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► **To cite this version:**

Julio Araujo, Nathann Cohen, Frédéric Giroire, Frédéric Havet. Good edge-labelling of graphs. LAGOS'09 - V Latin-American Algorithms, Graphs and Optimization Symposium, Nov 2009, Gramado, Brazil. Elsevier, Volume 35 (1), pp.275-280, 2009, Eletronic Notes in Discrete Mathematics. <10.1016/j.endm.2009.11.045>. <hal-00749194>

**HAL Id: hal-00749194**

**<https://hal.inria.fr/hal-00749194>**

Submitted on 12 Dec 2012

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# Good edge-labelling of graphs

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## Abstract

A *good edge-labelling* of a graph  $G$  is a labelling of its edges such that for any two distinct vertices  $u, v$ , there is at most one  $(u, v)$ -path with non-decreasing labels. This notion was introduced in [3] to solve wavelength assignment problems for specific categories of graphs. In this paper, we aim at characterizing the class of graphs that admit a good edge-labelling. First, we exhibit infinite families of graphs for which no such edge-labelling can be found. We then show that deciding if a graph admits a good edge-labelling is NP-complete. Finally, we give large classes of graphs admitting a good edge-labelling:  $C_3$ -free outerplanar graphs, planar graphs of girth at least 6, subcubic  $\{C_3, K_{2,3}\}$ -free graphs.

*Keywords:* edge-labelling, planar graphs, matching-cut, channel assignment.

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# 1 Motivation

A classical and widely studied problem in WDM (Wavelength Division Multiplexing) networks is the Routing and Wavelength Assignment (RWA) problem [8,9,2]. It consists in finding routes, and their associated wavelength as well, to satisfy a set of traffic requests while minimising the number of used wavelengths. This is a difficult problem which is, in general, NP-hard. Thus, it is often split into two distinct problems: First, routes are found for the requests. Then, in a second step, these routes are taken as an input. Wavelengths must be associated to them in such a way that two routes using the same fibre do not have the same wavelength. The last problem can be reformulated as follows: Given a digraph and a set of dipaths, corresponding to the routes for the requests, find the minimum number of wavelengths  $w$  needed to assign different wavelengths to dipaths sharing an edge. This problem may be seen as a colouring problem of the *conflict graph* which is defined as follows: It has one vertex per dipath and two vertices are linked by an edge if their corresponding dipaths share an edge. In [3], Bermond et al. studied the RWA problem for UPP-DAG which are acyclic digraphs (or DAG) in which there is at most one dipath from one vertex to another. In such digraph the routing is forced and thus the unique problem is the wavelength assignment one.

In their paper, they introduce the notion of good edge-labelling. An *edge-labelling* of a graph  $G$  is a function  $\phi : E(G) \rightarrow \mathbb{R}$ . A path is *increasing* if the sequence of its edges labels is non-decreasing. An edge-labelling of  $G$  is *good* if for any two distinct vertices  $u, v$ , there is at most one increasing  $(u, v)$ -path. Bermond et al.[3] showed that the conflict graph of a set of dipaths in a UPP-DAG has a *good edge-labelling*. Conversely, for any graph admitting a good edge-labelling one can exhibit a family of dipaths on a UPP-DAG whose conflict graph is precisely this graph. Bermond et al. [3] then use the existence of graphs with a good edge-labelling and large chromatic number to prove that there exist sets of requests on UPP-DAGs with load 2 (an edge is shared by at most two paths) requiring an arbitrarily large number of wavelengths. To obtain other results on this problem, it may be useful to identify the *good* graphs, which admit a good edge-labelling, and the *bad* ones, which do not.

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<sup>1</sup> Partially supported by the INRIA associated team EWIN between Mascotte and ParGO

<sup>2</sup> Partially supported by the European project FET AEOLUS

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## 2 Our results

In the following, proofs are omitted. The interested reader is referred to [1].

### 2.1 Bad graphs

Bermond et al.[3] noticed that  $C_3$  and  $K_{2,3}$  are bad. J.-S. Sereni [11] asked whether every  $\{C_3, K_{2,3}\}$ -free graph (i.e. with no  $C_3$  nor  $K_{2,3}$  as a subgraph) is good. We answer this question in the negative. We give an infinite family of bad graphs, none of which is the subgraph of another. The construction of this family is based on the bipartite graphs  $B_k$ ,  $k \geq 3$ , also known as ‘biwheels’, of order  $2k + 2$ , depicted below. and defined as follows :

$$\begin{aligned} V(B_k) &= \{u, v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_i \mid 1 \leq i \leq k\}, \\ E(B_k) &= \{uu_i \mid 1 \leq i \leq k\} \cup \{u_i v_i \mid 1 \leq i \leq k\} \cup \{v_i v \mid 1 \leq i \leq k\}, \\ &\quad \cup \{v_i u_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

with  $u_{k+1} = u_1$ .

For convenience, we denote by  $B_2$  a path of length 2 with endvertices  $u$  and  $v$ . All the  $H_k$  share the following property: *for every good edge-labelling,  $H_k$  has either an increasing  $(u, v)$ -path or an increasing  $(v, u)$ -path.* Hence the graph  $J_{i,j,k}$ ,  $i, j, k \geq 2$ , obtained from disjoint copies of  $B_i$ ,  $B_j$  and  $B_k$  by identifying the vertices  $u$  of these three copies and the vertices  $v$  of these three copies, is bad, even though  $\{C_3, K_{2,3}\}$ -free. Indeed, any good edge-labelling would have either two increasing  $(u, v)$ -paths or two increasing  $(v, u)$ -paths, which is a contradiction.

### 2.2 NP-completeness

We now prove that deciding if a graph has a good edge-labelling is NP-complete. We give a reduction from the NOT-ALL-EQUAL (NAE) 3-SAT Problem [10] which is defined as follows:

**Instance:** A set  $V$  of variables and a collection  $\mathcal{C}$  of clauses over  $V$  such that each clause has exactly 3 literals.

**Question:** Is there a truth assignment such that each clause has at least one true and at least one false literal?

The reduction works as follows. Let  $V = \{x_1, \dots, x_n\}$  and  $\mathcal{C} = \{C_1, \dots, C_m\}$  be an instance  $I$  of the NAE 3-SAT Problem. We shall construct a graph  $G_I$  such a way that  $I$  has an answer yes for the NAE 3-SAT Problem if and only if  $G$  has a good edge-labelling.

For each variable  $x_i$ ,  $1 \leq i \leq n$ , we create a variable graph  $VG_i$  defined as

follows (See Figure 1.):

$$\begin{aligned}
V(VG_i) &= \{v_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\
&\quad \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}. \\
E(VG_i) &= \{v_k^{i,j} v_{k+1}^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 3\} \cup \{v_4^{i,j} v_1^{i,j+1} \mid 1 \leq j \leq m-1\} \\
&\quad \cup \{v_k^{i,j} r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{v_k^{i,j} s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\
&\quad \cup \{v_4^{i,j} r_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} r_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\} \\
&\quad \cup \{v_4^{i,j} s_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} s_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\}.
\end{aligned}$$

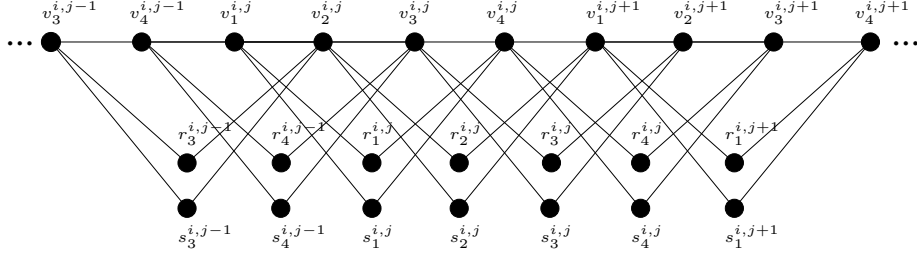


Fig. 1. The variable graph  $VG_i$

For each clause  $C_j = l_1 \vee l_2 \vee l_3$ ,  $1 \leq j \leq m$ , we create a clause graph  $CG_j$  defined by  $V(CG_j) = \{c^j, b_1^j, b_2^j, b_3^j\}$  and  $E(CG_j) = \{c^j b_1^j, c^j b_2^j, c^j b_3^j\}$

Now for each literal  $l_k$ ,  $1 \leq k \leq 3$ , if  $l_k$  is the non-negated variable  $x_i$ , we identify  $b_k^j$ ,  $c^j$  and  $b_{k+1}^j$  (index  $k$  is taken modulo 3) with  $v_1^{i,j}$ ,  $v_2^{i,j}$  and  $v_3^{i,j}$ , respectively. Otherwise, if  $l_k$  is the negated variable  $\bar{x}_i$ , we identify  $b_k^j$ ,  $c^j$  and  $b_{k+1}^j$  with  $v_3^{i,j}$ ,  $v_2^{i,j}$  and  $v_1^{i,j}$ , respectively.

### 2.3 Good graphs

We also show large classes of good graphs. To do so, we use the notion of *critical* graph which is a bad graph, every proper subgraph of which is good. Clearly, a good edge-labelling of a graph induces a good edge-labelling of all its subgraphs. So every bad graph must contain a critical subgraph.

We establish several properties of critical graphs. Firstly, as the union of good edge-labellings of the blocks of a graph is a good edge-labelling of this graph, critical graphs are 2-connected. In particular, every forest is good.

Secondly, a critical graph has no *matching-cut*, that is an edge-cut which is a matching. Indeed, let  $M$  be such an edge cut then if  $G \setminus M$  admits a good edge-labelling then assigning to all the edges of  $M$  a label greater than all the labels of  $G \setminus M$  we obtain a good edge-labelling of  $G$ . As a consequence,

**Theorem 2.1** *every  $C_3$ -free outerplanar graph is good.*

Indeed Eaton and Hull [5] showed that every such graph has either a 1-vertex or two adjacent 2-vertices and so has a matching-cut.

Secondly, every subcubic  $\{C_3, K_{2,3}\}$ -free graph has a matching cut. Indeed consider a smallest cycle  $C$ . Then the edge-cut between  $C$  and its complement is a matching. Hence,

**Theorem 2.2** *every subcubic  $\{C_3, K_{2,3}\}$ -free graph is good.*

Farley and Proskurowski [6] showed that a graph with less than  $\frac{3}{2}|V(G)| - \frac{3}{2}$  edges has a matching-cut. Hence a critical graph  $G$  has at least  $\frac{3}{2}|V(G)| - \frac{3}{2}$  edges. A well known consequence of Euler's Formula is that every planar graph  $G$  of girth at least 6 (the *girth* is the length of a smallest cycle) has at most  $\frac{3}{2}|V(G)| - 3$  edges. Thus

**Theorem 2.3** *every planar graph of girth at least 6 is good.*

Bonsma [4] characterized graphs with no matching-cut and  $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$  edges. We use this characterization to show that a critical graph  $G$  has at least  $\frac{3}{2}|V(G)| - \frac{1}{2}$  edges unless  $G$  is  $C_3$  or  $K_{2,3}$ .

### 3 Conclusions and further research

The *average degree* of a graph  $G$  is  $Ad(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$ . The above lower bound on the number of edges of a critical graphs implies that, for any  $c < 3/2$ , the number of critical graphs with average degree at most  $c$  is finite. Actually, we conjecture that the only ones are  $C_3$  and  $K_{2,3}$ .

**Conjecture 3.1** *Let  $G$  be a critical graph. Then  $Ad(G) \geq 3$  unless  $G \in \{C_3, K_{2,3}\}$ .*

More generally for any  $c < 4$ , we conjecture the following.

**Conjecture 3.2** *For any  $c < 4$ , there exists a finite list of graphs  $\mathcal{L}$  such that if  $G$  is a critical graph with  $Ad(G) \leq c$  then  $G \in \mathcal{L}$ .*

The constant 4 in the above conjecture would be tight because the graph  $J_{2,2,k}$  is critical and  $Ad(J_{2,2,k}) = \frac{8k+8}{2k+4} = 4 - \frac{4}{k+2}$ .

We have shown that a graph with no dense subgraphs is good. On the opposite direction one may wonder what is the minimum density ensuring a graph to be bad. Or equivalently,

**Problem 3.3** what is the maximum number  $g(n)$  of edges of a good graph on  $n$  vertices?

Clearly, we have  $g(n) = ex(n, \mathcal{C})$  where  $\mathcal{C}$  is the set of critical graphs. As  $K_{2,3}$  is critical then  $g(n) \leq ex(n, K_{2,3}) = \frac{1}{\sqrt{2}}n^{3/2} + O(n^{4/3})$  by a result of Füredi [7]. The hypercubes show that  $g(n) \geq \frac{1}{2}n \log n$ .

## References

- [1] Julio Araújo, Nathann Cohen, Frédéric Giroire, and Frédéric Havet. Good edge-labelling of graphs. Research Report RR-6934, INRIA, 2009.
- [2] K. Bala and T.E. Stern. Algorithms for routing in a linear lightwave network. In *IEEE INFOCOM'91. Proceedings. Tenth Annual Joint Conference of the IEEE Computer and Communications Societies. Networking in the 90s.*, pages 1–9, 1991.
- [3] J.-C. Bermond, M. Cosnard, and S. Pérennes. Directed acyclic graphs with unique path property. Technical Report RR-6932, INRIA, May 2009.
- [4] Paul Bonsma. A characterization of extremal graphs with no matching-cut. In *Proceedings of the European conference on combinatorics, graph theory and applications (EuroComb 05)*, pages 135–138, Nancy, 2005. Discrete Mathematics and Theoretical Computer Science Proceedings AE.
- [5] Nancy Eaton and Thomas Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.
- [6] A.M. Farley and A. Proskurowski. Extremal graphs with no disconnecting matching. In *Proceedings of the second West Coast conference on combinatorics, graph theory, and computing*, pages 153–165, 1984.
- [7] Zoltán Füredi. New asymptotics for bipartite Turán numbers. *J. Combin. Theory Ser. A*, 75(1):141–144, 1996.
- [8] B. Mukherjee. *Optical communication networks*. McGraw-Hill New York, 1997.
- [9] R. Ramaswami and K.N. Sivarajan. Routing and wavelength assignment in all-optical networks. *IEEE/ACM Transactions on Networking (TON)*, 3(5):489–500, 1995.
- [10] Thomas J. Schaefer. The complexity of satisfiability problems. In *STOC '78: Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 216–226, New York, NY, USA, 1978. ACM.
- [11] J.-S. Sereni. Private communication. 2006.