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Backbone colouring: tree backbones with small diameter in planar graphs

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Backbone colouring: tree backbones with small diameter in planar graphs

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Abstract: Given a graph $G$ and a spanning subgraph $T$ of $G$, a backbone $k$-colouring for $(G,T)$ is a mapping $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $|c(u) - c(v)| \geq 2$ for every edge $uv \in E(T)$ and $|c(u) - c(v)| \geq 1$ for every edge $uv \in E(G) \setminus E(T)$. The backbone chromatic number $BBC(G,T)$ is the smallest integer $k$ such that there exists a backbone $k$-colouring of $(G,T)$. In 2007, Broersma et al. conjectured that $BBC(G,T) \leq 6$ for every planar graph $G$ and every spanning tree $T$ of $G$. In this paper, we prove this conjecture when $T$ has diameter at most four.

Key-words: backbone colouring, planar graph, spanning tree

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Coloration dorsale : arbres dorsaux de petit diamètre dans les graphes planaires

Résumé : Pour un graphe $G$ et un sous-graphe $T$ de $G$, une $k$-coloration dorsale de $(G,T)$ est une application $c : V(G) \rightarrow \{1, \ldots, k\}$ telle que $|c(u) - c(v)| \geq 2$ pour tout arête $uv \in E(T)$ et $|c(u) - c(v)| \geq 1$ pour toute arête $uv \in E(G) \setminus E(T)$. Le nombre chromatique dorsal $BBC(G,T)$ est le plus petit entier $k$ tel qu’il existe une $k$-coloration dorsale de $(G,T)$. En 2007, Broersma et al. [2] ont conjecturé que $BBC(G,T) \leq 6$ pour tout graphe planaire $G$ et tout arbre couvrant $T$ de $G$. Dans ce rapport, nous montrons cette conjecture lorsque $T$ est de diamètre au plus 4.

Mots-clés : coloration dorsale, graphe planaire, arbre couvrant
1 Introduction

All the graphs considered in this paper are simple. Let \( G = (V,E) \) be a graph, and let \( H = (V,E(H)) \) be a spanning subgraph of \( G \). A \( k \)-colouring of \( G \) is a mapping \( f : V \rightarrow \{1,2,\ldots,k\} \). Let \( f \) be a \( k \)-colouring of \( G \). It is a proper colouring if \( |f(u) - f(v)| \geq 1 \). It is a backbone colouring for \((G,H)\) if \( f \) is a proper colouring of \( G \) and \( |f(u) - f(v)| \geq 2 \) for all edges \( uv \in E(H) \). The chromatic number \( \chi(G) \) is the smallest integer \( k \) for which there exists a proper \( k \)-colouring of \( G \). The backbone colouring number \( BBC(G,H) \) is the smallest integer \( l \) for which there exists a backbone \( k \)-colouring of \((G,H)\).

If \( f \) is a proper \( k \)-colouring of \( G \), then \( g \) defined by \( g(v) = 2f(v) - 1 \) is a backbone \((2k-1)\)-colouring of \((G,H)\) for any spanning subgraph \( H \) of \( G \). Hence, \( BBC(G,H) \leq 2\chi(G) - 1 \). In \cite{1,2}, Broersma et al. showed that for any integer \( k \) there is a graph \( G \) with a spanning tree \( T \) such that \( BBC(G,T) = 2k - 1 \).

The above inequality and the Four Colour Theorem implies that for any planar graph \( G \) and spanning subgraph \( H \) then \( BBC(G,H) \leq 7 \). However Broersma et al. \cite{2} conjectured that this is not best possible if \( T \) is a tree.

Conjecture 1. If \( G \) is a planar graph and \( T \) a spanning tree of \( G \), then \( BBC(G,T) \leq 6 \).

If true this conjecture would be best possible. Broersma et al. \cite{2} gave an example of a graph \( G^* \) with a spanning tree \( T^* \) such that \( BBC(G^*,T^*) = 6 \). See Figure 1.

![Figure 1: A planar graph \( G^* \) with a spanning tree \( T^* \) (bold edges) such that \( BBC(G^*,T^*) = 6 \).](image)

Bu and Zhang \cite{5} proved that, if \( G \) is a connected non-bipartite \( C_4 \)-free planar graph, then there exists a spanning tree \( T \) of \( G \) such that \( BBC(G,T) = 4 \). On the other hand, Bu and Li \cite{4} proved that, if \( G \) is a connected planar graph that is \( C_6 \)-free or \( C_7 \)-free and without adjacent triangles, then there exists a spanning tree \( T \) of \( G \) such that \( BBC(G,T) \leq 4 \). In \cite{6}, Wang et al. investigated backbone colouring for special graph classes such as Halin graphs, complete graphs, wheels, graphs with small maximum average degree and graphs with maximum degree 3.

The diameter of a graph is the maximum distance between two vertices in this graph. If \( T \) has diameter 2, then it is a star, that is a tree in which a vertex \( v \), called the center, is adjacent to every other. If \( G \) has a spanning star \( T \), with center \( v \), then \( G - v \) is an outerplanar graph which can be properly 3-coloured with \( \{1,2,3\} \). Thus assigning the colour 5 to \( v \), we obtain a backbone 5-colouring of \((G,T)\). This result may be extended if \( G \) has a spanning tree with diameter at most 3.

Proposition 2. Let \( G \) be a planar graph with a spanning tree \( T \). If \( T \) has diameter at most three, then \( BBC(G,T) \leq 5 \).
Proof. Free to add some edges, we may assume that $G$ is triangulated. If $T$ has diameter at most 3, then there exists two adjacent vertices $x$ and $y$ such that all edges of $T$ are incident to $x$ or $y$. Let $z_1, \ldots, z_p$ be the common neighbours of $x$ and $y$, ordered in clockwise order around $x$ (and so in anti-clockwise order around $y$). We consider an embedding of $G$ with outer face $x_0z_1$.

For $1 \leq i \leq p - 1$, let $G_i$ be the graph induced by the vertices in the cycle $xiz_{i+1}y$ and inside, and let $H_i = G_i \setminus \{x,y\}$. Since $G$ is triangulated, all the vertices are in at least one $G_i$. Furthermore, every $H_i$ is outerplanar, and every vertex in $V(H_i) \setminus \{z_i, z_{i+1}\}$ is adjacent to exactly one of $x, y$.

We shall now define a backbone 5-colouring $c$ of $(G, T)$.

First, we set $c(x) = 1$, $c(y) = 5$ and $c(z_1) = 3$. Next, we extend this colouring to the $H_i$ one after another. Since $H_i$ is outerplanar, it is 3-colourable. Let $c_i$ be a proper 3-colouring of $H_i$ in $\{2, 3, 4\}$ such that $c_i(z_i) = c(z_i)$ and $c_i(z_{i+1}) \in \{3, 4\}$ if $z_{i+1}x \in E(T)$ and $c_i(z_{i+1}) \in \{2, 3\}$ if $z_{i+1}y \in E(T)$. We set $c(z_{i+1}) = c_i(z_{i+1})$, and for every vertex $v$ of $V(H_i) \setminus \{z_i, z_{i+1}\}$, we define
\[
\begin{align*}
&c(v) = c_i(v), \text{ if } c_i(v) = 3, \text{ or } c_i(v) = 2 \text{ and } vy \in E(T), \text{ or } c_i(v) = 4 \text{ and } vx \in E(T); \\
&c(v) = 5, \text{ if } c_i(v) = 2 \text{ and } vx \in E(T); \\
&c(v) = 1, \text{ if } c_i(v) = 4 \text{ and } vy \in E(T).
\end{align*}
\]

It is easy to check that $c$ is a backbone 5-colouring of $(G, T)$.

Remark 3. Proposition 2 is best possible, because when $G$ is a complete graph on four vertices and $T$ a spanning star of $G$, $\text{BBC}(G, T) = 5$.

In this paper, we settle Conjecture 1 for tree with diameter at most 4.

Theorem 4. Let $G$ be a planar graph with a spanning tree $T$. If $T$ has diameter at most 4, then $\text{BBC}(G, T) \leq 6$.

Note that this result is best possible as the tree $T^*$ in the above example has diameter 4.

In the next section, we outline the proof of Theorem 5 whose details are postponed to Section 5.

2 The proof

We denote by $Z_6$ the set $\{1, 2, 3, 4, 5, 6\}$ and, for any integer $a \in Z_6$, we denote by $[a]$ the set $\{a - 1, a, a + 1\} \cap Z_6$.

Let $G = (V, E)$ be a planar graph and $T$ a spanning tree of $G$ with diameter at most 4. $T$ has a vertex $r$ such that every vertex is at distance two from it in $T$. We call such a vertex the root of $T$. A vertex of $V \setminus \{r\}$, is a twig if it is adjacent to $r$ in $T$ and a leaf otherwise.

We shall prove a slightly stronger result than the one of Theorem 4.

Theorem 5. $(G, T)$ admits a backbone colouring in $Z_6$ such that the root is assigned 1.

Proof. In the remaining, by $(G, T)$-colouring, one should understand a backbone 6-colouring of $(G, T)$ such that $r$ is assigned 1.

We will prove it by considering a minimum counterexample $(G, T)$ with respect to its number of vertices. An edge of $E \setminus E(T)$ is said to be thin. Free to add some more thin edges, we may assume that $G$ is triangulated.

If $T$ has a unique twig, then it has diameter 2, and we have the result by the proof of Proposition 2 (The root corresponds to $x_1$ and the twig to $x_2$.) Hence $T$ has at least two twigs. We consider an embedding of $G$ in the plane such that the outer face contains $r$ and a minimum number of thin edges.

The interior (resp. exterior) of a cycle $C$, denoted $C^\text{int}$ (resp. $C^\text{ext}$) is the subgraph of $G$ induced by $C$ and the vertices inside $C$ (resp. outside $C$).
Let \( e \) be a thin edge. The graph \( T \cup \{e\} \) has a unique cycle \( C_e \) (which contains \( e \)). The edge \( e \) is overstepping if there is a vertex inside \( C_e \). In other words, \( V(C_e^\text{int}) \neq V(C_e) \). Let \( O \) be the set of overstepping edges. There is a partial order \( \preceq \) on \( O \) defined as follows: \( e_1 \preceq e_2 \) if \( e_1 = e_2 \) or \( e_1 \) is inside \( C_e \). Observe that the Hasse diagram of such a partial order is a set of at most two disjoint trees, each one rooted at an overstepping thin edge in the outer face. Indeed, it is easy to see that every overstepping edge \( e \) that is not maximal has a unique successor for \( \preceq \) (i.e. overstepping edge \( f \) such that \( \preceq e \preceq f \) then \( e \preceq f \)). This successor is one of the two edges of the face containing \( e \) contained in \( C_e^\text{int} \). Furthermore, every edge \( e \) has at most two predecessors for \( \preceq \): the two other edges of the face containing \( e \) contained in \( C_e^\text{int} \).

The idea of the proof is to find a “good” overstepping edge \( e \), such that a backbone 6-colouring of the graph induced by \( V(C_e^\text{int}) \) (which exists by minimality of \( (G,T) \)) can be extended to \( V(C_e^\text{int}) \) to obtain a \( (G,T) \)-colouring. This will be a contradiction.

Natural candidates for such a good edge are overstepping edges \( e \) which are minimal for \( \preceq \) (i.e. such that \( e' \preceq e \) implies \( e' = e \)) or their predecessors. However we will need to consider a more precise partial ordering. If there are two overstepping edges \( e_3 = rv_1 \) and \( e_4 = v_1v_2 \) such that \( v_1 \) and \( v_2 \) are leaves and \( e_4 \not\preceq e_3 \), (i.e. \( e_4 \) is not inside \( e_3 \)), then we would like to have \( e_3 \) smaller than \( e_4 \) in the ordering. This leads to the following partial order \( \preceq \) : \( e_1 \preceq e_2 \) if \( e_1 \preceq e_2 \) or there exist two edges \( e_3 = rv_1 \) and \( e_4 = v_1v_2 \) such that \( v_1 \) and \( v_2 \) are leaves, \( e_4 \not\preceq e_3 \), \( e_1 \preceq e_3 \) and \( e_4 \preceq e_2 \).

In the remainder of the paper, we will only consider the partial order \( \preceq \). Hence the terms minimal, predecessor, successor, and so on refer to \( \preceq \).

We first show some properties of minimal overstepping edges and deduce in Lemma 13 that if \( e \) is a minimal overstepping edge, then \( C_e^\text{int} \) is isomorphic to one of the graphs \( A_1, A_2 \) and \( A_3 \), depicted in Figure 2. In addition, if

\[
C_e^\text{int} = A_1, \text{ then } rv_1 \in E(G).
\]

As any ordering, \( \preceq \) may be decomposed into levels. The first level \( L_1 \) the maximal edges for \( \preceq \) (i.e. such that \( e \preceq e' \) implies \( e' = e \)). This level contains at most two edges, depending on the number of thin overstepping edges in the outer face. Then, for every \( j \geq 1 \), the level \( L_{j+1} \) is the set of predecessors of elements of \( L_j \). The depth of \( \preceq \), denoted \( D \), is the maximum \( j \) such that \( L_j \) is not empty. An overstepping edge of \( L_D \) is said to be ultimate. An edge of \( L_{D-2} \) having at least one (ultimate) predecessor is said to be penultimate. An edge of \( L_{D-1} \) having at least one penultimate predecessor is said to be antepenultimate.

If \( f \) is a penultimate edge, then it has one or two predecessors. Furthermore each of this predecessors \( e \) is ultimate and so minimal. Thus \( C_f^\text{int} \) is isomorphic to \( A_1, A_2 \) or \( A_3 \). Analyzing all possible cases, we show (Corollary 16) that, if \( f \) is a penultimate edge, then \( C_f^\text{int} \) is isomorphic to \( B_1 \) or \( B_2 \), and that moreover \( rv_1 \in E(G) \) and \( rv_3 \not\in E(G) \).

Now if \( g \) is an antepenultimate edge, then it has one or two predecessors. Furthermore at least one of its predecessors \( f \) is penultimate (and so \( C_f^\text{int} \) is isomorphic to \( B_1 \) or \( B_2 \)), and the other predecessor \( f' \) (if it exists) is
either penultimate (so $\int C_e$ is isomorphic to $B_1$ or $B_2$) or ultimate (so $\int C_e$ is isomorphic to $A_1$, $A_2$ or $A_3$). Analyzing all the possibles cases again, we show that there are no antepenultimate edges (Corollary 23).

Now, suppose that $G$ contains at least one overstepping edge. If $e$ is a minimal edge, then $\int C_e$ is isomorphic to $A_i$. In any of these cases, there is at least one face containing the root and only one thin edge. Therefore, the partial order considered contains a unique maximal overstepping edge $e_0$. Furthermore, since $e_0$ is not antepenultimate, $\int C_{e_0}$ must be isomorphic to one of the $A_i$ or $B_j$ configurations. We get a contradiction as the outer face contains $r$ and the endpoints of $e_0$ and $e_0$ is the unique thin edge in this configuration and $T$ would not be a tree.

We proved that $G$ contains no overstepping edge. If the outer face of $G$ contains only one thin edge, then $G$ contains three vertices and the diameter of $G$ is 2. If the outer face contains two thin edges $e_1$ and $e_2$, then one thin edge (say $e_1$) is adjacent to $r$, since $r$ is on the outer face, and the other (say $e_2$) is adjacent to a twig $t$ while both are incident to a vertex $v$ in the outer face. Now, both $r$ and $v$ have a twig $v'$ as a common neighbour through edges of $T$ as $T$ is a spanning tree. Since neither $e_1$ nor $e_2$ are overstepping, then $V(G) = \{r, t, v, v'\}$ and $G$ has diameter 3. Both of these cases are solved using Proposition and both can give colour 1 to the root, a contradiction.

3 The details

Lemma 6. Let $x$ be a vertex of $G$. If $d_T(x) = 1$ and $d_G(x) \geq 3$.

Proof. Suppose for a contradiction that $d_T(x) = 1$ and $d_G(x) \leq 3$. By minimality of $(G, T)$, there is a $(G - x, T - x)$-colouring $c$. At $x$, at most 3 colours are forbidden by its neighbour in $T$ and at most 2 colours are forbidden by its two other neighbours. So one colour of $Z_6$ is still available to colour the vertex $x$. Hence one can extend $c$ to $(G, T)$, a contradiction.

3.1 Minimal overstepping edges

Lemma 7. Let $e = uv$ be a minimal overstepping edge. Then there are at most two vertices inside $\int C_e$. Moreover if there are two, then they are adjacent in $T$ and one of them is a twig and the other is a leaf.

Proof. Since $G$ is triangulated, $uv$ is incident to two triangular faces, one of which, say $F$, is included in $\int C_e$. Let $w$ be the third vertex incident to $F$. Let $P$ be the path joining $u$ to $v$ in $T$ and $Q$ be the path joining $w$ to $P$ in $T$. Since $T$ has diameter 4 and $r$ is on the outer face, then $Q$ has length at most 2.

Then $\int C_e$ is divided into at most three regions: $F$, $\int C_{uw}$ and $\int C_{vw}$ (the region $\int C_{uw}$ or $\int C_{vw}$ may not exist if $uw \in E(T)$ or $vw \in E(T)$ respectively). As $F$ is a face, its interior is empty, and there are no vertices inside $\int C_{uw}$.
and $C_{e}^{\text{int}}$ because $uw$ and $vw$ are not overstepping since $e$ is minimal. Hence the only possible vertices inside $C_e$ are those of $Q$. Therefore there are at most two vertices inside $C_e$ as $Q$ has length at most 2.

Furthermore, if there are two vertices inside $C_e$, they must be adjacent as they are in $Q$. In addition, since $r$ is on the outer face, none of these vertices is the root and thus one of them is a twig and the other is a leaf. \qedhere

Lemma 8. No minimal overstepping edge joins two leaves adjacent to a same twig.

Proof. Suppose for a contradiction that an edge $e = uv$ joins two leaves adjacent to a same twig $t$. Then $C_e = tuvt$. The root $r$ is not in $C_e^{\text{int}}$ as it is on the outer face. So by Lemma 7 and because $G$ is triangulated, $C_e^{\text{int}}$ is a $K_4$ and there is a unique vertex $x$ inside $C_e$. Hence, $x$ contradicts Lemma 6. \qedhere

Lemma 9. No minimal overstepping edge joins two twigs.

Proof. Suppose for a contradiction that two twigs $s$ and $t$ are joined by a minimal edge $e$. Then $C_e = rstr$. If there is a unique vertex $u$ inside $C_e^{\text{int}}$, then $u$ contradicts Lemma 6. So by Lemma 7, we may assume that the interior of $C_e$ contains two adjacent vertices $u_1$ and $u_2$ and that $u_1$ is a twig and $u_2$ a leaf. By minimality of $(G,T)$, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$. Set $c(u_2) = 2$ and choose $c(u_1) \in Z_6 \setminus \{1, 2, 3, c(s), c(t)\}$. This yields a $(G,T)$-colouring, a contradiction. \qedhere

Lemma 10. No minimal overstepping edge joins the root and a leaf.

Proof. Suppose for a contradiction that a minimal edge $e$ joins the root $r$ and a leaf $v$. Let $t$ be the twig adjacent to $v$.

Suppose there is a unique vertex $u$ inside $C_e$. Then this vertex has only 3 neighbours, and $d_T(u) = 1$. This contradicts Lemma 6. Hence by Lemma 7, we may assume that there are two adjacent vertices $u_1$ and $u_2$ inside $C_e$. Without loss of generality, $u_2$ is a leaf and $u_1$ is a twig. By Lemma 6, $d_G(u_2) \geq 4$, so $N_G(u_2) = \{u_1, r, v, t\}$. By minimality of $G$, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$. Let $c(u_2)$ be a colour in $\{2, 3\} \setminus \{c(v), c(t)\}$. (Such a colour exists because $|c(v) - c(t)| \geq 2$.) Now by planarity, $u_1$ has at most one neighbour $x$ in $\{v, t\}$ as $ru_2$ is an edge. The set of forbidden colours in $u_1$ is $I = [1] \cup \{c(u_2)\} \cup \{c(x)\}$ which has cardinality at most 5 by the choice of $c(u_2)$. Hence assigning to $u_1$ a colour $c(u_1) \in Z_6 \setminus I$, we obtain a $(G,T)$-colouring, a contradiction. \qedhere

Lemma 11. No minimal overstepping edge joins a leaf and a twig.

Proof. Suppose for a contradiction that a minimal overstepping edge $e = sv$ joins a twig $s$ and a leaf $v$. Then $C_e = svtrs$. By Lemma 7, there are at most two vertices inside $C_e$.

Suppose that there is a unique vertex $u$ inside $C_e$. As $d_T(u) = 1$, by Lemma 6, $d_G(u) \geq 4$. So $N_G(u) = \{r, s, t, v\}$. Note that $rv$ or $st$ is not an edge, by planarity. Then, removing $u$ and contracting $rv$ or $st$, we find by the minimality of $G$ a $(G - u, T - u)$-colouring $c$ such that $c(v) = 1$ or $c(s) = c(t)$. Since the set of forbidden colours for $u$ has at most 5 colours, one can extend $c$ into a $(G,T)$-colouring, a contradiction.

Hence by Lemma 7, inside $C_e$ there are a twig $u_1$ and leaf $u_2$ which are adjacent in $T$. As $d_T(u_2) = 1$, $d_G(u_2) \geq 4$ by Lemma 6.

- Suppose first that $r$ is not adjacent to $u_2$. By Lemma 6, $d_G(u_2) \geq 4$. So $N_G(u_2) = \{u_1, s, t, v\}$.

Hence $u_1$ is not adjacent to $v$ by planarity. By minimality of $(G,T)$, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$. Assign to $u_2$ a colour $c(u_2)$ in $\{1, 2\} \setminus \{c(v)\}$. Observe that it is valid since $s$ and $t$ are not coloured in $\{1, 2\}$. Then the set of forbidden colours in $u_1$ is included in $\{1, 2, 3, c(s), c(t)\}$ and so has cardinality at most 5. Hence one can extend $c$ into a $(G,T)$-colouring contradiction.
• Suppose now that \( r \) is adjacent to \( u_2 \).

By planarity, \( u_1 \) is adjacent to at most one vertex \( w \) in \( \{s,t\} \). By minimality of \((G,T)\), there is a \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\)-colouring \( c \).

If \( c(v) \neq 2 \), then set \( c(u_2) = 2 \). This it is valid since \( s \) and \( t \) are not coloured 2. Then the set of forbidden colours in \( u_1 \) is included in \( \{1, 2, 3, c(v), c(w)\} \) and so has cardinality at most 5. Hence one can extend \( c \) into a \((G,T)\)-colouring a contradiction. Hence we may assume that \( c(v) = 2 \).

If no neighbour of \( u_2 \) is coloured 6, then set \( c(u_2) = 6 \). The set of forbidden colours in \( u_1 \) is then \( \{1, 2, 5, 6, c(w)\} \) and so one can extend \( c \) into a \((G,T)\)-colouring a contradiction. Hence we may assume that a neighbour \( y \) of \( u_2 \) is coloured 6.

If no neighbour of \( u_2 \) is coloured 3, then set \( c(u_2) = 3 \). The set of forbidden colours in \( u_1 \) is then \( \{1, 2, 3, 4, c(w)\} \) and so one can extend \( c \) into a \((G,T)\)-colouring a contradiction. Hence we may assume that a neighbour \( y \) of \( u_2 \) is coloured 3. But this neighbour cannot be \( t \) since \( c(v) = 2 \). Thus \( c(s) = 3 \) and \( c(t) = 6 \).

If \( w = s \), that is if \( u_1 \) is not adjacent to \( t \), then setting \( c(u_1) = 6 \) and \( c(u_2) = 4 \) yields a \((G,T)\)-colouring, a contradiction.

If \( w = t \), then setting \( c(u_1) = 3 \) and \( c(u_2) = 5 \) yields a \((G,T)\)-colouring, a contradiction.

\[ \square \]

**Lemma 12.** If \( e \) is a minimal overstepping edge joining two leaves, then there is one vertex inside \( C_e \).

**Proof.** Let \( e = v_1 v_2 \) and for \( i = 1, 2 \), let \( t_i \) be the twig adjacent to \( v_i \). By Lemma 8, \( t_1 \neq t_2 \). Since \( e \) is minimal and \( G \) is triangulated, \( u_2 v_1, u_2 v_2 \in E(G) \).

Suppose for a contradiction that more than one vertex is inside \( C_e \). Then, by Lemma 7, inside \( C_e \), there are a twig \( u_1 \) and a leaf \( u_2 \) which are adjacent in \( T \). Moreover, by Lemma 6, \( d_G(u_2) \geq 4 \) and so \( d_G(u_1) \leq 5 \).

Let us first suppose that \( r u_2 \) is not an edge. By symmetry, we may assume that \( u_1 v_1 \) is not an edge. Set \( G' = (G - \{u_1, u_2\}) \cup \{r_1 v_1, r_2 v_2\} \). By minimality of \((G,T)\), there is a \((G', T - \{u_1, u_2\})\)-colouring, which is a \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\)-colouring \( c \) such that \( c(v_1) \neq 1 \) and \( c(v_2) \neq 1 \). Then setting \( c(u_2) = 1 \) and colouring \( u_1 \) with a colour in \( Z_6 \setminus \{1, 2, c(t_1), c(t_2), c(v_2)\} \), we obtain a \((G,T)\)-colouring, a contradiction. Hence we may assume that \( r u_2 \in E(G) \). Then, since \( e \) is minimal, \( u_1 v_1 \) is not an edge. By symmetry, we may assume that \( r u_2 \) is inside the cycle \( r_1 v_1 u_2 u_1 r \). Thus \( N(u_1) \subset \{r_1, v_2, u_2\} \).

Assume now that \( r v_1 \) is not an edge. Let \((G', T')\) be the graph pair obtained from \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\) by identifying \( r \) and \( v_1 \). By minimality of \((G, T)\), there is a \((G', T')\)-colouring which is a \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\)-colouring \( c \) such that \( c(v_1) = c(r) = 1 \). If \( c(v_2) \neq 2 \), then setting \( c(u_2) = 2 \) and colouring \( u_1 \) with a colour in \( Z_6 \setminus \{1, 2, 3, c(t_2), c(v_2)\} \), we obtain a \((G, T)\)-colouring, a contradiction. If \( c(v_2) = 2 \), then \( c(t_2) \geq 4 \). If \( c(t_1) \neq 3 \), then colour \( u_2 \) with 3 and \( u_1 \) with some colour in \( \{5, 6\} \setminus \{c(t_2)\} \); otherwise, colour \( u_1 \) with 3 and \( u_2 \) with a colour in \( \{5, 6\} \setminus \{c(t_2)\} \). In both cases, we obtain a \((G, T)\)-colouring, a contradiction. Hence we may assume that \( r v_1 \in E(G) \).

Assume that \( r v_2 \) is not an edge. Let \((G', T')\) be the graph pair obtained from \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\) by identifying \( r \) and \( v_2 \). By minimality of \((G, T)\), there is a \((G', T')\)-colouring which is a \((G - \{u_1, u_2\}, T - \{u_1, u_2\})\)-colouring \( c \) such that \( c(v_2) = c(r) = 1 \). If there is a colour \( \alpha \in \{2, 3, 6\} \) which does not appear on the neighbourhood of \( u_2 \), then setting \( c(u_2) = \alpha \) and colouring \( u_1 \) with a colour in \( Z_6 \setminus \{\{1, 2, c(t_2)\}\} \cup [\alpha] \), we obtain a \((G, T)\)-colouring, a contradiction. So all the colours of \( \{2, 3, 6\} \) appear on the neighbourhood of \( u_2 \). Necessarily, in this case, \( u_2 \) is adjacent to \( v_1 \), \( t_1 \) and \( t_2 \) and \( c(v_1) = 2 \), \( c(t_1) = 6 \) and \( c(t_2) = 3 \). Then setting \( c(u_2) = 4 \) and \( c(u_1) = 6 \), we obtain a \((G, T)\)-colouring, a contradiction. Hence we may assume that \( r v_2 \in E(G) \).

We now distinguish several cases depending on the position of \( r v_1 \) and \( r v_2 \) regarding \( C_e \).
1. Assume first that $rv_1$ and $rv_2$ are in $C^*_{e^1}$. Then $t_1t_2$ is not an edge by planarity. Let $(G', T')$ be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying $t_1$ and $t_2$. By minimality of $(G, T)$, there is a $(G', T')$-colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$ such that $c(t_1) = c(t_2) = \alpha$. If $2 \not\in \{c(v_1), c(v_2)\}$, then setting $c(u_2) = 2$ and colouring $u_1$ with a colour in $Z_6 \setminus \{1, 2, 3, \alpha, c(v_2)\}$, we obtain a $(G, T)$-colouring, a contradiction. Hence $2 \not\in \{c(v_1), c(v_2)\}$, so $\alpha \geq 4$. If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then setting $c(u_2) = 3$ and colouring $u_1$ with a colour in $\{5, 6\} \setminus \{\alpha, c(v_2)\}$, we obtain a $(G, T)$-colouring, a contradiction. Hence $\{c(v_1), c(v_2)\} = \{2, 3\}$, so $\alpha \geq 5$. If $c(v_2) \neq 3$ or $u_1v_2 \notin E(G)$, then setting $c(u_1) = 3$ and colouring $u_2$ with a colour in $\{5, 6\} \setminus \{\alpha\}$, we obtain a $(G, T)$-colouring, a contradiction. Hence $c(v_2) = 3$ and $u_1v_2 \in E(G)$. By planarity, this implies that $n_2t_2$ is not an edge.

Observe that at least one of the two edges $rv_1$ and $rv_2$ is not overstepping otherwise one of them would be smaller than $e$ in the order $\preceq$.

If $rv_1$ is not overstepping, then the interior of $rt_1v_1$ is empty. Hence $N_G(t_1) = \{rv, u_1, u_2\}$. Setting $c(u_1) = 4$, $c(u_2) = 6$ and recolouring $t_1$ with $5$, we obtain a $(G,T)$-colouring, a contradiction.

If $rv_2$ is not overstepping, then the interior of $rt_2v_2$ is empty. Hence $N_G(t_2) = \{rv, u_1, u_2\}$. Setting $c(u_1) = 6$, $c(u_2) = 4$ and recolouring $t_2$ with $5$, we obtain a $(G,T)$-colouring, a contradiction.

2. Assume that $rv_1$ and $rv_2$ are in $C^*_{e^1}$. Then $N_G(u_1) = \{rv, u_2, v_2\}$. By minimality of $(G, T)$, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$. Colour $u_2$ with a colour $c(u_2)$ in $\{2, 3, 6\} \setminus \{c(v_1), c(v_2)\}$. Then the set of forbidden colours in $u_1$ is $\{1, 2, c(v_2)\} \cup \{c(u_2)\}$ which has cardinality at most 5 because $\{1, 2\} \cup \{c(u_2)\}$ has cardinality at most 4. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction.

3. Assume that $rv_1$ is in $C^*_{e^1}$ and $rv_2$ is in $C^*_{e^2}$. Assume that $d_G(u_2) = 5$, so $N_G(u_2) = \{rv, u_1, v_1, v_2, t_2\}$ and $N_G(u_1) = \{rv, t_2, u_2\}$. By minimality of $(G, T)$, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$. If one can colour $u_2$ with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup \{c(u_2)\}$ has cardinality at most 4 and so at most 5 colours are forbidden for $u_1$. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction. So we may assume $\{c(t_2), c(v_1), c(v_2)\} = \{2, 3, 6\}$.

If $c(t_2) = 6$, then setting $c(u_1) = 3$ and $c(u_2) = 5$, we obtain a $(G,T)$-colouring, a contradiction. If $c(t_2) \neq 6$, then setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a $(G,T)$-colouring, a contradiction.

Henceforth we may assume that $d_G(u_2) = 4$, so $N_G(u_2) = \{rv, u_1, v_1, v_2\}$ and $N_G(u_1) = \{rv, t_2, u_2, v_2\}$.

If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then one can colour $u_2$ with a colour in $\{2, 3\}$ and $u_1$ with a colour in $\{5, 6\} \setminus \{c(t_2), c(v_2)\}$ to obtain a $(G,T)$-colouring, a contradiction.

If $\{c(v_1), c(v_2)\} = \{2, 3\}$, then colouring $u_1$ with a colour $c(u_1)$ in $\{4, 6\} \setminus \{c(t_2)\}$ and $u_2$ with the colour in $\{4, 6\} \setminus \{c(u_1)\}$, we obtain a $(G,T)$-colouring, a contradiction.

4. Assume $rv_2$ is in $C^*_{e^1}$ and $rv_1$ is in $C^*_{e^2}$. Then $N_G(u_1) = \{rv, u_2, v_2\}$ and $N_G(u_2) = \{rv, u_1, t_1, v_1, v_2\}$. By minimality of $(G, T)$, there exists a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring $c$.

If $c(v_2) = 2$, then colouring $u_2$ with a colour $c(u_2)$ in $Z_6 \setminus \{1, 2, c(t_1), c(v_1)\}$ and $u_1$ with a colour in $\{3, 4, 5, 6\} \setminus \{c(u_2)\}$, we obtain a $(G, T)$-colouring, a contradiction. So we may assume that $c(v_2) \neq 2$.

If one can colour $u_2$ with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup \{c(u_2)\}$ has cardinality at most 4 and so at most 5 colours are forbidden for $u_1$. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction.

So we may assume $\{c(t_1), c(v_1), c(v_2)\} = \{2, 3, 6\}$. Necessarily, $c(v_1) = 2$, $c(v_2) = 3$ and $c(t_1) = 6$. Setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a $(G, T)$-colouring, a contradiction.

\[ \square \]
Lemma 13. If $e$ is a minimal overstepping edge, then $C_e^{int}$ is one of the graphs depicted in Figure 2. In addition, if $C_e^{int} = A_1$, then $rv_1 \notin E(G)$.

Proof. Let $e$ be a minimal edge. According to the previous lemmas, it has to join two leaves $v_1$ and $v_2$ and there is a unique vertex $u$ inside $C_e$. For $i = 1, 2$, let $t_i$ be the twig adjacent to $v_i$. By Lemma 8, $t_1 \neq t_2$.

- Assume first that $u$ is a twig.

  If $d_G(u) \leq 4$, then consider a $(G - u, T - u)$-colouring $c$, which exists by minimality of $(G, T)$. In $u$, there are at most 5 colours forbidden as $r$ is coloured 1, and thus forbids only two colours. Hence, one can extend $c$ into a $(G, T)$-colouring, a contradiction.

  So we may assume that $d_G(u) \geq 5$, and thus $N_G(u) = \{r, t_1, t_2, v_1, v_2\}$.

  If $rv_1$ is not an edge, then let $(G', T')$ be the pair obtained from $(G - u, T - u)$ by identifying $r$ and $v_1$. By minimality of $(G, T)$, there is a $(G', T')$-colouring, which is a $(G - u, T - u)$-colouring such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in $u$ is included in $\{1, 2, c(t_1), c(t_2), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction.

  Hence we may assume that $rv_1$ is an edge. This edge must be in $C_e^{int}$ by planarity of $G$. Thus $t_1 t_2$ is not an edge of $G$. Let $(G', T')$ be the pair obtained from $(G - u, T - u)$ by identifying $t_1$ and $t_2$. By minimality of $(G, T)$, there is a $(G', T')$-colouring $c$ which is a $(G - u, T - u)$-colouring such that $c(t_1) = c(t_2)$. Then the set of forbidden colours in $u$ is included in $\{1, 2, c(t_1), c(v_1), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction.

- Assume now that $u$ is a leaf. By symmetry, we may assume that $u$ is adjacent to $t_1$. By Lemma 6 and since $G$ is triangulated, $C_e^{int}$ is one of the graphs $A_1, A_2$ or $A_3$.

  Assume now that $C_e^{int} = A_1$ and $rv_1 \notin E(G)$. Let $(G', T')$ be the pair obtained from $(G - u, T - u)$ by identifying $r$ and $v_1$. By minimality of $(G, T)$, there is a $(G', T')$-colouring which is a $(G - u, T - u)$-colouring $c$ such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in $u$ is included in $\{1, c(v_2)\} \cup [c(t_1)]$ and so has cardinality at most 5. Hence one can extend $c$ into a $(G, T)$-colouring, a contradiction.

\[\square\]

3.2 Penultimate edges

Lemma 14. Let $f$ be an edge which is the successor of a minimal edge $e$. If $e$ is the unique predecessor of $f$, then $C_f^{int}$ is one of the graphs depicted in Figure 3 and $rv_1 \in E(G)$. Moreover, if $C_f^{int} = B_2$, $rv_3 \notin E(G)$.

Proof. Let $e'$ be the third edge of the triangle bounded by $f$ and $e$ in $C_f^{int}$. Suppose, by way of contradiction, that $e$ is the unique predecessor of $f$. Then $e'$ is not overstepping. So all the vertices inside $C_f$ are in $C_e^{int}$. By Lemma 13, $C_e^{int}$ is one of the graphs $A_1, A_2$ or $A_3$.

One of the endvertices of $f$ must be $v_1$ and $v_2$ (as defined for $A_i$). We now distinguish many cases depending on $C_e^{int}$ and the possible endvertices of $f$.

1. Assume that $C_e^{int}$ is $A_1$.

1.1. Assume $f = rv_1$. Then the 4-cycle $rt_2 v_2 v_1$ has no chord, because $rv_2$ is in $C_e^{int}$ and $v_1 t_2$ is not an edge since $f$ is the successor of $e$. This contradicts the fact that $G$ is triangulated.

1.2. Observe that $f = t_1 v_2$ is impossible since $rv_1$ is an edge. Assume that $f = t_2 v_1$. Let $G' = (G \setminus \{u, v_2\}) \cup t_1 t_2$. By minimality of $(G, T)$, there exists a $(G', T \setminus \{u, v_2\})$-colouring which is a $(G \setminus \{u, v_2\}, T \setminus \{u, v_2\})$-colouring $c$ such that $c(t_1) \neq c(t_2)$. If $c(t_1) = 6$, then one can greedily extend $c$ to $v_2$ and then $u$ to get a $(G, T)$-colouring, a contradiction. If $c(t_1) \neq 6$, then colouring $v_2$ with a colour in $\{c(t_1) -
1. Assume that \( f = v_1 t_3 \) with \( t_3 \) a twig distinct from \( t_2 \). Since \( rv_1 \) is an edge, \( t_1 t_3 \) is not an edge. Let \( G' \) be the graph pair obtained from \((G - \{t_2, v_2 \}, T - \{t_2, v_2 \})\). If one can colour \( v_2 \) with a colour \( c(v_2) \) in \( \{2, 3, 6\} \), then \( \{1, 2\} \cup \{c(v_2)\} \) has cardinality at most 4 and so at most 5 colours are forbidden in \( t_2 \). Hence one can extend \( c \) into a \((G, T)\)-colouring , a contradiction. So we may assume that \( \{c(u), c(v_1), c(t_3) = \{2, 3, 6\} \). If \( c(t_3) = 3 \), set \( c(v_2) = 4 \) and \( c(t_2) = 6 \). If \( c(t_3) = 6 \), set \( c(v_2) = 5 \) and \( c(t_2) = 3 \). In both cases, we obtain a \((G, T)\)-colouring, a contradiction.

1.3. Assume that \( f = v_1 t_3 \) with \( t_3 \) a twig distinct from \( t_2 \). Since \( rv_1 \) is an edge, \( t_1 t_3 \) is not an edge. Let \( G' \) be the graph pair obtained from \((G - \{t_2, v_2 \}, T - \{t_2, v_2 \})\). If one can colour \( v_2 \) with a colour \( c(v_2) \) in \( \{2, 3, 6\} \), then \( \{1, 2\} \cup \{c(v_2)\} \) has cardinality at most 4 and so at most 5 colours are forbidden in \( t_2 \). Hence one can extend \( c \) into a \((G, T)\)-colouring , a contradiction. So we may assume that \( \{c(u), c(v_1), c(t_3) = \{2, 3, 6\} \). If \( c(t_3) = 3 \), set \( c(v_2) = 4 \) and \( c(t_2) = 6 \). If \( c(t_3) = 6 \), set \( c(v_2) = 5 \) and \( c(t_2) = 3 \). In both cases, we obtain a \((G, T)\)-colouring, a contradiction.

1.4. Assume that \( f = v_2 t_3 \) with \( t_3 \) a twig distinct from \( t_1 \). By minimality of \((G, T)\), there exists a \((G - \{u, t_1, v_1 \}, T - \{u, t_1, v_1 \})\)-colouring \( c \). Setting \( c(t_1) = 6 \) and choosing \( c(v_1) \) in \( Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\} \) and \( c(u) \) in \( Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\} \), we get a \((G, T)\)-colouring, a contradiction.

1.5. \( f \) cannot be \( v_2 t_3 \) with \( v_1 \) a leaf adjacent to \( t_1 \) because \( rv_1 \) is an edge.

1.6 Assume that \( f = v_1 t_3 \) with \( v_3 \) a leaf adjacent to \( t_2 \). Then \( C_{v_3} = B_{v_1} \). By Lemma[13] \( rv_1 \in E(G) \).

1.7. Assume \( f = v_2 t_3 \) with \( v_3 \) a leaf adjacent in \( T \) to a twig \( t_3 \) not in \( \{t_1, t_2\} \). Then \( v_1 v_3 \in E(G) \) and either \( rv_3 \in E(G) \) or \( t_1 v_1 \in E(G) \). Since \( rv_1 \) is an edge, we have that \( N(t_1) = \{r, u, v_1\} \). By minimality of \( G \), there exists a \((G - \{u, t_1, v_1 \}, T - \{u, t_1, v_1 \})\)-colouring \( c \).

If \( \{c(t_3), c(v_3), c(v_2)\} \neq \{2, 3, 4\} \), then setting \( c(t_1) = 6 \) and choosing \( c(v_1) \) in \( \{2, 3, 4\} \setminus \{c(t_3), c(v_3), c(v_2)\} \) and \( c(u) \) in \( \{2, 3, 4\} \setminus \{c(v_1), c(v_2)\} \), we obtain a \((G, T)\)-colouring, a contradiction. Hence \( \{c(t_3), c(v_3), c(v_2)\} = \{2, 3, 4\} \), and so \( c(t_3) = 4, c(v_3) = 2 \) and \( c(v_2) = 3 \). Then setting \( c(t_1) = 3 \), \( c(u) = 5 \) and \( c(v_1) = 6 \) yields a \((G, T)\)-colouring, a contradiction.

1.8. Assume \( f = v_1 t_3 \) with \( v_3 \) a leaf adjacent in \( T \) to a twig \( t_3 \) not in \( \{t_1, t_2\} \). Since \( rv_1 \in E(G) \), then \( t_1 t_3 \notin E(G) \).

Assume first that \( rv_3 \in C_{v_3} \). By minimality of \((G, T)\), there exists a \((G - \{t_2, v_2 \}, T - \{t_2, v_2 \})\)-colouring \( c \). One can choose a colour \( c(v_2) \) in \( Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\} \) such that \( I = \{c(v_2)\} \cup \{1, 2, c(v_3)\} \neq Z_6 \). Then choosing \( c(t_3) \in Z_6 \setminus I \), we obtain a \((G, T)\)-colouring, a contradiction.

Hence we may assume that \( rv_3 \) is not in \( C_{v_3} \). Let \((G', T')\) be the graph obtained from \((G - \{u, t_2, v_2 \}, T - \{u, t_2, v_2 \})\) by identifying \( t_1 \) and \( t_3 \). By minimality of \((G, T)\), there exists a \((G', T')\)-colouring which is a \((G - \{u, t_2, v_2 \}, T - \{u, t_2, v_2 \})\)-colouring \( c \) such that \( c(t_1) = c(t_3) \). If \( c(t_1) \neq 6 \), then one can choose a colour \( c(v_2) \in \{c(t_1) - 1, c(t_1) + 1\} \) such that \( I = \{c(v_2)\} \cup \{1, 2, c(v_3)\} \neq Z_6 \). Then choosing \( c(t_3) \in Z_6 \setminus I \) and \( c(u) \) in \( Z_6 \setminus \{c(t_1)\} \cup \{1, c(v_1)\} \), we obtain a \((G, T)\)-colouring, a contradiction. Hence we may suppose that \( c(t_1) = 6 \). If \( v_3 \notin E(G) \), then setting \( c(v_2) = 6 \) and choosing \( c(t_2) \in \{3, 4\} \setminus \{c(v_3) \) and \( c(u) \) in \( Z_6 \setminus \{1, 5, 6, c(v_1)\} \) yields a \((G, T)\)-colouring, a contradiction. If \( v_3 \in E(G) \), then setting \( c(v_2) = 5 \), \( c(t_3) = 3 \) and choosing \( c(u) \) in \( Z_6 \setminus \{1, 5, 6, c(v_1)\} \) yields a \((G, T)\)-colouring, a contradiction.

2. Assume that \( C_{v_3} \) is \( A_2 \).

2.1. Assume \( f = rv_1 \). Since \( f \) is the successor of \( e \), then \( v_3 \notin E(G) \) because \( G \) is triangulated. By minimality of \( G \), there exists a \((G - \{u, t_2, v_2 \}, T - \{u, t_2, v_2 \})\)-colouring \( c \). Setting \( c(u) = 1 \), one can then extend \( c \) greedily to \( t_2 \) and \( v_2 \) to get a \((G, T)\)-colouring, a contradiction.

2.2. Assume that \( f = rv_2 \). By minimality of \((G, T)\), there is a \((G - \{u, t_1, v_1 \}, T - \{u, t_1, v_1 \})\)-colouring.

Setting \( c(u) = 1 \), one can then extend \( c \) greedily to \( t_1 \) and \( v_1 \) to get a \((G, T)\)-colouring, a contradiction.

2.3 Assume that \( f = t_1 v_2 \). Since \( f \) is the successor of \( e \), the cycle \( t_1 v_1 v_2 \) is empty, so \( v_1 \) contradicts Lemma[6]. Similarly, if \( f = t_2 v_1 \), then \( v_2 \) contradicts Lemma[6].
2.4. Assume that \( f = v_1t_3 \) with \( t_3 \) a twig distinct from \( t_2 \). Since \( f \) is the successor of \( e \), \( t_2v_1 \) is not an edge. Then either \( rv_2 \) is an edge or \( rt_2v_1 \) is an edge. Set \( G' = (G - \{u, t_2, v_2\}) \cup rv_1 \). By minimality of \( (G, T) \), there is a \( (G', T - \{u, t_2, v_2\}) \)-colouring \( c \) which is a \( (G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\}) \)-colouring \( c \) such that \( c(v_1) \neq c(r) = 1 \). Set \( c(u) = 1 \).

If \( c(v_1) \neq 2 \), then setting \( c(v_2) = 2 \) and colouring \( t_2 \) with a colour in \( Z_6 \setminus \{1, 2, 3, c(t_1), c(t_3)\} \), we obtain a \( (G, T) \)-colouring, a contradiction. So \( c(v_1) = 2 \) and thus \( c(t_1) \geq 4 \).

If \( c(t_3) \neq 3 \), then setting \( c(t_2) = 3 \) and choosing \( c(v_2) \) in \( \{5, 6\} \setminus \{c(t_3)\} \), we obtain a \( (G, T) \)-colouring, a contradiction. So \( c(t_3) = 3 \).

Choosing \( c(t_2) \) in \( \{4, 6\} \setminus \{c(t_1)\} \) and \( c(v_2) \) in \( \{4, 6\} \setminus \{c(t_2)\} \), we obtain a \( (G, T) \)-colouring, a contradiction.

2.5. Assume that \( f = v_2t_3 \) with \( t_3 \) a twig distinct from \( t_1 \). Then either \( rv_1 \) is an edge or \( t_1v_3 \) is an edge. Set \( G' = (G - \{u, t_1, v_1\}) \cup rv_2 \). By minimality of \( (G, T) \), there exists a \( (G', T - \{u, t_1, v_1\}) \)-colouring which is a \( (G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\}) \)-colouring \( c \) such that \( c(v_2) \neq c(r) = 1 \). Set \( c(u) = 1 \).

If \( c(v_2) \neq 2 \), then setting \( c(v_1) = 2 \) and colouring \( t_1 \) with a colour in \( Z_6 \setminus \{1, 2, 3, c(t_2), c(t_3)\} \), we obtain a \( (G, T) \)-colouring, a contradiction. So \( c(v_2) = 2 \) and thus \( c(t_2) \geq 4 \).

If \( c(t_3) \neq 3 \), then setting \( c(t_1) = 3 \) and choosing \( c(v_1) \) in \( \{5, 6\} \setminus \{c(t_3)\} \), we obtain a \( (G, T) \)-colouring, a contradiction. So \( c(t_3) = 3 \).

Choosing \( c(t_1) \) in \( \{4, 6\} \setminus \{c(t_2)\} \) and \( c(v_1) \) in \( \{4, 6\} \setminus \{c(t_1)\} \), we obtain a \( (G, T) \)-colouring, a contradiction.

2.6. Assume that \( f = v_2v_3 \) with \( v_3 \) a leaf adjacent to \( t_1 \). Since \( f \) is the successor of \( e \), then \( t_1v_2 \) is not inside \( v_3t_1v_1v_2 \) and so \( v_1v_2 \in E(G) \). Set \( G' = (G - \{u, v_1\}) \cup t_2v_3 \). By minimality of \( (G, T) \), there is a \( (G', T - \{u, v_1\}) \)-colouring which is a \( (G - \{u, v_1\}, T - \{u, v_1\}) \)-colouring \( c \) such that \( c(t_1) \neq c(v_3) \). Setting \( c(u) = c(v_3) \) and colouring \( v_1 \) with a colour in \( Z_6 \setminus \{c(u), c(v_2) \} \), we obtain a \( (G, T) \)-colouring, a contradiction.

2.7. Assume that \( f = v_1v_3 \) with \( v_3 \) a leaf adjacent to \( t_2 \). Since \( f \) is the successor of \( e \), then \( t_2v_1 \) is not inside \( v_2t_3v_1v_3 \) and so \( v_2v_3 \in E(G) \). Set \( G' = (G - \{u, v_2\}) \cup t_1v_3 \). By minimality of \( (G, T) \), there is a \( (G', T - \{u, v_2\}) \)-colouring which is a \( (G - \{u, v_2\}, T - \{u, v_2\}) \)-colouring \( c \) such that \( c(t_1) \neq c(v_3) \). If \( c(t_2) \in \{c(t_1)\} \), then one can also extend \( c \) greedily to \( v_2 \) and then \( u \) to obtain a \( (G, T) \)-colouring, a contradiction. Hence \( |c(t_1) - c(t_2)| \geq 2 \). Thus one can colour \( v_2 \) with \( c(t_1) \) and then colour \( u \) with a colour in \( Z_6 \setminus \{c(t_1), c(t_2), c(v_1)\} \). This yields a \( (G, T) \)-colouring, a contradiction.

2.8. Assume \( f = v_2v_3 \) with \( v_3 \) a leaf adjacent in \( T \) to a twig \( t_3 \) not in \( \{t_1, t_2\} \).

Suppose first that \( rv_1 \notin E(G) \). By minimality of \( (G, T) \), there is a \( (G - \{u, t_1, v_1\} \cup rv_3, rv_2, T - \{u, t_1, v_1\}) \)-colouring which is a \( (G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\}) \)-colouring \( c \) such that \( c(v_2) \neq 1 \) and \( c(v_3) \neq 1 \). Colour \( v_1 \) with \( 1 \) and let \( L(t_1) \supseteq Z_6 \setminus \{1, 2, 3, c(v_3), c(t_1), c(t_2)\} \) and \( L(u) = Z_6 \setminus \{1, c(t_2), c(v_2)\} \) be the list of colours available for \( t_1 \) and \( u \), respectively. Note that there is at most one colour \( \alpha \) in \( Z_6 \) such that \( L(u) \setminus \{\alpha\} = \emptyset \). Thus, if there exists \( \beta \in L(t_1) \setminus \{\alpha\} \) if such \( \alpha \) exists, or in \( L(t_1) \) otherwise, then we can colour \( t_1 \) with \( \beta \) and \( u \) with a colour in \( L(u) \setminus \{\beta\} \) to obtain a \( (G, T) \)-colouring, a contradiction. So we may assume that no such \( \beta \) exists, that is \( L(t_1) = \{\alpha\} \) and \( L(u) \setminus \{\alpha\} = \emptyset \). Since \( |c(v_2) - c(t_2)| \geq 2 \), necessarily \( \alpha = 4 \), \( L(t_1) = \{4\} \), \( c(t_2) = 6 \), \( c(v_2) = 2 \), \( \{c(v_3), c(t_1)\} = \{3, 5\} \) and \( v_1, t_3 \in N(t_1) \). Then, recolouring \( v_1 \) with \( 6 \) and colouring \( t_1 \) with \( 4 \) and \( u \) with \( 1 \) yields a \( (G, T) \)-colouring, a contradiction.

Suppose now that \( rv_1 \in E(G) \). Then there is no vertex inside \( rt_1v_1r \). By minimality of \( (G, T) \), there is \( (G - u, T - u) \)-colouring \( c \). If \( c(v_2) \neq 1 \), then we can colour \( u \) with \( 1 \); so, suppose otherwise. If there is no colour available for \( u \) to extend \( c \), then \( F_1 = \{1, c(t_2), c(v_1)\} \cup \{c(t_1)\} \) is equal to \( Z_6 \); thus, \( c(t_1) \in \{3, 4, 5\} \). If \( c(t_1) = 3 \), then \( \{c(v_1), c(t_2)\} = \{5, 6\} \). If \( c(t_1) = 4 \), then \( \{c(v_1), c(t_2)\} = \{2, 6\} \). If
c(t_1) = 5, then \(\{c(v_1), c(t_2)\} = \{2, 3\}\). If the colour of \(t_1\) can be changed, we obtain a \((G - u, T - u)\)-colouring \(c'\) such that \(F_{c'} \neq Z_6\) which can be extended in a \((G, T)\)-colouring, a contradiction. Hence, \(c(t_1) = i\) is the sole colour in \(Z_6 \setminus \{(1,2, c(t_2)) \cup \{c(v_1)\}\}\). Thus, \(c(v_1) \neq 2\) and \(\{c(v_1), c(t_2)\} \neq (6,5)\).

Then, necessarily (*) \(c(v_1) = 5, c(t_1) = 3\) and \(c(t_2) = 6\). If \(c(t_3), c(v_3) \neq 3\), then recolour \(t_1\) with 5 and \(v_1\) with 3. Otherwise, if \(c(t_3), c(v_3) \neq 3\), then recolour \(v_1\) with 6. Otherwise (i.e., \(\{c(t_3), c(v_3)\} = \{3, 6\}\)), recolour \(v_1\) with 2. In any case, the resulting colouring \(c_1\) does not satisfy (*). Hence, either \(F_{c_1} \neq Z_6\) or \(t_1\) can be recoloured to get a colouring \(c_1'\) such that \(F_{c_1'} \neq Z_6\). Hence one of \(c_1, c_1'\) can be extended in a \((G, T)\)-colouring, a contradiction.

2.9. Assume \(f = v_1v_3\) with \(v_3\) a leaf adjacent in \(T\) to a twig \(t_3\) not in \(\{t_1, t_2\}\).

Suppose first that \(rv_2 \in E(G)\). Set \(G' = (G - \{u, t_2, v_2\}) \cup \{t_1t_3, v_1v_3\}\). By minimality of \((G, T)\), there is a \((G', T - \{u, t_2, v_2\})\)-colouring, which is a \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\)-colouring \(c\) such that \(c(t_1) \neq c(t_3)\) and \(c(t_1) \neq c(v_3)\). Set \(c(v_2) = c(t_1)\). Then choosing \(c(t_2)\) in \(\{3, 4, 5, 6\} \setminus \{c(t_1)\}\) and \(c(u)\) in \(Z_6 \setminus (\{c(t_1)\} \cup \{c(t_2), c(v_1)\})\), we obtain a \((G, T)\)-colouring, a contradiction. Hence \(rv_2 \notin E(G)\).

Suppose now that \(rv_3 \notin E(G)\). Let \((G', T')\) be the graph pair obtained from \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\) by identifying \(v_3\) and \(r\). By minimality of \((G, T)\), there is a \((G', T')\)-colouring, which is a \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\)-colouring \(c\) such that \(c(v_3) = 1\). Set \(c(u) = c(v_3)\). For \(t_2\), there at least two possible colours, namely the ones not in \(\{1, 2, c(t_1), c(t_3)\}\). One of them, say \(\alpha\), is such that \(I = [\alpha] \cup \{1, c(v_1), c(t_3)\}\) is not equal to \(Z_6\). Thus, setting \(c(t_2) = \alpha\) and choosing \(c(v_2)\) in \(Z_6 \setminus I\), we obtain a \((G, T)\)-colouring, a contradiction. Hence \(rv_3 \notin E(G)\).

Assume that \(rv_1\) is inside \(C_f\). Then \(t_1t_3 \notin E(G)\). By minimality of \((G, T)\), there is a \((G - \{u, t_2, v_2\} \cup rv_1, T\{u, t_2, v_2\})\)-colouring, which is a \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\)-colouring \(c\) such that \(c(v_1) \neq 1\). If \(c(v_1) = 2\), then setting \(c(u) = 2\) and \(c(v_2) = 1\), and choosing \(c(t_2)\) in \(Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}\), we obtain a \((G, T)\)-colouring, a contradiction. So we may assume that \(c(v_1) = 2\), and so \(c(t_1) \geq 4\). If \(c(v_3) \neq 3\), then setting \(c(u) = 1, c(t_2) = 3\) and choosing \(c(v_2)\) in \(\{5, 6\} \setminus \{c(v_3)\}\), we obtain a \((G, T)\)-colouring, a contradiction. If \(c(v_3) = 3\), then setting \(c(u) = 1\), and choosing \(c(t_2)\) in \(\{4, 6\} \setminus \{c(t_1)\}\) and \(c(v_2)\) in \(\{4, 6\} \setminus \{c(t_2)\}\), we obtain a \((G, T)\)-colouring, a contradiction. Hence we may assume that \(rv_3\) is outside \(C_f\).

So, by planarity, \(t_1t_3 \notin E(G)\). Let \((G', T')\) be the graph pair obtained from \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\) by identifying \(t_1\) and \(t_3\). By minimality of \((G, T)\), there is a \((G', T')\)-colouring, which is a \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\)-colouring \(c\) such that \(c(t_1) = c(t_3)\). Set \(c(u) = c(v_3)\). Let \(\alpha\) be a colour of \(Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}\) such that \(I = [\alpha] \cup \{c(v_1), c(v_3), c(t_3)\}\) is not \(Z_6\). Then setting \(c(t_2) = \alpha\) and choosing \(c(v_2)\) in \(Z_6 \setminus I\), we obtain a \((G, T)\)-colouring, a contradiction.

3. Assume that \(C^m_{c_1}\) is \(A_3\).

3.1. Assume \(f = rv_1\). Then \(rv_2\) is an edge. By minimality of \(G\), there exists a \((G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})\)-colouring \(c\). Colour \(u\) with a colour \(c(u)\) in \(Z_6 \setminus (\{c(t_1), c(v_1)\} \cup \{c(t_1)\})\). Set \(c(t_2) = 6\) if \(c(u) \neq 6\) and \(c(t_2) = 5\) otherwise. In both cases, at most five colours are forbidden for \(v_2\), and one can extend greedily the colouring into a \((G, T)\)-colouring, a contradiction.

3.2. Assume that \(f = rv_2\). By minimality of \(G\), there exists a \((G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})\)-colouring \(c\). Set \(c(t_1) = 6\), then colour \(u\) with any colour in \(Z_6 \setminus \{1, 5, 6, c(t_2), c(v_2)\}\), and \(v_1\) with any colour in \(Z_6 \setminus \{1, 5, 6, c(v_2), c(u)\}\). This yields a \((G, T)\)-colouring, a contradiction.

3.3. Assume that \(f = t_1v_2\). Then the cycle \(t_1v_1v_2\) is empty, an so \(v_1\) contradicts Lemma \(\square\). Similarly, if \(f = t_2v_1\), then \(v_2\) contradicts Lemma \(\square\).
3.4. Assume that $f = v_1t_3$ with $t_3$ a twig distinct from $t_2$.

Assume first that $t_2t_3 \in E(G)$. Set $G' = (G - \{u, t_2, v_2\}) \cup rv_1$. By minimality of $(G, T)$, there is a $(G', T - \{u, t_2, v_2\})$-colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring $c$ such that $c(v_1) \neq c(r) = 1$.

Setting $c(v_2) = 1$ and choosing $c(u)$ in $Z_6 \setminus \{(1, c(v_1)) \cup [c(t_1)]\}$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, c(u), c(t_3)\}$, we get a $(G, T)$-colouring, a contradiction. So $t_2t_3 \notin E(G)$ and thus $rv_2 \in E(G)$.

By minimality of $G$, there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring $c$.

Assume that $c(v_1) \neq 2$. If $c(t_1) = 3$, then setting $c(v_2) = 2$ and choosing $c(u)$ in $Z_6 \setminus \{1, 2, 3, c(v_1)\}$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, 3, c(u)\}$ yields a $(G, T)$-colouring, a contradiction. If $c(t_1) \geq 4$, then setting $c(u) = 2$ and choosing $c(v_2)$ in $Z_6 \setminus \{1, 2, c(v_1), c(t_3)\}$ and $c(t_2)$ in $Z_6 \setminus \{(1, 2) \cup [c(v_2)]\}$, we obtain a $(G, T)$-colouring, a contradiction. Hence $c(v_1) = 2$.

If $c(t_1) \neq 4$, then colouring $v_2$ with $c(v_2) \in \{4, 6\} \setminus \{c(t_1)\}$, $t_2$ with $c(t_2) \in \{4, 6\} \setminus \{c(v_2)\}$ and $u$ with $c(u)$ in $\{3, 5\} \setminus [c(t_1)]$, we get a $(G, T)$-colouring, a contradiction. So $c(t_1) = 4$.

Colouring $u$ with $6$, $v_2$ with $c(v_2) \in \{3, 5\} \setminus \{c(t_1)\}$ and $t_2$ with $c(t_2) \in \{3, 5\} \setminus \{c(v_2)\}$, we get a $(G, T)$-colouring, a contradiction.

3.5. Assume that $f = v_2t_3$ with $t_3$ a twig distinct from $t_1$.

Assume first that $t_1t_3$ is an edge. Set $G' = (G - \{u, t_1, v_1\}) \cup rv_2$. By minimality of $(G, T)$, there is a $(G', T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring $c$ such that $c(v_2) \neq c(r) = 1$. Set $c(v_1) = 1$. If $c(v_2) \neq 2$, then setting $c(u) = 2$ and assigning to $t_1$ a colour in $Z_6 \setminus \{1, 2, 3, c(t_3)\}$, we obtain a $(G, T)$-colouring, a contradiction. So $c(v_2) = 2$ and $c(t_2) \geq 4$. Setting $c(u) = 3$ and assigning to $t_1$ a colour in $Z_6 \setminus \{1, 2, 3, 4, c(t_3)\}$, we obtain a $(G, T)$-colouring, a contradiction. Hence $t_1t_3$ is not an edge.

So $rv_1$ is an edge. Let $G'$ be the graph from $G - \{u, t_1, v_1\}$ by adding the edge $t_2t_3$ if it does not exist. By minimality of $(G, T)$, there is a $(G', T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring $c$ such that $c(t_2) \neq c(t_3)$. Set $c(t_1) = 6$. If $c(t_2) \notin \{5, 6\}$, then set $c(v_1) = c(t_2)$ (this is possible because $c(t_3) \neq c(t_2)$), otherwise colour $v_1$ with any colour in $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$. Then colouring $u$ with a colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$, we get a $(G, T)$-colouring, a contradiction.

3.6. Assume that $f = v_2v_3$ with $v_3$ a leaf adjacent to $t_1$. Set $G' = (G - \{u, v_1\} \cup \{t_2v_3, rv_3\}$.

By minimality of $(G, T)$, there is a $(G', T - \{u, v_1\})$-colouring which is a $(G - \{u, v_1\}, T - \{u, v_1\})$-colouring $c$ such that $c(v_3) \notin \{c(r), c(t_2)\}$. Setting $c(u) = c(v_3)$ and colouring $v_1$ with a colour in $Z_6 \setminus \{(c(u), c(v_2)) \cup [c(t_1)]\}$, we obtain a $(G, T)$-colouring, a contradiction.

3.7. Assume that $f = v_1v_3$ with $v_3$ a leaf adjacent to $t_2$. Then $c_{f,v}^\alpha = B_2$.

Assume first that $rv_1 \notin E(G)$. Let $(G', T')$ be the graph pair obtained from $(G - u, T - u)$ by identifying $v_1$ and $r$. By minimality of $(G, T)$, there is a $(G', T')$-colouring, which yields a $(G - u, T - u)$-colouring such that $c(v_1) = c(r) = 1$. Then choosing $c(u)$ in $Z_6 \setminus \{(1, c(v_2)) \cup [c(t_1)]\}$ yields a $(G, T)$-colouring, a contradiction. Hence $rv_1 \in E(G)$.

Assume now that $rv_3 \in E(G)$. Let $(G', T')$ be the graph pair obtained from $(G - u, T - u)$ by identifying $t_1$ and $t_2$. By minimality of $(G, T)$, there is a $(G', T')$-colouring, which yields a $(G - u, T - u)$-colouring such that $c(t_1) = c(t_2)$. Then setting $c(u) = c(v_3)$, we obtain a $(G, T)$-colouring, a contradiction.

3.8. Assume $f = v_2v_3$ with $v_3$ a leaf adjacent to $T$ to a twig $t_3$ not in $\{t_1, t_2\}$.

Assume first that $rv_1$ is not an edge. By minimality of $(G, T)$, there is a $(G - \{u, t_1, v_1\} \cup \{rv_2, rv_3\}, T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring $c$ such that $c(v_2) \neq 1$ and $c(v_3) \neq 1$. Colour $v_1$ with 1. Colour $t_1$ with a colour $\alpha$ in $A = Z_6 \setminus \{1, 2, c(t_3), c(v_3)\}$ such that $\alpha \neq Z_6 \setminus [c(t_1)]$. Then setting $c(u) = c(v_2)$, we get a $(G, T)$-colouring, a contradiction.
\{1, c(t_2), c(v_2)\}. This is possible since \(|A| \geq 2\). Then colouring \(u\) with a colour in \(Z_6 \setminus \{(1, c(t_2), c(v_2)\} \cup \{\alpha\}\), we obtain a \((G, T)\)-colouring, a contradiction.

Suppose now that \(rv_1\) is an edge. Then, since \(e\) is minimal, \(rv_1\) is not overstepping and so there is no vertex inside \(rv_1 t_1 r\). Let \((G', T')\) be the graph pair obtained from \((G - u, T - u)\) by identifying \(t_1\) and \(t_2\). By minimality of \((G, T)\), there is a \((G', T')\)-colouring, which yields a \((G - u, T - u)\)-colouring such that \(c(t_1) = c(t_2)\). Set \(c(u) = 1\) if \(c(v_2) \neq 1\), and choose \(c(u)\) in in \(Z_6 \setminus \{(1, c(v_1),) \cup \{c(t_1)\}\) otherwise. This gives a \((G, T)\)-colouring, a contradiction.

3.9. Assume \(f = v_1 v_3\) with \(v_3\) a leaf adjacent in \(T\) to a twig \(t_3\) not in \(\{t_1, t_2\}\).

Suppose first that \(rv_2 \notin E(G)\). By minimality of \((G, T)\), there is a \((G - \{t_2, v_2\} \cup \{rv_1, rv_3\}, T - \{t_2, v_2\}\)-colouring which is a \((G - \{t_2, v_2\}, T - \{t_2, v_2\})\)-colouring \(c\) such that \(c(v_1) \neq 1\) and \(c(v_3) \neq 1\). Setting \(c(v_2) = 1\) and choosing \(c(t_2)\) in \(Z_6 \setminus \{1, 2, c(u), c(t_2), c(v_3)\}\) yields a \((G, T)\)-colouring, a contradiction. Hence \(rv_2 \in E(G)\).

Assume that \(rv_3 \notin E(G)\). By minimality of \((G, T)\), there is a \((G - \{t_2, v_2\}, T - \{t_2, v_2\})\)-colouring. We can choose \(c(v_2)\) in \(Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}\) such that \(I = c(v_2) \cup \{1, 2, c(u)\} \neq Z_6\) and \(c(t_2) \in Z_6 \setminus I\) to obtain a \((G, T)\)-colouring, a contradiction. Hence \(rv_3 \in E(G)\).

By minimality of \((G, T)\), there is a \((G - \{u, t_2, v_2\}, T - \{t_1, v_2, v_3\})\)-colouring which is a \((G - \{u, t_2, v_2\}, T - \{t_1, v_2, v_3\})\)-colouring \(c\) such that \(c(t_1) \neq c(v_3)\). Set \(c(u) = c(v_3)\). We can choose \(c(v_2)\) in \(Z_6 \setminus \{1, c(t_1), c(v_1), c(v_3)\}\) such that \(I = c(v_2) \cup \{1, 2, c(u)\} \neq Z_6\) and \(c(t_2) \in Z_6 \setminus I\) to obtain a \((G, T)\)-colouring, a contradiction.

\[\square\]

**Lemma 15.** Every penultimate edge has a unique predecessor.

**Proof.** By contradiction. Suppose that a penultimate edge \(f\) has two predecessors \(e\) and \(e'\). Then \(e\) and \(e'\) are ultimate and so minimal. According to Lemma 13, \(C^\text{int}_e\) and \(C^\text{int}_{e'}\) are isomorphic to some of \(A_1, A_2\) and \(A_3\). Let us denote the vertices of \(C^\text{int}_e\) by their names in Figure 2 and the vertices of \(C^\text{int}_{e'}\) by their names in Figure 2 augmented with a prime.

Since \(f, e\) and \(e'\) are bounding the face incident to \(f\) in \(C^\text{int}_e\), the edge \(f\) is \(v_1 v_2', v_1 v_1', v_2 v_2'\) or \(v_2' v_1\). If \(f = v_2 v_1'\), swapping the names of \(e\) and \(e'\), we are left with \(f = v_1 v_2\). Hence we may assume that \(f \in \{v_1 v_2', v_1 v_1', v_2 v_2'\}\).

Note that if \(f = v_1 v_2\), then \(t_2 = t_1\) and \(v_2 = v_1'\), if \(f = v_1 v_1'\), then \(t_2 = t_2'\) and \(v_2 = v_2'\), and if \(f = v_2 v_2'\), then \(t_1 = t_1'\) and \(v_1 = v_1'\).

Observe that if \(C^\text{int}_e\) is isomorphic to \(A_1\), then \(f\) cannot be \(v_2 v_2'\) because \(rv_1\) must be an edge that would cross \(f\). Moreover if \(C^\text{int}_e\) and \(C^\text{int}_{e'}\) are both isomorphic to \(A_1\), then \(f\) cannot be \(v_1 v_1'\) since \(G\) has no multiple edges. Hence must be in one of the following cases:

- \(C^\text{int}_e\) and \(C^\text{int}_{e'}\) are isomorphic to \(A_1\) and \(f = v_1 v_2'\).

By minimality of \(G\), there is a \((G - \{u', t_2, v_2\}, T - \{u', t_2, v_2\})\)-colouring \(c\). Let \(L(v_2) = Z_6 \setminus \{1, c(u), c(v_1), c(v_2')\}\) be the colours available for \(v_2\) and \(L(u') = Z_6 \setminus \{1, c(v_2')\}\) be the colours available for \(u'\). Observe that \(|L(v_2)| \geq 2\) and at least one colour in \(L(v_2)\), say \(\alpha\), is such that one integer \(\beta \in \{\alpha - 1, \alpha + 1\}\) is in \(L(u')\).

If \(\{\alpha, \beta\} \neq \{4, 5\}\), then \(\{1, 2\} \cup [\alpha] \cup [\beta] \neq Z_6\). Hence colouring \(v_2\) with \(\alpha\), \(u'\) with \(\beta\) and \(t_2\) with a colour in \(Z_6 \setminus \{(1, 2) \cup \{\alpha\} \cup [\beta]\}\), we obtain a \((G, T)\)-colouring, a contradiction.

If \(\alpha = 4\) and \(\beta = 5\), then \(c(v_2') = 3\) for otherwise we could have chosen \(\beta = 3\) and got a contradiction as above. Then setting \(c(v_2) = 4\), \(c(u') = 2\) and \(c(t_2) = 6\), we get a \((G, T)\)-colouring, a contradiction.

If \(\alpha = 5\) and \(\beta = 4\), then \(c(v_2') = 6\) for otherwise we could have chosen \(\beta = 6\) and got a contradiction as above. Then setting \(c(v_2) = 5\), \(c(u') = 3\) and \(c(t_2) = 6\), we get a \((G, T)\)-colouring, a contradiction.
• $C^m_e$ is isomorphic to $A_1$, $C^n_e$ is isomorphic to $A_2$ and $f = v_1v'_1$ for $i \in \{1, 2\}$.

By minimality of $(G, T)$, there is a $(G - \{t_2, v_2, u\} \cup rv, T - \{t_2, v_2, u\})$-colouring which is a $(G - \{t_2, v_2, u\}, T - \{t_2, v_2, u\})$-colouring $c$ such that $c(v'_1) \neq 1$. Colour $u'$ with 1. If $\{c(v_1), c(u), c(v'_1)\} \backslash \{2, 3, 6\}$, then assign to $v_2$ an element $c(v_2)$ of this set. Otherwise assign to $v_2$ a colour in $\{4, 5\} \cap [c(v'_1)]$. Then one can choose $c(t_2)$ in $Z_6 \backslash \{(1, 2, c(t'_1)) \cup [c(v_2)]\}$ to obtain a $(G, T)$-colouring, a contradiction.

• $C^m_e$ is isomorphic to $A_1$, $C^n_e$ is isomorphic to $A_3$ and $f = v_1v'_1$ for $i \in \{1, 2\}$.

Suppose first that $rv'_1$ is not an edge. Let $(G', T')$ be the graph pair obtained from $(G - \{t_2, v_2, u\}, T - \{t_2, v_2, u\})$ by identifying $v'_1$ and $r$. By minimality of $(G, T)$, there exists a $G', T'$-colouring which is a $(G - \{t_2, v_2, u\}, T - \{t_2, v_2, u\})$-colouring such that $c(v'_1) = 1$. We can choose $c(F)$ in $\{3, 4, 6\} \backslash \{c(u), c(v_1)\}$ and $c(t_2)$ in $\{4, 5, 6\} \backslash [\alpha]$. If $i = 2$ or $c(t'_2) \neq 3$, then setting $c(u') = 1$ yields a $(G, T)$-colouring, a contradiction. If $i = 1$ and $c(t'_2) = 3$, then choosing $c(u')$ in $\{5, 6\} \backslash [c(t_2), c(v_2)]$ yields a $(G, T)$-colouring, a contradiction.

Hence $rv'_1$ is an edge. The two edges $rv_1$ and $rv'_1$ cannot cross. Thus, for at least one edge $e^* \in \{rv_1, rv'_1\}$, $v_1v'_1$ is not inside $C^m_e$. Hence $e^*$ is not overstepping for otherwise $e^* = e$ contradicting the minimality of $e$. Hence at least one of the two edges $rv_1$ and $rv'_1$ is not overstepping. We analyze the two possible cases.

– $rv_1$ is not overstepping. In this case, there is no vertex inside $C_{rv_1}$ and thus the only neighbours of $t_1$ are $r, v_1$ and $u$.

Let us first prove (∗): there is no $(G - \{u\}, T - \{u\})$-colouring such that $c(v_2) \notin \{5, 6\}$. Indeed suppose there is such a colouring $c$. The set of colours forbidden for $u$ is $F = \{1, c(v_1), c(v_2)\} \cup [c(t_1)]$. Then $F = Z_6$, for otherwise colouring $u$ with a colour in $Z_6 \backslash F$ yields a $(G, T)$-colouring, a contradiction. Thus because $c(v_2) \leq 4$, the colour triple $(c(t_1), c(v_1), c(v_2))$ is either $(4, 6, 2)$, or $(5, 3, 2)$, or $(5, 2, 3)$. In the last two cases, recolouring $t_1$ with 6 and colouring $u$ with 4 we obtain a $(G, T)$-colouring, a contradiction. In the first case, recolouring $t_1$ with 3 and colouring $u$ with 5 we obtain a $(G, T)$-colouring, a contradiction. This proves (∗).

Suppose that $i = 2$, that $t_2 = t'_2$ and $v_2 = v'_1$. By minimality of $G$, there is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$-colouring $c$. Choose $c(u')$ in $\{2, 3\} \backslash [c(t'_2), c(v'_2)]$. Then setting $c(t_2) = 6$ and $c(v_2) = 5$, we get a contradiction to (∗). Hence $(c(u'), c(v_1), c(v'_2)) = \{2, 3, 4\}$. Observe that we can suppose that $c(v_1) = 4$, for otherwise we could have chosen $c(v'_2)$ to be equal to $c(v_1)$. Thus $c(t_1) = 6$. Then recolour $u'$ with some colour in $\{5, 6\} \backslash \{c(t'_2)\}$, colour $v_2$ with the colour in $\{5, 6\} \backslash \{c(u')\}$, $t_2$ with 3 and $u$ with 2. This gives a $(G, T)$-colouring, a contradiction.

Suppose now that $i = 1$, that is $t_2 = t'_2$ and $v_2 = v'_1$. By minimality of $G$, there is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring $c$.

• Assume first that $c(u') = 6$, then $(c(t'_1), c(v'_1))$ is either $(4, 2)$ or $(3, 5)$. If $c(v_1) \neq 3$, then setting $c(v_2) = 3$ and $c(t_2) = 5$, we get a $(G - \{u\}, T - \{u\})$-colouring contradicting (∗). So $c(v_1) = 3$ and thus $c(t_1) \geq 5$. If $(c(t'_1), c(v'_1)) = (3, 5)$, then setting $c(v_2) = 2$ and $c(t_2) = 4$, we also obtain a $(G - \{u\}, T - \{u\})$-colouring contradicting (∗). Hence $(c(t'_1), c(v'_1)) = (4, 2)$. Setting $c(v_2) = 5$, $c(t_2) = 3$ and $c(u) = 2$, we obtain a $(G, T)$-colouring, a contradiction.

• Suppose now that $c(u') \neq 6$. If there exists $\alpha \in \{2, 3, 4\} \backslash \{c(u'), c(v_1), c(v'_1)\}$, then colouring $v_2$ with $\alpha$ and $t_2$ with 6, we obtain a $(G - \{u\}, T - \{u\})$-colouring contradicting (∗). If not, then one can recolour $t_1$ with 6. Then colouring $v_2$ with 6, $t_2$ with some colour in $\{3, 4\} \backslash \{c(u')\}$ and $u$ with a colour in $Z_6 \backslash \{1, 5, 6, c(v_1)\}$, we obtain a $(G, T)$-colouring, a contradiction.
- \(rv'_i\) is not overstepping. In this case, there is no vertex inside \(C_{rv'_i}\) and thus the only neighbours of \(t'_i\) are \(r, v'_i\) and \(u'\).

Suppose first that \(i = 2\). By minimality of \(G\), there is a \((G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})\)-colouring \(c\). Colour \(u\) with a colour \(c(u)\) in \(Z_6 \setminus \{(1, c(v_1)) \cup [c(t_1)]\}\) such that \(c(u), c(v_1)) \neq \{3, 4\}\). (This is possible since if \(c(u), c(v_1)) = \{3, 4\}\), then \(c(t_1) = 6\) and so \(u\) can be recoloured 2.) If \(c(v_2) \leq 4\), then recolour \(r'\) with 6, colour \(v_2\) with 6, \(u'\) with 5 and \(t_2\) with 3. If \(c(v_2) \in \{5, 6\}\), then colour \(v_2\) with some colour in \(\{3, 4\} \setminus \{c(u), c(v_1))\}) \cup [c(t_2)]\} \cup [u']\) with 2 and \(t_2\) with 6. In both cases, we obtain a \((G, T)\)-colouring, a contradiction.

Suppose now that \(i = 1\), that is \(t_2 = t'_1\) and \(v_2 = v'_2\). By minimality of \((G, T)\), there exists a \((G - \{u, u', t_2, v_2\} \cup \{t_1v'_1\}, T - \{u, u', t_2, v_2\})\)-colouring \(c\) which is a \((G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})\)-colouring such that \(c(t_1) \neq c(v'_1)\). Colour \(u\) with some colour in \(Z_6 \setminus \{(1, c(v_1)) \cup [c(t_1)]\}\) and \(u'\) with some colour in \(Z_6 \setminus \{(1, c(v'_1)) \cup [c(t'_1)]\}\). Let \(L(v_2) = Z_6 \setminus \{(1, c(v_1)) \cup [c(t'_1)]\}\). Hence, if we can recolour of \(u'\), neither (a) nor (b) occurs anymore. So, we may assume that \(u'\) cannot be recoloured. If (a) occurs, this implies that \(c(v'_1) = 2\) and \(c(t'_1) = 5\) and \(c(t'_1) = 5\). But then, we recolour \(t'_1\) with 6 and \(u'\) with 4; clearly, neither (a) nor (b) occurs anymore and we get a contradiction as above. If (b) occurs, then either \(c(v'_1) = 2\) and \(c(t'_1) = 4\), or \(c(v'_1) = 5\) and \(c(t'_1) = 3\). In the former case, we recolour \(t'_1\) with 6 and \(u'\) with 4; clearly, neither (a) nor (b) occurs anymore and we get a contradiction as above. If the later case occurs, since \(L(v_2) = \{4\}\), necessarily \(c(v_1), c(u)) = \{2, 3\}\). If it is possible to recolour \(u\), then \(L(v_2)\) would have more than one colour and, consequently, neither (a) nor (b) would occur yielding a contradiction. Hence \(c(v_1), c(u)) = 1 = Z_6 \setminus c(t_1))\), so \(c(t_1) = 5 = c(v'_1)\) which is impossible by our choice of \(c\).

- \(C_{rv'_i}\) is isomorphic to \(A_2\) or \(A_3\), \(C_{rv'_i}\) is isomorphic to \(A_2\) or \(A_3\) and \(f = v_1v'_i\).

By minimality of \((G, T)\), there is a \((G - \{u, u', t_2, v_2\} \cup \{rv_1, rv'_1\}, T - \{u, u', t_2, v_2\})\)-colouring \(c\) which is a \((G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})\)-colouring such that \(c(t_1) \neq c(v'_1)\). Colour \(u\) with some colour in \(Z_6 \setminus \{(1, c(v_1)) \cup [c(t_1)]\}\) and \(u'\) with some colour in \(Z_6 \setminus \{(1, c(v'_1)) \cup [c(t'_1)]\}\). Then, colour \(v_2\) with 1. If either \(t_2\) is adjacent to at most one of \(t_1\) and \(t'_1\) or \(c(t_1), c(u), c(t'_1), c(u')) \neq (3, 4, 5, 6)\), then we can assign to \(t_2\) a colour in \(\{3, 4, 5, 6\}\) not assigned to any of its neighbours to get a \((G, T)\)-colouring, a contradiction.

So \(t_2\) is adjacent to \(t_1\) and \(t'_1\) and \(c(t_1), c(u), c(t'_1), c(u') = (3, 4, 5, 6)\). By symmetry, we may assume that \(c(t_1) = 3\). If \(c(v'_1) = 2\), then we can recolour \(u\) with 2 and colour \(t_2\) with the colour in \(\{3, 4, 5, 6\} \setminus \{c(t_1), c(t'_1), c(u')\}\) to get a \((G, T)\)-colouring, a contradiction. Hence \(c(v'_1) = 3\), then setting \(c(t_1) = 3\) and \(c(v'_2) = 6\) yields a \((G, T)\)-colouring, a contradiction. Hence \(c(v'_1) = 4\), so \(c(t'_1) \in (3, 6)\). If \(c(t'_1) = 3\), then we set \(c(v_2) = 3\) and choose \(c(t_2)\) in \(\{5, 6\} \setminus c(t_1)\). If \(c(t'_1) = 6\), then \(c(v'_1) = 4\), so we set \(c(v_2) = 5\) and \(c(t_2) = 3\). In both cases, we obtain a \((G, T)\)-colouring, a contradiction.

- \(C_{rv'_i}\) is isomorphic to \(A_2\) or \(A_3\), \(C_{rv'_i}\) is isomorphic to \(A_2\) or \(A_3\) and \(f = v_2v'_2\).

By minimality of \((G, T)\), there is a \((G - \{u, u', t_2, v_2\} \cup \{rv_2, rv'_2\}, T - \{u, u', t_2, v_2\})\)-colouring which is a \((G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})\)-colouring such that \(c(v_1), c(v'_1) \neq 1\). Choose \(c(u)\) in \(\{2, 3\} \setminus \{c(v_2), c(t_2)\}\) and \(c(u')\) in \(\{2, 3\} \setminus \{c(v'_2), c(t'_2)\}\) and set \(c(v_1) = 1\). If \(t_1\) has at most one neighbour in \(\{t_2, t_2\} or \{c(t_2), c(t'_2)\} = (5, 6)\), then we can colour \(t_1\) with a colour in \(\{5, 6\}\) not appearing on any of its neighbours to get a \((G, T)\)-colouring, a contradiction. Hence \(t_1\) is adjacent to \(t_2\) and \(t'_2\) (that is \(C_{rv'_i}\) and \(C_{rv'_i}\) are isomorphic to \(A_2\) and \(c(t_2), c(t'_2)\) \(= (5, 6)\)). Recolouring \(u\) with \(c(t'_2)\) and \(u'\) with \(c(t_2)\) and colouring \(t_1\) with 3, we obtain a \((G, T)\)-colouring, a contradiction.
• $C_f^m$ is isomorphic to $A_2$ or $A_3$, $C_f^m$ is isomorphic to $A_2$ or $A_3$ and $f = v_1v_2$.

By minimality of $(G,T)$, there exists a $G - \{u,u',t_2,v_2\} \cup \{rv_1,rv_2\}, T - \{u,u',t_2,v_2\}$-colouring $c$ which is a $(G - \{u,u',t_2,v_2\}, T - \{u,u',t_2,v_2\})$-colouring such that $c(v_1) \neq 1$ and $c(v'_2) \neq 1$. Colour $u$ with some colour in $Z_6 \setminus \{(1,c(v_1)) \cup (c(t_1)), 2\} \cup (c(t'_2),c(v'_2))$ and set $c(v_2) = 1$. Note that the set $F$ of forbidden colours for $t_2$ is the union of $\{1,2,c(u)\} \cup |c(u')|$ and the set of colours of the neighbours of $t_2$ in $\{t_1,t'_2\}$. Moreover $F = Z_6$ for otherwise we could colour $t_2$ with a colour in $Z_6 \setminus F$ to obtain a $(G,T)$-colouring, a contradiction.

If $c(u') = 2$, then, as $|F| = 6$, $\{t_1,t'_2\} \subseteq N_G(t_2)$ and $\{c(u),c(t_1),c(t'_2)\} = \{4,5,6\}$. Since $|c(u) - c(t_1)| \geq 2$, necessarily $\{c(t_1),c(u)\} = \{4,6\}$ and $c(t'_2) = 5$. If $c(t_1) = 6$, then recolouring $u$ with a colour in $\{2,3\} \setminus c(v_1)$ and assigning 4 to $t_2$, we obtain a $(G,T)$-colouring, a contradiction. Hence $c(t_1) = 4$ and $c(u) = 6$. So $c(v_1) = 2$, and thus $c(v'_2) = 3$. Then recolour $u$ and $u'$ with 1 and $v_2$ with 4 and colour $t_2$ with 6 to get a $(G,T)$-colouring, a contradiction.

Now, suppose that $c(u') = 3$. Then, as $|F| = 6$, $\{c(u),c(N(t_2) \cap \{t_1,t'_2\})\} \supseteq \{5,6\}$. Assume that $c(u) \notin \{5,6\}$, then $\{t_1,t'_2\} \subseteq N_G(t_2)$ and $\{c(t_1),c(t'_2)\} = \{5,6\}$. Note that, in this case, $c(v_1) \leq 4$ and $c(v'_2) \leq 4$. Recolour $u$ and $u'$ with 1, $v_2$ with 6 and colour $t_2$ with 3 to get a $(G,T)$-colouring, a contradiction. Hence $c(u) \in \{5,6\}$. Thus $c(t_1) \leq 4$ and so $c(t'_2) \in \{5,6\}$ and $c(v'_2) \leq 4$. Thus, $t'_2 \in N_G(t_2)$ and $c(t'_2) \in \{\{5,6\} \setminus c(u)|$. Recolour $u'$ with $c(u)$. If $t_1 \notin N_G(t_2)$ or $c(t_1) \neq 3$, colouring $t_2$ with 3 yields a $(G,T)$-colouring, a contradiction. So $t_1 \in N_G(t_2)$ and $c(t_1) = 3$. Then, recolour $u$ and $u'$ with 3 (note that $c(v_1) \geq 5$ and $c(v'_2) \neq 3$ as $u'$ was coloured 3) and $t_2$ with $i \in \{5,6\} \setminus \{c(t'_2)\}$. This gives a $(G,T)$-colouring, a contradiction.

\[\square\]

Lemmas 14 and 15 immediately imply the following.

**Corollary 16.** If $f$ is a penultimate edge, then $C_f^m$ is isomorphic to $B_1$ or $B_2$. Moreover $rv_1 \in E(G)$ and $rv_3 \notin E(G)$.

### 3.3 Antepenultimate edges

To deal with antepenultimate edges, we need the following two auxiliary results.

**Lemma 17.** Suppose that $(G,T)$ contains a configuration isomorphic to $B_1$ (see Figure 3). If there is a $(G - \{u,v_2\}, T - \{u,v_2\})$-colouring $c$ satisfying one of the following conditions:

- (a) $c(t_2) = 6$ and $(c(t_1),c(v_3)) \neq (5,4)$;
- (b) $c(v_3) = 1$ and $(c(t_1) \neq c(t_2)$;

Then there is a $(G,T)$-colouring.

**Proof.** Let $L(u) = Z_6 \setminus \{1,|c(t_1)|,c(v_1)\}$ and $L(v_2) = Z_6 \setminus \{1,|c(t_2)|,c(v_1),c(v_3)\}$ be the set of colours available for $u$ and $v_2$ respectively. Clearly $L(u) \neq \emptyset$. Observe that the conditions (a) and (b) also imply that $L(v_2) \neq \emptyset$. So, if $|L(u)| \geq 2$, $|L(v_2)| \geq 2$ or $L(u) \neq L(v_2)$, one can choose distinct colours $c(u) \in L(u)$ and $c(v_2) \in L(v_2)$ to obtain a $(G,T)$-colouring.

It is simple matter to check that in both cases, this condition is fulfilled. \[\square\]

**Lemma 18.** There is no antepenultimate edge $g$ with only one penultimate predecessor $f$ such that $C_f^m$ is $B_2$.

In order to prove Lemma 18, we first prove an auxiliary result.

**Lemma 19.** Suppose that $(G,T)$ contains a configuration isomorphic to $B_2$ (see Figure 3). If there is a $(G - \{u,v_2\}, T - \{u,v_2\})$-colouring $c$ satisfying one of the following conditions:
(a) \( c(t_1) = c(t_2) \) and \( c(v_3) \neq 1 \);

(b) \( c(t_1) \neq c(t_2) \) and

(b1) \( c(t_1) = 6; \) or

(b2) \( c(v_1) = c(t_2); \) or

(b3) \( c(t_2) \in [c(t_1)]. \)

Then \( G \) has a \((G, T)\)-colouring.

Proof. Let \( L(u) = Z_6 \setminus \{1, [c(t_1)], c(t_2), c(v_1)\} \) and \( L(v_2) = Z_6 \setminus \{[c(t_2)], c(v_1), c(v_3)\} \) be the set of colours available for \( u \) and \( v_2 \) respectively. Clearly \( L(v_2) \neq \emptyset \). Observe that the conditions (a), (b1), (b2) and (b3) also imply that \( L(u) \neq \emptyset \). So, if \( |L(u)| \geq 2, |L(v_2)| \geq 2 \) or \( L(u) \neq L(v_2) \), one can choose distinct colours \( c(u) \in L(u) \) and \( c(v_2) \in L(v_2) \) to obtain a \((G, T)\)-colouring.

It is simple matter to check that in each case, this condition is fulfilled.

Lemma 20. Every antepenultimate edge has a unique penultimate predecessor.

Proof. By contradiction. Suppose that an antepenultimate edge \( g \) has two penultimate predecessors \( f \) and \( f' \).

According to Corollary 16, \( C^*_{j'f} \) and \( C^*_{jg} \) are isomorphic to one of the graphs \( B_1 \) and \( B_2 \). Let us denote the vertices of \( C^*_{jg} \) by their names in Figure 3 and the vertices of \( C^*_{j'f} \) by their names in Figure 3 augmented with a prime.

Since \( g, f \) and \( f' \) are bounding the face incident to \( g \) in \( C^*_{jg} \), the edge \( g \) is \( v_1v'_1, v_1v'_2, v_3v'_2 \) or \( v_3v'_1 \). Since \( rv_1 \) and \( rv'_1 \) are edges, then \( g = v_1v'_1 \), for otherwise \( rv_1 \) would cross \( g \).

First, suppose that \( C^*_{jg} \) is isomorphic to \( B_1 \), i.e., \( rv'_2 \in E \). By minimality of \((G, T)\), there is a \((G - \{v_2, v_3\}) \cup \{v'_2u, v_2v_3\}, T - \{v_2, v_3\}\)-colouring, which is a \((G - \{v_2, v_3\}, T - \{v_2, v_3\})\)-colouring such that \( c(v'_2) \notin \{c(u), c(v_1)\} \). Setting \( c(v_2) = c(v'_2) \) and \( c(v_3) = 1 \) gives a \((G, T)\)-colouring, a contradiction.

The case \( C^*_{jg} \) is isomorphic to \( B_1 \) is symmetric, so we may assume that both are \( C^*_{jg} \) and \( C^*_{j'f} \) are isomorphic to \( B_2 \). By minimality of \((G, T)\), there exists a \((G - \{v_2, v_3\}, T - \{v_2, v_3\})\)-colouring. Set \( c(v_2) = c(v'_2) = 1 \). Then, one can choose \( c(t_2) \) in \( L = Z_6 \setminus \{1, 2, c(u), c(u')\} \) such that \( I = [c(t_2)] \cup \{1, c(v_1), c(v'_1)\} \neq Z_6 \) because \( |L| \geq 2 \). Hence colouring \( v_3 \) with a colour in \( Z_6 \setminus I \), we obtain a \((G, T)\)-colouring, a contradiction.

Lemma 21. Every antepenultimate edge has a unique predecessor.

Proof. By contradiction. Suppose that an antepenultimate edge \( g \) has two predecessors \( f \) and \( f' \). By Lemma 20, one of those is not penultimate. So, without loss of generality, \( f \) is penultimate, and \( f' \) is not. Hence \( f' \) is minimal.

According to Corollary 16, \( C^*_{jg} \) is isomorphic to \( B_1 \) or \( B_2 \), and according to Lemma 13, \( C^*_{j'f} \) is isomorphic to one of \( A_1, A_2 \) and \( A_3 \). Let us denote the vertices of \( C^*_{jg} \) by their names in Figure 3 and the vertices of \( C^*_{j'f} \) by their names in Figure 3 augmented with a prime.

Since \( g, f \) and \( f' \) are bounding the face incident to \( g \) in \( C^*_{jg} \), the edge \( g \) is \( v_1v'_1, v_1v'_2, v_3v'_2 \) or \( v_3v'_1 \). Moreover, since \( rv_1 \) is an edge, \( rv_3 \) is not an edge if \( C^*_{jg} \) is isomorphic to \( B_1 \), and \( rv'_1 \) is an edge if \( C^*_{j'f} \) is isomorphic to \( A_1 \), we must be in one of the following cases:

- \( C^*_{jg} \) is isomorphic to \( B_1 \), \( C^*_{j'f} \) is isomorphic to \( A_2 \) or \( A_3 \) and \( g = v_1v'_2 \).

By minimality of \((G, T)\), there is a \((G - \{v_2, v_3, u', v_3\}) \cup \{v'_2f', T - \{v_2, v_3, u', v_3\}\})\)-colouring \( c \) which is a \((G - \{v_2, v_3, u', v_3\}, T - \{v_2, v_3, u', v_3\})\)-colouring such that \( c(v'_2) \neq 1 \). Set \( c(v_3) = 1 \). Let \( L(t_2) \supseteq Z_6 \setminus \{1, 2, c(t'_2), L(v_2) = Z_6 \setminus \{1, c(u), c(v_1)\} \) and \( L(u') = Z_6 \setminus \{1, c(t'_2), c(v'_2)\} \). Clearly, there exists at most one \( i \in Z_6 \) such that \( L(u') = [i] \) and at most one \( j \in Z_6 \) such that \( L(v_2) = [j] \). Thus, as \( |L(t_2)| \geq 3 \), there exists \( k \in L(t_2) \) such that \( L(u') \setminus [k] \neq \emptyset \) and \( L(v_2) \setminus [k] \neq \emptyset \). Setting \( c(t_2) = k \) and colouring \( u' \) and \( v_2 \) by colours in \( L(u') \setminus [k] \) and \( L(v_2) \setminus [k] \), respectively, we obtain a \((G, T)\)-colouring, a contradiction.
\[ C^m_i \] is isomorphic to \( B_2 \), \( C^m_j \) is isomorphic to \( A_2 \) or \( A_3 \) and \( g = v_1v_2' \).

By minimality of \( (G, T) \), there is a \( \langle (G - \{v_2, t_2, v_3, u'\}) \cup \{t'_2u, t'_2v_1\}, T - \{v_2, t_2, v_3, u'\}\rangle \)-colouring \( c \) which is a \( \langle (G - \{v_2, t_2, v_3, u'\}), T - \{v_2, t_2, v_3, u'\}\rangle \)-colouring such that \( c(t'_2) \not\in \{c(u), c(v_1)\} \).

Suppose that \( t_2t'_2 \in E \). If we can colour \( t_2 \) with \( \beta \in [c(v_1)] \cup \{6\} \), then we can colour \( u' \) with some colour in \( Z_6 \setminus \langle (c(t'_2), c(v_2')) \cup \{\beta\} \rangle \) and \( v_3 \) with some colour in \( Z_6 \setminus \langle (c(u'), c(v_2'), c(v_1)) \cup \{\beta\} \rangle \) and \( v_2 \) with some colour in \( Z_6 \setminus \langle (c(v_3), c(u), c(v_1)) \cup \{\beta\} \rangle \), a contradiction. So, there is no available colour in \( [c(v_1)] \cup \{6\} \) for \( t_2 \); that is, \( c(v_1) \cup \{6\} \not\subset \{1, 2, c(u), c(t'_2)\} \). Since \( c(v_1) \not\subset \{1, c(u), c(t'_2)\} \), we must have \( c(v_1) = 2 \) and \( \{c(u), c(t'_2)\} = \{3, 6\} \). Colour \( u' \) with 2 (since \( v_2' \in N(v_1) \) we know that \( c(v_2') \neq c(v_1) \)), \( v_2 \) with 1 and \( v_3 \) with \( c(t'_2) \). Colour \( t_2 \) with 4 if \( c(u) = 3 \) and with 5 otherwise. This gives a \( (G, T) \)-colouring, a contradiction.

Now, suppose that \( ru' \in E \). If \( c(u) \neq 6 \), then we can colour \( t_2 \) with 6 and \( u' \), \( v_3 \) and \( v_2 \) can be greedily coloured in this order, a contradiction; thus, \( c(u) = 6 \). Let \( L(u') = Z_6 \setminus \{1, c(t'_2), c(v_2')\} \) be the colours available for \( u' \); note that if \( L(u') = [i] \) for some \( i \in Z_6 \), then \( c(t'_2) = 6 \) and \( c(v_1') = 2 \), a contradiction since \( c(t'_2) \neq c(u) \). Clearly, there exists \( \beta \in [c(v_1)] \setminus \{1, 2\} \) so we can colour \( t_2 \) with \( \beta \), \( u' \) with any colour in \( L(u') \setminus \{\beta\} \) (recall that \( L(u') \neq [i] \) for all \( i \in Z_6 \)). Then colour \( v_3 \) and \( v_2 \) greedily gives a \( (G, T) \)-colouring, a contradiction.

\[ C^m_i \] is isomorphic to \( B_1 \) or \( B_2 \). \( C^m_j \) is isomorphic to \( A_2 \) or \( A_3 \) and \( g = v_1v'_1 \).

By minimality of \( (G, T) \), there is a \( \langle (G - \{v_2, t_2, v_3\}) \cup \{uv'_1, rv'_1, rv'_1\}, T - \{v_2, t_2, v_3\}\rangle \)-colouring, which is a \( \langle (G - \{v_2, t_2, v_3\}), T - \{v_2, t_2, v_3\}\rangle \)-colouring such that \( c(v'_1), c(u') \neq 1 \) and \( c(v'_1) \neq c(u) \).

Suppose that we can colour \( t_2 \) with \( \beta \in [c(v_1)] \cup \{6\} \). We know that there is at least one colour \( i \in Z_6 \setminus \{1, c(u), c(v_1)\} \cup \{\beta\} \) available for \( v_3 \) and at least one colour \( j \in Z_6 \setminus \langle (c(v'_1), c(u'), c(v_1)) \cup \{\beta\} \rangle \) available for \( v_3 \). Since \( c(v'_1) \not\subset \{1, c(u)\} \), then \( i \neq j \) and we can colour \( v_2 \) with \( i \) and \( v_3 \) with \( j \) to obtain a \( (G, T) \)-colouring, a contradiction.

So, suppose that the colours of \( [c(v_1)] \cup \{6\} \) all appear in \( N(t_2, v_2, v_3) \); since \( [c(v_1)] \cup \{6\} \not\subset \{1, 2\} \geq 2 \), \( t_2 \) must be adjacent to at least one of \( u \) and \( t'_1 \).

Assume first \( t'_2 \in E \). Then recolour \( u' \) with 1. If \( ut_2 \not\in E \) or \( [c(v_1)] \cup \{6\} \not\subset \{1, 2, c(u), c(t'_2)\} \), note that we can apply the same argument as before since it holds even if \( c(u') = 1 \); so suppose otherwise. In this case we must have: either \( c(v_1) = 6 \), \( c(u) = 5 \) and \( c(t'_2) = 6 \); or \( b) c(v_1) = 2 \) and \( c(u), c(t'_2) = \{3, 6\} \), colour \( v_2 \) with 1. If (a) occurs, then colour \( t_2 \) with 3 and \( v_3 \) with 5; if (b) occurs and \( c(t'_2) = 6 \), then colour \( t_2 \) with 4 and \( v_3 \) with 6; if (b) occurs and \( c(t'_2) = 3 \), then colour \( t_2 \) with 5 and \( v_3 \) with 3.

Hence \( t'_2 \not\in E \), and so \( t_2 \in E \). The possible situations are: (c) \( c(v_1) = 6 \), \( c(u) = 5 \) and \( c(u') = 6 \); or (d) \( c(v_1) = 2 \) and \( c(u), c(t'_2) = \{3, 6\} \). If (c) occurs, then colour \( v_3 \) with \( \{2, 5\} \setminus \{c(v'_1)\} \) and \( t_2 \) with \( \{3, 4\} \setminus \{c(v_3)\} \). If (d) occurs, then colour \( v_3 \) with \( c(u) \) (recall that \( c(v'_1) \neq c(u) \)) and \( t_2 \) with \( \{4, 5\} \setminus \{c(v_3)\} \). In both cases we get a \( (G, T) \)-colouring, a contradiction.

\[ C^m_i \] is isomorphic to \( B_1 \). \( C^m_j \) is isomorphic to \( A_1 \) and \( g = v_1v'_1 \).

Let \( (G', T') \) be the graph pair obtained from \( G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, v_3, u'\}\) by identifying \( t_1 \) and \( t'_1 \). By minimality of \( (G, T) \), there exists a \( (G', T') \)-colouring which is a \( \langle G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\}\rangle \)-colouring such that \( c(t_1) = c(t'_1) \). Set \( c(t_2) = 6 \) and \( c(u') = c(v_1) \). Let \( L = \{2, 3, 4\} \setminus \{c(t_1), c(v'_1)\} \).

If \( c(t_1) \neq 5 \), then choosing \( c(v_3) \in L \), and applying Lemma [17] we obtain a \( (G, T) \)-colouring, a contradiction. Hence \( c(t_1) = c(t'_1) = 5 \).

If \( L \neq \{4\} \), then we can choose \( c(v_3) \in L \setminus \{4\} \), and apply Lemma [17] to get a \( (G, T) \)-colouring, a contradiction. Hence \( \{c(v_1), c(v'_1)\} = \{2, 3\} \).
Now setting $c(v_3) = 5$, $c(v_2) = 6$, $c(u) = c(v'_1)$ and recolouring $t_2$ with 3, we obtain a $(G, T)$-colouring, a contradiction.

• $C^m$ is isomorphic to $B_1$, $C^m$ is isomorphic to $A_1$ and $g = v_3v'_1$.

Let $(G', T')$ be the graph pair obtained from $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$ by identifying $t_2$ and $t'_2$. By minimality of $(G, T)$, there exists a $(G', T')$-colouring which is a $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$-colouring such that $c(t_2) = c(t'_2)$. Set $c(t_1) = 6$.

If $c(t_2) = c(t'_2) = 6$, then set $c(u) = c(v_3)$. One can then greedily extend the colouring to $v_1$, $v_2$ and $u'$ in this order, a contradiction.

If $c(t_2) \neq 6$, then one can choose $c(v_1) \in [c(t_2)] \setminus \{5, 6\}$. This is valid since $c(v_3)$ and $c(v'_1)$ are not in $[c(t_2)]$. On can then greedily extend the colouring to $v_2$, $u$ and $u'$ in this order, a contradiction.

• $C^m$ is isomorphic to $B_2$, $C^m$ is isomorphic to $A_1$ and $g = v_3v'_1$.

By minimality of $(G, T)$, there exists a $(G - \{t_1, u, v_1, v_2\} \cup \{rv_3\}, T - \{t_1, u, v_1, v_2\})$-colouring $c$. Set $c(v_2) = 1$ and let $L(v_1) = Z_6 \setminus \{1, c(v_3), c(u'), c(v'_1)\}$ be the colours available for $v_1$.

If $L(v_1) \neq \{5, 6\}$, then colouring $t_1$ with 6, $v_1$ with some colour in $L(v_1) \setminus \{5, 6\}$ and $u$ with some colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(t_2)\}$, we obtain a $(G, T)$-colouring, a contradiction.

If $L(v_1) = \{5, 6\}$, then $c(u'), c(v'_1) \in \{2, 3, 4\}$ and consequently $c(t'_2) \in \{5, 6\}$. We can suppose that $c(t'_2) = 5$ and $c(v_3) = 4$ for otherwise we can recolour $u'$ with $c(v_3)$ and fall in the case $L(v_1) \neq \{5, 6\}$.

So, $c(u'), c(v'_1) \in \{2, 3\}$ and $c(t_2) \geq 6$. Setting $c(t_1) = 3$, $c(u) = 5$ and $c(v_1) = 6$, we obtain a $(G, T)$-colouring, a contradiction.

\[ \square \]

**Lemma 22.** There is no antepenultimate edge $g$ with only one penultimate predecessor $f$ such that $C^m$ is $B_1$.

**Proof.** One of the endvertices of $g$ must be $v_1$ or $v_3$ (see Figure 3). We now distinguish some cases depending on the possible endvertices of $g$.

(a) Assume $g = v'v_3$ with $v'$ a leaf with twig $t'$. Since $rv_1 \in E(G)$, by planarity, $t' \neq t_1$. Let $(G', T')$ be the graph pair obtained from $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$ by identifying $t'$ and $t_2$. By minimality of $(G, T)$, there is a $(G', T')$-colouring which is a $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$-colouring such that $c(t') = c(t_2)$. Set $c(v_2) = c(v')$ and $c(t_1) = 6$.

If $c(t') \in \{5, 6\}$, then setting $c(u) = c(v_3)$ and choosing $c(v_1) \in \{2, 3, 4\} \setminus \{c(v'), c(v_3)\}$, we obtain a $(G, T)$-colouring, a contradiction.

If $c(t') \in \{3, 4\}$, then setting $c(u) = c(v_3) - 1$ and choosing $c(u) \in \{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$, we obtain a $(G, T)$-colouring, a contradiction.

(b) Assume $g = t'v_3$ with $t'$ a twig. We can apply an argument similar to (a) choosing $c(v_2) \in Z_6 \setminus \{(1, c(v_3)) \cup \{c(t_2)\}\}$.

(c) Assume $g = v_1t_2$. Since $G$ is triangulated, the edge $v_1t_2$ must exist. This is a contradiction, since $f$ is the successor of $e$.

(d) Assume $g = t_1v_3$. Then $v_3$ is a leaf of degree at most 3, a contradiction from Lemma 6.

(e) Assume $g = v_1v'_1$ with $v'$ a leaf with twig $t_2$. Let $(G', T')$ be the graph pair obtained from $(G - \{v_3\}, T - \{v_3\})$ by identifying $v_2$ and $v'$. By minimality of $(G, T)$, there is a $(G', T')$-colouring which is a $(G - \{v_3\}, T - \{v_3\})$-colouring such that $c(v_2) = c(v')$. Hence one can colour $v_3$ with a colour from $Z_6 \setminus \{c(t_2), c(v_2), c(v_1)\}$ to obtain a $(G, T)$-colouring, a contradiction.
Proof. One of the endvertices of \( g \) must be \( v_1 \) or \( v_3 \) (see Figure 3). We now distinguish some cases depending on the possible endvertices of \( g \).

(a) Assume \( g = v'v_3 \) with \( v' \) a leaf with twig \( t' \). Since \( rv_1 \in E(G) \), by planarity, \( t' \neq t_1 \). By minimality of \( (G, T) \), there is a \( ((G - \{ u_2, v_3 \}) \cup t_1 t_2, T - \{ u_2, v_3 \}) \)-colouring which is a \( (G - \{ u_2, v_3 \}, T - \{ u_2, v_3 \}) \)-colouring such that \( c(t_1) \neq c(t_2) \). Since \( c(t'), c(v_1) \neq 1 \), we can colour \( v_3 \) with 1. Then, by Lemma 17 (b), there is a \((G, T)\)-colouring, a contradiction.

(b) Assume \( g = t'v_3 \) with \( t' \) a twig. We can apply an argument similar to (a).

(c) Assume \( g = v_1r \). Since \( G \) is triangulated, the edge \( v_1t_2 \) must exist. This is a contradiction, since \( f \) is the successor of \( e \).

(d) Assume \( g = v_1t_2 \). Then \( v_3 \) is a leaf of degree at most 3, a contradiction from Lemma 6.

(e) Assume \( g = v_1v' \) with \( v' \) a leaf adjacent to \( t_2 \) in \( T \). Since \( f \) is the successor of \( v_1v_3 \), \( v_1t_2 \notin E(G) \) and so \( v_2v_3 \in E(G) \) because \( G \) is triangulated. Let \( (G', T') \) be the graph pair obtained from \( (G - \{ v_3 \}, T - \{ v_3 \}) \) by identifying \( v_2 \) and \( v' \). By minimality of \( (G, T) \), there is a \( (G', T')\)-colouring which is a \( (G - \{ v_3 \}, T - \{ v_3 \})\)-colouring such that \( c(v_2) = c(v') \). Hence one can colour \( v_3 \) with a colour from \( Z_6 \setminus \{ c(t_2), c(t_1), c(v_1) \} \) to obtain a \((G, T)\)-colouring, a contradiction.

(f) Assume \( g = v_1t' \) with \( t' \neq t_2 \) a twig. Since \( g \) is the successor of \( f \), \( v_1t_2 \) is not an edge and \( v_1r \) is not inside \( C_g \), so \( v_3t' \in E \).

Assume first that \( rv_3 \notin E \). By minimality of \( (G, T) \), there is a \( ((G - \{ u, v_2, v_3 \}) \cup t_1 t_2, T - \{ u, v_2, v_3 \}) \)-colouring which is a \( (G - \{ u, v_2, v_3 \}, T - \{ u, v_2, v_3 \}) \)-colouring such that \( c(t_1) \neq c(t_2) \). Since \( c(t'), c(v_1) \neq 1 \), we can colour \( v_3 \) with 1. Then, by Lemma 17 (b), there is a \((G, T)\)-colouring, a contradiction.

Assume now that \( rv_3 \in E \). Then \( t' \notin E \) by planarity. Let \((G', T')\) be the graph pair obtained from \( (G - \{ u_2, v_1, v_3 \}, T - \{ u_2, v_1, v_3 \}) \) by identifying \( t_1 \) and \( t' \). By minimality of \((G, T)\), there is a \((G', T')\)-colouring, which is a \( ((G - \{ u_2, v_1, v_3 \}, T - \{ u_2, v_1, v_3 \}) \)-colouring such that \( c(t_1) = c(t') \). Set \( c(t_2) = 6 \). One can choose \( c(v_3) \in \{ 2, 3 \} \setminus \{ c(t'), c(v_1) \} \) because \( |c(v_1) - c(t')| \geq 2 \). Then by Lemma 17 (a), there is a \((G', T')\)-colouring, a contradiction.

Assume now that \( rv_3 \notin E \). Let \((G', T')\) be the graph pair obtained from \( (G - \{ u_2, v_1, v_3 \}, T - \{ u_2, v_1, v_3 \}) \) by identifying \( v_1 \) and \( v' \). By minimality of \((G, T)\), there is a \((G', T')\)-colouring, which is a \( (G - \{ u_2, v_1, v_3 \}, T - \{ u_2, v_1, v_3 \}) \)-colouring such that \( c(t_1) = c(t') \). If \( \{ c(v_1), c(v') \} \neq \{ 2, 3 \} \), then one can choose \( c(v_3) \in \{ 2, 3 \} \setminus \{ c(t'), c(v'), c(v_1) \} \). Then setting \( c(t_2) = 6 \) and applying Lemma 17 (a), we obtain a \((G, T)\)-colouring, a contradiction. Thus \( \{ c(v_1), c(v') \} = \{ 2, 3 \} \), and so \( c(t_1) \geq 5 \). Setting \( c(u) = c(v'), c(t_2) = 3 \) and choosing \( c(v_3) \) in \( \{ 5, 6 \} \setminus \{ c(t'), c(v_1) \} \) and \( c(v_2) \) in \( \{ 5, 6 \} \setminus \{ c(v_3) \} \) yields a \((G, T)\)-colouring, a contradiction.
(f) Assume \( g = v_1t' \) with \( t' \neq t_2 \) a twig. Let \((G', T')\) be the graph pair obtained from \((G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})\) by identifying \( t_1 \) and \( t' \). This is possible since \( t_1t' \) is not an edge by planarity. By minimality of \((G, T)\), there is a \((G', T')\)- colouring which is a \((G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})\)-colouring such that \( c(t') = c(t_1) \). Let \( L(t_2) = Z_6 \setminus \{1, 2, c(t')\} \). If \( c(t_1) = 6 \), then colouring \( v_3 \) with 1 and \( t_2 \) with a colour from \( L(t_2) \setminus \{6\} \) and using Lemma [19](b1), we obtain a \((G, T)\)- colouring, a contradiction. So \( c(t_1) \neq 6 \), that is \( c(t_1) \in \{3, 4, 5\} \). We can colour \( t_2 \) with a colour from \( \{c(t_1)\} \setminus \{c(t'), c(t_1)\} \subseteq L(v_2) \). By Lemma [19](b3), there is a \((G, T)\)- colouring, a contradiction.

(g) Assume \( g = v_1v' \) with \( v' \) a leaf with twig \( t' \neq t_2 \). By minimality of \((G, T)\), there is a \(((G - \{u, t_2, v_2, v_3\}) \cup \{v_1t'\}, T - \{u, t_2, v_2, v_3\}\)\)((\(G', T')\))- colouring which is a \((G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})\)- colouring such that \( c(v_1) \neq c(t') \). If \( c(v_1) \neq 2 \), we can colour \( t_2 \) with \( c(v_1) \), since \( c(t'), c(v') \neq c(v_1) \). Then, colouring \( v_3 \) with a colour from \( Z_6 \setminus \{c(v_1), c(t'), c(v')\} \), and applying Lemma [19](b2), we obtain a \((G, T)\)- colouring, a contradiction. So, \( c(v_1) = 2 \).

Suppose that \( c(v') \neq 1 \). Since \( c(t'), c(v') \notin \{1, 2\} \), then \( \{c(t'), c(v')\} \notin \{3, 5\} \cup \{3, 6\} \cup \{4, 6\} \). Let \( L(v_3) = Z_6 \setminus \{2, c(t'), c(v')\} \) and let \( L(t_2) = Z_6 \setminus \{1, 2, c(t'), c(v')\} \). If \( L(v_2) \cap \{c(t_1)\} \setminus \{c(t_1)\} \neq \emptyset \), then choosing \( c(t_2) = c(t_1) \) and \( c(v_3) \) in \( L(v_3) \setminus \{c(t_2)\} \) (observe that \([c(t_2)] \cap \{c(t_2)\} \leq 2 \), since \( L(v_3) \) has no three consecutive integers), and using Lemma [19](b3), we obtain a \((G, T)\)- colouring, a contradiction. Then \( L(v_2) \cap \{c(t_1)\} \setminus \{c(t_1)\} = \emptyset \). If \( c(t_1) = 3 \), then \( \{c(t'), c(v')\} = \{4, 6\} \). In this case, colouring \( t_2 \) with 3, \( v_3 \) and \( u \) with 5 and \( v_2 \) with 6, we can obtain a \((G, T)\)- colouring, a contradiction. If \( c(t_1) = 4 \), then \( \{c(t'), c(v')\} = \{3, 5\} \) and \( c(t_1) = 5 \), then \( \{c(t'), c(v')\} = \{4, 6\} \). In both cases, setting \( c(t_2) = c(t_1) \), choosing \( c(v_3) \) in \( Z_6 \setminus \{1, 2, c(t_1)\} \) and using Lemma [19](a), we obtain a \((G, T)\)- colouring, a contradiction. Hence \( c(v') = 1 \). If \( c(t_1) \in \{3, 4\} \), colour \( t_2 \) with 3 (if \( c(t') \neq 3 \) or 4 (otherwise). If \( c(t_1) = 5 \), colour \( t_2 \) with 6 (if \( c(t') \neq 6 \) or 5 (otherwise). These cases satisfy the conditions \( c(t_2) \in [c(t_1)] \) and \( Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\} \neq \emptyset \). Then, colouring \( v_3 \) with a colour from \( Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\} \), and using Lemma [19](a) or (b3), we obtain \((G, T)\)- colouring, a contradiction.

□

Lemmas [20] [21] [22] and [18] directly imply the following.

**Corollary 23.** \((G, T)\) has no antepenultimate edges.

**References**


