

Backbone colouring: tree backbones with small diameter in planar graphs

Victor Campos, Frédéric Havet, Rudini Sampaio, Ana Silva

► **To cite this version:**

Victor Campos, Frédéric Havet, Rudini Sampaio, Ana Silva. Backbone colouring: tree backbones with small diameter in planar graphs. [Research Report] RR-8151, INRIA. 2012. <hal-00758548>

HAL Id: hal-00758548

<https://hal.inria.fr/hal-00758548>

Submitted on 28 Nov 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Backbone colouring: tree backbones with small diameter in planar graphs

Victor Campos — Frédéric Havet — Rudini Sampaio* — Ana Silva*

N° 8151

November 2012

Thème COM

A large blue rectangle occupies the lower half of the page. On the left side of this rectangle, there is a large, light grey, stylized letter 'R'. To the right of the 'R', the words 'Rapport de recherche' are written in a white, italicized serif font. A horizontal grey brushstroke is positioned below the text.

R *apport
de recherche*

Backbone colouring: tree backbones with small diameter in planar graphs

Victor Campos^{*}, Frédéric Havet[†], Rudini Sampaio^{*}, Ana Silva^{*}

Thème COM — Systèmes communicants
Équipe-Projet Mascotte

Rapport de recherche n° 8151 — November 2012 — 24 pages

Abstract: Given a graph G and a spanning subgraph T of G , a *backbone k -colouring* for (G, T) is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that $|c(u) - c(v)| \geq 2$ for every edge $uv \in E(T)$ and $|c(u) - c(v)| \geq 1$ for every edge $uv \in E(G) \setminus E(T)$. The *backbone chromatic number* $BBC(G, T)$ is the smallest integer k such that there exists a backbone k -colouring of (G, T) . In 2007, Broersma et al. [2] conjectured that $BBC(G, T) \leq 6$ for every planar graph G and every spanning tree T of G . In this paper, we prove this conjecture when T has diameter at most four.

Key-words: backbone colouring, planar graph, spanning tree

^{*} Universidade Federal do Ceará, Fortaleza, Brazil. Partly supported by CNPq/Universal and FUNCAP/Pronem. Email: {campos, rudini}@lia.ufc.br, anasilva@mat.ufc.br

[†] Projet Mascotte, I3S (CNRS, UNS) and INRIA, Sophia Antipolis. Partly supported by the French *Agence Nationale de la Recherche* under Grant GRATEL ANR-09-blanc-0373-01. Email: Frederic.Havet@inria.fr

Coloration dorsale : arbres dorsaux de petit diamètre dans les graphes planaires

Résumé : Pour un graphe G et un sous-graphe T de G , une k -coloration dorsale de (G, T) est une application $c : V(G) \rightarrow \{1, \dots, k\}$ telle que $|c(u) - c(v)| \geq 2$ pour toute arête $uv \in E(T)$ et $|c(u) - c(v)| \geq 1$ pour toute arête $uv \in E(G) \setminus E(T)$. Le nombre chromatique dorsal $BBC(G, T)$ est le plus petit entier k tel qu'il existe une k -coloration dorsale de (G, T) . En 2007, Broersma et al. [2] ont conjecturé que $BBC(G, T) \leq 6$ pour tout graphe planaire G et tout arbre couvrant T de G . Dans ce rapport, nous montrons cette conjecture lorsque T est de diamètre au plus 4.

Mots-clés : coloration dorsale, graphe planaire, arbre couvrant

1 Introduction

All the graphs considered in this paper are simple. Let $G = (V, E)$ be a graph, and let $H = (V, E(H))$ be a spanning subgraph of G . A k -colouring of G is a mapping $f : V \rightarrow \{1, 2, \dots, k\}$. Let f be a k -colouring of G . It is a *proper colouring* if $|f(u) - f(v)| \geq 1$. It is a *backbone colouring* for (G, H) if f is a proper colouring of G and $|f(u) - f(v)| \geq 2$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k -colouring of G . The *backbone colouring number* $BBC(G, H)$ is the smallest integer l for which there exists a backbone k -colouring of (G, H) .

If f is a proper k -colouring of G , then g defined by $g(v) = 2f(v) - 1$ is a backbone $(2k - 1)$ -colouring of (G, H) for any spanning subgraph H of G . Hence, $BBC(G, H) \leq 2\chi(G) - 1$. In [1, 2], Broersma et al. showed that for any integer k there is a graph G with a spanning tree T such that $BBC(G, T) = 2k - 1$.

The above inequality and the Four Colour Theorem implies that for any planar graph G and spanning subgraph H then $BBC(G, H) \leq 7$. However Broersma et al. [2] conjectured that this is not best possible if T is a tree.

Conjecture 1. *If G is a planar graph and T a spanning tree of G , then $BBC(G, T) \leq 6$.*

If true this conjecture would be best possible. Broersma et al. [2] gave an example of a graph G^* with a spanning tree T^* such that $BBC(G, T) = 6$. See Figure 1.

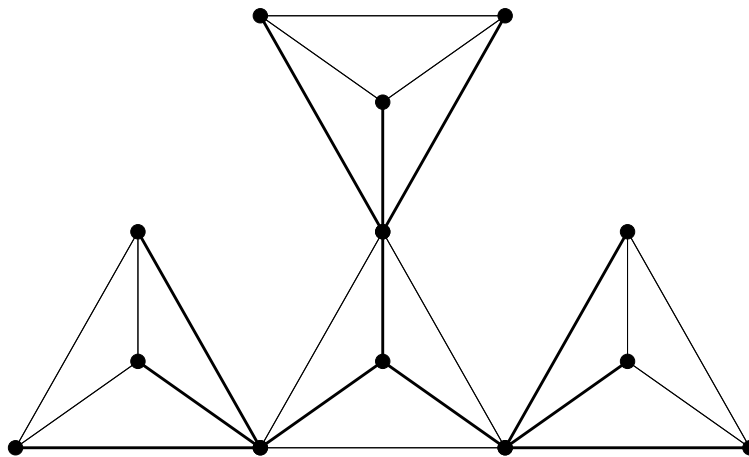


Figure 1: A planar graph G^* with a spanning tree T^* (bold edges) such that $BBC(G^*, T^*) = 6$.

Bu and Zhang [5] proved that, if G is a connected non-bipartite C_4 -free planar graph, then there exists a spanning tree T of G such that $BBC(G, T) = 4$. On the other hand, Bu and Li [4] proved that, if G is a connected planar graph that is C_6 -free or C_7 -free and without adjacent triangles, then there exists a spanning tree T of G such that $BBC(G, T) \leq 4$. In [6], Wang et al. investigated backbone colouring for special graph classes such as Halin graphs, complete graphs, wheels, graphs with small maximum average degree and graphs with maximum degree 3.

The *diameter* of a graph is the maximum distance between two vertices in this graph. If T has diameter 2, then it is a *star*, that is a tree in which a vertex v , called the *center*, is adjacent to every other. If G has a spanning star T , with center v , then $G - v$ is an outerplanar graph which can be properly 3-coloured with $\{1, 2, 3\}$. Thus assigning the colour 5 to v , we obtain a backbone 5-colouring of (G, T) . This result may be extended if G has a spanning tree with diameter at most 3.

Proposition 2. *Let G be a planar graph with a spanning tree T . If T has diameter at most three, then $BBC(G, T) \leq 5$.*

Proof. Free to add some edges, we may assume that G is triangulated. If T has diameter at most 3, then there exists two adjacent vertices x and y such that all edges of T are incident to x or y . Let z_1, \dots, z_p be the common neighbours of x and y , ordered in clockwise order around x (and so in anti-clockwise order around y). We consider an embedding of G with outer face xyz_1 .

For $1 \leq i \leq p-1$, let G_i be the graph induced by the vertices in the cycle $xz_i z_{i+1} y$ and inside, and let $H_i = G_i \setminus \{x, y\}$. Since G is triangulated, all the vertices are in at least one G_i . Furthermore, every H_i is outerplanar, and every vertex in $V(H_i) \setminus \{z_i, z_{i+1}\}$ is adjacent to exactly one of x, y .

We shall now define a backbone 5-colouring c of (G, T) .

First, we set $c(x) = 1$, $c(y) = 5$ and $c(z_1) = 3$. Next, we extend this colouring to the H_i one after another. Since H_i is outerplanar, it is 3-colourable. Let c_i be a proper 3-colouring of H_i in $\{2, 3, 4\}$ such that $c_i(z_i) = c(z_i)$ and $c_i(z_{i+1}) \in \{3, 4\}$ if $z_{i+1}x \in E(T)$ and $c_i(z_{i+1}) \in \{2, 3\}$ if $z_{i+1}y \in E(T)$. We set $c(z_{i+1}) = c_i(z_{i+1})$, and for every vertex v of $V(H_i) \setminus \{z_i, z_{i+1}\}$, we define

- $c(v) = c_i(v)$, if $c_i(v) = 3$, or $c_i(v) = 2$ and $vy \in E(T)$, or $c_i(v) = 4$ and $vx \in E(T)$;
- $c(v) = 5$, if $c_i(v) = 2$ and $vx \in E(T)$;
- $c(v) = 1$, if $c_i(v) = 4$ and $vy \in E(T)$.

It is easy to check that c is a backbone 5-colouring of (G, T) . □

Remark 3. Proposition 2 is best possible, because when G is a complete graph on four vertices and T a spanning star of G , $BBC(G, T) = 5$.

In this paper, we settle Conjecture 1 for tree with diameter at most 4.

Theorem 4. *Let G be a planar graph with a spanning tree T . If T has diameter at most 4, then $BBC(G, T) \leq 6$.*

Note that this result is best possible as the tree T^* in the above example has diameter 4.

In the next section, we outline the proof of Theorem 4 whose details are postponed to Section 3.

2 The proof

We denote by Z_6 the set $\{1, 2, 3, 4, 5, 6\}$ and, for any integer $a \in Z_6$, we denote by $[a]$ the set $\{a-1, a, a+1\} \cap Z_6$.

Let $G = (V, E)$ be a planar graph and T a spanning tree of G with diameter at most 4. T has a vertex r such that every vertex is at distance two from it in T . We call such a vertex the *root* of T . A vertex of $V \setminus \{r\}$, is a *twig* if it is adjacent to r in T and a *leaf* otherwise.

We shall prove a slightly stronger result than the one of Theorem 4.

Theorem 5. *(G, T) admits a backbone colouring in Z_6 such that the root is assigned 1.*

Proof. In the remaining, by (G, T) -colouring, one should understand a backbone 6-colouring of (G, T) such that r is assigned 1.

We will prove it by considering a minimum counterexample (G, T) with respect to its number of vertices. An edge of $E \setminus E(T)$ is said to be *thin*. Free to add some more thin edges, we may assume that G is triangulated.

If T has a unique twig, then it has diameter 2, and we have the result by the proof of Proposition 2. (The root corresponds to x_1 and the twig to x_2 .) Hence T has at least two twigs. We consider an embedding of G in the plane such that the outer face contains r and a minimum number of thin edges.

The *interior* (resp. *exterior*) of a cycle C , denoted C^{int} (resp. C^{ext}) is the subgraph of G induced by C and the vertices inside C (resp. outside C).

Let e be a thin edge. The graph $T \cup \{e\}$ has a unique cycle C_e (which contains e). The edge e is *overstepping* if there is a vertex inside C_e . In other words, $V(C_e^{int}) \neq V(C_e)$. Let O be the set of overstepping edges. There is a partial order \leq on O defined as follows: $e_1 \leq e_2$ if $e_1 = e_2$ or e_1 is inside C_{e_2} . Observe that the Hasse diagram of such a partial order is a set of at most two disjoint trees, each one rooted at an overstepping thin edge in the outer face. Indeed, it is easy to see that every overstepping edge e that is not maximal has a unique *successor* for \leq (i.e. overstepping edge f such that if $e \leq e' \leq f$ then $e' \in \{e, f\}$). This successor is one of the two edges of the face containing e contained in C_e^{ext} . Furthermore, every edge e has at most two *predecessors* for \leq : the two other edges of the face containing e contained in C_e^{int} .

The idea of the proof is to find a “good” overstepping edge e , such that a backbone 6-colouring of the graph induced by $V(C_e^{ext})$ (which exists by minimality of (G, T)) can be extended to $V(C_e^{int})$ to obtain a (G, T) -colouring. This will be a contradiction.

Natural candidates for such a good edge are overstepping edges e which are *minimal* for \leq (i.e. such that $e' \leq e$ implies $e' = e$) or their predecessors. However we will need to consider a more precise partial ordering. If there are two overstepping edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves and $e_4 \not\leq e_3$, (i.e. e_4 is not inside e_3), then we would like to have e_3 smaller than e_4 in the ordering. This leads to the following partial order \preceq : $e_1 \preceq e_2$ if $e_1 \leq e_2$ or there exist two edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves, $e_4 \not\leq e_3$, $e_1 \leq e_3$ and $e_4 \leq e_2$.

In the remainder of the paper, we will only consider the partial order \preceq . Hence the terms minimal, predecessor, successor, and so on refer to \preceq .

We first show some properties of minimal overstepping edges and deduce in Lemma 13 that if e is a minimal overstepping edge, then C_e^{int} is isomorphic to one of the graphs A_1 , A_2 and A_3 , depicted in Figure 2. In addition, if

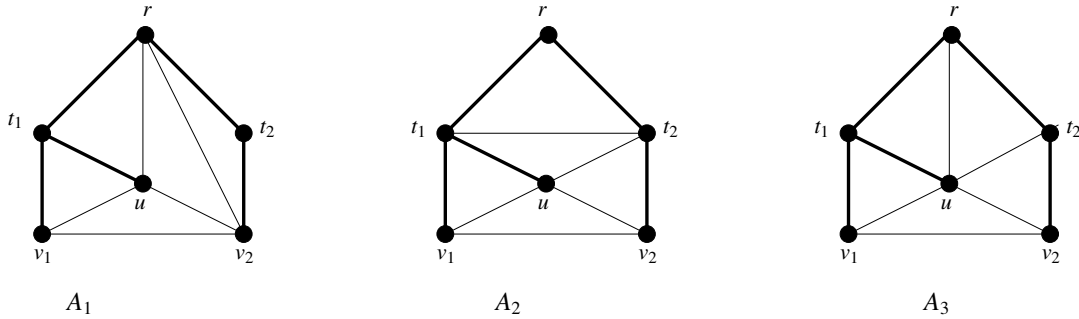


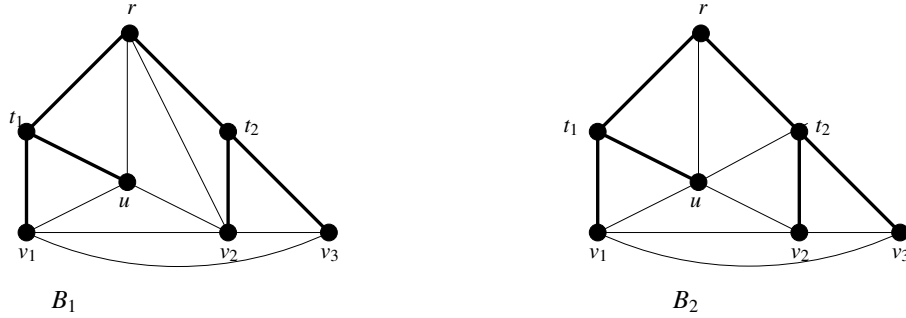
Figure 2: Configurations A_1 , A_2 and A_3

$C_e^{int} = A_1$, then $rv_1 \in E(G)$.

As any ordering, \preceq may be decomposed into levels. The first level L_1 the *maximal* edges for \preceq (i.e. such that $e \preceq e'$ implies $e' = e$). This level contains at most two edges, depending on the number of thin overstepping edges in the outer face. Then, for every $j \geq 1$, the level L_{j+1} is the set of predecessors of elements of L_j . The *depth* of \preceq , denoted D , is the maximum j such that L_j is not empty. An overstepping edge of L_D is said to be *ultimate*. An edge of L_{D-1} having at least one (ultimate) predecessor is said to be *penultimate*. An edge of L_{D-2} having at least one penultimate predecessor is said to be *antepenultimate*.

If f is a penultimate edge, then it has one or two predecessors. Furthermore each of this predecessors e is ultimate and so minimal. Thus C_e^{int} is isomorphic to A_1 , A_2 or A_3 . Analyzing all possible cases, we show (Corollary 16) that, if f is a penultimate edge, then C_f^{int} is isomorphic to B_1 or B_2 , and that moreover $rv_1 \in E(G)$ and $rv_3 \notin E(G)$.

Now if g is an antepenultimate edge, then it has one or two predecessors. Furthermore at least one of its predecessors f is penultimate (and so C_f^{int} is isomorphic to B_1 or B_2), and the other predecessor f' (if it exists) is

Figure 3: Configurations B_1 and B_2

either penultimate (so $C_{f'}^{int}$ is isomorphic to B_1 or B_2) or ultimate (so $C_{f'}^{int}$ is isomorphic to A_1, A_2 or A_3). Analyzing all the possible cases again, we show that there are no antepenultimate edges (Corollary 23).

Now, suppose that G contains at least one overstepping edge. If e is a minimal edge, then C_e^{int} is isomorphic to some configuration A_i . In any of these cases, there is at least one face containing the root and only one thin edge. Therefore, the partial order considered contains a unique maximal overstepping edge e_0 . Furthermore, since e_0 is not antepenultimate, $C_{e_0}^{int}$ must be isomorphic to one of the A_i or B_j configurations. We get a contradiction as the outer face contains r and the endpoints of e_0 and e_0 is the unique thin edge in this configuration and T would not be a tree.

We proved that G contains no overstepping edge. If the outer face of G contains only one thin edge, then G contains three vertices and the diameter of G is 2. If the outer face contains two thin edges e_1 and e_2 , then one thin edge (say e_1) is adjacent to r , since r is on the outer face, and the other (say e_2) is adjacent to a twig t while both are incident to a vertex v in the outer face. Now, both r and v have a twig v' as a common neighbour through edges of T as T is a spanning tree. Since neither e_1 nor e_2 are overstepping, then $V(G) = \{r, t, v, v'\}$ and G has diameter 3. Both of these cases are solved using Proposition 2 and both can give colour 1 to the root, a contradiction. \square

3 The details

Lemma 6. *Let x be a vertex of G . If $d_T(x) = 1$, then $d_G(x) \geq 4$.*

Proof. Suppose for a contradiction that $d_T(x) = 1$ and $d_G(x) \leq 3$. By minimality of (G, T) , there is a $(G-x, T-x)$ -colouring c . At x , at most 3 colours are forbidden by its neighbour in T and at most 2 colours are forbidden by its two other neighbours. So one colour of Z_6 is still available to colour the vertex x . Hence one can extend c to (G, T) , a contradiction. \square

3.1 Minimal overstepping edges

Lemma 7. *Let $e = uv$ be a minimal overstepping edge. Then there are at most two vertices inside C_e . Moreover if there are two, then they are adjacent in T and one of them is a twig and the other is a leaf.*

Proof. Since G is triangulated, uv is incident to two triangular faces, one of which, say F , is included in C_e^{int} . Let w be the third vertex incident to F . Let P be the path joining u to v in T and Q be the path joining w to P in T . Since T has diameter 4 and r is on the outer face, then Q has length at most 2.

Then C_e^{int} is divided into at most three regions: F , C_{uw}^{int} and C_{vw}^{int} (the region C_{uw}^{int} or C_{vw}^{int} may not exist if $uw \in E(T)$ or $vw \in E(T)$ respectively). As F is a face, its interior is empty, and there are no vertices inside C_{uw}^{int}

and C_{vw}^{int} because uw and vw are not overstepping since e is minimal. Hence the only possible vertices inside C_e are those of Q . Therefore there are at most two vertices inside C_e as Q has length at most 2.

Furthermore, if there are two vertices inside C_e , they must be adjacent as they are in Q . In addition, since r is on the outer face, none of these vertices is the root and thus one of them is a twig and the other is a leaf. \square

Lemma 8. *No minimal overstepping edge joins two leaves adjacent to a same twig.*

Proof. Suppose for a contradiction that an edge $e = uv$ joins two leaves adjacent to a same twig t . Then $C_e = tuvt$. The root r is not in C_e^{int} as it is on the outer face. So by Lemma 7 and because G is triangulated, C_e^{int} is a K_4 and there is a unique vertex x inside C_e . Hence, x contradicts Lemma 6. \square

Lemma 9. *No minimal overstepping edge joins two twigs.*

Proof. Suppose for a contradiction that two twigs s and t are joined by a minimal edge e . Then $C_e = rstr$. If there is a unique vertex u inside C_e^{int} , then u contradicts Lemma 6. So by Lemma 7, we may assume that the interior of C_e contains two adjacent vertices u_1 and u_2 and that u_1 is a twig and u_2 a leaf. By minimality of (G, T) , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c . Set $c(u_2) = 2$ and choose $c(u_1)$ in $Z_6 \setminus \{1, 2, 3, c(s), c(t)\}$. This yields a (G, T) -colouring, a contradiction. \square

Lemma 10. *No minimal overstepping edge joins the root and a leaf.*

Proof. Suppose for a contradiction that a minimal edge e joins the root r and a leaf v . Let t be the twig adjacent to v .

Suppose there is a unique vertex u inside C_e . Then this vertex has only 3 neighbours, and $d_T(u) = 1$. This contradicts Lemma 6. Hence by Lemma 7, we may assume that there are two adjacent vertices u_1 and u_2 inside C_e . Without loss of generality, u_2 is a leaf and u_1 is a twig. By Lemma 6, $d_G(u_2) \geq 4$, so $N_G(u_2) = \{u_1, r, v, t\}$. By minimality of G , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c . Let $c(u_2)$ be a colour in $\{2, 3\} \setminus \{c(v), c(t)\}$. (Such a colour exists because $|c(v) - c(t)| \geq 2$.) Now by planarity, u_1 has at most one neighbour x in $\{v, t\}$ as ru_2 is an edge. The set of forbidden colours in u_1 is $I = [1] \cup [c(u_2)] \cup \{c(x)\}$ which has cardinality at most 5 by the choice of $c(u_2)$. Hence assigning to u_1 a colour $c(u_1)$ in $Z_6 \setminus I$, we obtain a (G, T) -colouring, a contradiction. \square

Lemma 11. *No minimal overstepping edge joins a leaf and a twig.*

Proof. Suppose for a contradiction that a minimal overstepping edge $e = sv$ joins a twig s and a leaf v . Then $C_e = svtrs$. By Lemma 7 there are at most two vertices inside C_e .

Suppose that there is a unique vertex u inside C_e . As $d_T(u) = 1$, by Lemma 6, $d_G(u) \geq 4$. So $N_G(u) = \{r, s, t, v\}$. Note that rv or st is not an edge, by planarity. Then, removing u and contracting rv or st , we find by the minimality of G a $(G - u, T - u)$ -coloring c such that $c(v) = 1$ or $c(s) = c(t)$. Since the set of forbidden colours for u has at most 5 colours, one can extend c into a (G, T) -colouring, a contradiction.

Hence by Lemma 7, inside C_e there are a twig u_1 and leaf u_2 which are adjacent in T . As $d_T(u_2) = 1$, $d_G(u_2) \geq 4$ by Lemma 6.

- Suppose first that r is not adjacent to u_2 . By Lemma 6, $d_G(u_2) \geq 4$. So $N_G(u_2) = \{u_1, s, t, v\}$.

Hence u_1 is not adjacent to v by planarity. By minimality of (G, T) , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c . Assign to u_2 a colour $c(u_2)$ in $\{1, 2\} \setminus \{c(v)\}$. Observe that it is valid since s and t are not coloured in $\{1, 2\}$. Then the set of forbidden colours in u_1 is included in $\{1, 2, 3, c(s), c(t)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T) -colouring a contradiction.

- Suppose now that r is adjacent to u_2 .

By planarity, u_1 is adjacent to at most one vertex w in $\{s, t\}$. By minimality of (G, T) , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c .

If $c(v) \neq 2$, then set $c(u_2) = 2$. This is valid since s and t are not coloured 2. Then the set of forbidden colours in u_1 is included in $\{1, 2, 3, c(v), c(w)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T) -colouring a contradiction. Hence we may assume that $c(v) = 2$.

If no neighbour of u_2 is coloured 6, then set $c(u_2) = 6$. The set of forbidden colours in u_1 is then $\{1, 2, 5, 6, c(w)\}$ and so one can extend c into a (G, T) -colouring a contradiction. Hence we may assume that a neighbour y of u_2 is coloured 6.

If no neighbour of u_2 is coloured 3, then set $c(u_2) = 3$. The set of forbidden colours in u_1 is then $\{1, 2, 3, 4, c(w)\}$ and so one can extend c into a (G, T) -colouring a contradiction. Hence we may assume that a neighbour y of u_2 is coloured 3. But this neighbour cannot be t since $c(v) = 2$. Thus $c(s) = 3$ and $c(t) = 6$.

If $w = s$, that is if u_1 is not adjacent to t , then setting $c(u_1) = 6$ and $c(u_2) = 4$ yields a (G, T) -colouring, a contradiction.

If $w = t$, then setting $c(u_1) = 3$ and $c(u_2) = 5$ yields a (G, T) -colouring, a contradiction.

□

Lemma 12. *If e is a minimal overstepping edge joining two leaves, then there is one vertex inside C_e .*

Proof. Let $e = v_1v_2$ and for $i = 1, 2$, let t_i be the twig adjacent to v_i . By Lemma 8, $t_1 \neq t_2$. Since e is minimal and G is triangulated, $u_2v_1, u_2v_1 \in E(G)$.

Suppose for a contradiction that more than one vertex is inside C_e . Then, by Lemma 7, inside C_e , there are a twig u_1 and a leaf u_2 which are adjacent in T . Moreover, by Lemma 6, $d_G(u_2) \geq 4$ and so $d_G(u_1) \leq 5$.

Let us first suppose that ru_2 is not an edge. By symmetry, we may assume that u_1v_1 is not an edge. Set $G' = (G - \{u_1, u_2\}) \cup \{rv_1, rv_2\}$. By minimality of (G, T) , there is a $(G', T - \{u_1, u_2\})$ -colouring, which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c such that $c(v_1) \neq 1$ and $c(v_2) \neq 1$. Then setting $c(u_2) = 1$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, c(t_1), c(t_2), c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction. Hence we may assume that $ru_2 \in E(G)$. Then, since e is minimal, u_1v_1 is not an edge. By symmetry, we may assume that ru_2 is inside the cycle $rt_1v_1u_2u_1r$. Thus $N(u_1) \subset \{r, t_2, v_2, u_2\}$.

Assume now that rv_1 is not an edge. Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying r and v_1 . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c such that $c(v_1) = c(r) = 1$. If $c(v_2) \neq 2$, then setting $c(u_2) = 2$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_2), c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction. If $c(v_2) = 2$, then $c(t_2) \geq 4$. If $c(t_1) \neq 3$, then colour u_2 with 3 and u_1 with some colour in $\{5, 6\} \setminus \{c(t_2)\}$; otherwise, colour u_1 with 3 and u_2 with a colour in $\{5, 6\} \setminus \{c(t_2)\}$. In both cases, we obtain a (G, T) -colouring, a contradiction. Hence we may assume that $rv_1 \in E(G)$.

Assume that rv_2 is not an edge. Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying r and v_2 . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c such that $c(v_2) = c(r) = 1$. If there is a colour $\alpha \in \{2, 3, 6\}$ which does not appear on the neighbourhood of u_2 , then setting $c(u_2) = \alpha$ and colouring u_1 with a colour in $Z_6 \setminus (\{1, 2, c(t_2)\} \cup [\alpha])$, we obtain a (G, T) -colouring, a contradiction. So all the colours of $\{2, 3, 6\}$ appear on the neighbourhood of u_2 . Necessarily, in this case, u_2 is adjacent to v_1, t_1 and t_2 and $c(v_1) = 2, c(t_1) = 6$ and $c(t_2) = 3$. Then setting $c(u_2) = 4$ and $c(u_1) = 6$, we obtain a (G, T) -colouring, a contradiction. Hence we may assume that $rv_2 \in E(G)$.

We now distinguish several cases depending on the position of rv_1 and rv_2 regarding C_e .

1. Assume first that rv_1 and rv_2 are in C_e^{ext} . Then t_1t_2 is not an edge by planarity.

Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying t_1 and t_2 . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c such that $c(t_1) = c(t_2) = \alpha$. If $2 \notin \{c(v_1), c(v_2)\}$, then setting $c(u_2) = 2$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, 3, \alpha, c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction. Hence $2 \in \{c(v_1), c(v_2)\}$, so $\alpha \geq 4$.

If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then setting $c(u_2) = 3$ and colouring u_1 with a colour in $\{5, 6\} \setminus \{\alpha, c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction. Hence $\{c(v_1), c(v_2)\} = \{2, 3\}$, so $\alpha \geq 5$.

If $c(v_2) \neq 3$ or $u_1v_2 \notin E(G)$, then setting $c(u_1) = 3$ and colouring u_2 with a colour in $\{5, 6\} \setminus \{\alpha\}$, we obtain a (G, T) -colouring, a contradiction. Hence $c(v_2) = 3$ and $u_1v_2 \in E(G)$. By planarity, this implies that u_2t_2 is not an edge.

Observe that at least one of the two edges rv_1 and rv_2 is not overstepping otherwise one of them would be smaller than e in the order \preceq .

If rv_1 is not overstepping, then the interior of rt_1v_1 is empty. Hence $N_G(t_1) = \{r, v_1, u_2\}$. Setting $c(u_1) = 4$, $c(u_2) = 6$ and recolouring t_1 with 5, we obtain a (G, T) -colouring, a contradiction.

If rv_2 is not overstepping, then the interior of rt_2v_2 is empty. Hence $N_G(t_2) = \{r, u_1, v_2\}$. Setting $c(u_1) = 6$, $c(u_2) = 4$ and recolouring t_2 with 5, we obtain a (G, T) -colouring, a contradiction.

2. Assume that rv_1 and rv_2 are in C_e^{int} . Then $N_G(u_1) = \{r, u_2, v_2\}$. By minimality of (G, T) , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c . Colour u_2 with a colour $c(u_2)$ in $\{2, 3, 6\} \setminus \{c(v_1), c(v_2)\}$. Then the set of forbidden colours in u_1 is $\{1, 2, c(v_2)\} \cup [c(u_2)]$ which has cardinality at most 5 because $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4. Hence one can extend c into a (G, T) -colouring, a contradiction.
3. Assume that rv_1 is in C_e^{int} and rv_2 is in C_e^{ext} .

Assume that $d_G(u_2) = 5$, so $N_G(u_2) = \{r, u_1, v_1, v_2, t_2\}$ and $N_G(u_1) = \{r, t_2, u_2\}$. By minimality of (G, T) , there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c . If one can colour u_2 with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden for u_1 . Hence one can extend c into a (G, T) -colouring, a contradiction. So we may assume $\{c(t_2), c(v_1), c(v_2)\} = \{2, 3, 6\}$. If $c(t_2) = 6$, then setting $c(u_1) = 3$ and $c(u_2) = 5$, we obtain a (G, T) -colouring, a contradiction. If $c(t_2) \neq 6$, then setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a (G, T) -colouring, a contradiction.

Henceforth we may assume that $d_G(u_2) = 4$, so $N_G(u_2) = \{r, u_1, v_1, v_2\}$ and $N_G(u_1) = \{r, t_2, v_2, u_2\}$.

If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then one can colour u_2 with a colour in $\{2, 3\}$ and u_1 with a colour in $\{5, 6\} \setminus \{c(t_2), c(v_2)\}$ to obtain a (G, T) -colouring, a contradiction.

If $\{c(v_1), c(v_2)\} = \{2, 3\}$, then colouring u_1 with a colour $c(u_1)$ in $\{4, 6\} \setminus \{c(t_2)\}$ and u_2 with the colour in $\{4, 6\} \setminus \{c(u_1)\}$, we obtain a (G, T) -colouring, a contradiction.

4. Assume rv_2 is in C_e^{int} and rv_1 is in C_e^{ext} . Then $N_G(u_1) = \{r, u_2, v_2\}$ and $N_G(u_2) = \{r, u_1, t_1, v_1, v_2\}$. By minimality of (G, T) , there exists a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring c .

If $c(v_2) = 2$, then colouring u_2 with a colour $c(u_2)$ in $Z_6 \setminus \{1, 2, c(t_1), c(v_1)\}$ and u_1 with a colour in $\{3, 4, 5, 6\} \setminus [c(u_2)]$, we obtain a (G, T) -colouring, a contradiction. So we may assume that $c(v_2) \neq 2$.

If one can colour u_2 with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden in u_1 . Hence one can extend c into a (G, T) -colouring, a contradiction.

So we may assume $\{c(t_1), c(v_1), c(v_2)\} = \{2, 3, 6\}$. Necessarily, $c(v_1) = 2$, $c(v_2) = 3$ and $c(t_1) = 6$. Setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a (G, T) -colouring, a contradiction.

□

Lemma 13. *If e is a minimal overstepping edge, then C_e^{int} is one of the graphs depicted in Figure 2. In addition, if $C_e^{int} = A_1$, then $rv_1 \in E(G)$.*

Proof. Let e be a minimal edge. According to the previous lemmas, it has to join two leaves v_1 and v_2 and there is a unique vertex u inside C_e . For $i = 1, 2$, let t_i be the twig adjacent to v_i . By Lemma 8, $t_1 \neq t_2$.

- Assume first that u is a twig.

If $d_G(u) \leq 4$, then consider a $(G - u, T - u)$ -colouring c , which exists by minimality of (G, T) . In u , there are at most 5 colours forbidden as r is coloured 1, and thus forbids only two colours. Hence, one can extend c into a (G, T) -colouring, a contradiction.

So we may assume that $d_G(u) \geq 5$, and thus $N_G(u) = \{r, t_1, t_2, v_1, v_2\}$.

If rv_1 is not an edge, then let (G', T') be the pair obtained from $(G - u, T - u)$ by identifying r and v_1 . By minimality of (G, T) , there is a (G', T') -colouring, which is a $(G - u, T - u)$ -colouring such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in u is included in $\{1, 2, c(t_1), c(t_2), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T) -colouring, a contradiction.

Hence we may assume that rv_1 is an edge. This edge must be in C_e^{ext} by planarity of G . Thus $t_1 t_2$ is not an edge of G . Let (G', T') be the pair obtained from $(G - u, T - u)$ by identifying t_1 and t_2 . By minimality of (G, T) , there is a (G', T') -colouring c which is a $(G - u, T - u)$ -colouring such that $c(t_1) = c(t_2)$. Then the set of forbidden colours in u is included in $\{1, 2, c(t_1), c(v_1), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T) -colouring, a contradiction.

- Assume now that u is a leaf. By symmetry, we may assume that u is adjacent to t_1 . By Lemma 6 and since G is triangulated, C_e^{int} is one of the graphs A_1, A_2 or A_3 .

Assume now that $C_e^{int} = A_1$ and $rv_1 \notin E(G)$. Let (G', T') be the pair obtained from $(G - u, T - u)$ by identifying r and v_1 . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - u, T - u)$ -colouring c such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in u is included in $\{1, c(v_2)\} \cup [c(t_1)]$ and so has cardinality at most 5. Hence one can extend c into a (G, T) -colouring, a contradiction. \square

3.2 Penultimate edges

Lemma 14. *Let f be an edge which is the successor of a minimal edge e . If e is the unique predecessor of f , then C_f^{int} is one of the graphs depicted in Figure 3, and $rv_1 \in E(G)$. Moreover, if $C_f^{int} = B_2$, $rv_3 \notin E(G)$.*

Proof. Let e' be the third edge of the triangle bounded by f and e in C_f^{int} . Suppose, by way of contradiction, that e is the unique predecessor of f . Then e' is not overstepping. So all the vertices inside C_f are in C_e^{int} . By Lemma 13, C_e^{int} is one of the graphs A_1, A_2 or A_3 .

One of the endvertices of f must be v_1 and v_2 (as defined for A_i). We now distinguish many cases depending on C_e^{int} and the possible endvertices of f .

1. Assume that C_e^{int} is A_1 .
 - 1.1. Assume $f = rv_1$. Then the 4-cycle $rt_2v_2v_1$ has no chord, because rv_2 is in C_e^{int} and v_1t_2 is not an edge since f is the successor of e . This contradicts the fact that G is triangulated.
 - 1.2 Observe that $f = t_1v_2$ is impossible since rv_1 is an edge. Assume that $f = t_2v_1$. Let $G' = (G - \{u, v_2\}) \cup t_1t_2$. By minimality of (G, T) , there exists a $(G', T - \{u, v_2\})$ -colouring which is a $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring c such that $c(t_1) \neq c(t_2)$. If $c(t_1) = 6$, then one can greedily extend c to v_2 and then u to get a (G, T) -colouring, a contradiction. If $c(t_1) \neq 6$, then colouring v_2 with a colour in $\{c(t_1) -$

- $1, c(t_1) + 1\} \setminus [c(t_2)]$ and u with a colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ we obtain a (G, T) -colouring, a contradiction.
- 1.3. Assume that $f = v_1 t_3$ with t_3 a twig distinct from t_2 . Since rv_1 is an edge, $t_1 t_3$ is not an edge. Let G' be the graph pair obtained from $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$. If one can colour v_2 with a colour $c(v_2)$ in $\{2, 3, 6\}$, then $\{1, 2\} \cup [c(v_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden in t_2 . Hence one can extend c into a (G, T) -colouring, a contradiction. So we may assume that $\{c(u), c(v_1), c(t_3)\} = \{2, 3, 6\}$. If $c(t_3) = 3$, set $c(v_2) = 4$ and $c(t_2) = 6$. If $c(t_3) = 6$, set $c(v_2) = 5$ and $c(t_2) = 3$. In both cases, we obtain a (G, T) -colouring, a contradiction.
- 1.4. Assume that $f = v_2 t_3$ with t_3 a twig distinct from t_1 . By minimality of (G, T) , there exists a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c . Setting $c(t_1) = 6$ and choosing $c(v_1)$ in $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$ and $c(u)$ in $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$, we get a (G, T) -colouring, a contradiction.
- 1.5. f cannot be $v_2 v_3$ with v_3 a leaf adjacent to t_1 because rv_1 is an edge.
- 1.6. Assume that $f = v_1 v_3$ with v_3 a leaf adjacent to t_2 . Then $C_e^{int} = B_1$. By Lemma 13, $rv_1 \in E(G)$.
- 1.7. Assume $f = v_2 v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$. Then $v_1 v_3 \in E(G)$ and either $rv_3 \in E(G)$ or $t_3 v_1 \in E(G)$. Since rv_1 is an edge, we have that $N(t_1) = \{r, u, v_1\}$. By minimality of G , there exists a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c .
- If $\{c(t_3), c(v_3), c(v_2)\} \neq \{2, 3, 4\}$, then setting $c(t_1) = 6$ and choosing $c(v_1)$ in $\{2, 3, 4\} \setminus \{c(t_3), c(v_3), c(v_2)\}$ and $c(u)$ in $\{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction. Hence $\{c(t_3), c(v_3), c(v_2)\} = \{2, 3, 4\}$, and so $c(t_3) = 4$, $c(v_3) = 2$ and $c(v_2) = 3$. Then setting $c(t_1) = 3$, $c(u) = 5$ and $c(v_1) = 6$ yields a (G, T) -colouring, a contradiction.
- 1.8. Assume $f = v_1 v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$. Since $rv_1 \in E(G)$, then $t_1 t_3 \notin E(G)$.
- Assume first that $rv_3 \in C_f^{int}$. By minimality of (G, T) , there exists a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring c . One can choose a colour $c(v_2)$ in $Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}$ such that $I = [c(v_2)] \cup \{1, 2, c(v_3)\} \neq Z_6$. Then choosing $c(t_2) \in Z_6 \setminus I$, we obtain a (G, T) -colouring, a contradiction.
- Hence we may assume that rv_3 is not in C_f^{int} . Let (G', T') be the graph obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying t_1 and t_3 . By minimality of (G, T) , there exists a (G', T') -colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(t_1) = c(t_3)$. If $c(t_1) \neq 6$, then one can choose a colour $c(v_2) \in \{c(t_1) - 1, c(t_1) + 1\}$ such that $I = [c(v_2)] \cup \{1, 2, c(v_3)\} \neq Z_6$. Then choosing $c(t_2) \in Z_6 \setminus I$ and $c(u)$ in $Z_6 \setminus ([c(t_1)] \cup \{1, c(v_1)\})$, we obtain a (G, T) -colouring, a contradiction. Hence we may suppose that $c(t_1) = 6$. If $v_2 t_3 \notin E(G)$, then setting $c(v_2) = 6$ and choosing $c(t_2) \in \{3, 4\} \setminus c(v_3)$ and $c(u)$ in $Z_6 \setminus \{1, 5, 6, c(v_1)\}$ yields a (G, T) -colouring, a contradiction. If $v_2 t_3 \in E(G)$, then setting $c(v_2) = 5$, $c(t_2) = 3$ and choosing $c(u)$ in $Z_6 \setminus \{1, 5, 6, c(v_1)\}$ yields a (G, T) -colouring, a contradiction.
2. Assume that C_e^{int} is A_2 .
- 2.1. Assume $f = rv_1$. Since f is the successor of e , then $v_1 t_2$ is not an edge and so $rv_2 \in E(G)$ because G is triangulated. By minimality of G , there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c . Setting $c(u) = 1$, one can then extend c greedily to t_2 and v_2 to get a (G, T) -colouring, a contradiction.
- 2.2. Assume that $f = rv_2$. By minimality of (G, T) , there is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring. Setting $c(u) = 1$, one can then extend c greedily to t_1 and v_1 to get a (G, T) -colouring, a contradiction.
- 2.3. Assume that $f = t_1 v_2$. Since f is the successor of e , the cycle $t_1 v_1 v_2$ is empty, and so v_1 contradicts Lemma 6. Similarly, if $f = t_2 v_1$, then v_2 contradicts Lemma 6.

- 2.4. Assume that $f = v_1t_3$ with t_3 a twig distinct from t_2 . Since f is the successor of e , t_2v_1 is not an edge. Then either rv_2 is an edge or t_2t_3 is an edge. Set $G' = (G - \{u, t_2, v_2\}) \cup rv_1$. By minimality of (G, T) , there is a $(G', T - \{u, t_2, v_2\})$ -colouring c which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(v_1) \neq c(r) = 1$. Set $c(u) = 1$.
- If $c(v_1) \neq 2$, then setting $c(v_2) = 2$ and colouring t_2 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_1), c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. So $c(v_1) = 2$ and thus $c(t_1) \geq 4$.
- If $c(t_3) \neq 3$, then setting $c(t_2) = 3$ and choosing $c(v_2)$ in $\{5, 6\} \setminus \{c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. So $c(t_3) = 3$.
- Choosing $c(t_2)$ in $\{4, 6\} \setminus \{c(t_1)\}$ and $c(v_2)$ in $\{4, 6\} \setminus \{c(t_2)\}$, we obtain a (G, T) -colouring, a contradiction.
- 2.5. Assume that $f = v_2t_3$ with t_3 a twig distinct from t_1 . Then either rv_1 is an edge or t_1t_3 is an edge. Set $G' = (G - \{u, t_1, v_1\}) \cup rv_2$. By minimality of (G, T) , there exists a $(G', T - \{u, t_1, v_1\})$ -colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c such that $c(v_2) \neq c(r) = 1$. Set $c(u) = 1$.
- If $c(v_2) \neq 2$, then setting $c(v_1) = 2$ and colouring t_1 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_2), c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. So $c(v_2) = 2$ and thus $c(t_2) \geq 4$.
- If $c(t_3) \neq 3$, then setting $c(t_1) = 3$ and choosing $c(v_1)$ in $\{5, 6\} \setminus \{c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. So $c(t_3) = 3$.
- Choosing $c(t_1)$ in $\{4, 6\} \setminus \{c(t_2)\}$ and $c(v_1)$ in $\{4, 6\} \setminus \{c(t_1)\}$, we obtain a (G, T) -colouring, a contradiction.
- 2.6. Assume that $f = v_2v_3$ with v_3 a leaf adjacent to t_1 . Since f is the successor of e , then t_1v_2 is not inside $v_3t_1v_1v_2$ and so $v_1v_3 \in E(G)$. Set $G' = (G - \{u, v_1\}) \cup t_2v_3$. By minimality of (G, T) , there is a $(G', T - \{u, v_1\})$ -colouring which is a $(G - \{u, v_1\}, T - \{u, v_1\})$ -colouring c such that $c(t_2) \neq c(v_3)$. Setting $c(u) = c(v_3)$ and colouring v_1 with a colour in $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$, we obtain a (G, T) -colouring, a contradiction.
- 2.7. Assume that $f = v_1v_3$ with v_3 a leaf adjacent to t_2 . Since f is the successor of e , then t_2v_1 is not inside $v_2t_2v_3v_1$ and so $v_2v_3 \in E(G)$. Set $G' = (G - \{u, v_2\}) \cup t_1v_3$. By minimality of (G, T) , there is a $(G', T - \{u, v_2\})$ -colouring, which is a $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring c such that $c(t_1) \neq c(v_3)$. If $c(t_2) \in [c(t_1)]$, then one can also extend c greedily to v_2 and then u to obtain a (G, T) -colouring, a contradiction. Hence $|c(t_1) - c(t_2)| \geq 2$. Thus one can colour v_2 with $c(t_1)$ and then colour u with a colour in $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$. This yields a (G, T) -colouring, a contradiction.
- 2.8. Assume $f = v_2v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.
- Suppose first that $rv_1 \notin E(G)$. By minimality of (G, T) , there is a $(G - \{u, t_1, v_1\} \cup \{rv_3, rv_2\}, T - \{u, t_1, v_1\})$ -colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c such that $c(v_2) \neq 1$ and $c(v_3) \neq 1$. Colour v_1 with 1 and let $L(t_1) \supseteq Z_6 \setminus \{1, 2, c(v_3), c(t_3), c(t_2)\}$ and $L(u) = Z_6 \setminus \{1, c(t_2), c(v_2)\}$ be the list of colours available for t_1 and u , respectively. Note that there is at most one colour α in Z_6 such that $L(u) \setminus [\alpha] = \emptyset$. Thus, if there exists β in $L(t_1) \setminus \{\alpha\}$ if such α exists, or in $L(t_1)$ otherwise, then we can colour t_1 with β and u with a colour in $L(u) \setminus [\beta]$ to obtain a (G, T) -colouring, a contradiction. So we may assume that no such β exists, that is $L(t_1) = \{\alpha\}$ and $L(u) \setminus [\alpha] = \emptyset$. Since $|c(v_2) - c(t_2)| \geq 2$, necessarily $\alpha = 4$, $L(t_1) = \{4\}$, $c(t_2) = 6$, $c(v_2) = 2$, $\{c(v_3), c(t_3)\} = \{3, 5\}$ and $v_3, t_3 \in N(t_1)$. Then, recolouring v_1 with 6 and colouring t_1 with 4 and u with 1 yields a (G, T) -colouring, a contradiction.
- Suppose now that $rv_1 \in E(G)$. Then there is no vertex inside rt_1v_1r . By minimality of (G, T) , there is $(G - u, T - u)$ -colouring c . If $c(v_2) \neq 1$, then we can colour u with 1; so, suppose otherwise. If there is no colour available for u to extend c , then $F_c = \{1, c(t_2), c(v_1)\} \cup [c(t_1)]$ is equal to Z_6 ; thus, $c(t_1) \in \{3, 4, 5\}$. If $c(t_1) = 3$, then $\{c(v_1), c(t_2)\} = \{5, 6\}$. If $c(t_1) = 4$, then $\{c(v_1), c(t_2)\} = \{2, 6\}$. If

$c(t_1) = 5$, then $\{c(v_1), c(t_2)\} = \{2, 3\}$. If the colour of t_1 can be changed, we obtain a $(G - u, T - u)$ -colouring c' such that $F_{c'} \neq Z_6$ which can be extended in a (G, T) -colouring, a contradiction. Hence, $c(t_1) = i$ is the sole colour in $Z_6 \setminus (\{1, 2, c(t_2)\} \cup [c(v_1)])$. Thus, $c(v_1) \neq 2$ and $(c(v_1), c(t_2)) \neq (6, 5)$. Then, necessarily (*) $c(v_1) = 5$, $c(t_1) = 3$ and $c(t_2) = 6$. If $c(t_3), c(v_3) \neq 3$, then recolour t_1 with 5 and v_1 with 3. Otherwise, if $c(t_3), c(v_3) \neq 6$, then recolour v_1 with 6. Otherwise (i.e., $\{c(t_3), c(v_3)\} = \{3, 6\}$), recolour v_1 with 2. In any case, the resulting colouring c_1 does not satisfy (*). Hence, either $F_{c_1} \neq Z_6$ or t_1 can be recoloured to get a colouring c'_1 such that $F_{c'_1} \neq Z_6$. Hence one of c_1, c'_1 can be extended in a (G, T) -colouring, a contradiction.

2.9. Assume $f = v_1 v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Suppose first that $rv_2 \in E(G)$. Set $G' = (G - \{u, t_2, v_2\}) \cup \{t_1 t_3, t_1 v_3\}$. By minimality of (G, T) , there is a $(G', T - \{u, t_2, v_2\})$ -colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(t_1) \neq c(t_3)$ and $c(t_1) \neq c(v_3)$. Set $c(v_2) = c(t_1)$. Then choosing $c(t_2)$ in $\{3, 4, 5, 6\} \setminus [c(t_1)]$ and $c(u)$ in $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$, we obtain a (G, T) -colouring, a contradiction. Hence $rv_2 \notin E(G)$.

Suppose now that $rv_3 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying v_3 and r . By minimality of (G, T) , there is a (G', T') -colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(v_3) = 1$. Set $c(u) = c(v_3)$. For t_2 , there at least two possible colours, namely the ones not in $\{1, 2, c(t_1), c(t_3)\}$. One of them, say α , is such that $I = [\alpha] \cup \{1, c(v_1), c(t_3)\}$ is not equal to Z_6 . Thus, setting $c(t_2) = \alpha$ and choosing $c(v_2)$ in $Z_6 \setminus I$, we obtain a (G, T) -colouring, a contradiction. Hence $rv_3 \in E(G)$.

Assume that rv_3 is inside C_f . Then $t_2 t_3 \notin E(G)$. By minimality of (G, T) , there is a $(G - \{u, t_2, v_2\}) \cup rv_1, T \setminus \{u, t_2, v_2\}$ -colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(v_1) \neq 1$. If $c(v_1) \neq 2$, then setting $c(u) = 2$ and $c(v_2) = 1$, and choosing $c(t_2)$ in $Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}$, we obtain a (G, T) -colouring, a contradiction. So we may assume that $c(v_1) = 2$, and so $c(t_1) \geq 4$. If $c(v_3) \neq 3$, then setting $c(u) = 1$, $c(t_2) = 3$ and choosing $c(v_2)$ in $\{5, 6\} \setminus \{c(v_3)\}$, we obtain a (G, T) -colouring, a contradiction. If $c(v_3) = 3$, then setting $c(u) = 1$, and choosing $c(t_2)$ in $\{4, 6\} \setminus \{c(t_1)\}$ and $c(v_2)$ in $\{4, 6\} \setminus \{c(t_2)\}$, we obtain a (G, T) -colouring, a contradiction. Hence we may assume that rv_3 is outside C_f .

So, by planarity, $t_1 t_3 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying t_1 and t_3 . By minimality of (G, T) , there is a (G', T') -colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(t_1) = c(t_3)$. Set $c(u) = c(v_3)$. Let α be a colour of $Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}$ such that $I = [\alpha] \cup \{c(v_1), c(v_3), c(t_3)\}$ is not Z_6 . Then setting $c(t_2) = \alpha$ and choosing $c(v_2)$ in $Z_6 \setminus I$, we obtain a (G, T) -colouring, a contradiction.

3. Assume that C_e^{int} is A_3 .

3.1. Assume $f = rv_1$. Then rv_2 is an edge. By minimality of G , there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c . Colour u with a colour $c(u)$ in $Z_6 \setminus (\{c(r), c(v_1)\} \cup [c(t_1)])$. Set $c(t_2) = 6$ if $c(u) \neq 6$ and $c(t_2) = 5$ otherwise. In both cases, at most five colours are forbidden for v_2 , and one can extend greedily the colouring into a (G, T) -colouring, a contradiction.

3.2. Assume that $f = rv_2$. By minimality of G , there exists a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c . Set $c(t_1) = 6$, then colour u with any colour in $Z_6 \setminus \{1, 5, 6, c(t_2), c(v_2)\}$ and v_1 with any colour in $Z_6 \setminus \{1, 5, 6, c(v_2), c(u)\}$. This yields a (G, T) -colouring, a contradiction.

3.3 Assume that $f = t_1 v_2$. Then the cycle $t_1 v_1 v_2$ is empty, an so v_1 contradicts Lemma 6. Similarly, if $f = t_2 v_1$, then v_2 contradicts Lemma 6.

3.4. Assume that $f = v_1t_3$ with t_3 a twig distinct from t_2 .

Assume first that $t_2t_3 \in E(G)$. Set $G' = (G - \{u, t_2, v_2\}) \cup rv_1$. By minimality of (G, T) , there is a $(G', T - \{u, t_2, v_2\})$ -colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(v_1) \neq c(r) = 1$. Setting $c(v_2) = 1$ and choosing $c(u)$ in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, c(u), c(t_3)\}$, we get a (G, T) -colouring, a contradiction. So $t_2t_3 \notin E(G)$ and thus $rv_2 \in E(G)$.

By minimality of G , there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c .

Assume that $c(v_1) \neq 2$. If $c(t_1) = 3$, then setting $c(v_2) = 2$ and choosing $c(u)$ in $Z_6 \setminus \{1, 2, 3, 4, c(v_1)\}$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, 3, c(u)\}$ yields a (G, T) -colouring, a contradiction. If $c(t_1) \geq 4$, then setting $c(u) = 2$ and choosing $c(v_2)$ in $Z_6 \setminus \{1, 2, c(v_1), c(t_3)\}$ and $c(t_2)$ in $Z_6 \setminus (\{1, 2\} \cup [c(v_2)])$, we obtain a (G, T) -colouring, a contradiction. Hence $c(v_1) = 2$.

If $c(t_1) \neq 4$, then colouring v_2 with $c(v_2) \in \{4, 6\} \setminus \{c(t_3)\}$, t_2 with $c(t_2) \in \{4, 6\} \setminus \{c(v_2)\}$ and u with $c(u)$ in $\{3, 5\} \setminus [c(t_1)]$, we get a (G, T) -colouring, a contradiction. So $c(t_1) = 4$.

Colouring u with 6, v_2 with $c(v_2) \in \{3, 5\} \setminus \{c(t_3)\}$ and t_2 with $c(t_2)$ in $\{3, 5\} \setminus [c(v_2)]$, we get a (G, T) -colouring, a contradiction.

3.5. Assume that $f = v_2t_3$ with t_3 a twig distinct from t_1 .

Assume first that t_1t_3 is an edge. Set $G' = (G - \{u, t_1, v_1\}) \cup rv_2$. By minimality of (G, T) , there is a $(G', T - \{u, t_1, v_1\})$ -colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c such that $c(v_2) \neq c(r) = 1$. Set $c(v_1) = 1$. If $c(v_2) \neq 2$, then setting $c(u) = 2$ and assigning to t_1 a colour in $Z_6 \setminus \{1, 2, 3, c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. So $c(v_2) = 2$ and $c(t_2) \geq 4$. Setting $c(u) = 3$ and assigning to t_1 a colour in $Z_6 \setminus \{1, 2, 3, 4, c(t_3)\}$, we obtain a (G, T) -colouring, a contradiction. Hence t_1t_3 is not an edge.

So rv_1 is an edge. Let G' be the graph from $G - \{u, t_1, v_1\}$ by adding the edge t_2t_3 if it does not exist. By minimality of (G, T) , there is a $(G', T - \{u, t_1, v_1\})$ -colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c such that $c(t_2) \neq c(t_3)$. Set $c(t_1) = 6$. If $c(t_2) \notin \{5, 6\}$, then set $c(v_1) = c(t_2)$ (this is possible because $c(t_3) \neq c(t_2)$), otherwise colour v_1 with any colour in $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$. Then colouring u with a colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$, we get a (G, T) -colouring, a contradiction.

3.6. Assume that $f = v_2v_3$ with v_3 a leaf adjacent to t_1 . Set $G' = (G - \{u, v_1\}) \cup \{t_2v_3, rv_3\}$. By minimality of (G, T) , there is a $(G', T - \{u, v_1\})$ -colouring which is a $(G - \{u, v_1\}, T - \{u, v_1\})$ -colouring c such that $c(v_3) \notin \{c(r), c(t_2)\}$. Setting $c(u) = c(v_3)$ and colouring v_1 with a colour in $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$, we obtain a (G, T) -colouring, a contradiction.

3.7. Assume that $f = v_1v_3$ with v_3 a leaf adjacent to t_2 . Then $C_f^{int} = B_2$.

Assume first that $rv_1 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - u, T - u)$ by identifying v_1 and r . By minimality of (G, T) , there is a (G', T') -colouring, which yields a $(G - u, T - u)$ -colouring such that $c(v_1) = c(r) = 1$. Then choosing $c(u)$ in $Z_6 \setminus (\{1, c(v_2)\} \cup [c(t_1)])$ yields a (G, T) -colouring, a contradiction. Hence $rv_1 \in E(G)$.

Assume now that $rv_3 \in E(G)$. Let (G', T') be the graph pair obtained from $(G - u, T - u)$ by identifying t_1 and t_2 . By minimality of (G, T) , there is a (G', T') -colouring, which yields a $(G - u, T - u)$ -colouring such that $c(t_1) = c(t_2)$. Then setting $c(u) = c(v_3)$, we obtain a (G, T) -colouring, a contradiction.

3.8. Assume $f = v_2v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Assume first that rv_1 is not an edge. By minimality of (G, T) , there is a $(G - \{u, t_1, v_1\}) \cup \{rv_2, rv_3\}, T - \{u, t_1, v_1\})$ -colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring c such that $c(v_2) \neq 1$ and $c(v_3) \neq 1$. Colour v_1 with 1. Colour t_1 with a colour α in $A = Z_6 \setminus \{1, 2, c(t_3), c(v_3)\}$ such that $[\alpha] \neq Z_6 \setminus$

$\{1, c(t_2), c(v_2)\}$. This is possible since $|A| \geq 2$. Then colouring u with a colour in $Z_6 \setminus (\{1, c(t_2), c(v_2)\} \cup [\alpha])$, we obtain a (G, T) -colouring, a contradiction.

Suppose now that rv_1 is an edge. Then, since e is minimal, rv_1 is not overstepping and so there is no vertex inside rv_1t_1r . Let (G', T') be the graph pair obtained from $(G - u, T - u)$ by identifying t_1 and t_2 . By minimality of (G, T) , there is a (G', T') -colouring, which yields a $(G - u, T - u)$ -colouring such that $c(t_1) = c(t_2)$. Set $c(u) = 1$ if $c(v_2) \neq 1$, and choose $c(u)$ in $Z_6 \setminus (\{1, c(v_1), c(v_2)\} \cup [c(t_1)])$ otherwise. This gives a (G, T) -colouring, a contradiction.

3.9. Assume $f = v_1v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Suppose first that $rv_2 \notin E(G)$. By minimality of (G, T) , there is a $(G - \{t_2, v_2\} \cup \{rv_1, rv_3\}, T - \{t_2, v_2\})$ -colouring which is a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring c such that $c(v_1) \neq 1$ and $c(v_3) \neq 1$. Setting $c(v_2) = 1$ and choosing $c(t_2)$ in $Z_6 \setminus \{1, 2, c(u), c(t_3), c(v_3)\}$ yields a (G, T) -colouring, a contradiction. Hence $rv_2 \in E(G)$.

Assume that $v_2t_3 \notin E(G)$. By minimality of (G, T) , there is a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring. We can choose $c(v_2)$ in $Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}$ such that $I = [c(v_2)] \cup \{1, 2, c(u)\} \neq Z_6$ and $c(t_2) \in Z_6 \setminus I$ to obtain a (G, T) -colouring, a contradiction. Hence $v_2t_3 \in E(G)$.

By minimality of (G, T) , there is a $(G - \{u, t_2, v_2\} \cup \{t_1v_3\}, T - \{u, t_2, v_2\})$ -colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c such that $c(t_1) \neq c(v_3)$. Set $c(u) = c(v_3)$. We can choose $c(v_2)$ in $Z_6 \setminus \{1, c(t_3), c(v_1), c(v_3)\}$ such that $I = [c(v_2)] \cup \{1, 2, c(u)\} \neq Z_6$ and $c(t_2) \in Z_6 \setminus I$ to obtain a (G, T) -colouring, a contradiction. □

Lemma 15. *Every penultimate edge has a unique predecessor.*

Proof. By contradiction. Suppose that a penultimate edge f has two predecessors e and e' . Then e and e' are ultimate and so minimal. According to Lemma 13, C_e^{int} and $C_{e'}^{int}$ are isomorphic to some of A_1, A_2 and A_3 . Let us denote the vertices of C_e^{int} by their names in Figure 2 and the vertices of $C_{e'}^{int}$ by their names in Figure 2 augmented with a prime.

Since f, e and e' are bounding the face incident to f in C_f^{int} , the edge f is $v_1v'_2, v_1v'_1, v_2v'_2$ or $v_2v'_1$. If $f = v_2v'_1$, then swapping the names of e and e' , we are left with $f = v_1v'_2$. Hence we may assume that $f \in \{v_1v'_2, v_1v'_1, v_2v'_2\}$. Note that if $f = v_1v'_2$, then $t_2 = t'_1$ and $v_2 = v'_1$, if $f = v_1v'_1$, then $t_2 = t'_2$ and $v_2 = v'_2$, and if $f = v_2v'_2$, then $t_1 = t'_1$ and $v_1 = v'_1$.

Observe that if C_e^{int} is isomorphic to A_1 , then f cannot be $v_2v'_2$ because rv_1 must be an edge that would cross f . Moreover if C_e^{int} and $C_{e'}^{int}$ are both isomorphic to A_1 , then f cannot be $v_1v'_1$ since G has no multiple edges. Hence must be in one of the following cases:

- C_e^{int} and $C_{e'}^{int}$ are isomorphic to A_1 and $f = v_1v'_2$.

By minimality of G , there is a $(G - \{u', t_2, v_2\}, T - \{u', t_2, v_2\})$ -colouring c . Let $L(v_2) = Z_6 \setminus \{1, c(u), c(v_1), c(v'_2)\}$ be the colours available for v_2 and $L(u') = Z_6 \setminus \{1, c(v'_2)\}$ be the colours available for u' . Observe that $|L(v_2)| \geq 2$ and at least one colour in $L(v_2)$, say α , is such that one integer $\beta \in \{\alpha - 1, \alpha + 1\}$ is in $L(u')$.

If $\{\alpha, \beta\} \neq \{4, 5\}$, then $\{1, 2\} \cup [\alpha] \cup [\beta] \neq Z_6$. Hence colouring v_2 with α , u' with β and t_2 with a colour in $Z_6 \setminus (\{1, 2\} \cup [\alpha] \cup [\beta])$, we obtain a (G, T) -colouring, a contradiction.

If $\alpha = 4$ and $\beta = 5$, then $c(v'_2) = 3$ for otherwise we could have chosen $\beta = 3$ and got a contradiction as above. Then setting $c(v_2) = 4$, $c(u') = 2$ and $c(t_2) = 6$, we get a (G, T) -colouring, a contradiction.

If $\alpha = 5$ and $\beta = 4$, then $c(v'_2) = 6$ for otherwise we could have chosen $\beta = 6$ and got a contradiction as above. Then setting $c(v_2) = 5$, $c(u') = 3$ and $c(t_2) = 6$, we get a (G, T) -colouring, a contradiction.

- C_e^{int} is isomorphic to A_1 , $C_{e'}^{int}$ is isomorphic to A_2 and $f = v_1v'_i$ for $i \in \{1, 2\}$.

By minimality of (G, T) , there is a $(G - \{t_2, v_2, u'\} \cup rv'_i, T - \{t_2, v_2, u'\})$ -colouring which is a $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$ -colouring c such that $c(v'_i) \neq 1$. Colour u' with 1. If $\{c(v_1), c(u), c(v'_i)\} \setminus \{2, 3, 6\}$, then assign to v_2 an element $c(v_2)$ of this set. Otherwise assign to $c(v_2)$ a colour in $\{4, 5\} \cap [c(t'_i)]$. Then one can choose $c(t_2)$ in $Z_6 \setminus (\{1, 2, c(t'_i)\} \cup [c(v_2)])$ to obtain a (G, T) -colouring, a contradiction.

- C_e^{int} is isomorphic to A_1 , $C_{e'}^{int}$ is isomorphic to A_3 and $f = v_1v'_i$ for $i \in \{1, 2\}$.

Suppose first that rv'_i is not an edge. Let (G', T') be the graph pair obtained from $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$ by identifying v'_i and r . By minimality of (G, T) , there exists a (G', T') -colouring which is a $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$ -colouring such that $c(v'_i) = 1$. We can choose $c(v_2)$ in $\{3, 4, 6\} \setminus \{c(u), c(v_1)\}$ and $c(t_2)$ in $\{4, 5, 6\} \setminus [\alpha]$. If $i = 2$ or $c(t'_2) \neq 3$, then setting $c(u') = 1$ yields a (G, T) -colouring, a contradiction. If $i = 1$ and $c(t'_2) = 3$, then choosing $c(u')$ in $\{5, 6\} \setminus \{c(t_2), c(v_2)\}$ yields a (G, T) -colouring, a contradiction.

Hence rv'_i is an edge. The two edges rv_1 and rv'_i cannot cross. Thus, for at least one edge $e^* \in \{rv_1, rv'_i\}$, $v_1v'_i$ is not inside $C_{e^*}^{int}$. Hence e^* is not overstepping for otherwise $e^* \preceq e$ or $e^* \preceq e'$ contradicting the minimality of e . Hence at least one of the two edges rv_1 and rv'_i is not overstepping. We analyze the two possible cases.

- rv_1 is not overstepping. In this case, there is no vertex inside C_{rv_1} and thus the only neighbours of t_1 are r, v_1 and u .

Let us first prove (\star) : *there is no $(G - \{u\}, T - \{u\})$ -colouring such that $c(v_2) \notin \{5, 6\}$* . Indeed suppose there is such a colouring c . The set of colours forbidden for u is $F = \{1, c(v_1), c(v_2)\} \cup [c(t_1)]$. Then $F = Z_6$, for otherwise colouring u with a colour in $Z_6 \setminus F$ yields a (G, T) -colouring, a contradiction. Thus because $c(v_2) \leq 4$, the colour triple $(c(t_1), c(v_1), c(v_2))$ is either $(4, 6, 2)$, or $(5, 3, 2)$, or $(5, 2, 3)$. In the last two cases, recolouring t_1 with 6 and colouring u with 4 we obtain a (G, T) -colouring, a contradiction. In the first case, recolouring t_1 with 3 and colouring u with 5 we obtain a (G, T) -colouring, a contradiction. This proves (\star) .

Suppose that $i = 2$, that is $t_2 = t'_1$ and $v_2 = v'_1$. By minimality of G , there is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring c . Choose $c(u')$ in $\{2, 3\} \setminus \{c(t'_2), c(v'_2)\}$. If there exists $\alpha \in \{2, 3, 4\} \setminus \{c(u'), c(v_1), c(v'_2)\}$, then setting $c(t_2) = 6$ and $c(v_2) = \alpha$, we get a contradiction to (\star) . Hence $\{c(u'), c(v_1), c(v'_2)\} = \{2, 3, 4\}$. Observe that we can suppose that $c(v_1) = 4$, for otherwise we could have chosen $c(u')$ to be equal to $c(v_1)$. Thus $c(t_1) = 6$. Then recolour u' with some colour in $\{5, 6\} \setminus \{c(t'_2)\}$, colour v_2 with the colour in $\{5, 6\} \setminus \{c(u')\}$, t_2 with 3 and u with 2. This gives a (G, T) -colouring, a contradiction.

Suppose now that $i = 1$, that is $t_2 = t'_2$ and $v_2 = v'_2$. By minimality of G , there is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring c .

- * Assume first that $c(u') = 6$, then $(c(t'_1), c(v'_1))$ is either $(4, 2)$ or $(3, 5)$. If $c(v_1) \neq 3$, then setting $c(v_2) = 3$ and $c(t_2) = 5$, we get a $(G - \{u\}, T - \{u\})$ -colouring contradicting (\star) . So $c(v_1) = 3$ and thus $c(t_1) \geq 5$.

If $(c(t'_1), c(v'_1)) = (3, 5)$, then setting $c(v_2) = 2$ and $c(t_2) = 4$, we also obtain a $(G - \{u\}, T - \{u\})$ -colouring contradicting (\star) .

Hence $(c(t'_1), c(v'_1)) = (4, 2)$. Setting $c(v_2) = 5$, $c(t_2) = 3$ and $c(u) = 2$, we obtain a (G, T) -colouring, a contradiction.

- * Suppose now that $c(u') \neq 6$. If there exists $\alpha \in \{2, 3, 4\} \setminus \{c(u'), c(v_1), c(v'_1)\}$, then colouring v_2 with α and t_2 with 6, we obtain a $(G - \{u\}, T - \{u\})$ -colouring contradicting (\star) . If not, then one can recolour t_1 with 6. Then colouring v_2 with 6, t_2 with some colour in $\{3, 4\} \setminus \{c(u')\}$ and u with a colour in $Z_6 \setminus \{1, 5, 6, c(v_1)\}$, we obtain a (G, T) -colouring, a contradiction.

- rv'_i is not overstepping. In this case, there is no vertex inside $C_{rv'_i}$ and thus the only neighbours of t'_i are r , v'_i and u' .

Suppose first that $i = 2$. By minimality of G , there is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring c . Colour u with a colour $c(u)$ in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ such that $\{c(u), c(v_1)\} \neq \{3, 4\}$. (This is possible since if $\{c(u), c(v_1)\} = \{3, 4\}$, then $c(t_1) = 6$ and so u can be recoloured 2.) if $c(v'_2) \leq 4$, then recolour t'_2 with 6, colour v_2 with 6, u' with 5 and t_2 with 3. If $c(v'_2) \in \{5, 6\}$, then colour v_2 with some colour in $\{3, 4\} \setminus \{c(u), c(v_1)\}$, u' with 2 and t_2 with 6. In both cases, we obtain a (G, T) -colouring, a contradiction.

Suppose now that $i = 1$, that is $t_2 = t'_2$ and $v_2 = v'_2$. By minimality of (G, T) , there exists a $(G - \{u, u', t_2, v_2\} \cup \{t_1 v'_1\}, T - \{u, u', t_2, v_2\})$ -colouring c which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring such that $c(t_1) \neq c(v'_1)$. Colour u with some colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and u' with some colour in $Z_6 \setminus (\{1, c(v'_1)\} \cup [c(t'_1)])$. Let $L(v_2) = Z_6 \setminus \{1, c(v_1), c(v'_1), c(u), c(u')\}$ be the set of colours available for v_2 . If there exists $\alpha \in L(v_2)$ such that $L(t_2) = Z_6 \setminus (\{1, 2, c(u')\} \cap [\alpha]) \neq \emptyset$, then we can colour v_2 with α and t_2 with a colour in $L(t_2)$ to get a (G, T) -colouring, a contradiction. Hence either (a) $c(u') = 3$ and $L(v_2) = \{5\}$ or (b) $c(u') = 6$ and $L(v_2) = \{4\}$. Note that $\{3, 6\} \not\subseteq Z_6 \setminus (\{1, c(v'_1)\} \cup [c(t'_1)])$; hence, if we can recolour of u' , neither (a) nor (b) occurs anymore. So, we may assume that u' cannot be recoloured. If (a) occurs, this implies that $c(v'_1) = 2$ and $c(t'_1) = 5$. But then, we recolour t'_1 with 6 and u' with 4; clearly, neither (a) nor (b) occurs anymore and we get a contradiction as above. If (b) occurs, then either $c(v'_1) = 2$ and $c(t'_1) = 4$, or $c(v'_1) = 5$ and $c(t'_1) = 3$. In the former case, we recolour t'_1 with 6 and u' with 4; clearly, neither (a) nor (b) occurs anymore and we get a contradiction as above. If the later case occurs, since $L(v_2) = \{4\}$, necessarily $\{c(v_1), c(u)\} = \{2, 3\}$. If it is possible to recolour u , then $L(v_2)$ would have more than one colour and, consequently, neither (a) nor (b) would occur yielding a contradiction. Hence $\{c(v_1), c(u), 1\} = Z_6 \setminus [c(t_1)]$, so $c(t_1) = 5 = c(v'_1)$ which is impossible by our choice of c .

- C_e^{int} is isomorphic to A_2 or A_3 , $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_1 v'_1$.

By minimality of (G, T) , there is a $(G - \{u, u', t_2, v_2\} \cup \{rv_1, rv'_1\}, T - \{u, u', t_2, v_2\})$ -colouring c which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring c such that $c(v_1), c(v'_1) \neq 1$. Colour u with some colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and u' with some colour in $Z_6 \setminus (\{1, c(v'_1)\} \cup [c(t'_1)])$. Then, colour v_2 with 1. If either t_2 is adjacent to at most one of t_1 and t'_1 or $\{c(t_1), c(u), c(t'_1), c(u')\} \neq \{3, 4, 5, 6\}$, then we can assign to t_2 a colour in $\{3, 4, 5, 6\}$ not assigned to any of its neighbours to get a (G, T) -colouring, a contradiction.

So t_2 is adjacent to t_1 and t'_1 and $\{c(t_1), c(u), c(t'_1), c(u')\} = \{3, 4, 5, 6\}$. By symmetry, we may assume that $c(t_1) \neq 3$. If $c(v_1) \neq 2$, then we can recolour u with 2 and colour t_2 with the colour in $\{3, 4, 5, 6\} \setminus \{c(t_1), c(t'_1), c(u')\}$ to get a (G, T) -colouring, a contradiction. Hence $c(v_1) = 2$ and so $c(t_1) \geq 4$. If $c(v'_1) = 3$, then setting $c(t_2) = 3$ and $c(v_2) = 6$ yields a (G, T) -colouring, a contradiction. Hence $c(v'_1) \geq 4$, so $c(t'_1) \in \{3, 6\}$. If $c(t'_1) = 3$, then we set $c(v_2) = 3$ and choose $c(t_2)$ in $\{5, 6\} \setminus c(t_1)$. If $c(t'_1) = 6$, then $c(v'_1) = 4$, so we set $c(v_2) = 5$ and $c(t_2) = 3$. In both cases, we obtain a (G, T) -colouring, a contradiction.

- C_e^{int} is isomorphic to A_2 or A_3 , $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_2 v'_2$.

By minimality of (G, T) , there is a $(G - \{u, u', t_2, v_2\} \cup \{rv_2, rv'_2\}, T - \{u, u', t_2, v_2\})$ -colouring which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring c such that $c(v_1), c(v'_1) \neq 1$. Choose $c(u)$ in $\{2, 3\} \setminus \{c(v_2), c(t_2)\}$ and $c(u')$ in $\{2, 3\} \setminus \{c(v'_2), c(t'_2)\}$ and set $c(v_1) = 1$. If t_1 has at most one neighbour in $\{t_2, t'_2\}$ or $\{c(t_2), c(t'_2)\} \neq \{5, 6\}$, then we can colour t_1 with a colour in $\{5, 6\}$ not appearing on any of its neighbours to get a (G, T) -colouring, a contradiction. Hence t_1 is adjacent to t_2 and t'_2 (that is C_e^{int} and $C_{e'}^{int}$) are isomorphic to A_2 and $\{c(t_2), c(t'_2)\} = \{5, 6\}$. Recolouring u with $c(t'_2)$ and u' with $c(t_2)$ and colouring t_1 with 3, we obtain a (G, T) -colouring, a contradiction.

- C_e^{int} is isomorphic to A_2 or A_3 , $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_1 v'_2$.

By minimality of (G, T) , there exists a $G - \{u, u', t_2, v_2\} \cup \{rv_1, rv'_2\}, T - \{u, u', t_2, v_2\}$ -colouring c which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring such that $c(v_1) \neq 1$ and $c(v'_2) \neq 1$. Colour u with some colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$, u' with some colour in $\{2, 3\} \setminus \{c(t'_2), c(v'_2)\}$ and set $c(v_2) = 1$. Note that the set F of forbidden colours for t_2 is the union of $\{1, 2, c(u)\} \cup [c(u')]$ and the set of colours of the neighbours of t_2 in $\{t_1, t'_2\}$. Moreover $F = Z_6$ for otherwise we could colour t_2 with a colour in $Z_6 \setminus F$ to obtain a (G, T) -colouring, a contradiction.

If $c(u') = 2$, then, as $|F| = 6$, $\{t_1, t'_2\} \subseteq N_G(t_2)$ and $\{c(u), c(t_1), c(t'_2)\} = \{4, 5, 6\}$. Since $|c(u) - c(t_1)| \geq 2$, necessarily $\{c(t_1), c(u)\} = \{4, 6\}$ and $c(t'_2) = 5$. If $c(t_1) = 6$, then recolouring u with a colour in $\{2, 3\} \setminus c(v_1)$ and assigning 4 to t_2 , we obtain a (G, T) -colouring, a contradiction. Hence $c(t_1) = 4$ and $c(u) = 6$. So $c(v_1) = 2$, and thus $c(v'_2) = 3$. Then recolour u and u' with 1 and v_2 with 4 and colour t_2 with 6 to get a (G, T) -colouring, a contradiction.

Now, suppose that $c(u') = 3$. Then, as $|F| = 6$, $\{c(u), c(N(t_2) \cap \{t_1, t'_2\})\} \supseteq \{5, 6\}$. Assume that $c(u) \notin \{5, 6\}$, then $\{t_1, t'_2\} \subseteq N(t_2)$ and $\{c(t_1), c(t'_2)\} = \{5, 6\}$. Note that, in this case, $c(v_1) \leq 4$ and $c(v'_2) \leq 4$. Recolour u and u' with 1, v_2 with 6 and colour t_2 with 3 to get a (G, T) -colouring, a contradiction. Hence $c(u) \in \{5, 6\}$. Thus $c(t_1) \leq 4$ and so $c(t'_2) \in \{5, 6\}$ and $c(v'_2) \leq 4$. Thus, $t'_2 \in N(t_2)$ and $c(t'_2) \in (\{5, 6\} \setminus \{c(u)\})$. Recolour u' with $c(u)$. If $t_1 \notin N(t_2)$ or $c(t_1) \neq 3$, colouring t_2 with 3 yields a (G, T) -colouring, a contradiction. So $t_1 \in N(t_2)$ and $c(t_1) = 3$. Then, recolour u and u' with 3 (note that $c(v_1) \geq 5$ and $c(v'_2) \neq 3$ as u' was coloured 3) and t_2 with $i \in \{5, 6\} \setminus \{c(t'_2)\}$. This gives a (G, T) -colouring, a contradiction. □

Lemmas 14 and 15 immediately imply the following.

Corollary 16. *If f is a penultimate edge, then C_f^{int} is isomorphic to B_1 or B_2 . Moreover $rv_1 \in E(G)$ and $rv_3 \notin E(G)$.*

3.3 Antepenultimate edges

To deal with antepenultimate edges, we need the following two auxiliary results.

Lemma 17. *Suppose that (G, T) contains a configuration isomorphic to B_1 (see Figure 3). If there is a $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring c satisfying one of the following conditions :*

- (a) $c(t_2) = 6$ and $(c(t_1), c(v_3)) \neq (5, 4)$;
- (b) $c(v_3) = 1$ and $c(t_1) \neq c(t_2)$;

Then there is a (G, T) -colouring.

Proof. Let $L(u) = Z_6 \setminus \{1, [c(t_1)], c(v_1)\}$ and $L(v_2) = Z_6 \setminus \{1, [c(t_2)], c(v_1), c(v_3)\}$ be the set of colours available for u and v_2 respectively. Clearly $L(u) \neq \emptyset$. Observe that the conditions (a) and (b) also imply that $L(v_2) \neq \emptyset$. So, if $|L(u)| \geq 2$, $|L(v_2)| \geq 2$ or $L(u) \neq L(v_2)$, one can choose distinct colours $c(u) \in L(u)$ and $c(v_2) \in L(v_2)$ to obtain a (G, T) -colouring.

It is simple matter to check that in both cases, this condition is fulfilled. □

Lemma 18. *There is no antepenultimate edge g with only one penultimate predecessor f such that C_f^{int} is B_2 .*

In order to prove Lemma 18, we first prove an auxiliary result.

Lemma 19. *Suppose that (G, T) contains a configuration isomorphic to B_2 (see Figure 3). If there is a $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring c satisfying one of the following conditions :*

(a) $c(t_1) = c(t_2)$ and $c(v_3) \neq 1$;

(b) $c(t_1) \neq c(t_2)$ and

(b1) $c(t_1) = 6$; or

(b2) $c(v_1) = c(t_2)$; or

(b3) $c(t_2) \in [c(t_1)]$.

Then G has a (G, T) -colouring.

Proof. Let $L(u) = Z_6 \setminus \{1, [c(t_1)], c(t_2), c(v_1)\}$ and $L(v_2) = Z_6 \setminus \{[c(t_2)], c(v_1), c(v_3)\}$ be the set of colours available for u and v_2 respectively. Clearly $L(v_2) \neq \emptyset$. Observe that the conditions (a), (b1), (b2) and (b3) also imply that $L(u) \neq \emptyset$. So, if $|L(u)| \geq 2$, $|L(v_2)| \geq 2$ or $L(u) \neq L(v_2)$, one can choose distinct colours $c(u) \in L(u)$ and $c(v_2) \in L(v_2)$ to obtain a (G, T) -colouring.

It is simple matter to check that in each case, this condition is fulfilled. \square

Lemma 20. *Every antepenultimate edge has a unique penultimate predecessor.*

Proof. By contradiction. Suppose that an antepenultimate edge g has two penultimate predecessors f and f' .

According to Corollary 16, C_f^{int} and $C_{f'}^{int}$ are isomorphic to one of the graphs B_1 and B_2 . Let us denote the vertices of C_f^{int} by their names in Figure 3 and the vertices of $C_{f'}^{int}$ by their names in Figure 3 augmented with a prime.

Since g , f and f' are bounding the face incident to g in C_g^{int} , the edge g is $v_1v'_1$, $v_1v'_3$, $v_3v'_3$ or $v_3v'_1$. Since rv_1 and rv'_1 are edges, then $g = v_1v'_1$, for otherwise rv_1 would cross g .

First, suppose that $C_{f'}^{int}$ is isomorphic to B_1 , i.e., $rv'_2 \in E$. By minimality of (G, T) , there is a $((G - \{v_2, v_3\}) \cup \{v'_2u, v'_2v_1\}, T - \{v_2, v_3\})$ -colouring, which is a $(G - \{v_2, v_3\}, T - \{v_2, v_3\})$ -colouring such that $c(v'_2) \notin \{c(u), c(v_1)\}$. Setting $c(v_2) = c(v'_2)$ and $c(v_3) = 1$ gives a (G, T) -colouring, a contradiction.

The case C_f^{int} is isomorphic to B_1 is symmetric, so we may assume that both are C_e^{int} and $C_{e'}^{int}$ are isomorphic to B_2 . By minimality of (G, T) , there exists a $(G - \{v_2, v'_2, v_3, t_2\}, T - \{v_2, v'_2, v_3, t_2\})$ -colouring. Set $c(v_2) = c(v'_2) = 1$. Then, one can choose $c(t_2)$ in $L = Z_6 \setminus \{1, 2, c(u), c(u')\}$ such that $I = [c(t_2)] \cup \{1, c(v_1), c(v'_1)\} \neq Z_6$ because $|L| \geq 2$. Hence colouring v_3 with a colour in $Z_6 \setminus I$, we obtain a (G, T) -colouring, a contradiction. \square

Lemma 21. *Every antepenultimate edge has a unique predecessor.*

Proof. By contradiction. Suppose that an antepenultimate edge g has two predecessors f and f' . By Lemma 20, one of those is not penultimate. So, without loss of generality, f is penultimate, and f' is not. Hence f' is minimal.

According to Corollary 16, C_f^{int} is isomorphic to B_1 or B_2 , and according to Lemma 13, $C_{f'}^{int}$ is isomorphic to some of A_1 , A_2 and A_3 . Let us denote the vertices of C_f^{int} by their names in Figure 3 and the vertices of $C_{f'}^{int}$ by their names in Figure 2 augmented with a prime.

Since g , f and f' are bounding the face incident to g in C_g^{int} , the edge g is $v_1v'_1$, $v_1v'_2$, $v_3v'_1$ or $v_3v'_2$. Moreover, since rv_1 is an edge, rv_3 is not an edge if C_f^{int} is isomorphic to B_1 , and rv'_1 is an edge if $C_{f'}^{int}$ is isomorphic to A_1 , we must be in one of the following cases:

- C_f^{int} is isomorphic to B_1 , $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v'_2$.

By minimality of (G, T) , there is a $(G - \{v_2, t_2, u', v_3\} \cup \{v'_2r\}, T - \{v_2, t_2, u', v_3\})$ -colouring c which is a $(G - \{v_2, t_2, u', v_3\}, T - \{v_2, t_2, u', v_3\})$ -colouring such that $c(v'_2) \neq 1$. Set $c(v_3) = 1$. Let $L(t_2) \supseteq Z_6 \setminus \{1, 2, c(t'_2)\}$, $L(v_2) = Z_6 \setminus \{1, c(u), c(v_1)\}$ and $L(u') = Z_6 \setminus \{1, c(t'_2), c(v'_2)\}$. Clearly, there exists at most one $i \in Z_6$ such that $L(u') = [i]$ and at most one $j \in Z_6$ such that $L(v_2) = [j]$. Thus, as $|L(t_2)| \geq 3$, there exists $k \in L(t_2)$ such that $L(u') \setminus [k] \neq \emptyset$ and $L(v_2) \setminus [k] \neq \emptyset$. Setting $c(t_2) = k$ and colouring u' and v_2 by colours in $L(u') \setminus [k]$ and $L(v_2) \setminus [k]$, respectively, we obtain a (G, T) -colouring, a contradiction.

- C_f^{int} is isomorphic to B_2 , $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v_2'$.

By minimality of (G, T) , there is a $((G - \{v_2, t_2, v_3, u'\}) \cup \{t_2' u, t_2' v_1\}, T - \{v_2, t_2, v_3, u'\})$ -colouring c which is a $(G - \{v_2, t_2, v_3, u'\}, T - \{v_2, t_2, v_3, u'\})$ -colouring such that $c(t_2') \notin \{c(u), c(v_1)\}$.

Suppose that $t_2t_2' \in E$. If we can colour t_2 with $\beta \in [c(v_1)] \cup \{6\}$, then we can colour u' with some colour in $Z_6 \setminus (\{c(t_2'), c(v_2')\} \cup [\beta])$, v_3 with some colour in $Z_6 \setminus (\{c(u'), c(v_2'), c(v_1)\} \cup [\beta])$ and v_2 with some colour in $Z_6 \setminus (\{c(v_3), c(u), c(v_1)\} \cup [\beta])$, a contradiction. So, there is no available colour in $[c(v_1)] \cup \{6\}$ for t_2 ; that is, $[c(v_1)] \cup \{6\} \subseteq \{1, 2, c(u), c(t_2')\}$. Since $c(v_1) \notin \{1, c(u), c(t_2')\}$, we must have $c(v_1) = 2$ and $\{c(u), c(t_2')\} = \{3, 6\}$. colour u' with 2 (since $v_2' \in N(v_1)$ we know that $c(v_2') \neq c(v_1)$), v_2 with 1 and v_3 with $c(t_2')$. Colour t_2 with 4 if $c(u) = 3$ and with 5 otherwise. This gives a (G, T) -colouring, a contradiction.

Now, suppose that $ru' \in E$. If $c(u) \neq 6$, then we can colour t_2 with 6 and u' , v_3 and v_2 can be greedily coloured in this order, a contradiction; thus, $c(u) = 6$. Let $L(u') = Z_6 \setminus \{1, c(t_2'), c(v_2')\}$ be the colours available for u' ; note that if $L(u') = [i]$ for some $i \in Z_6$ then $c(t_2') = 6$ and $c(v_2') = 2$, a contradiction since $c(t_2') \neq c(u)$. Clearly, there exists $\beta \in [c(v_1)] \setminus \{1, 2\}$ so we can colour t_2 with β , u' with any colour in $L(u') \setminus [\beta]$ (recall that $L(u') \neq [i]$ for all $i \in Z_6$). Then colour v_3 and v_2 greedily gives a (G, T) -colouring, a contradiction.

- C_f^{int} is isomorphic to B_1 or B_2 , $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v_1'$.

By minimality of (G, T) , there is a $((G - \{t_2, v_2, v_3\}) \cup \{uv_1', ru', rv_1'\}, T - \{t_2, v_2, v_3\})$ -colouring, which is a $(G - \{t_2, v_2, v_3\}, T - \{t_2, v_2, v_3\})$ -colouring such that $c(v_1'), c(u') \neq 1$ and $c(v_1') \neq c(u)$.

Suppose that we can colour t_2 with $\beta \in [c(v_1)] \cup \{6\}$. We know that there is at least one colour $i \in Z_6 \setminus (\{1, c(u), c(v_1)\} \cup [\beta])$ available for v_2 and at least one colour $j \in Z_6 \setminus (\{c(v_1'), c(u'), c(v_1)\} \cup [\beta])$ available for v_3 . Since $c(v_1') \notin \{1, c(u)\}$, then $i \neq j$ and we can colour v_2 with i and v_3 with j to obtain a (G, T) -colouring, a contradiction.

So, suppose that the colours of $[c(v_1)] \cup \{6\}$ all appear in $N(t_2)\{v_2, v_3\}$; since $|([c(v_1)] \cup \{6\}) \setminus \{1, 2\}| \geq 2$, t_2 must be adjacent to at least one of u and t_1' .

Assume first $t_2t_1' \in E$. Then recolour u' with 1. If $ut_2 \notin E$ or $[c(v_1)] \cup \{6\} \not\subseteq \{1, 2, c(u), c(t_1')\}$, note that we can apply the same argument as before since it holds even if $c(u') = 1$; so suppose otherwise. In this case we must have: either (a) $c(v_1) = 6$, $c(u) = 5$ and $c(t_1') = 6$; or (b) $c(v_1) = 2$ and $\{c(u), c(t_1')\} = \{3, 6\}$. colour v_2 with 1. If (a) occurs, then colour t_2 with 3 and v_3 with 5; if (b) occurs and $c(t_1') = 6$, then colour t_2 with 4 and v_3 6; if (b) occurs and $c(t_1') = 3$, then colour t_2 with 5 and v_3 with 3.

Hence $t_2t_1' \notin E$, and so $t_2u \in E$. The possible situations are: (c) $c(v_1) = 6$, $c(u) = 5$ and $c(u') = 6$; or (d) $c(v_1) = 2$ and $\{c(u), c(u')\} = \{3, 6\}$. If (c) occurs, then colour v_3 with $\{2, 5\} \setminus \{c(v_1')\}$ and t_2 with $\{3, 4\} \setminus [c(v_3)]$. If (d) occurs, then colour v_3 with $c(u)$ (recall that $c(v_1') \neq c(u)$) and t_2 with $\{4, 5\} \setminus [c(v_3)]$. In both cases we get a (G, T) -colouring, a contradiction.

- C_f^{int} is isomorphic to B_1 , $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_1v_1'$.

Let (G', T') be the graph pair obtained from $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$ by identifying t_1 and t_1' . By minimality of (G, T) , there exists a (G', T') -colouring which is a $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$ -colouring such that $c(t_1) = c(t_1')$. Set $c(t_2) = 6$ and $c(u') = c(v_1)$. Let $L = \{2, 3, 4\} \setminus \{c(v_1), c(v_1')\}$.

If $c(t_1) \neq 5$, then choosing $c(v_3) \in L$, and applying Lemma 17, we obtain a (G, T) -colouring, a contradiction. Hence $c(t_1) = c(t_1') = 5$.

If $L \neq \{4\}$, then we can choose $c(v_3) \in L \setminus \{4\}$, and apply Lemma 17 to get a (G, T) -colouring, a contradiction. Hence $\{c(v_1), c(v_1')\} = \{2, 3\}$.

Now setting $c(v_3) = 5$, $c(v_2) = 6$, $c(u) = c(v'_1)$ and recolouring t_2 with 3, we obtain a (G, T) -colouring, a contradiction.

- C_f^{int} is isomorphic to B_1 , $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_3v'_1$.

Let (G', T') be the graph pair obtained from $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$ by identifying t_2 and t'_1 . By minimality of (G, T) , there exists a (G', T') -colouring which is a $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$ -colouring such that $c(t_2) = c(t'_1)$. Set $c(t_1) = 6$.

If $c(t_2) = c(t'_1) = 6$, then set $c(u) = c(v_3)$. One can then greedily extend the colouring to v_1, v_2 and u' in this order, a contradiction.

If $c(t_2) \neq 6$, then one can choose $c(v_1) \in [c(t_2)] \setminus \{5, 6\}$. This is valid since $c(v_3)$ and $c(v'_1)$ are not in $[c(t_2)]$. One can then greedily extend the colouring to v_2, u and u' in this order, a contradiction.

- C_f^{int} is isomorphic to B_2 , $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_3v'_1$.

By minimality of (G, T) , there exists a $(G - \{t_1, u, v_1, v_2\} \cup \{rv_3\}, T - \{t_1, u, v_1, v_2\})$ -colouring c . Set $c(v_2) = 1$ and let $L(v_1) = Z_6 \setminus \{1, c(v_3), c(u'), c(v'_1)\}$ be the colours available for v_1 .

If $L(v_1) \neq \{5, 6\}$, then colouring t_1 with 6, v_1 with some colour in $L(v_1) \setminus \{5, 6\}$ and u with some colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(t_2)\}$, we obtain a (G, T) -colouring, a contradiction.

If $L(v_1) = \{5, 6\}$, then $c(u'), c(v'_1) \in \{2, 3, 4\}$ and consequently $c(t'_1) \in \{5, 6\}$. We can suppose that $c(t'_1) = 5$ and $c(v_3) = 4$ for otherwise we can recolour u' with $c(v_3)$ and fall in the case $L(v_1) \neq \{5, 6\}$. So, $\{c(u'), c(v'_1)\} = \{2, 3\}$ and $c(t_2) \geq 6$. Setting $c(t_1) = 3$, $c(u) = 5$ and $c(v_1) = 6$, we obtain a (G, T) -colouring, a contradiction. □

Lemma 22. *There is no antepenultimate edge g with only one penultimate predecessor f such that C_f^{int} is B_1 .*

Proof. One of the endvertices of g must be v_1 or v_3 (see Figure 3). We now distinguish some cases depending on the possible endvertices of g .

- Assume $g = v'v_3$ with v' a leaf with twig t' . Since $rv_1 \in E(G)$, by planarity, $t' \neq t_1$. Let (G', T') be the graph pair obtained from $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$ by identifying t' and t_2 . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$ -colouring such that $c(t') = c(t_2)$. Set $c(v_2) = c(v')$ and $c(t_1) = 6$.

If $c(t') \in \{5, 6\}$, then setting $c(u) = c(v_3)$ and choosing $c(v_1)$ in $\{2, 3, 4\} \setminus \{c(v'), c(v_3)\}$, we obtain a (G, T) -colouring, a contradiction.

If $c(t') \in \{3, 4\}$, then setting $c(v_1) = c(t') - 1$ and choosing $c(u)$ in $\{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$, we obtain a (G, T) -colouring, a contradiction.

- Assume $g = t'v_3$ with t' a twig. We can apply an argument similar to (a) choosing $c(v_2) \in Z_6 \setminus (\{1, c(v_3)\} \cup [c(t_2)])$.
- Assume $g = v_1r$. Since G is triangulated, the edge v_1t_2 must exist. This is a contradiction, since f is the successor of e .
- Assume $g = v_1t_2$. Then v_3 is a leaf of degree at most 3, a contradiction from Lemma 6.
- Assume $g = v_1v'$ with v' a leaf with twig t_2 . Let (G', T') be the graph pair obtained from $(G - \{v_3\}, T - \{v_3\})$ by identifying v_2 and v' . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{v_3\}, T - \{v_3\})$ -colouring such that $c(v_2) = c(v')$. Hence one can colour v_3 with a colour from $Z_6 \setminus \{[c(t_2)], c(v_2), c(v_1)\}$ to obtain a (G, T) -colouring, a contradiction.

- (f) Assume $g = v_1 t'$ with $t' \neq t_2$ a twig. Since g is the successor of f , $v_1 t_2$ is not an edge and $v_1 r$ is not inside C_g , so $v_3 t' \in E$.

Assume first that $rv_3 \notin E$. By minimality of (G, T) , there is a $((G - \{u, v_2, v_3\}) \cup t_1 t_2, T - \{u, v_2, v_3\})$ -colouring which is a $(G - \{u, v_2, v_3\}, T - \{u, v_2, v_3\})$ -colouring such that $c(t_1) \neq c(t_2)$. Since $c(t'), c(v_1) \neq 1$, we can colour v_3 with 1. Then, by Lemma 17 (b), there is a (G, T) -colouring, a contradiction.

Assume now that $rv_3 \in E$. Then $t' t_2 \notin E$ by planarity. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t' . By minimality of (G, T) , there is a (G', T') -colouring, which is a $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$ -colouring such that $c(t_1) = c(t')$. Set $c(t_2) = 6$. One can choose $c(v_3) \in \{2, 3\} \setminus \{c(t'), c(v_1)\}$ because $|c(v_1) - c(t')| \geq 2$. Then by Lemma 17 (a), there is a (G, T) -colouring, a contradiction.

- (g) Assume $g = v_1 v'$ with v' a leaf with twig $t' \neq t_2$. Since g is the successor of f , $v_1 t_2$ is not an edge and $v_1 r$ and $v_1 t'$ are not inside C_g , so $v_3 t' \in E$.

Assume first that $rv_3 \notin E$. By minimality of (G, T) , there is a $((G - \{u, t_2, v_2, v_3\}) \cup rv', T - \{u, t_2, v_2, v_3\})$ -colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that $c(v') \neq 1$. Since $c(t'), c(v'), c(v_1) \neq 1$, we can colour v_3 with 1. Then, colouring t_2 with a colour from $Z_6 \setminus \{1, 2, c(t'), c(v'), c(t_1)\}$ and using Lemma 17 (b), we obtain a (G, T) -colouring, a contradiction.

Assume now that $rv_3 \in E$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t' . By minimality of (G, T) , there is a (G', T') -colouring, which is a $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$ -colouring such that $c(t_1) = c(t')$. If $\{c(v_1), c(v')\} \neq \{2, 3\}$, then one can choose $c(v_3)$ in $\{2, 3\} \setminus \{c(t'), c(v'), c(v_1)\}$. Then setting $c(t_2) = 6$ and applying Lemma 17 (a), we obtain a (G, T) -colouring, a contradiction. Thus $\{c(v_1), c(v')\} = \{2, 3\}$, and so $c(t_1) \geq 5$. Setting $c(u) = c(v')$, $c(t_2) = 3$ and choosing $c(v_3)$ in $\{5, 6\} \setminus c(t')$ and $c(v_2)$ in $\{5, 6\} \setminus c(v_3)$ yields a (G, T) -colouring, a contradiction. □

Proof. One of the endvertices of g must be v_1 or v_3 (see Figure 3). We now distinguish some cases depending on the possible endvertices of g .

- (a) Assume $g = v' v_3$ with v' a leaf with twig t' . Since $rv_1 \in E(G)$, by planarity, $t' \neq t_1$. By minimality of (G, T) , there is a $((G - \{t_1, v_1, u\}) \cup \{t_2 v', t_2 t'\}, T - \{t_1, v_1, u\})$ -colouring which is a $(G - \{t_1, v_1, u\}, T - \{t_1, v_1, u\})$ -colouring such that $c(v') \neq c(t_2)$ and $c(t') \neq c(t_2)$. Hence one can colour v_1 with $c(t_2)$ and colour t_1 with a colour in $Z_6 \setminus ([c(t_2)] \cup \{1, 2\})$. From Lemma 19 (b2), we obtain a (G, T) -colouring, a contradiction.
- (b) Assume $g = t' v_3$ with t' a twig. We can apply an argument similar to (a).
- (c) Assume $g = v_1 r$. Since G is triangulated, the edge $v_1 t_2$ must exist. This is a contradiction, since f is the successor of e .
- (d) Assume $g = v_1 t_2$. Then v_3 is a leaf of degree at most 3, a contradiction from Lemma 6.
- (e) Assume $g = v_1 v'$ with v' a leaf adjacent to t_2 in T . Since f is the successor of $v_1 v_3$, $v_1 t_2 \notin E(G)$ and so $v_2 v_3 \in E(G)$ because G is triangulated. Let (G', T') be the graph pair obtained from $(G - \{v_3\}, T - \{v_3\})$ by identifying v_2 and v' . By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{v_3\}, T - \{v_3\})$ -colouring such that $c(v_2) = c(v')$. Hence one can colour v_3 with a colour from $Z_6 \setminus \{c(t_2), c(v_2), c(v_1)\}$ to obtain a (G, T) -colouring, a contradiction.

- (f) Assume $g = v_1 t'$ with $t' \neq t_2$ a twig. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t' . This is possible since $t_1 t'$ is not an edge by planarity. By minimality of (G, T) , there is a (G', T') -colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that $c(t') = c(t_1)$. Let $L(t_2) = Z_6 \setminus \{1, 2, c(t')\}$. If $c(t_1) = 6$, then, colouring v_3 with 1 and t_2 with a colour from $L(t_2) \setminus \{6\}$, and using Lemma 19 (b1), we obtain a (G, T) -colouring, a contradiction. So $c(t_1) \neq 6$, that is $c(t_1) \in \{3, 4, 5\}$. We can colour t_2 with a colour from $[c(t_1)] \setminus \{c(t'), c(t_1)\} \subseteq L(v_2)$. By Lemma 19 (b3), there is a (G, T) -colouring, a contradiction.
- (g) Assume $g = v_1 v'$ with v' a leaf with twig $t' \neq t_2$. By minimality of (G, T) , there is a $((G - \{u, t_2, v_2, v_3\}) \cup \{v_1 t'\}, T - \{u, t_2, v_2, v_3\})$ -colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that $c(v_1) \neq c(t')$. If $c(v_1) \neq 2$, we can colour t_2 with $c(v_1)$, since $c(t'), c(v') \neq c(v_1)$. Then, colouring v_3 with a colour from $Z_6 \setminus \{c(v_1), c(t'), c(v')\}$, and applying Lemma 19 (b2), we obtain a (G, T) -colouring, a contradiction. So, $c(v_1) = 2$.

Suppose that $c(v') \neq 1$. Since $c(t'), c(v') \notin \{1, 2\}$, then $\{c(t'), c(v')\} \in \{\{3, 5\}, \{3, 6\}, \{4, 6\}\}$. Let $L(v_3) = Z_6 \setminus \{2, c(t'), c(v')\}$ and let $L(t_2) = Z_6 \setminus \{1, 2, c(t'), c(v')\}$. If $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) \neq \emptyset$, then choosing $c(t_2)$ in $[c(t_1)] \setminus \{c(t_1)\}$ and $c(v_3)$ in $L(v_3) \setminus [c(t_2)]$ (observe that $|[c(t_2)] \cap L(v_3)| \leq 2$, since $L(v_3)$ has no three consecutive integers), and using Lemma 19 (b3), we obtain a (G, T) -colouring, a contradiction. Then $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) = \emptyset$. If $c(t_1) = 3$, then $\{c(t'), c(v')\} = \{4, 6\}$. In this case, colouring t_2 with 3, v_3 and u with 5 and v_2 with 6, we can obtain a (G, T) -colouring, a contradiction. If $c(t_1) = 4$, then $\{c(t'), c(v')\} = \{3, 5\}$, and if $c(t_1) = 5$, then $\{c(t'), c(v')\} = \{4, 6\}$. In both cases, setting $c(t_2) = c(t_1)$, choosing $c(v_3)$ in $Z_6 \setminus \{1, 2, [c(t_1)]\}$ and using Lemma 19 (a), we obtain a (G, T) -colouring, a contradiction.

Hence $c(v') = 1$. If $c(t_1) \in \{3, 4\}$, colour t_2 with 3 (if $c(t') \neq 3$) or 4 (otherwise). If $c(t_1) = 5$, colour t_2 with 6 (if $c(t') \neq 6$) or 5 (otherwise). These cases satisfy the conditions $c(t_2) \in [c(t_1)]$ and $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\} \neq \emptyset$. Then, colouring v_3 with a colour from $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\}$, and using Lemma 19 ((a) or (b3)), we obtain (G, T) -colouring, a contradiction. □

Lemmas 20, 21, 22 and 18 directly imply the following.

Corollary 23. (G, T) has no antepenultimate edges.

References

- [1] H. Broersma, F. V. Fomin, P. A. Golovach and G. J. Woeginger. Backbone colourings for networks. In *Proceedings of the 29th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2003)*, LNCS:2880:131–142, 2003.
- [2] H. Broersma, F. V. Fomin, P. A. Golovach and G. J. Woeginger. Backbone colourings for graphs: tree and path backbones. *Journal of Graph Theory* 55(2):137–152, 2007.
- [3] H.J. Broersma, J. Fujisawa, L. Marchal, D. Paulusma, A.N.M. Salman, K. Yoshimoto. λ -backbone colorings along pairwise disjoint stars and matchings. *Discrete Mathematics* 309:5596–5609, 2009.
- [4] Y. Bu and Y. Li. Backbone coloring of planar graphs without special circles. *Theoretical Computer Science* 412 (46) (2011), 6464–6468.
- [5] Y. Bu and S. Zhang. Backbone coloring for C_4 -free planar graphs. *Science China Mathematics* 41 (2) (2011), 197–206.

- [6] W. Wang, Y. Bu, M. Montassier and A. Raspaud. On backbone coloring of graphs. *Journal of Combinatorial Optimization* 23 (1): 79–93, 2012.



Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399