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Good edge-labelling of graphs

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Abstract

A *good edge-labelling* of a graph G is a labelling of its edges such that for any two distinct vertices u, v, there is at most one (u, v)-path with non-decreasing labels. This notion was introduced in [3] to solve wavelength assignment problems for specific categories of graphs. In this paper, we aim at characterizing the class of graphs that admit a good edge-labelling. First, we exhibit infinite families of graphs for which no such edge-labelling can be found. We then show that deciding if a graph admits a good edge-labelling is NP-complete. Finally, we give large classes of graphs admitting a good edge-labelling: C_3-free outerplanar graphs, planar graphs of girth at least 6, subcubic $\{C_3, K_{2,3}\}$-free graphs.

Keywords: edge-labelling, planar graphs, matching-cut, channel assignment.

1 Motivation

A classical and widely studied problem in WDM (Wavelength Division Multiplexing) networks is the Routing and Wavelength Assignment (RWA) problem [8,9,2]. It consists in finding routes, and their associated wavelength as well, to satisfy a set of traffic requests while minimising the number of used wavelengths. This is a difficult problem which is, in general, NP-hard. Thus, it is often split into two distinct problems: First, routes are found for the requests. Then, in a second step, these routes are taken as an input. Wavelengths must be associated to them in such a way that two routes using the same fibre do not have the same wavelength. The last problem can be reformulated as follows: Given a digraph and a set of dipaths, corresponding to the routes for the requests, find the minimum number of wavelengths w needed to assign different wavelengths to dipaths sharing an edge. This problem may be seen as a colouring problem of the *conflict graph* which is defined as follows: It has one vertex per dipath and two vertices are linked by an edge if their corresponding dipaths share an edge. In [3], Bermond et al. studied the RWA problem for UPP-DAG which are acyclic digraphs (or DAG) in which there is at most one dipath from one vertex to another. In such digraph the routing is forced and thus the unique problem is the wavelength assignment one.

In their paper, they introduce the notion of good edge-labelling. An *edge-labelling* of a graph G is a function $\phi : E(G) \to \mathbb{R}$. A path is *increasing* if the sequence of its edges labels is non-decreasing. An edge-labelling of G is *good* if for any two distinct vertices u, v, there is at most one increasing (u, v)-path. Bermond et al.[3] showed that the conflict graph of a set of dipaths in a UPP-DAG has a *good edge-labelling*. Conversely, for any graph admitting a good edge-labelling one can exhibit a family of dipaths on a UPP-DAG whose conflict graph is precisely this graph. Bermond et al. [3] then use the existence of graphs with a good edge-labelling and large chromatic number to prove that there exist sets of requests on UPP-DAGs with load 2 (an edge is shared by at most two paths) requiring an arbitrarily large number of wavelengths. To obtain other results on this problem, it may be useful to identify the *good* graphs, which admit a good edge-labelling, and the *bad* ones, which do not.

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2 Our results

In the following, proofs are omitted. The interested reader is referred to [1].

2.1 Bad graphs

Bermond et al.[3] noticed that C_3 and $K_{2,3}$ are bad. J.-S. Sereni [11] asked whether every $\{C_3, K_{2,3}\}$-*free graph* (i.e. with no C_3 nor $K_{2,3}$ as a subgraph) is good. We answer this question in the negative. We give an infinite family of bad graphs, none of which is the subgraph of another. The construction of this family is based on the bipartite graphs B_k, $k \geq 3$, also known as 'biwheels', of order $2k+2$, depicted below. and defined as follows :

$$\begin{aligned} V(B_k) &= \{u, v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_i \mid 1 \leq i \leq k\}, \\ E(B_k) &= \{uu_i \mid 1 \leq i \leq k\} \cup \{u_i v_i \mid 1 \leq i \leq k\} \cup \{v_i v \mid 1 \leq i \leq k\}, \\ &\quad \cup \{v_i u_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

with $u_{k+1} = u_1$.

For convenience, we denote by B_2 a path of length 2 with endvertices u and v. All the H_k share the following property: *for every good edge-labelling, H_k has either an increasing (u, v)-path or an increasing (v, u)-path.* Hence the graph $J_{i,j,k}$, $i, j, k \geq 2$, obtained from disjoint copies of B_i, B_j and B_k by identifying the vertices u of these three copies and the vertices v of these three copies, is bad, even though $\{C_3, K_{2,3}\}$-free. Indeed, any good edge-labelling would have either two increasing (u, v)-paths or two increasing (v, u)-paths, which is a contradiction.

2.2 NP-completeness

We now prove that deciding if a graph has a good edge-labelling is NP-complete. We give a reduction from the NOT-ALL-EQUAL (NAE) 3-SAT Problem [10] which is defined as follows:

Instance: A set V of variables and a collection $\mathcal{C}$ of clauses over V such that each clause has exactly 3 literals.

Question: Is there a truth assignment such that each clause has at least one true and at least one false literal?

The reduction works as follows. Let $V = \{x_1, \ldots, x_n\}$ and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be an instance I of the NAE 3-SAT Problem. We shall construct a graph G_I such a way that I has an answer yes for the NAE 3-SAT Problem if and only if G has a good edge-labelling.

For each variable x_i, $1 \leq i \leq n$, we create a variable graph VG_i defined as follows (See Figure 1.):

$$
\begin{aligned}
V(VG_i) = &\{v_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \cup \{r_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \\
&\cup \{s_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\}. \\
E(VG_i) = &\{v_k^{i,j} v_{k+1}^{i,j} \mid 1 \le j \le m, 1 \le k \le 3\} \cup \{v_4^{i,j} v_1^{i,j+1} \mid 1 \le j \le m-1\} \\
&\cup \{v_k^{i,j} r_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \cup \{v_k^{i,j} s_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \\
&\cup \{v_4^{i,j} r_1^{i,j} \mid 1 \le j \le m\} \cup \{v_k^{i,j+1} r_{k+1}^{i,j} \mid 1 \le j \le m-1, 1 \le k \le 3\} \\
&\cup \{v_4^{i,j} s_1^{i,j} \mid 1 \le j \le m\} \cup \{v_k^{i,j+1} s_{k+1}^{i,j} \mid 1 \le j \le m-1, 1 \le k \le 3\}.
\end{aligned}
$$

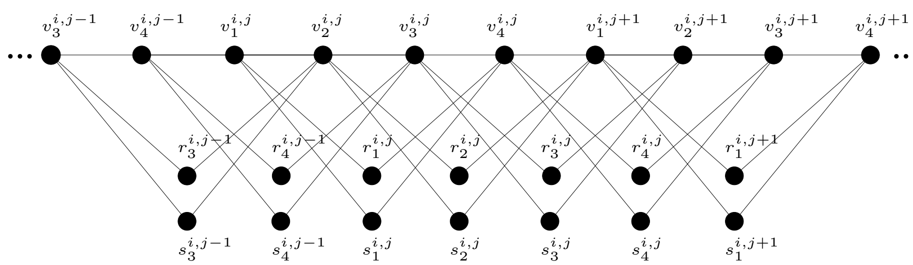

Fig. 1. The variable graph VG_i

For each clause $C_j = l_1 \vee l_2 \vee l_3$, $1 \le j \le m$, we create a clause graph CG_j defined by $V(CG_j) = \{c^j, b_1^j, b_2^j, b_3^j\}$ and $E(CG_j) = \{c^j b_1^j, c^j b_2^j, c^j b_3^j\}$

Now for each literal l_k, $1 \le k \le 3$, if l_k is the non-negated variable x_i, we identify b_k^j, c^j and b_{k+1}^j (index k is taken modulo 3) with $v_1^{i,j}$, $v_2^{i,j}$ and $v_3^{i,j}$, respectively. Otherwise, if l_k is the negated variable $\bar{x}_i$, we identify b_k^j, c^j and b_{k+1}^j with $v_3^{i,j}$, $v_2^{i,j}$ and $v_1^{i,j}$, respectively.

2.3 Good graphs

We also show large classes of good graphs. To do so, we use the notion of *critical* graph which is a bad graph, every proper subgraph of which is good. Clearly, a good edge-labelling of a graph induces a good edge-labelling of all its subgraphs. So every bad graph must contain a critical subgraph.

We establish several properties of critical graphs. Firstly, as the union of good edge-labellings of the blocks of a graph is a good edge-labelling of this graph, critical graphs are 2-connected. In particular, every forest is good.

Secondly, a critical graph has no *matching-cut*, that is an edge-cut which is a matching. Indeed, let M be such an edge cut then if $G \setminus M$ admits a good edge-labelling then assigning to all the edges of M a label greater than all the labels of $G \setminus M$ we obtain a good edge-labelling of G. As a consequence,

Theorem 2.1 *every C_3-free outerplanar graph is good.*

Indeed Eaton and Hull [5] showed that every such graph has either a 1-vertex or two adjacent 2-vertices and so has a matching-cut.

Secondly, every subcubic $\{C_3, K_{2,3}\}$-free graph has a matching cut. Indeed consider a smallest cycle C. Then the edge-cut between C and its complement is a matching. Hence,

Theorem 2.2 *every subcubic $\{C_3, K_{2,3}\}$-free graph is good.*

Farley and Proskurowski [6] showed that a graph with less than $\frac{3}{2}|V(G)|-\frac{3}{2}$ edges has a matching-cut. Hence a critical graph G has at least $\frac{3}{2}|V(G)|-\frac{3}{2}$ edges. A well known consequence of Euler's Formula is that every planar graph G of girth at least 6 (the *girth* is the length of a samllest cycle) has at most $\frac{3}{2}|V(G)|-3$ edges. Thus

Theorem 2.3 *every planar graph of girth at least* 6 *is good.*

Bonsma [4] characterized graphs with no matching-cut and $\lceil \frac{3}{2}|V(G)|-\frac{3}{2} \rceil$ edges. We use this characterization to show that a critical graph G has at least $\frac{3}{2}|V(G)|-\frac{1}{2}$ edges unless G is C_3 or $K_{2,3}$.

3 Conclusions and further research

The *average degree* of a graph G is $Ad(G)=\frac{\sum_{v\in V(G)} d(v)}{|V(G)|}=\frac{2|E(G)|}{|V(G)|}$. The above lower bound on the number of edges of a critical graphs implies that, for any $c<3/2$, the number of critical graphs with average degree at most c is finite. Actually, we conjecture that the only ones are C_3 and $K_{2,3}$.

Conjecture 3.1 *Let G be a critical graph. Then $Ad(G)\geq 3$ unless $G\in \{C_3, K_{2,3}\}$.*

More generally for any $c<4$, we conjecture the following.

Conjecture 3.2 *For any $c<4$, there exists a finite list of graphs $\mathcal{L}$ such that if G is a critical graph with $Ad(G)\leq c$ then $G\in\mathcal{L}$.*

The constant 4 in the above conjecture would be tight because the graph $J_{2,2,k}$ is critical and $Ad(J_{2,2,k})=\frac{8k+8}{2k+4}=4-\frac{4}{k+2}$.

We have shown that a graph with no dense subgraphs is good. On the opposite direction one may wonder what is the minimum density ensuring a graph to be bad. Or equivalently,

Problem 3.3 what is the maximum number $g(n)$ of edges of a good graph on n vertices?

Clearly, we have $g(n) = ex(n, \mathcal{C})$ where $\mathcal{C}$ is the set of critical graphs. As $K_{2,3}$ is critical then $g(n) \leq ex(n, K_{2,3}) = \frac{1}{\sqrt{2}} n^{3/2} + O(n^{4/3})$ by a result of Füredi [7]. The hypercubes show that $g(n) \geq \frac{1}{2} n \log n$.

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