sum_{|\alpha|=d} U_\alpha X^\alpha \in A[X_1, \dots, X_n]$$

*be the generic homogeneous polynomial of degree  $d$  over the ring  $k$ . If  $n$  is odd or if 2 is a nonzero divisor in  $k$  then the determinant*

$$\det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} \tag{4.6.1}$$

*is a nonzero divisor in the quotient ring  $A[X_1, \dots, X_n] / \text{TF}_m(\mathcal{D})$ .*

*Proof.* The case  $n = 1$  being trivially correct, we assume that  $n \geq 2$ . We first prove the claimed result under the assumption that  $k$  is a domain. In this case,  $\text{TF}_m(\mathcal{D})$  is a prime ideal by Corollary 4.17 and hence we have to show that

$$\det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} \notin \text{TF}_m(\mathcal{D}) \tag{4.6.2}$$

But it is enough to exhibit a particular specialization for which this property holds. So consider the specialization the sends  $f$  to the polynomial

$$h := \left( \sum_{1 \leq i < j \leq n-1} U_{i,j} X_i X_j \right) X_n^{d-2} \in k[U_{i,j} \mid 1 \leq i < j \leq n-1][X_1, \dots, X_n].$$

Denoting  $g := \sum_{1 \leq i \leq j \leq n-1} U_{i,j} X_i X_j$ , we have

$$\det \left( \frac{\partial^2 h}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} = \text{Hess}(g) X_n^{(d-2)(n-1)}.$$

Therefore, specializing further the variable  $X_n$  to 1, we see that to prove (4.6.2) it is sufficient to prove that

$$\begin{aligned} \text{Hess}(g) &\notin (g, \partial_1 g, \dots, \partial_{n-1} g, (d-2)g) \\ &= (g, \partial_1 g, \dots, \partial_{n-1} g) \subset k[U_{i,j} \mid 1 \leq i, j \leq n-1][X_1, \dots, X_{n-1}]. \end{aligned}$$

But this holds because the ideal  $(g, \partial_1 g, \dots, \partial_{n-1} g)$  is nonzero and it is contained in the ideal  $(X_1, \dots, X_{n-1})$ , whereas  $\text{Hess}(g)$  belongs to  $k[U_{i,j} \mid 1 \leq i, j \leq n-1]$  and is nonzero by Lemma 4.27.

We now turn to the proof in the case  $k$  is an arbitrary commutative ring. Let  $D$  stands for the determinant (4.6.1). We begin with the case where  $n$  is odd. By (4.4.1),  ${}_Z B_{X_n}$  is a free abelian group. Moreover, from what we have just proved under the assumption that  $k$  is a domain, we deduce that the multiplication by  $D$  in  ${}_Z B_{X_n}$  and  ${}_Z/pZ B_{X_n}$ ,  $p$  a prime integer, are all injective maps. Denoting by  ${}_Z Q$  the quotient abelian group of the multiplication by  $D$  in  ${}_Z B_{X_n}$ , that is to say we have the exact sequence of abelian groups

$$0 \rightarrow {}_Z B_{X_n} \xrightarrow{\times D} {}_Z B_{X_n} \rightarrow {}_Z Q \rightarrow 0,$$

we deduce that  ${}_Z Q$  is torsion free (for  $\text{Tor}_1^Z(\mathbb{Z}/p\mathbb{Z}, {}_Z Q) = 0$  for all prime integers  $p$ ) and hence is flat. By a classical property of flatness we obtain that  $\text{Tor}_1^Z({}_Z Q, k) = 0$  and therefore that the multiplication by  $D$  in  ${}_k B_{X_n}$  is an injective map, i.e.  $D$  is a nonzero divisor in  ${}_k B_{X_n}$ . Finally, since

$$\text{TF}_m(\mathcal{D}) = \ker({}_k C \rightarrow {}_k B_{X_n}) \quad (4.6.3)$$

by Corollary 4.17, it follows that  $D$  is a nonzero divisor in  ${}_k C / \text{TF}_m(\mathcal{D})$ .

We can proceed similarly to prove the claimed result in the case where  $n$  is even. The multiplication by  $D$  in  ${}_Z B_{X_n}$  and  ${}_Z/pZ B_{X_n}$ ,  $p$  a prime but odd integer, are all injective maps. It follows that after inversion of 2 we obtain the exact sequence

$$0 \rightarrow {}_{\mathbb{Z}[\frac{1}{2}]} B_{X_n} \xrightarrow{\times D} {}_{\mathbb{Z}[\frac{1}{2}]} B_{X_n} \rightarrow {}_{\mathbb{Z}[\frac{1}{2}]} Q \rightarrow 0$$

where the  $\mathbb{Z}[\frac{1}{2}]$ -module  ${}_{\mathbb{Z}[\frac{1}{2}]} Q$  is torsion free and is hence flat. Consequently, if 2 is a unit in  $k$  we immediately deduce by tensorization by  $k$  over  $\mathbb{Z}[\frac{1}{2}]$  that the multiplication by  $D$  in  ${}_k B_{X_n}$  is an injective map. Now, if 2 is a nonzero divisor in  $k$  then  $k$  can be embedded in  $k[\frac{1}{2}]$ . This induces the inclusion of  ${}_k B_{X_n}$  in  ${}_{k[\frac{1}{2}]} B_{X_n}$ . But we have just proved that  $D$  is a nonzero divisor in  ${}_{k[\frac{1}{2}]} B_{X_n}$ , so we deduce that it is also a nonzero divisor in  ${}_k B_{X_n}$  and hence also a nonzero divisor in  ${}_k C / \text{TF}_m(\mathcal{D})$  by (4.6.3).  $\square$

**Theorem 4.29.** *Set  $A := k[U_\alpha \mid |\alpha| = d]$  and let*

$$f := \sum_{|\alpha|=d} U_\alpha X^\alpha \in A[X_1, \dots, X_n]$$

*be the generic homogeneous polynomial of degree  $d$  over  $k$ . If  $n$  is odd or if 2 is a nonzero divisor in  $k$  then*

$$\text{TF}_m(f, \partial_1 f, \dots, \partial_n f) \cap A \subset (\partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})^2 + (\tilde{f}),$$

*where for all polynomial  $P(X_1, \dots, X_n)$  the notation  $\tilde{P}$  stands for  $P(X_1, \dots, X_{n-1}, 1)$ .*

*Proof.* Let  $a \in \text{TF}_m(f, \partial_1 f, \dots, \partial_n f) \cap A$ . There exists an integer  $N$  such that  $X_n^{N-1}a$  belongs to the ideal  $(f, \partial_1 f, \dots, \partial_n f)$ . Moreover, using the Euler identity  $df = \sum_{i=1}^n X_i \partial_i f$ , we obtain that  $X_n^N a$  belongs to the ideal  $(f, \partial_1 f, \dots, \partial_{n-1} f)$  and therefore that there exist polynomials  $P_1, \dots, P_{n-1}$  and  $Q$  in  $A[X_1, \dots, X_n]$  such that

$$X_n^N a = P_1 \partial_1 f + \dots + P_{n-1} \partial_{n-1} f + Qf. \quad (4.6.4)$$

By applying the derivation  $\partial_j(-)$  for all  $j = 1, \dots, n-1$ , we obtain the following equalities:

$$\forall j \in \{1, \dots, n-1\}, \quad \sum_{i=1}^{n-1} P_i \frac{\partial^2 f}{\partial X_i \partial X_j} = 0 \pmod{(f, \partial_1 f, \dots, \partial_{n-1} f)}.$$

By Cramer's rules, it follows that for all  $i = 1, \dots, n-1$  we have

$$P_i \cdot \det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} \in (f, \partial_1 f, \dots, \partial_n f) \subset \text{TF}_m(f, \partial_1 f, \dots, \partial_n f).$$

But by Proposition 4.28, the determinant

$$\det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1}$$

is not a zero divisor in the quotient ring

$$A[X_1, \dots, X_n] / \text{TF}_m(f, \partial_1 f, \dots, \partial_n f).$$

Therefore, we deduce that  $P_i \in \text{TF}_m(f, \partial_1 f, \dots, \partial_n f)$  for all  $i = 1, \dots, n-1$  and hence, using again Euler identity, that

$$\tilde{P}_i \in (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f}).$$

Coming back to the definition (4.6.4) of the  $P_i$ 's, the claimed result is proved.  $\square$

An immediate consequence of this theorem is the

**Corollary 4.30.** *For any commutative ring  $k$  and any homogeneous polynomial  $f \in k[X_1, \dots, X_n]$ , we have*

$$\text{Disc}(f) \in (\partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})^2 + (\tilde{f}).$$

We end this paragraph with the computation of the module of relative differentials  $\Omega_{B(X_n)/A}$  induced by the canonical inclusion  $A \rightarrow B(X_n)$ .

**Lemma 4.31.** *For any commutative ring  $k$ , the module  $\Omega_{B(X_n)/A}$  of relative differential of  $B(X_n)$  over  $A$  is isomorphic to the cokernel of the map*

$$\bigoplus_{i=1}^{n-1} \frac{A[X_1, \dots, X_{n-1}]}{(\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})} \xrightarrow{\text{Hess}(\tilde{f})} \bigoplus_{i=1}^{n-1} \frac{A[X_1, \dots, X_{n-1}]}{(\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})}$$

whose matrix in the canonical basis is given by the Hessian matrix  $\mathbb{H}(\tilde{f})$ .

*Proof.* By definition of  $B$ , it is clear that

$$B(X_n) \simeq A[X_1, \dots, X_{n-1}] / (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f}).$$

We need to introduce some notation. We can decompose  $f$  as a sum

$$f = f_d + f_{d-1} X_n + \dots + f_{d-2} X_n^2 + f_1 X_n^{d-1} + f_0 X_n^d$$

where the  $f_i$ 's are homogeneous polynomials in  $X_1, \dots, X_{n-1}$  of degree  $d-i$ . We set

$$h := f - f_1 X_n^{d-1} - f_0 X_n^d$$

and we rename the coefficients  $U_\alpha$ ,  $\alpha_n \geq d-1$ , of  $f$  by setting

$$f_1 = \mathcal{E}_1 X_1 + \mathcal{E}_2 X_2 \cdots + \mathcal{E}_{n-1} X_{n-1}, \quad f_0 = \mathcal{E}_n.$$

Setting  $D := k[X_1, \dots, X_{n-1}][U_\alpha \mid \alpha_n \leq d-2]$ , we define a  $k$ -linear map  $\lambda$  from  $B_{(X_n)}$  to  $D$  as follows:

$$\begin{aligned} X_i &\mapsto X_i, \quad i = 1, \dots, n-1 \\ U_\alpha &\mapsto U_\alpha, \quad \alpha_n \leq d-2 \\ \mathcal{E}_i &\mapsto -\partial_i \tilde{h}, \quad i = 1, \dots, n-1 \\ \mathcal{E}_n &\mapsto -\tilde{h} + \sum_{i=1}^{n-1} X_i \partial_i \tilde{h} \end{aligned} \tag{4.6.5}$$

It is clear that  $\lambda$  is surjective. Moreover, observe that  $\tilde{f} = \tilde{h} + \mathcal{E}_n + \sum_{i=1}^{n-1} \mathcal{E}_i X_i$ , so that  $\partial_i \tilde{f} = \partial_i \tilde{h} + \mathcal{E}_i$  for all  $i = 1, \dots, n-1$ , and hence we deduce that  $\lambda$  is an isomorphism.

Now,  $B_{(X_n)}$  is an  $A$ -algebra by the canonical inclusion of  $A$  in  $B_{(X_n)}$ . Using the isomorphism  $\lambda$ , we get that  $\Omega_{B_{(X_n)}/A} \simeq \Omega_{D/A}$  and  $A \rightarrow D$  is given by (4.6.5) (without the  $X_i$ 's that have been removed). Setting  $\bar{A} = k[U_\alpha \mid \alpha_n \leq d-2]$ , so that  $A = \bar{A}[\mathcal{E}_1, \dots, \mathcal{E}_n]$ , we get maps of rings  $\bar{A} \rightarrow A \rightarrow D$  and the relative cotangent sequence

$$D \otimes_A \Omega_{A/\bar{A}} \xrightarrow{can} \Omega_{D/\bar{A}} \rightarrow \Omega_{D/A} \rightarrow 0$$

which is exact. Since  $\Omega_{A/\bar{A}} \simeq \bigoplus_{i=1}^n A d\mathcal{E}_i$  and  $\Omega_{D/\bar{A}} \simeq \bigoplus_{i=1}^{n-1} D dX_i$ , the map  $can$  in this sequence can be represented by a matrix in the basis  $d\mathcal{E}_1, \dots, d\mathcal{E}_n$  and  $dX_1, \dots, dX_{n-1}$  respectively. By straightforward computations, we get

$$can(d\mathcal{E}_i) = - \left( \sum_{j=1}^{n-1} \frac{\partial^2 \tilde{h}}{\partial X_i \partial X_j} dX_j \right) = - \left( \sum_{j=1}^{n-1} \frac{\partial^2 \tilde{f}}{\partial X_i \partial X_j} dX_j \right), \quad i = 1, \dots, n-1,$$

and

$$\begin{aligned} can(d\mathcal{E}_n) &= \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} X_j \frac{\partial^2 \tilde{h}}{\partial X_i \partial X_j} \right) dX_i = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} X_j \frac{\partial^2 \tilde{f}}{\partial X_i \partial X_j} \right) dX_i \\ &= - \sum_{j=1}^{n-1} X_j can(d\mathcal{E}_j) \end{aligned}$$

so that the first  $n-1$  columns of this matrix corresponds to  $-\text{Hess}(\tilde{f})$  and its last column is the span of the  $n-1$  first ones. Therefore, the image of  $can$  is isomorphic to the image of the map  $D^{n-1} \rightarrow D^{n-1}$  defined by the matrix  $-\text{Hess}(\tilde{f})$ , and the claimed result follows.  $\square$

The computation we have done in this lemma shows that the unramified points of  $\text{Proj}(B)$  over  $\text{Spec}(A)$  are the non-degenerated quadratic points, that is to say the points where the Hessian of  $\tilde{f}$  does not vanish. We used it at the end of the proof of Theorem 4.26 to show that the canonical projection of  $\text{Proj}(B)$  over  $\text{Spec}(A)$  is not birational if  $\text{char}(k) = 2$  and  $n$  is even under the assumption that  $k$  is a domain. If  $n$  is odd or 2 is a nonzero divisor in  $k$  then this projection is birational (without assuming that  $k$  is a domain). The purpose of the next section is to prove this fact by providing an explicit blowup structure to  $\text{Proj}(B)$ .

#### 4.7. Effective blow-up structure

For the sake of simplicity in the text, we introduce a particular notation for some coefficients  $U_\alpha$  of the generic homogeneous polynomial  $f \in {}_k A$  of degree  $d \geq 2$ :

$$f(X_1, \dots, X_n) = \mathcal{E}_1 X_1 X_n^{d-1} + \mathcal{E}_2 X_2 X_n^{d-1} + \dots + \mathcal{E}_{n-1} X_{n-1} X_n^{d-1} + \mathcal{E}_n X_n^d + \dots$$

Moreover, we introduce  $n - 1$  polynomials

$$g_i(X_1, \dots, X_n) = \sum_{|\beta|=d-1} V_{i\beta} X^\beta, \quad i = 1, \dots, n-1$$

and define the coefficient ring

$${}_k A' = {}_k A[V_{i\beta} \mid 1 \leq i \leq n-1, |\beta| = d-1]$$

so that  $f$  and  $g_1, \dots, g_{n-1}$  belong to  ${}_k A'[X_1, \dots, X_n]$ . For the sake of simplicity, we will omit the subscript  $k$  in the notation whenever there is no possible confusion.

The resultant  $S := \text{Res}(\partial_1 f, \dots, \partial_{n-1} f, f) \in A$  can be obtained by specialization of the resultant  $R := \text{Res}(g_1, \dots, g_{n-1}, f) \in A'$ . More precisely, for all integers  $i = 1, \dots, n-1$  we have

$$\frac{\partial f}{\partial X_i} = \sum_{|\alpha|=d, \alpha_i \geq 1} \alpha_i U_\alpha \frac{X^\alpha}{X_i} = \sum_{|\beta|=d-1} (\beta_i + 1) U_{\beta+e_i} X^\beta$$

where  $e_i$  stands for the multi-index such that  $X^{e_i} = X_i$  for all  $i = 1, \dots, n-1$ . Thus, we define the specialization

$$\begin{aligned} \rho : A' &\rightarrow A \\ V_{i\beta} &\mapsto (\beta_i + 1) U_{\beta+e_i}, \quad i = 1, \dots, n-1 \\ U_\beta &\mapsto U_\beta \end{aligned}$$

so that  $\rho(R) = S$ . Notice that we also have  $\rho(\partial R / \partial \mathcal{E}_n) = \partial S / \partial \mathcal{E}_n$ . Now, set  $D := \text{Disc}(f) \in A$  and recall that  $\bar{f}(X_1, \dots, X_{n-1}) := f(X_1, \dots, X_{n-1}, 0)$ .

**Proposition 4.32.** *There exist polynomials  $\Delta_1(f), \dots, \Delta_n(f) \in \mathbb{Z}A$  such that*

$$\text{Disc}(\bar{f}) \Delta_i(f) = \rho \left( \frac{\partial R}{\partial \mathcal{E}_i} \right) \in \mathbb{Z}A.$$

For any commutative ring  $k$ , we define the polynomials  $\Delta_1(f), \dots, \Delta_n(f) \in {}_k A$  by change of basis  $\mathbb{Z} \rightarrow k$ .

Moreover,

$$\Delta_n(f) = \frac{\partial D}{\partial \mathcal{E}_n} \in {}_k A \tag{4.7.1}$$

and for all  $1 \leq i, j \leq n$  we have

$$\Delta_i(f) X_j - \Delta_j(f) X_i \in \text{TF}_m(\partial_1 f, \dots, \partial_{n-1} f, f) \subset {}_k A[X_1, \dots, X_n]. \tag{4.7.2}$$

*Proof.* We begin by proving the claim about  $\Delta_n(f)$ . For that purpose, introduce a new indeterminate  $T$ . By Taylor expansion we have

$$\text{Res}(g_1, \dots, g_{n-1}, f + T X_n^d) - R = T \frac{\partial R}{\partial \mathcal{E}_n} \pmod{(T^2)} \in {}_k A'[T].$$

Applying the specialization  $\rho$  and the definition of the discriminant, we obtain

$$\text{Disc}(\bar{f})(\text{Disc}(f + T X_n^d) - \text{Disc}(f)) = T \rho \left( \frac{\partial R}{\partial \mathcal{E}_n} \right) \pmod{(T^2)} \in {}_k A[T]. \tag{4.7.3}$$

But the Taylor expansion also yields the equality

$$\text{Disc}(f + T X_n^d) - \text{Disc}(f) = T \frac{\partial D}{\partial \mathcal{E}_n} \pmod{(T^2)} \in {}_k A[T]. \tag{4.7.4}$$

Therefore, combining (4.7.3) and (4.7.4) we deduce that

$$\text{Disc}(\bar{f}) \frac{\partial D}{\partial \mathcal{E}_n} = \rho \left( \frac{\partial R}{\partial \mathcal{E}_n} \right) = \frac{\partial S}{\partial \mathcal{E}_n} \in {}_k A \quad (4.7.5)$$

so that the claim  $\Delta_n(f) = \partial D / \partial \mathcal{E}_n$  in  ${}_k A$  is proved since  $\text{Disc}(\bar{f})$  is a nonzero divisor by Corollary 4.9.

Now, we turn to the polynomials  $\Delta_1(f), \dots, \Delta_{n-1}(f)$  and hence we assume that  $k = \mathbb{Z}$ . From [Jou91, Lemme 4.6.1], we know that for all multi-index  $\alpha$  such that  $|\alpha| = d$  we have

$$X_n^d \frac{\partial R}{\partial U_\alpha} - X^\alpha \frac{\partial R}{\partial \mathcal{E}_n} \in \text{TF}_m(g_1, \dots, g_n).$$

Moreover, [Jou91, Lemma 4.6.6] then shows that the specialization of  $X_i$  by  $\partial R / \partial \mathcal{E}_i$  for all  $i = 1, \dots, n$  yields

$$\left( \frac{\partial R}{\partial \mathcal{E}_n} \right)^d \frac{\partial R}{\partial U_\alpha} - \left( \frac{\partial R}{\partial \mathcal{E}_1} \right)^{\alpha_1} \cdots \left( \frac{\partial R}{\partial \mathcal{E}_n} \right)^{\alpha_n} \frac{\partial R}{\partial \mathcal{E}_n} \in R_{\cdot \mathbb{Z}} A'$$

By the properties of the resultant,  $R$  is irreducible,  $\partial R / \partial \mathcal{E}_n \neq 0$  and  $\partial R / \partial \mathcal{E}_n \notin R_{\cdot \mathbb{Z}} A'$  so we deduce that

$$\left( \frac{\partial R}{\partial \mathcal{E}_n} \right)^{d-1} \frac{\partial R}{\partial U_\alpha} - \left( \frac{\partial R}{\partial \mathcal{E}_1} \right)^{\alpha_1} \cdots \left( \frac{\partial R}{\partial \mathcal{E}_n} \right)^{\alpha_n} \in R_{\cdot \mathbb{Z}} A'.$$

Taking suitable choices for the multi-index  $\alpha$ , we finally obtain that for all integers  $i = 1, \dots, n-1$

$$\left( \frac{\partial R}{\partial \mathcal{E}_n} \right)^{d-1} \frac{\partial R}{\partial U_\alpha} - \left( \frac{\partial R}{\partial \mathcal{E}_i} \right)^d \in R_{\cdot \mathbb{Z}} A'. \quad (4.7.6)$$

Now, since  $\text{Disc}(\bar{f})$  divides  $S = \rho(R)$ , by definition of the discriminant and divides  $\rho(\partial R / \partial \mathcal{E}_n)$  by (4.7.5), we deduce that it also divides  $\rho(\partial R / \partial \mathcal{E}_i)^d$  for all integers  $i = 1, \dots, n-1$  by specialization of (4.7.6) under  $\rho$ . But  $\text{Disc}(\bar{f})$  is irreducible in  ${}_{\mathbb{Z}} A$ , so we finally deduce that  $\text{Disc}(\bar{f})$  divides  $\rho(\partial R / \partial \mathcal{E}_i)$  for all  $i = 1, \dots, n-1$  and hence we obtain the existence of the polynomials  $\Delta_1(f), \dots, \Delta_{n-1}(f) \in {}_{\mathbb{Z}} A$ .

It remains to prove (4.7.2). Recall from [Jou91, Lemma 4.6.1, (4.6.3)] that for all  $1 \leq i, j \leq n$  we have

$$\frac{\partial R}{\partial \mathcal{E}_i} X_j - \frac{\partial R}{\partial \mathcal{E}_j} X_i \in \text{TF}_m(g_1, \dots, g_{n-1}, f) \subset {}_k A'[X_1, \dots, X_n].$$

Applying the specialization  $\rho$ , we deduce that

$$\text{Disc}(\bar{f}) (\Delta_i(f) X_j - \Delta_j(f) X_i) \in \text{TF}_m(\mathcal{D}).$$

Therefore, we deduce that (4.7.2) holds if  $k = \mathbb{Z}$  because in this case  $\text{Disc}(\bar{f})$  is irreducible and does not divide  $\text{Disc}(f)$ , hence does not belong to the prime ideal  $\text{TF}_m(\mathcal{D})$ . Finally, (4.7.2) holds for any  $k$  by change of basis  $\mathbb{Z} \rightarrow k$ .  $\square$

We are now ready to define a map from  ${}_k C$  to a Rees algebra. Recall that  $\mathfrak{P} := \text{TF}_m({}_k \mathcal{D})_0 \subset {}_k A$  and denote by  $\bar{\Delta}_i$  the image of  $\Delta_i(f)$  by the canonical map  $A \rightarrow A/\mathfrak{P}$  for all  $i = 1, \dots, n$ . Introducing a new indeterminate  $T$ , we define the  $A$ -algebra morphism

$$\begin{aligned} \varphi : {}_k C &= {}_k A[X_1, \dots, X_n] \rightarrow \text{Rees}_{A/\mathfrak{P}}(\bar{\Delta}_1, \dots, \bar{\Delta}_n) \subset A/\mathfrak{P}[T] \\ h &= \sum_{\nu \in \mathbb{N}} h_\nu(X_1, \dots, X_n) \mapsto \sum_{\nu \in \mathbb{N}} h_\nu(\bar{\Delta}_1, \dots, \bar{\Delta}_n) T^\nu \end{aligned}$$

where the notation  $h_\nu$  stands for the homogeneous part of degree  $\nu$  of  $h \in {}_k C$ . Notice that it is a graded and surjective map.

**Lemma 4.33.** *With the above notation,  $\varphi$  vanishes on  $\text{TF}_m(\mathcal{D})$ .*



*Proof.* Since  $\varphi$  is graded, it is sufficient to check the claimed property on graded parts. Let  $h \in C_\nu$ . By using (4.7.2), we obtain that

$$X_n^\nu h(\Delta_1(f), \dots, \Delta_n(f)) - \Delta_n(f)^\nu h(X_1, \dots, X_n) \in \mathrm{TF}_m(\mathcal{D}). \quad (4.7.7)$$

It follows that if  $h \in C_\nu \cap \mathrm{TF}_m(\mathcal{D})$  then  $X_n^\nu h(\Delta_1(f), \dots, \Delta_n(f)) \in \mathrm{TF}_m(\mathcal{D})$ . Similarly, we get that  $X_j^\nu h(\Delta_1(f), \dots, \Delta_n(f)) \in \mathrm{TF}_m(\mathcal{D})$  for all  $j = 1, \dots, n$  and consequently, we deduce that

$$h(\Delta_1(f), \dots, \Delta_n(f)) \in \mathrm{TF}_m(\mathcal{D})_0 = \mathfrak{P} \subset {}_k A,$$

hence  $h_\nu(\bar{\Delta}_1, \dots, \bar{\Delta}_n) = 0 \in A/\mathfrak{P}$ .  $\square$

As a consequence of this lemma, the morphism  $\varphi$  induces

$$\bar{\varphi} : {}_k C/\mathrm{TF}_m(\mathcal{D}) = {}_k B/H_m^0(B) \rightarrow \mathrm{Rees}_{A/\mathfrak{P}}(\bar{\Delta}_1, \dots, \bar{\Delta}_n)$$

From a geometric point of view,  $\bar{\varphi}$  defines a map from a blow-up variety to the discriminant variety. Below, we will prove that this map is an isomorphism under suitable assumptions. As a consequence, it will follow that the scheme morphism

$$\mathrm{Proj}(B) = \mathrm{Proj}(B/H_m^0(B)) \rightarrow \mathrm{Spec}(A/\mathfrak{P})$$

is birational since  $\bar{\varphi}$  identifies  $\mathrm{Proj}(B)$  to the blow-up of  $\mathrm{Spec}(A/\mathfrak{P})$  along the ideal  $(\bar{\Delta}_1, \dots, \bar{\Delta}_n)$ .

**Lemma 4.34.** *Assume that  $k$  is a domain and that  $n$  is odd or  $2 \neq 0$  in  $k$ . Let  $a \in \mathfrak{P} = \mathrm{TF}_m(\mathcal{D})_0$ . If  $\partial a/\partial \mathcal{E}_n = 0$  then  $a \in \mathrm{TF}_m(\mathcal{D}^2)_0$ . In particular, if  $\partial a/\partial \mathcal{E}_n = 0$  then  $\partial a/\partial U_\alpha \in \mathfrak{P}$  for all multi-index  $\alpha$  such that  $|\alpha| = d$ .*

*Proof.* Let  $a \in \mathfrak{P} = \mathrm{TF}_m(\mathcal{D})_0$  such that  $\partial a/\partial \mathcal{E}_n = 0$ . by Corollary 4.17, there exists an integer  $N$  and polynomials  $P_1, \dots, P_{n-1}, Q \in {}_k A[X_1, \dots, X_n]$  such that

$$X_n^N a = P_1 \partial_1 f + \partial_2 f + \dots + \partial_{n-1} f + Qf. \quad (4.7.8)$$

Since  $\partial_i f$  does not depend on  $\mathcal{E}_n$  for all  $1 \leq i \leq n-1$ , by derivation with respect to  $\mathcal{E}_n$  we get

$$0 = X_n^N \frac{\partial a}{\partial \mathcal{E}_n} = \frac{\partial P_1}{\partial \mathcal{E}_n} \partial_1 f + \dots + \frac{\partial P_{n-1}}{\partial \mathcal{E}_n} \partial_{n-1} f + \frac{\partial Q}{\partial \mathcal{E}_n} f + Q X_n^d.$$

It follows immediately that

$$X_n^d Q \in (f, \partial_1 f, \dots, \partial_{n-1} f)$$

and hence, by comparing with (4.7.8), we deduce that there exists polynomials  $L_1, \dots, L_{n-1}, M \in {}_k A[X_1, \dots, X_n]$  such that

$$X_n^{N+d} a = L_1 \partial_1 f + L_2 \partial_2 f + \dots + L_{n-1} \partial_{n-1} f + M f^2. \quad (4.7.9)$$

Computing the derivatives with respect to  $X_j$  for all  $1 \leq j \leq n-1$ , we get the equalities

$$0 = \sum_{i=1}^{n-1} L_i \frac{\partial^2 f}{\partial X_i \partial X_j} + \sum_{i=1}^{n-1} \frac{\partial L_i}{\partial X_j} \frac{\partial f}{\partial X_i} + 2M f \frac{\partial f}{\partial X_j} + \frac{\partial M}{\partial X_j} f^2, \quad 1 \leq j \leq n-1.$$

Hence, for all  $1 \leq j \leq n-1$  we have

$$\sum_{i=1}^{n-1} L_i \frac{\partial^2 f}{\partial X_i \partial X_j} \in (f^2, \partial_1 f, \dots, \partial_{n-1} f)$$

and Cramer's rules show that for all  $1 \leq l \leq n-1$  we have

$$\det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} L_l \in (f^2, \partial_1 f, \dots, \partial_{n-1} f).$$

Finally, by comparison with (4.7.9) we obtain

$$X_n^{N+d} a \det \left( \frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{1 \leq i, j \leq n-1} \in (f, \partial_1 f, \partial_2 f, \dots, \partial_{n-1} f)^2.$$

In other words, using the notation  $\tilde{f}(X_1, \dots, X_{n-1}) := f(X_1, \dots, X_{n-1}, 1)$ , we obtained that

$$\text{Hess}(\tilde{f}).a \in (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})^2. \quad (4.7.10)$$

Now, Proposition 4.28 implies that  $\text{Hess}(\tilde{f})$  is a nonzero divisor in the quotient ring  ${}_k \tilde{B} := {}_k A[X_1, \dots, X_{n-1}] / (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})$ . Moreover, Proposition 4.2, (ii) shows that the sequence  $\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f}$  is a regular sequence in  ${}_k A[X_1, \dots, X_{n-1}]$  and hence

$$(\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f}) / (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})^2$$

is a free  $\tilde{B}$ -module. Therefore, this and (4.7.10) show that

$$a \in (\tilde{f}, \partial_1 \tilde{f}, \dots, \partial_{n-1} \tilde{f})^2.$$

Finally, using Corollary 4.17 we conclude that  $a \in \text{TF}_{(X_n)}(\mathcal{D}^2) = \text{TF}_m(\mathcal{D}^2)$ .  $\square$

**Corollary 4.35.** *If  $n$  is odd or if 2 is a nonzero divisor in  $k$  then  $\partial D / \partial \mathcal{E}_n$  is not a zero divisor in the quotient ring  ${}_k C / \text{TF}_m(\mathcal{D})$ .*

*Proof.* We first assume that  $k$  is a domain. Then, observe that we can assume without loss of generality that  $k$  is actually a field by extension to the fraction field of  $k$ . Now, if  $\partial D / \partial \mathcal{E}_n \neq 0$  then Lemma 4.34 implies that  $D$  divides  $\partial D / \partial U_\alpha$  for all multi-index  $\alpha$  such that  $|\alpha| = d$  and hence that  $\partial D / \partial U_\alpha = 0$  for all  $\alpha$  such that  $|\alpha| = d$  by inspecting the degrees. If  $k$  has characteristic zero then we deduce that  $D = 0$ , a contradiction with Theorem 4.26. If  $k$  has characteristic  $p > 0$ , then passing to the algebraic closure of  $k$  (which is a perfect field) we get that  $D$  must be some polynomial raised to the power  $p$ , again a contradiction with Theorem 4.26.

It remains to prove that the claimed property holds for an arbitrary ring  $k$ , knowing that it is valid for a domain. To do this, we can proceed exactly as in the proof of Proposition 4.28.  $\square$

We are now ready to prove the main result of this section.

**Theorem 4.36.** *If  $n$  is odd or 2 is a nonzero divisor in  $k$ , then  $\tilde{\varphi}$  is an isomorphism.*

*Proof.* Since  $\tilde{\varphi}$  is graded and surjective, it is sufficient to show that it is injective on graded parts. So let  $h \in C_\nu$  and assume that  $h(\Delta_1(f), \dots, \Delta_n(f)) \in \mathfrak{P}$ . Then, (4.7.7) and (4.7.1) shows that

$$\left( \frac{\partial D}{\partial \mathcal{E}_n} \right)^\nu h(X_1, \dots, X_n) \in \text{TF}_m(\mathcal{D}). \quad (4.7.11)$$

But by Corollary 4.35,  $\partial D / \partial \mathcal{E}_n$  is not a zero divisor in the quotient ring  ${}_k C / \text{TF}_m(\mathcal{D})$ . Therefore (4.7.11) implies that  $h(X_1, \dots, X_n) \in \text{TF}_m(\mathcal{D})$  and from here we deduce that  $\tilde{\varphi}$  is an isomorphism.  $\square$

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## Appendix - Two formulas of F. Mertens

In this appendix, we give rigorous proofs of two outstanding formulas that were given by Frantz Mertens around 1890 in its study of the resultant of homogeneous polynomials [Mer86].

Let  $R$  be a commutative ring and suppose given  $n \geq 1$  homogeneous polynomials  $f_1, \dots, f_n$  in  $R[X_1, \dots, X_n]$  with positive degree  $d_1, \dots, d_n$  respectively, such that  $\prod_{i=1}^n d_i > 1$ . Introducing news indeterminates  $U_1, \dots, U_n$ , we define

$$\theta(U_1, \dots, U_n) := \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n U_i X_i) \in R[U_1, \dots, U_n]$$

and  $\theta_i(U_1, \dots, U_n) := \partial\theta/\partial U_i \in R[U_1, \dots, U_n]$  for all  $i = 1, \dots, n$ . In addition, let  $V_1, \dots, V_n, W_1, \dots, W_n, X, Y$  be a collection of some other new indeterminates and consider the ring morphisms

$$\begin{aligned} \rho : R[U_1, \dots, U_n] &\rightarrow R[V_1, \dots, V_n, W_1, \dots, W_n][X_1, \dots, X_n] \\ U_i &\mapsto V_i \left( \sum_{j=1}^n W_j X_j \right) - W_i \left( \sum_{j=1}^n V_j X_j \right). \end{aligned}$$

and

$$\begin{aligned} \bar{\rho} : R[U_1, \dots, U_n] &\rightarrow R[V_1, \dots, V_n, W_1, \dots, W_n][X, Y] \\ U_i &\mapsto V_i X + W_i Y \end{aligned}$$

**First Mertens' formula:**

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(f_n(\theta_1, \dots, \theta_n))) = (-1)^{d_1 \cdots d_n} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_n} \text{Res}(f_1, \dots, f_n).$$

**Second Mertens' formula:**

$$\text{Res}(f_1, \dots, f_{n-1}, \rho(f_n(\theta_1, \dots, \theta_n))) = (-1)^{d_1 \cdots d_n} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_n} \text{Res}(f_1, \dots, f_n).$$

Notice that the subscript  $X, Y$  is written to emphasize that the discriminant, or the resultant, is taken with respect to these two variables.

*Proof.* We begin with the proof of the first formula and then we will deduce the second formula from the first one. Observe that we can assume that  $R$  is actually the universal ring of coefficients of the polynomials  $f_1, \dots, f_n$  that we will denote by  $A$ .

From definition,  $\theta$  is an inertia form of the polynomials  $f_1, \dots, f_{n-1}, \sum_{i=1}^n U_i X_i$  with respect to  $(X_1, \dots, X_n)$ : there exists an integer, say  $N$ , and polynomials  $h_1, \dots, h_{n-1}, h$  in the polynomial ring  $A[U_1, \dots, U_n][X_1, \dots, X_n]$  such that

$$X_n^N \theta = h_1 f_1 + \cdots + h_{n-1} f_{n-1} + h \left( \sum_{i=1}^n U_i X_i \right).$$

A simple computation then shows that  $X_i \theta_j - X_j \theta_i$  is an inertia form of the same polynomials for all pairs  $(i, j)$ . By successive iterations, we deduce that  $X_n^{d_n} f_n(\theta_1, \dots, \theta_n) - \theta_n^{d_n} f_n(X_1, \dots, X_n)$  is also such an inertia form. Finally, we obtain that  $f_n(\theta_1, \dots, \theta_n)$  is an inertia forms of the polynomials

$$f_1, \dots, f_{n-1}, f_n, \sum_{i=1}^n U_i X_i$$

with respect to  $(X_1, \dots, X_n)$ . Obviously, the same holds for  $\theta$ .

Set  $R := \text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(f_n(\theta_1, \dots, \theta_n)))$ . There exists an integer  $N_1$  such that

$$X^{N_1} R \in (\bar{\rho}(\theta), \bar{\rho}(f_n(\theta_1, \dots, \theta_n))) \subset A[V_1, \dots, V_n, W_1, \dots, W_n][X, Y]$$

and therefore we deduce that there exists an integer  $N_2$  such that

$$\begin{aligned} X^{N_1} X_n^{N_2} R &\in (f_1, \dots, f_n, \bar{\rho}(\sum_{i=1}^n U_i X_i)) \\ &\subset A[V_1, \dots, V_n, W_1, \dots, W_n][X, Y][X_1, \dots, X_n]. \end{aligned}$$

Now, specializing  $X$  to  $\sum_{i=1}^n W_i X_i$  and  $Y$  to  $-\sum_{i=1}^n V_i X_i$  we obtain that

$$\left( \sum_{i=1}^n W_i X_i \right)^{N_1} X_n^{N_2} R \in (f_1, \dots, f_n) \subset A[V_1, \dots, V_n, W_1, \dots, W_n][X_1, \dots, X_n].$$

In other words,  $(\sum_{i=1}^n W_i X_i)^{N_1} R$  is an inertia form of the polynomials  $f_1, \dots, f_n$  with respect to  $(X_1, \dots, X_n)$ . Moreover, since  $\sum_{i=1}^n W_i X_i$  is obviously not such an inertia form, we deduce that  $R$  is. Consequently, there exists

$$M \in A[V_1, \dots, V_n, W_1, \dots, W_n]$$

such that

$$R := \text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(f_n(\theta_1, \dots, \theta_n))) = M \text{Res}(f_1, \dots, f_n). \quad (\text{A.1})$$

Looking at this equation, we see that both  $R$  and  $\text{Res}(f_1, \dots, f_n)$  are homogeneous with respect to the coefficients of the polynomial  $f_n$  of the same degree  $d_1 \dots d_{n-1}$ . Therefore,  $M$  must be independent of these coefficients, but it could depend on the degree  $d_n$ . To emphasize this property, we use the notation  $M(f_1, \dots, f_{n-1}, d_n)$ . If we specialize  $f_n$  to  $X_n^{d_n}$  in (A.1), we obtain

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(\theta_n))^{d_n} = M(f_1, \dots, f_{n-1}, d_n) \text{Res}(f_1, \dots, f_{n-1}, X_n)^{d_n}.$$

But on the other hand, from the definition of  $M$ , we have

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(\theta_n)) = M(f_1, \dots, f_{n-1}, 1) \text{Res}(f_1, \dots, f_{n-1}, X_n).$$

By comparison, it follows that  $M(f_1, \dots, f_{n-1}, d_n) = M(f_1, \dots, f_{n-1}, 1)^{d_n}$  and hence it remains to determine  $M(f_1, \dots, f_{n-1}, 1)$ . For that purpose, noticing that  $\partial \bar{\rho}(\theta) / \partial Y = \sum_{i=1}^n W_i \bar{\rho}(\theta_i)$ , we choose to specialize  $f_n$  to the linear form  $\sum_{i=1}^n W_i X_i$ . We obtain

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), \frac{\partial \bar{\rho}(\theta)}{\partial Y}) = M(f_1, \dots, f_{n-1}, 1) \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n W_i X_i).$$

Now, by definition of  $\text{Disc}_{X,Y}(\bar{\rho}(\theta))$ , we have

$$\begin{aligned} \text{Res}_{X,Y}(\bar{\rho}(\theta), \frac{\partial \bar{\rho}(\theta)}{\partial Y}) &= \text{Disc}_{X,Y}(\bar{\rho}(\theta)) \text{Res}_{X,Y}(\bar{\rho}(\theta), X) \\ &= \text{Disc}_{X,Y}(\bar{\rho}(\theta)) \cdot \bar{\rho}(\theta)(0, -1) \\ &= \text{Disc}_{X,Y}(\bar{\rho}(\theta)) \text{Res}(f_1, \dots, f_{n-1}, -\sum_{i=1}^n W_i X_i) \\ &= (-1)^{d_1 \dots d_{n-1}} \text{Disc}_{X,Y}(\bar{\rho}(\theta)) \text{Res}(f_1, \dots, f_{n-1}, \sum_{i=1}^n W_i X_i). \end{aligned}$$

It follows that  $M(f_1, \dots, f_{n-1}, 1) = (-1)^{d_1 \dots d_{n-1}} \text{Disc}_{X,Y}(\bar{\rho}(\theta))$  and the first formula is proved.

We turn to the proof of the second formula. For the sake of simplicity, define

$$h := \rho(f_n(\theta_1, \dots, \theta_n)) \in A[V_1, \dots, V_n, W_1, \dots, W_n][X_1, \dots, X_n]$$

and denote by  $d_h$  its degree with respect to the variables  $X_1, \dots, X_n$ . It is not hard to check that  $d_h = d_n(d_1 \dots d_{n-1} - 1)$  which is a positive integer by our assumption  $\prod_{i=1}^n d_i > 1$ .

By applying Mertens' first formula, we obtain the equality

$$\begin{aligned} \text{Res}_{X,Y}(\bar{\theta}, \bar{\rho}(h(\theta_1, \dots, \theta_n))) &= (-1)^{d_1 \dots d_{n-1} d_h} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h} \text{Res}(f_1, \dots, f_{n-1}, h) \quad (\text{A.2}) \\ &= \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h} \text{Res}(f_1, \dots, f_{n-1}, h) \end{aligned}$$

From the definitions we have

$$\begin{aligned} \bar{\rho}(h(\theta_1, \dots, \theta_n)) &= \bar{\rho} \left( [f_n(\theta_1, \dots, \theta_n)] \left( \dots, V_i \left( \sum_{j=1}^n W_j \theta_j \right) - W_i \left( \sum_{j=1}^n V_j \theta_j \right), \dots \right) \right) \\ &= [f_n(\theta_1, \dots, \theta_n)] \left( \dots, V_i \left( \sum_{j=1}^n W_j \bar{\rho}(\theta_j) \right) - W_i \left( \sum_{j=1}^n V_j \bar{\rho}(\theta_j) \right), \dots \right) \\ &= [f_n(\theta_1, \dots, \theta_n)] \left( \dots, V_i \frac{\partial \bar{\rho}(\theta)}{\partial Y} - W_i \frac{\partial \bar{\rho}(\theta)}{\partial X}, \dots \right). \end{aligned}$$

Thus, if we define

$$F(X, Y) := \bar{\rho}(f_n(\theta_1, \dots, \theta_n)) = [f_n(\theta_1, \dots, \theta_n)] (\dots, V_i X + W_i Y, \dots),$$

then

$$\bar{\rho}(h(\theta_1, \dots, \theta_n)) = F \left( \frac{\partial \bar{\rho}(\theta)}{\partial Y}, -\frac{\partial \bar{\rho}(\theta)}{\partial X} \right) = (-1)^{d_h} F \left( -\frac{\partial \bar{\rho}(\theta)}{\partial Y}, \frac{\partial \bar{\rho}(\theta)}{\partial X} \right)$$

where the last equality holds because  $\deg(F) = d_h$ . Now, from Proposition 3.7, recall that

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), F(-\frac{\partial \bar{\rho}(\theta)}{\partial Y}, \frac{\partial \bar{\rho}(\theta)}{\partial X})) = \text{Res}_{X,Y}(\bar{\rho}(\theta), F(X, Y)) \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h}.$$

Therefore, we have (observe that  $(-1)^{d_1 \dots d_n - 1} = 1$ )

$$\text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(h(\theta_1, \dots, \theta_n))) = \text{Res}_{X,Y}(\bar{\rho}(\theta), F(X, Y)) \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h}$$

and using again the first Mertens' formula for  $\text{Res}_{X,Y}(\bar{\rho}(\theta), F(X, Y))$ , we obtain

$$\begin{aligned} \text{Res}_{X,Y}(\bar{\rho}(\theta), \bar{\rho}(h(\theta_1, \dots, \theta_n))) &= \\ &= (-1)^{d_1 \dots d_n} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_n} \text{Res}(f_1, \dots, f_n) \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h}. \end{aligned} \quad (\text{A.3})$$

Now, the comparison of the equations (A.2) and (A.3) yields

$$\begin{aligned} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h} \text{Res}(f_1, \dots, f_{n-1}, h) \\ = (-1)^{d_1 \dots d_n} \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_n} \text{Res}(f_1, \dots, f_n) \text{Disc}_{X,Y}(\bar{\rho}(\theta))^{d_h}. \end{aligned}$$

We conclude the proof by observing that  $\text{Disc}(\bar{\rho}(\theta)) \in A[V_1, \dots, V_n, W_1, \dots, W_n]$  is nonzero, a fact that we show in the following lemma.  $\square$

**Lemma A**  $\text{Disc}_{X,Y}(\bar{\rho}(\theta))$  is nonzero in  $A[V_1, \dots, V_n, W_1, \dots, W_n]$ , where  $A$  is the universal ring of the coefficients of the polynomials  $f_1, \dots, f_{n-1}$ .

*Proof.* We exhibit a specialization for which  $\text{Disc}(\bar{\rho}(\theta))$  is easily seen to be nonzero. We start by specializing each polynomial  $f_i$ ,  $i = 1, \dots, n-1$  to the product of  $d_i$  generic linear forms

$$l_{i,j} := U_{i,j,1} X_1 + U_{i,j,2} X_2 + \dots + U_{i,j,n} X_n = \sum_{r=1}^{d_i} U_{i,j,r} X_r, \quad i = 1, \dots, n, \quad j = 1, \dots, d_i.$$

Set  $A' = \mathbb{Z}[U_{i,j,r} : i = 1, \dots, n, j = 1, \dots, d_i, r = 1, \dots, n]$ . After this specialization, we get

$$\theta = \prod_{\substack{1 \leq j_i \leq d_i \\ i=1, \dots, n-1}} \det(l_{1,j_1}, l_{2,j_2}, \dots, l_{n-1,j_{n-1}}, U_1 X_1 + \dots, U_n X_n)$$

in  $A'[U_1, \dots, U_n]$ . For each  $(n-1)$ -uple  $\lambda := (j_1, \dots, j_{n-1})$  in the above product we denote by  $\Delta_\lambda(U_1, \dots, U_n)$  the corresponding determinant. We deduce that

$$\frac{\partial \bar{\rho}(\theta)}{\partial Y} = \sum_{\lambda} \left( \Delta_\lambda(W_1, \dots, W_n) \prod_{\mu, \mu \neq \lambda} \bar{\rho}(\Delta_\mu) \right).$$

Now, on the one hand we have (the resultant and the discriminant are taken with respect to  $X, Y$ )

$$\begin{aligned} \text{Res} \left( \bar{\rho}(\theta), \frac{\partial \bar{\rho}(\theta)}{\partial Y} \right) &= \\ \text{Disc}(\bar{\rho}(\theta)) \cdot \bar{\rho}(\theta)(0, -1) &= (-1)^{d_1 \dots d_{n-1}} \text{Disc}(\bar{\rho}(\theta)) \prod_{\lambda} \Delta_\lambda(W_1, \dots, W_n), \end{aligned}$$

and on the other hand

$$\begin{aligned} \text{Res} \left( \bar{\rho}(\theta), \frac{\partial \bar{\rho}(\theta)}{\partial Y} \right) &= \text{Res} \left( \prod_{\lambda} \bar{\rho}(\Delta_\lambda), \sum_{\lambda} \left( \Delta_\lambda(W_1, \dots, W_n) \prod_{\mu, \mu \neq \lambda} \bar{\rho}(\Delta_\mu) \right) \right) \\ &= \prod_{\lambda} \text{Res} \left( \bar{\rho}(\Delta_\lambda), \sum_{\omega} \left( \Delta_\omega(W_1, \dots, W_n) \prod_{\mu, \mu \neq \omega} \bar{\rho}(\Delta_\mu) \right) \right) \\ &= \prod_{\lambda} \text{Res} \left( \bar{\rho}(\Delta_\lambda), \Delta_\lambda(W_1, \dots, W_n) \prod_{\mu, \mu \neq \lambda} \bar{\rho}(\Delta_\mu) \right) \\ &= \left( \prod_{\lambda} \Delta_\lambda(W_1, \dots, W_n) \right) \prod_{\substack{\lambda, \mu \\ \lambda \neq \mu}} \text{Res}(\bar{\rho}(\Delta_\lambda), \bar{\rho}(\Delta_\mu)). \end{aligned}$$

Therefore, choosing an order for the  $(n-1)$ -uples  $\lambda$ , we deduce that

$$\text{Disc}(\bar{\rho}(\theta)) = (-1)^{\frac{N^2+N}{2}} \prod_{\lambda < \mu} \text{Res}(\bar{\rho}(\Delta_\lambda), \bar{\rho}(\Delta_\mu))^2$$

with  $N = d_1 \dots d_{n-1}$ . Moreover, for any  $(n-1)$ -uple  $\lambda$ , it is easy to see that

$$\bar{\rho}(\Delta_\lambda) = \Delta_\lambda(V_1, \dots, V_n)X + \Delta_\lambda(W_1, \dots, W_n)Y.$$

It follows that in  $A'[V_1, \dots, V_n, W_1, \dots, W_n]$  we have the equality

$$\begin{aligned} \text{Disc}(\bar{\rho}(\theta)) &= (-1)^{\frac{N^2+N}{2}} \times \\ &\prod_{\lambda < \mu} (\Delta_\lambda(V_1, \dots, V_n) \Delta_\mu(W_1, \dots, W_n) - \Delta_\lambda(W_1, \dots, W_n) \Delta_\mu(V_1, \dots, V_n))^2. \end{aligned} \quad (\text{A.4})$$

To finish the proof, we specialize a little more our polynomials  $f_1, \dots, f_{n-1}$  by specializing each linear form  $l_{i,j}$  to  $X_i - U_{i,j,n} X_n$ . Then, it is not hard to check that

$$\Delta_{\lambda=(j_1, \dots, j_{n-1})}(V_1, \dots, V_n) = U_{1,j_1} V_1 + U_{2,j_2} V_2 + \dots + U_{n-1,j_{n-1}} V_{n-1} + V_n \quad (\text{A.5})$$

and hence that  $\Delta_\lambda(0, \dots, 0, 1) = 1$ . Therefore, we deduce that for any pair  $(\lambda, \mu)$  such that  $\lambda \neq \mu$  we have

$$\begin{aligned} \Delta_\lambda(0, \dots, 0, 1) \Delta_\mu(W_1, \dots, W_n) - \Delta_\lambda(W_1, \dots, W_n) \Delta_\mu(0, \dots, 0, 1) &= \\ \Delta_\mu(W_1, \dots, W_n) - \Delta_\lambda(W_1, \dots, W_n) & \end{aligned}$$

and this quantity is clearly nonzero in view of (A.5).  $\square$

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