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# $b$ -coloring of tight graphs \*

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## Abstract

A coloring  $c$  of a graph  $G = (V, E)$  is a  $b$ -coloring if in every color class there is a vertex whose neighborhood intersects every other color classes. The  $b$ -chromatic number of  $G$ , denoted  $\chi_b(G)$ , is the greatest integer  $k$  such that  $G$  admits a  $b$ -coloring with  $k$  colors. A graph  $G$  is *tight* if it has exactly  $m(G)$  vertices of degree  $m(G) - 1$ , where  $m(G)$  is the largest integer  $m$  such that  $G$  has at least  $m$  vertices of degree at least  $m - 1$ . Determining the  $b$ -chromatic number of a tight graph  $G$  is NP-hard even for a connected bipartite graph [15]. In this paper we show that it is also NP-hard for a tight chordal graph. We also show that the  $b$ -chromatic number of a split graph can be computed is polynomial. Then we define the  $b$ -closure and the partial  $b$ -closure of a tight graph, and use these concepts to give a characterization of tight graphs whose  $b$ -chromatic number is equal to  $m(G)$ . This characterization is used to develop polynomial time algorithms for deciding whether  $\chi_b(G) = m(G)$ , for tight graphs that are complement of bipartite graphs,  $P_4$ -sparse and block graphs. We generalize the concept of pivoted tree introduced by Irving and Manlove [12] and show its relation with the  $b$ -chromatic number of tight graphs.

Graph coloring,  $b$ -coloring, precoloring extension, tight graphs.

## 1 Introduction

A  $k$ -coloring of a graph  $G = (V, E)$  is a function  $c : V \rightarrow \{1, 2, \dots, k\}$ , such that  $c(u) \neq c(v)$  for all  $uv \in E(G)$ . The *color class*  $c_i$  is the subset of vertices

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of  $G$  that are assigned to color  $i$ . The *chromatic number* of  $G$ , denoted  $\chi(G)$ , is the least integer  $k$  such that  $G$  admits a  $k$ -coloring. Given a  $k$ -coloring  $c$ , a vertex  $v$  is a *b-vertex* of color  $i$ , if  $c(v) = i$  and  $v$  has at least one neighbor in every color class  $c_j$ ,  $j \neq i$ . A coloring of  $G$  is a *b-coloring* if every color class has a  $b$ -vertex. The *b-chromatic number* of a graph  $G$ , denoted  $\chi_b(G)$ , is the largest integer  $k$  such that  $G$  admits a  $b$ -coloring with  $k$  colors. These concepts were first defined in [12]. In that paper, Irving and Manlove proved that the problem of determining the  $b$ -chromatic number of a graph is NP-Hard. In fact, it was shown in [15] that deciding whether a graph admits a  $b$ -coloring with a given number of colors is an NP-complete problem, even for connected bipartite graphs. The following upper bound for the  $b$ -chromatic number of a graph, presented in [12], has been proved to be very useful. If  $G$  admits a  $b$ -coloring with  $m$  colors, then  $G$  must have at least  $m$  vertices with degree at least  $m - 1$  (since each color class has one  $b$ -vertex). The *m-degree* of a graph  $G$ , denoted by  $m(G)$ , is the largest integer  $m$  such that  $G$  has  $m$  vertices of degree at least  $m - 1$ . It is easy to see that  $\chi_b(G) \leq m(G)$  for every graph  $G$ . A vertex of  $G$  with degree at least  $m(G)$  is called a *dense* vertex. The preceding upper bound leads us to the definition of a class of graphs which are tight with respect to the number and degree of their dense vertices:

**Definition 1 (tight graph)** A graph  $G$  is *tight* if it has exactly  $m(G)$  dense vertices, each of which has degree  $m(G) - 1$ .

In this paper, we mainly investigate the following decision problem:

TIGHT  $b$ -CHROMATIC PROBLEM

Instance: A tight graph  $G$ .

Question: Does  $\chi_b(G)$  equals  $m(G)$ ?

A direct consequence of the NP-completeness result shown in [15] is the following:

**Theorem 2** *The TIGHT  $b$ -CHROMATIC PROBLEM is NP-complete for connected bipartite graphs.*

For any positive  $k$ ,  $P_k$  denotes a path with  $k$  vertices. A graph  $G$  is  $P_4$ -sparse if every set of five vertices of  $G$  induces at most one  $P_4$ . Bonomo et al. [1] proved that the  $b$ -chromatic number of  $P_4$ -sparse graphs can be determined in polynomial time. They asked if this result could be extended to *distance-hereditary graphs*, that are graphs in which every induced path is a shortest path. We answer in the negative to this question by showing the following stronger result (Theorem 3). *The TIGHT  $b$ -CHROMATIC PROBLEM is NP-complete for chordal distance-hereditary graphs.* We recall that a graph is *chordal* if it does not contain any induced cycle of size greater than 3.

The proof of our NP-completeness result is a reduction from 3-EDGE-COLORABILITY. We reduce an instance of this problem to a graph which is

slightly more than a *split graph*, i.e. a graph whose vertex set may be partitioned into a clique and an independent set. Hence a natural question is to ask about the complexity of finding the  $b$ -chromatic number of a given split graph. We show in Theorem 5 that it can be solved in polynomial time.

In Section 3, we introduce the  $b$ -closure  $G^*$  of a graph  $G$ . We show that for a tight graph  $G$ ,  $\chi_b(G) = m(G)$  if and only if  $\chi(G^*) = m(G)$ . Hence if one can determine the chromatic number of the closure in polynomial time, one can also solve the TIGHT  $b$ -CHROMATIC PROBLEM in polynomial time. We show that it is the case for (tight) complement of bipartite graphs. Indeed, we prove that the closures of such graphs are also complements of bipartite graphs and the chromatic number of the complement of a bipartite graph can be determined in polynomial time. This was unknown since the characterization of complements of bipartite graphs with  $\chi_b(G) = k$  given by [13] does not lead to a polynomial algorithm for determining their  $b$ -chromatic number.

Moreover, we introduce the definition of pivoted tight graph and use this definition to give a sufficient condition for a tight graph to satisfy  $\chi_b(G) < m(G)$ .

The method of computing the  $b$ -closure of a graph and then the chromatic number of it does not yield polynomial-time algorithms to solve the TIGHT  $b$ -CHROMATIC PROBLEM for all classes of tight graphs. However, for some of them, we show in Section 4, that the TIGHT  $b$ -CHROMATIC PROBLEM may be solved in polynomial time using a slight modification of the closure, the *partial closure*. It is the case for block graphs and  $P_4$ -sparse graphs. It is already known that deciding if  $\chi_b(G) = m(G)$  is polynomial time solvable for  $P_4$ -sparse graphs [1]. However, our linear-time algorithm for tight  $P_4$ -sparse graphs is faster than the  $O(|V|^3)$  algorithm of [1]. It is also interesting to see how our general method can be used to solve these problems.

## 2 Chordal graphs

**Theorem 3** *The TIGHT  $b$ -CHROMATIC PROBLEM is NP-complete for chordal distance-hereditary graphs.*

**Proof.** The problem belongs to NP since a  $b$ -coloring with  $m(G)$  colors is a certificate. To show that it is also NP-complete, we present a reduction from 3-EDGE-COLORABILITY of 3-regular graphs, which is known to be NP-complete [10]. Let  $G$  be a 3-regular graph with  $n$  vertices. Set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $E(G) = \{e_1, \dots, e_m\}$ . Let  $I$  be the vertex-edge incidence graph of  $G$ , that is the bipartite graph with vertex set  $V(I) = V(G) \cup E(G)$  in which an edge of  $G$  is adjacent to its two end-vertices. We construct from  $I$  a new graph  $H$  as follows. First, we add an edge between every pair of vertices in  $V(G)$  and then, we add three disjoint copies of  $K_{1, n+2}$ . One can easily see that  $d_H(v) = n - 1 + 3 = n + 2$ , for  $v \in V(G)$ , and that  $d_H(u) = 2$ , for  $u \in E(G)$ . Moreover, each copy of  $K_{1, n+2}$  has exactly one vertex with degree equal to  $n+2$ . Consequently,  $m(H) = n + 3$  and  $H$  is tight. In  $H$ ,  $V(G)$  is a clique and  $E(G)$  is an independent set, so  $H[A \cup B]$  is a split graph, and so it is chordal. As the

disjoint copies of  $K_{1,n+2}$  are themselves chordal graphs, we get that the entire graph  $H$  is chordal. One can easily check that  $H$  is also distance-hereditary. We now prove that  $G$  admits a 3-edge-coloring if and only if  $\chi_b(H) = m(H) = n+3$ .

Let  $c$  be a 3-edge-coloring of  $E(G)$  that uses colors  $\{1, 2, 3\}$ . We shall construct a  $b$ -coloring  $c'$  of  $H$  with  $n+2$  colors. Let  $c'(u) = c(u)$ , for  $u \in E(G)$ , and  $c'(v_i) = i+3$ , for  $1 \leq i \leq n$ . Note that in this partial coloring, the vertices in  $V(G)$  are  $b$ -vertices of their respective colors. To obtain the remaining  $b$ -vertices, one just have to appropriately color the copies of  $K_{1,n+2}$ , which can be easily done. Then,  $c'$  is a  $b$ -coloring of  $H$  with  $m(H) = n+3$  colors.

Now, let  $c'$  be a  $b$ -coloring of  $H$  that uses  $n+3$  colors. Since  $V(G)$  is a clique, we may assume that  $c'(v_i) = i+3$ , for  $1 \leq i \leq n$ . Since there are only  $n+3$  vertices of degree  $n+2$  in  $H$ , each vertex in  $V(G)$  is an  $b$ -vertex. But then, since every vertex in  $V(G)$  has degree exactly  $n+2$  in  $H$ , all its neighbors must have distinct colors. As a consequence, since no vertex in  $V(G)$  is colored with one of the colors in  $\{1, 2, 3\}$ , for every vertex in  $V(G)$ , its 3 neighbours in  $E(G)$  are colored with distinct colors in  $\{1, 2, 3\}$ . This implies that  $G$  admits a 3-edge-coloring of  $G$ , and completes the proof.  $\square$

**Remark 4** A graph  $G = (V, E)$  is  $P_4$ -laden [5] (resp. *extended  $P_4$ -laden* if for every set  $S \subseteq V$  of six vertices, the subgraph induced by  $S$  contains at most one induced  $P_4$  or is a split graph (resp. a *pseudo-split graph*, i.e. a  $\{C_4, K_2\}$ -free graph. By definition, every  $P_4$ -laden graph is extended  $P_4$ -laden. One can check that the graph in the reduction of Theorem 3 is  $P_4$ -laden, so the TIGHT  $b$ -CHROMATIC PROBLEM is NP-complete for  $P_4$ -laden graphs.

The class of the extended  $P_4$ -laden graphs contains many graph classes with few induced  $P_4$ 's. In particular, it contains the class of  $P_4$ -tidy graphs [4] which in turn contains the ones of  $P_4$ -lite [7],  $P_4$ -extendible [6] and  $P_4$ -reducible graphs [8]. A graph is  $P_4$ -tidy if for every set  $A$  inducing a  $P_4$  there is at most one vertex  $x$  such that the subgraph induced by  $A \cup \{x\}$  has at most one induced  $P_4$ . Bonomo, Koch, and Velasquez [2] proved that the  $b$ -chromatic number of a  $P_4$ -tidy graph can be determined in polynomial time, thus extending the result in [1].

The three copies of  $K_{1,n+2}$  play an important role in the reduction of the proof of Theorem 3, since one can show the following

**Theorem 5** *If  $G$  is a split graph then  $\chi_b(G) = m(G)$ . Hence, the  $b$ -chromatic number of a split graph can be determined in polynomial time.*

**Proof.** Let  $G$  be a split graph and  $(K, S)$  a partition of  $V(G)$  with  $K$  a clique and  $S$  an independent set such that  $|K|$  is maximum. Every vertex in  $K$  has degree at least  $|K| - 1$  and every vertex  $s$  in  $S$  has degree at most  $|K| - 1$  otherwise  $(K \cup \{s\}, S \setminus \{s\})$  would contradict the maximality of  $|K|$ . Hence  $m(G) = |K|$ .

Coloring the vertices in  $K$  with  $|K|$  distinct colors and then extend it greedily to the vertices of  $S$  (This is possible since every vertex in  $S$  has degree smaller than  $|K|$ .) gives a  $b$ -coloring of  $G$  with  $m(G) = |K|$  colors.  $\square$

### 3 $b$ -closure

**Definition 6 ( $b$ -closure)** Let  $G$  be a tight graph. The  $b$ -closure of  $G$ , denoted by  $G^*$ , is the graph with vertex set  $V(G^*) = V(G)$  and edge set  $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are non-adjacent dense vertices}\} \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$ .

The next theorem proves the relation, for a tight graph  $G$ , between the parameters  $\chi_b(G)$  and  $\chi(G^*)$ :

**Lemma 7** *Let  $G$  be a tight graph. Then  $\chi_b(G) = m(G)$  if and only if  $\chi(G^*) = m(G)$ .*

**Proof.** Set  $m = m(G)$ . Suppose that  $\chi_b(G) = m$ , and let  $c$  be a  $b$ -coloring of  $G$  with  $m$  colors. It is easy to see that the  $m$  dense vertices form a clique in  $G^*$  and so  $\chi(G^*) \geq m$ . Let us show that  $c$  is a proper coloring for  $G^*$ . Let  $uv \notin E$  be such that  $uv \in E(G^*)$ . If both  $u$  and  $v$  are dense, as there are exactly  $m$  dense vertices in  $G$ , they must have distinct colors in  $c$ . Now, suppose that  $u$  or  $v$  is not a dense vertex. By the definition of  $G^*$ ,  $u$  and  $v$  have a common dense neighbor, say  $d$ , in  $G$ . Since all dense vertices of  $G$  have degree  $m - 1$  and  $c$  is a  $b$ -coloring,  $u$  and  $v$  must have been assigned distinct colors in  $c$ . Hence,  $\chi(G^*) = m$ .

Conversely, let  $c'$  be a proper coloring of  $G^*$  with  $m$  colors. In this case, since  $E(G) \subseteq E(G^*)$ ,  $c'$  is also a proper coloring of  $G$ . It only remains to show that every color of  $c'$  has a  $b$ -vertex. As the dense vertices of  $G$  form a clique in  $G^*$ , they have distinct colors in  $c'$ . Moreover, for a dense vertex  $d$  of  $G$ , we have that  $N_{G^*}(d)$  is a clique. As a consequence,  $d$  is a  $b$ -vertex. Therefore,  $\chi_b(G) = m$ .  $\square$

Since  $\omega(G^*) > m$  implies that  $\chi(G^*) > m$ , it follows:

**Corollary 8** *Let  $G$  be a tight graph. If  $\chi_b(G) = m(G)$ , then  $\omega(G^*) = \chi(G^*) = m(G)$ .*

#### 3.1 Complement of bipartite graphs

By Lemma 7, it is interesting to consider the  $b$ -closure of a tight graph  $G$  if the chromatic number of its closure can be determined in polynomial time. Indeed if so, one can decide in polynomial time if  $\chi_b(G) = m(G)$ . We now show that it is the case if  $G$  is the complement of a bipartite graph.

**Lemma 9** *The  $b$ -closure of the complement of a bipartite graph is a complement of a bipartite graph.*

**Proof.** Let  $G$  be a tight complement of a bipartite graph. Let  $V(G) = X \cup Y$  where  $X$  and  $Y$  are two disjoint cliques in  $G$ . As  $V(G^*) = V(G)$ , and since  $E(G) \subseteq E(G^*)$ , the sets  $X$  and  $Y$  are cliques in  $G^*$ . So they also form a partition of  $V(G^*)$  into two cliques.  $\square$

Computing the chromatic number of the complement  $G$  of a bipartite graph  $\overline{G}$  is equivalent to compute the maximum size of a matching in this bipartite graph. Hence it can be done in  $O(\sqrt{|V(G)|} \cdot |E(\overline{G})|)$  by the algorithm of Hopcroft and Karp [11] and in  $O(|V(G)|^{2.376})$  using an approach based on the fast matrix multiplication algorithm [17].

**Corollary 10** *Let  $G$  be a tight complement of a bipartite graph. It can be decided in  $O(\max\{\sqrt{|V(G)|} \cdot |E(\overline{G})|, |V(G)|^{2.376}\})$  if  $\chi_b(G) = m(G)$ .*

## 3.2 Pivoted graphs

In the study of the  $b$ -chromatic number of trees, Irving and Manlove [12] introduced the notion of a *pivoted tree*, and showed that a tree  $T$  satisfies  $\chi_b(T) < m(T)$  if and only if it is pivoted. We generalize this notion and show how our generalization is related to the  $b$ -chromatic number of tight graphs.

**Definition 11 (Pivoted Graph)** Let  $G$  be a tight graph. We say that  $G$  is *pivoted* if there is a set  $N$  of non-dense vertices, with  $|N| = k$ , and a set of dense vertices  $D$ , with  $|D| = m(G) - k + 1$ , satisfying:

1. For every pair  $u, v \in N$ ,  $u$  is adjacent to  $v$ , or there is a dense vertex  $w$  that is adjacent to both  $u$  and  $v$ .
2. For every pair  $u \in N, d \in D$ , either  $u$  is adjacent to  $d$  or  $u$  and  $d$  are both adjacent to a dense vertex  $w$  (not necessarily in  $D$ ).

**Theorem 12** *Let  $G$  be a tight graph. Then  $G$  is a pivoted graph if and only if  $\omega(G^*) > m(G)$ .*

**Proof.** First, assume that  $G$  is a pivoted graph. Then Definitions 6 and 11 immediately imply that  $N \cup D$  is a clique of size  $m + 1$  in  $G^*$ .

Reciprocally, assume that  $\omega(G^*) > m$ . Let  $S \subseteq V(G^*)$  be a clique of size  $m + 1$  in  $G^*$ . Let  $N = \{v \in S \mid v \text{ is not dense in } G\}$  and  $D = \{v \in S \mid v \text{ is dense in } G\}$ . Let  $u, v \in S$ . If  $u, v \in D$ , there is nothing to show, since Definition 11 imposes no restrictions between dense vertices in  $G$ . If  $u \in N, v \in D \cup N$ , we have that either  $uv \in E(G)$ , or  $ud, vd \in E(G)$ , for a dense vertex  $d \in V(G)$ . So, it is easy to see that the sets  $N$  and  $D$  satisfy the requirements of Definition 11.  $\square$

Lemma 7 and Theorem 12 have the following corollary.

**Corollary 13** *Let  $G$  be a tight graph. If  $G$  is a pivoted graph, then  $\chi_b(G) < m(G)$ .*

**Proof.** As  $G$  is pivoted, Theorem 12 implies that  $\omega(G^*) > m(G)$ , and therefore  $\chi(G^*) > m(G)$ . Then, by Lemma 7,  $\chi_b(G) < m(G)$ .  $\square$

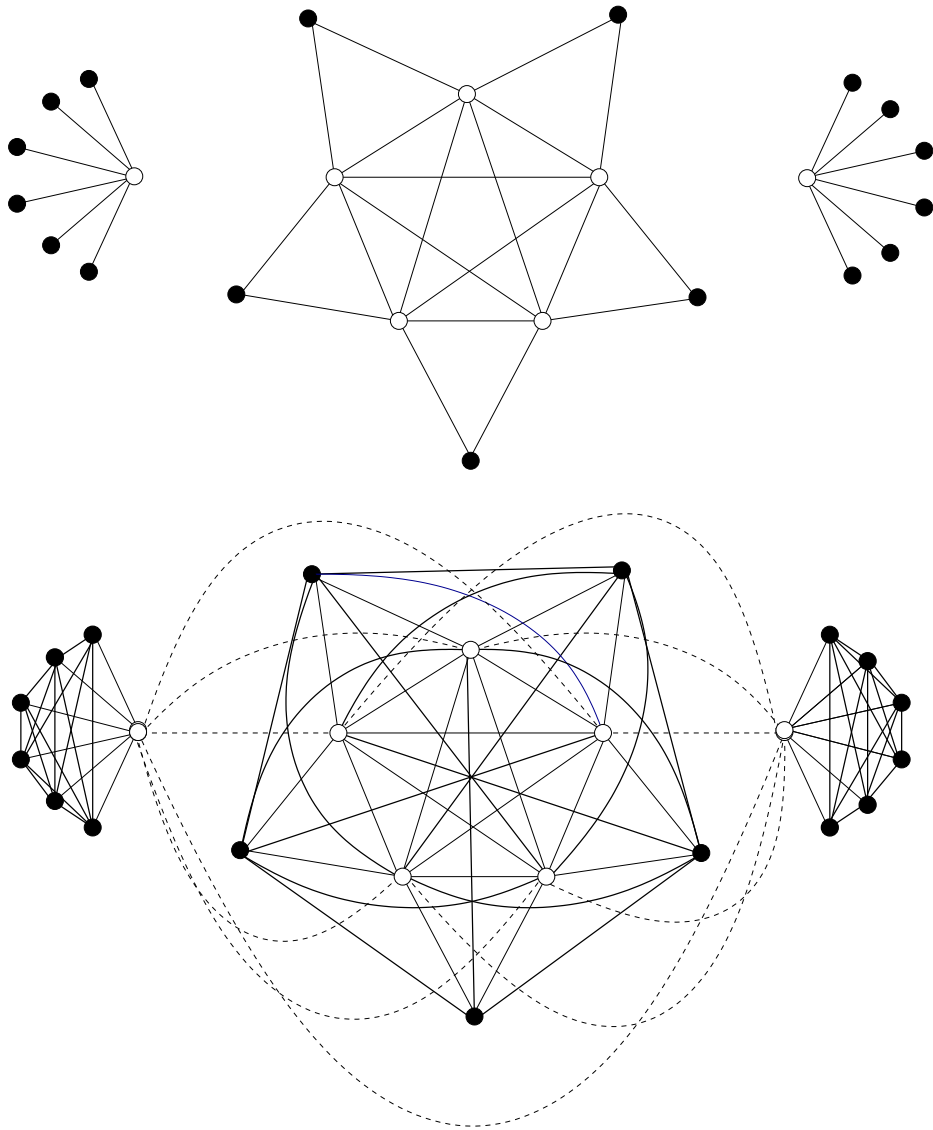


Figure 1: A non-pivoted chordal graph, satisfying  $\chi_b(G) < m(G)$ , and its  $b$ -closure  $G^*$ , satisfying  $\chi(G^*) > \omega(G^*) = m(G)$  (the new edges between the dense vertices are dashed).



There are graphs satisfying  $\chi(G^*) > m(G)$  but not  $\omega(G^*) > m(G)$ . Figure 1 shows a chordal non-pivoted graph  $G$  with exactly  $m(G) = 7$  dense vertices, each of degree 6, such that  $\chi_b(G) < m(G)$ .

In contrast to what happens with pivoted graphs, where a clique of size greater than  $m$  is formed in their  $b$ -closures, the graph of Figure 1 has clique number 7, but its  $b$ -closure produces an odd hole (by the five non-dense vertices in the bigger component) which causes  $\chi(G^*) > 7$ .

## 4 Partial $b$ -closure

**Definition 14 (partial  $b$ -closure)** Let  $G$  be a tight graph. The *partial  $b$ -closure* of  $G$ , denoted  $G_p^*$ , is the graph with vertex set  $V(G^*) = V(G)$  and edge set  $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$ .

**Lemma 15** *Let  $G_p^*$  be the partial  $b$ -closure of a graph  $G$ , and let  $D$  be the set of  $m(G)$  dense vertices of  $G$ . Then  $\chi_b(G) = m(G)$  if and only if  $G_p^*$  admits a  $m(G)$ -coloring where all the vertices in  $D$  have distinct colors.*

**Proof.** The proof is similar to the one of Lemma 7. In this case, since we do not add edges between all the pairs of dense vertices in  $G_p^*$ , we need the requirement that the  $m(G)$ -coloring of  $G_p^*$  is such that all dense vertices have distinct colors.  $\square$

By Lemma 15, one can decide in polynomial time if  $\chi_b(G) = m$  wherever it can be decided in polynomial time if the constrained coloring of its partial closure  $G_p^*$  exists. In particular, it is the case if the precoloring extension problem can be decided in polynomial time for  $G$ . We show that this is the case for block graphs and  $P_4$ -sparse graphs.

### 4.1 Block graphs

The result in this subsection was obtained in cooperation with Ana Silva.

A graph  $G = (V, E)$  is a *block graph* if every of its blocks (maximal 2-connected subgraphs) is a complete graph. For an example, see Figure 2.

**Lemma 16** *The partial  $b$ -closure of a block graph is chordal.*

**Proof.** By contradiction, assume that the partial  $b$ -closure  $G_p^*$  of a block graph  $G$  is not chordal. Then it has an induced cycle  $C = (v_1, v_2, \dots, v_k)$  of length  $k \geq 4$ . For every edge  $v_i v_{i+1}$  of  $C$  (indices must be taken modulo  $k$ ) either  $v_i v_{i+1} \in E(G)$  or there is a dense vertex  $w_i \in V(G)$  such that  $v_i w_i, w_i v_{i+1} \in E(G)$ . In the latter case, the vertex  $w_i$  is adjacent to no  $v_j$  for  $j \notin \{i, i+1\}$  in  $G$ , otherwise both  $v_j v_i$  and  $v_j v_{i+1}$  would be edges of  $G_p^*$  and  $C$  would not be induced. Furthermore, this implies that all the existing  $w_i$ 's are distinct. Let  $C'$  be the cycle obtained from  $C$  by replacing each edge  $v_i v_{i+1}$  by  $v_i w_i v_{i+1}$  whenever  $v_i v_{i+1} \notin E(G)$ . Observe that  $C'$  is a cycle of  $G$ .

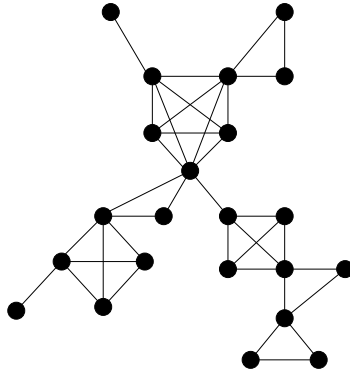


Figure 2: A block graph.

But, since  $G$  is a block graph, the vertices of any cycle (in particular,  $C'$ ) form a clique in  $G$  and thus also in  $G_p^*$ . Hence the vertices of  $C$  form a clique in  $G_p^*$ , a contradiction.  $\square$

Marx [16] showed that the precoloring extension problem when all the  $C$  colors are used at most once is solvable in time  $O(C \cdot |V(G)|^3)$  for a chordal graph  $G$ . Hence,

**Corollary 17** *the TIGHT  $b$ -CHROMATIC PROBLEM can be decided in time  $O(m(G)|V(G)|^3)$  for tight block graphs.*

**Remark 18** A tree is a block graph, so using the partial closure method the TIGHT  $b$ -CHROMATIC PROBLEM for tight trees can be solved in time  $O(m(G)|V(G)|^3)$ . However, Irving and Manlove [12] gave a linear time algorithm to compute the  $b$ -chromatic number of any tree. Hence the TIGHT  $b$ -CHROMATIC PROBLEM can be solved in linear time for trees.

## 4.2 $P_4$ -sparse graphs

**Lemma 19** *The partial  $b$ -closure of a  $P_4$ -sparse graph is  $P_4$ -sparse.*

**Proof.** Let  $G$  be a  $P_4$ -sparse graph. Suppose, by way of contradiction, that  $G_p^*$  is not  $P_4$ -sparse. Then there is at least one induced  $P_4$  in  $G_p^*$  that is not in  $G$ . Let  $P = (v_1, v_2, v_3, v_4)$  be such a  $P_4$  in  $G_p^*$ . We will show that there are 5 vertices that induces two  $P_4$ 's in  $G$ , thus getting a contradiction. By symmetry, it is enough to consider the following five cases.

Case 1 :  $v_1v_2 \in E(G)$ ,  $v_2v_3 \in E(G)$  and  $v_3v_4 \notin E(G)$ .

Then,  $v_3$  and  $v_4$  are both adjacent to a dense vertex  $w \in V(G)$  (by the

definition of the partial  $b$ -closure). Note that  $v_1w \notin E(G)$  (resp.  $v_2w \notin E(G)$ ) otherwise  $v_1v_4 \in E(G_p^*)$  (resp.  $v_2v_4 \in E(G_p^*)$ ) and  $P$  would not be an induced  $P_4$  in  $G_p^*$ . Hence,  $\{v_1, v_2, v_3, w, v_4\}$  induces a  $P_5$  which contains two induced  $P_4$ .

Case 2 :  $v_1v_2 \in E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \in E(G)$ .

In this case,  $v_2$  and  $v_3$  are both adjacent to a dense vertex  $w \in V(G)$  (again, by the definition of the  $b$ -closure). Note that  $v_1w, v_4w \notin E(G)$ , for otherwise, this would imply that  $v_1v_3 \in E(G_p^*)$  ( $v_2v_4 \in E(G_p^*)$ ), by the definition of the partial  $b$ -closure. But then,  $\{v_1, v_2, w, v_3, v_4\}$  is an induced  $P_5$  in  $G$ .

Case 3 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \in E(G)$  and  $v_3v_4 \notin E(G)$ .

As  $v_1v_2 \notin E(G)$ , the vertices  $v_1$  and  $v_2$  are both adjacent to a dense vertex  $w_1 \in V(G)$ . Moreover,  $w_1v_3 \notin E(G)$  (resp.  $w_1v_4 \notin E(G)$ ), since for otherwise  $v_1v_3 \in E(G_p^*)$  (resp.  $v_1v_4 \in E(G_p^*)$ ) and  $P$  would not be an induced  $P_4$  in  $G_p^*$ . By a similar argument,  $v_3$  and  $v_4$  are both adjacent to a dense vertex  $w_2 \in V(G)$ , which is not adjacent to  $v_1$  and  $w_2$ . Note that  $w_1$  and  $w_2$  are distinct since  $w_1v_4 \notin E(G)$ . If  $w_1w_2 \notin E(G)$ , then  $\{v_1, w_1, v_2, v_3, w_2\}$  is an induced  $P_5$  in  $G$ . If  $w_1w_2 \in E(G)$ , then  $\{v_1, w_1, v_2, w_2, v_4\}$  induces two  $P_4$ 's in  $G$ .

Case 4 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \in E(G)$ .

Using arguments similar to the ones in the previous cases, we obtain that there are distinct dense vertices  $w_1, w_2 \in V(G)$  satisfying  $v_1w_1, v_2w_1, v_2w_2, v_3w_2 \in E(G)$ , and  $v_1w_2, v_4w_2, v_3w_1, v_4w_1 \notin E(G)$ . If  $w_1w_2 \in E(G)$  then  $\{v_1, w_1, w_2, v_3, v_4\}$  induces a  $P_5$  in  $G$ . If  $w_1w_2 \notin E(G)$ , then the set  $\{v_1, w_1, v_2, w_2, v_3\}$  induces a  $P_5$  in  $G$ .

Case 5 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \notin E(G)$ .

Again, by similar arguments to the ones used in the previous cases, there are distinct dense vertices  $w_1, w_2, w_3 \in V(G)$  such that  $v_1w_1, v_2w_1, v_2w_2, v_3w_2, v_3w_3, v_4w_3 \in E(G)$ , and  $v_3w_1, v_4w_1, v_1w_2, v_4w_2, v_1w_3, v_2w_3 \notin E(G)$ . If  $w_1w_3 \in E(G)$ , the set  $\{v_1, w_1, w_3, v_3, v_4\}$  induces two  $P_4$ 's in  $G$ . Henceforth we may assume that  $w_1w_3 \notin E(G)$ . If  $w_1w_2, w_2w_3 \in E(G)$ , then the set  $\{v_1, w_1, w_2, w_3, v_4\}$  induces a  $P_5$  in  $G$ . Hence by symmetry, we may assume that  $w_2w_3 \in E(G)$ . If  $w_1w_2 \in E(G)$ , then the set  $\{v_1, w_1, v_2, w_2, v_3\}$  induces two  $P_4$ 's in  $G$ . If  $w_1w_2 \notin E(G)$  the set  $\{v_1, w_1, v_2, w_2, w_3\}$  induces two  $P_4$ 's in  $G$ .

□

Babel et al. [14] showed that the precoloring extension problem is linear-time solvable for  $(q, q-4)$ -graphs, which are graphs where no set of at most  $q$  vertices induces more than  $q-4$  different  $P_4$ 's. Hence,

**Corollary 20** *The TIGHT  $b$ -CHROMATIC PROBLEM can be decided in linear time for tight  $P_4$ -sparse graphs.*

Consequently, for tight  $P_4$ -sparse graphs, this algorithm is faster than the  $O(|V|^3)$  algorithm given in [1], that solves the more general case where the input graph is not necessarily tight.

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