Higher Order Moreau's Sweeping Process

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Formulation and numerical simulation

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Abstract In this chapter we present a mathematical formulation of complementarity dynamical systems with arbitrary dimension and arbitrary relative degree between the complementary slackness variables. The proposed model incorporates the state jumps via high-order distributions through the extension of Moreau's sweeping process, which is a special type of differential inclusion. The time-discretization of these nonsmooth systems, which is non-trivial, is also presented. Applications of such high-order sweeping processes can be found in dynamic optimization under state constraints and electrical circuits with ideal diodes, where it may be helpful for a better understanding of the closed-loop dynamics induced by some feedback laws.

Keywords: Complementarity systems, Hybrid systems, Convex analysis, Differential inclusions, Variational inequalities, Numerical simulation, Zero dynamics, Relative degree.

Introduction

The general objective of this chapter is the study of complementarity dynamical systems with arbitrary relative degree. As we shall briefly see below, such systems possess complex dynamics and their well-posedness, numerical time integration, and analysis for control, have not yet been understood except in some particular cases. It is proposed here to settle a general dynamical framework for such higher relative degree complementarity systems, using the concept of differential inclusions and Moreau's sweeping process. Besides showing the coherence of the presented dynamics and its usefulness in designing a numerical time-stepping scheme (which in particular paves the way for well-posedness studies), numerous problems like optimal control with state inequality constraints and feedback control of circuits can benefit from the approach.

1. The ZD canonical representation

In this section several state space representations are derived, which will prove to be useful to formalize the extended sweeping process.

1.1 Canonical state space representations

Let us consider the following linear complementarity system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t), \quad x(0^-) = x_0 \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) \geq 0 \end{cases} \tag{22.1}$$

where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. If the pair (A, B) is controllable, then the system in (22.1) has a relative degree $r^{w\lambda} \leq n$ (Sontag, 1998). Let us note that it is implicitly assumed in (22.1) that the relative degree is strictly larger than zero. Actually, the framework that is presented next is essentially linked to systems with $r^{w\lambda} \geq 1$. As a consequence it is of little interest for so-called relay systems, whose relative degree $r^{w\lambda}$ is always zero (Camlibel, 2001, Chapter 7). This allows one to perform a state space transformation, with new state vector $z = Wx$, W square full-rank, and $z^T = (w, \dot{w}, \ddot{w}, ..., w^{(r^{w\lambda}-1)}, \xi^T) = (\bar{z}^T, \xi^T)$, $\xi \in \mathbb{R}^{n-r^{w\lambda}}$ such that the new state space representation is (Sannuti, 1983)

$$
\begin{cases}
\dot z_1(t) = z_2(t) \\
\dot z_2(t) = z_3(t) \\
\dot z_3(t) = z_4(t) \\
\\
\vdots \\
\\
\dot z_{r^{w\lambda}-1}(t) = z_{r^{w\lambda}}(t) \\
\dot z_{r^{w\lambda}}(t) = CA^{r^{w\lambda}}W^{-1}z(t) + CA^{r^{w\lambda}-1}B\lambda(t) \\
\\
\dot\xi(t) = A_\xi \xi(t) + B_\xi z_1(t) \\
\\
0 \leq z_1(t) \perp \lambda(t) \geq 0, \quad z(0^-) = z_0
\end{cases}
\tag{22.2}
$$

In Systems and Control theory, the dynamics $\dot\xi = A_\xi \xi$ is called the *zero dynamics*, so we shall denote the state space form in (22.2) the ZD representation. The notion of relative degree is quite similar to that of index in DAE theory (Campbell and Gear, 1995), or to what is called the state constraint order in optimal control (Hartl et al., 1995). We note that the formalism in (22.2) continues to hold if $\lambda, w \in I\!R^m$. The system has a uniform relative degree if all the Markov parameter $C_iA^{r^{w\lambda}-1}B_j$ are non zero for some $r^{w\lambda}$ and all integers $i, j \in \{1, m\}$, where C_i is the ith row of C while B_j is the jth column of B. Then $CA^{r^{w\lambda}-1}B$ is an $m \times m$ matrix. One has in the multivariable case $z_1^i(\cdot) = w_i(\cdot)$, $1 \leq i \leq m$ and

$$
\begin{cases}
\dot z_1^i(t) = z_2^i(t) \\
\dot z_2^i(t) = z_3^i(t) \\
\vdots \\
\dot z_{r^{w\lambda}}^i(t) = C_iA^{r^{w\lambda}}W^{-1}z(t) + C_iA^{r^{w\lambda}-1}B\lambda \\
\\
\dot\xi_i(t) = A_{\xi_i}\xi_i(t) + B_{\xi_i}z_1^i(t), \quad 1 \leq i \leq m \\
\\
0 \leq z_1(t) \perp \lambda(t) \geq 0
\end{cases}
\tag{22.3}
$$

Grouping terms together $z_1 = (z_1^1, ..., z_1^m)^T$, and so on, one gets the same expression as in (22.2) but all z_i, $1 \leq i \leq r^{w\lambda}$, are $m-$dimensional. One has also $mr^{w\lambda} \leq n$.

1.2 Distributional dynamics

Until now possible state $x(\cdot)$ jumps have not been introduced. It is of utmost importance to notice that in general, the solutions of (22.2) (equivalently of (22.1)) will not be differentiable. Consider for instance the initial data $z_i(0^-) \leq -\delta$ for some $\delta > 0$ and all $1 \leq i \leq r^{w\lambda}$. Then obviously all the z_i, $1 \leq i \leq r^{w\lambda}$, need to jump to some non-negative value so that the unilateral constraint $z_1(t) \geq 0$ is satisfied on $(0, \epsilon)$ for some $\epsilon > 0$. At this stage we can just say that a jump mapping is needed. Its form will depend on the type of system one handles (in Mechanics, this is the realm of impact mechanics (Brogliato, 1999)). If one considers (22.2) as an equality of distributions, then we can rewrite it as

$$\left\{\begin{array}{l} Dz_1 = z_2 \\ Dz_2 = z_3 \\ Dz_3 = z_4 \\ \\ \vdots \\ \\ Dz_{r^{w\lambda}-1} = z_{r^{w\lambda}} \\ Dz_{r^{w\lambda}} = CA^{r^{w\lambda}} W^{-1} z + CA^{r^{w\lambda}-1} B\lambda \\ \\ D\xi = A_\xi \xi + B_\xi z_1 \end{array}\right. \tag{22.4}$$

where D denotes the distributional derivative (Ferreira (1997)). At a reinitialization time one has $z(t_k^+) = \mathcal{F}[z(t_k^-)]$, where $\mathcal{F}(\cdot)$ is an operator that will be defined later. Let us denote the jump of a function $f(\cdot)$, with right and left limits at time t, as $\sigma_f(t) = f(t^+) - f(t^-)$. Consider the above initial conditions on $z(\cdot)$. Then Dz_1 is a distribution of degree 2 and we get $Dz_1 = \{\dot{z}_1\} + \sigma_{z_1}(0)\delta_0 = z_2$. Consequently Dz_2 is a distribution of degree 3 (Ferreira (1997), Theorem 1.1), and $Dz_2 = D^2 z_1 = D\{\dot{z}_1\} + \sigma_{z_1}(0)\dot{\delta}_0 = \{\dot{z}_2\} + \sigma_{\{\dot{z}_1\}}(0)\delta_0 + \sigma_{z_1}(0)\dot{\delta}_0 = z_3$, and $\{\dot{z}_1\} = \{z_2\}$. Then Dz_3 is a distribution of degree 4, and we get $Dz_3 = D\{\dot{z}_2\} + \sigma_{\{\dot{z}_1\}}(0)\dot{\delta}_0 + \sigma_{z_1}(0)\ddot{\delta}_0 = \{\dot{z}_3\} + \sigma_{\{\dot{z}_2\}}(0)\delta_0 + \sigma_{\{\dot{z}_1\}}(0)\dot{\delta}_0 + \sigma_{z_1}(0)\ddot{\delta}_0 = z_4$, and $\{\dot{z}_2\} = \{z_3\}$. Thus $\sigma_{\{\dot{z}_1\}}(0) = z_2(0^+) - z_2(0^-)$, and $\sigma_{\{\dot{z}_2\}}(0) = z_3(0^+) - z_3(0^-)$. And so on. Until now we have decomposed only the left hand side of the dynamics as distributions of some degrees. Now let us get back to the distributional dynamics in (22.4). Starting from $Dz_1 = z_2$, one deduces that the right hand side has to be of the same degree than the left hand side. This means that the right hand side is equal to $\{z_2\} + \mu_1$, where μ_1 is a distribution of degree 2, i.e. a measure. Similarly from $Dz_2 = z_3$ one deduces that $z_3 = \{z_3\} + \mu_2'$, where μ_2' has degree 3 and can therefore further be decomposed as $\mu_2 + \mu_1'$, with $\deg(\mu_2) = 2$ and $\deg(\mu_1') = 3$. It is not difficult to see that $\mu_1' = \dot{\mu}_1$, using similar arguments as in (Brogliato, 1999, §1.1). Therefore $Dz_2 = \{z_3\} + \mu_2 + \dot{\mu}_1$. The variables μ_1 and μ_2 are slack variables (or Lagrange multipliers), and are measures of the form $\mu_i = g_i(t)dt + d\mu_i$, $g_i(\cdot)$ being Lebesgue integrable function, and $d\mu_i$ an atomic measure. We will see later that one cannot merge $g_i(t)$ with $\{z_{i+1}\}$, because $\mathrm{supp}(g_i(t)dt) \subset \{t|\ z_1(t) = 0\}$, and the measure μ_i will obey specific sign conditions. It happens that when no external inputs (functions of time) act on the system, the non-atomic part of μ_i, $1 \leq i \leq r^{w\lambda} - 1$, will always be zero, see lemma 22.10. Continuing the reasoning until $Dz_{r^{w\lambda}}$, we obtain $Dz_{r^{w\lambda}} = CA^{r^{w\lambda}}W^{-1}\{z\}+CA^{r^{w\lambda}-1}B\lambda$ where $\deg(\lambda) = \deg(Dz_{r^{w\lambda}}) = r^{w\lambda}+1$. Consequently from (22.4) one gets

$$\left\{\begin{array}{l} Dz_1 = \{z_2\} + \mu_1 \\ Dz_2 = \{z_3\} + \dot{\mu}_1 + \mu_2 \\ Dz_3 = \{z_4\} + \ddot{\mu}_1 + \dot{\mu}_2 + \mu_3 \\ \\ \vdots \\ Dz_i = \{z_{i+1}\} + \mu_1^{(i-1)} + \mu_2^{(i-2)} + \ldots + \dot{\mu}_{i-1} + \mu_i \\ \vdots \\ Dz_{r^{w\lambda}-1} = \{z_{r^{w\lambda}}\} + \mu_1^{(r^{w\lambda}-1)} + \ldots + \mu_{r^{w\lambda}-1} \\ Dz_{r^{w\lambda}} = CA^{r^{w\lambda}}W^{-1}\{z\} + CA^{r^{w\lambda}-1}B\lambda \end{array}\right. \tag{22.5}$$

We keep the notation λ for the multiplier which appears in the last line, and whose expression will be given in the next section. It is important at this stage to realize that λ is the unique source of higher degree distributions in the system, which will allow the state to jump. Therefore the measures μ_i have themselves to be considered as "sub-multipliers". In (22.5) we have separated the regular (functions) parts denoted as $\{\cdot\}$ and the atomic distributional parts. Notice that $\{z_{i+1}\} = \{\dot{z}_i\}$. Also $D\{z_i\} = \{z_{i+1}\} + \mu_i$. From this last fact it is convenient to extract the "measure" part of (22.5) as

$$\left\{\begin{array}{l} Dz_1 = \{z_2\} + \mu_1 \\ D\{z_2\} = \{z_3\} + \mu_2 \\ D\{z_3\} = \{z_4\} + \mu_3 \\ \\ \vdots \\ D\{z_i\} = \{z_{i+1}\} + \mu_i \\ \vdots \\ D\{z_{r^{w\lambda}-1}\} = \{z_{r^{w\lambda}}\} + \mu_{r^{w\lambda}-1} \\ D\{z_{r^{w\lambda}}\} = CA^{r^{w\lambda}}W^{-1}\{z\} + CA^{r^{w\lambda}-1}B\mu_{r^{w\lambda}} \end{array}\right. \tag{22.6}$$

where the terms $D\{z_i\}$ can now be interpreted as the differential measures of $\{z_i\}$ (Marques, 1993). It is noteworthy that (22.6) is not at all equivalent to (22.4). It will be quite useful, however, for the characterization of the extended sweeping process and some of its properties, as well as for time-discretization. Roughly speaking, (22.6) represents the system before and after a state reinitialization, whereas (22.4) intends to also represent the dynamics at jump times.

The measures μ_i and the distribution λ in (22.5) play a similar role to the Lagrange multiplier in Mechanics with unilateral contact. Viewing the dynamics as an equality of distributions as in (22.5) paves the way for time-discretization with time-stepping algorithms, i.e. numerical schemes working without event detection procedures and constant time-step.

Positivity of λ: Only the Dirac measures μ_i and time functions are signed. Consequently writing $\lambda \geq 0$ is meaningless. The correct writing of the complementarity $0 \leq z_1 \perp \lambda \geq 0$ (see corollary 22.9 below) requires to reformulate the dynamics under a suitable representation and will be done through several steps. Another point of view is to assert that $\lambda \geq 0$ implies that λ **is** a measure. However as we shall see this is not sufficient to assure $z_1(t) \geq 0$. Consequently one has to resort to higher degree distributions to give a reasonably general meaning to the dynamics in (22.2).

2. The extended Moreau's sweeping process

In order to simplify the presentation we shall assume in many places that $m = 1$. When the statements or results also obviously hold for $m \geq 2$ and uniform relative degree (see (22.3)) this will be pointed out. Starting from (22.4) (22.5) the extended sweeping process is written as follows (in order to lighten the writing we will denote the non-singular part of a distribution z as $z(t)$)

$$\mu_i \in -\partial\psi_{T_\Phi^{i-1}(z_1(t^-),...,z_{i-1}(t^-))}(z_i(t^+)) \text{ for all } 1 \leq i \leq r^{w\lambda} \tag{22.7}$$

where $T_\Phi^i(z_1, ..., z_i) = T_{T_\Phi^{i-1}(z_1,...,z_{i-1})}(z_i)$, $T_\Phi^1(z_1) = T_\Phi(z_1)$, $T_\Phi^0(z_1) = \Phi$ and $T_\Phi(x) = \{v | \ v \geq 0 \text{ if } x \leq 0, v \in I\!R \text{ if } x > 0\}$ is the tangent cone to $\Phi = I\!R^+$ at x (extended outside Φ), defined as in (Moreau, 1988) to take into account constraint violations. We shall keep the notation Φ noting that in general when one starts from the ZD dynamics, one gets $\Phi = (I\!R^+)^m$. Moreover we also keep in mind the extension of the material that follows towards formalisms involving convex sets $\Phi(t)$ and not necessarily being the ZD dynamics of a given system. The sets

$\partial\psi_{T_\Phi^{i-1}(z_1(t^-),...,z_{i-1}(t^-))}(z_i(t^+))$ are cones and therefore the inclusions in (22.7) make sense: since $\mu_i = g_i(t)dt + d\mu_i$, (22.7) means that either $g_i(t) \in -\partial\psi_{T_\Phi^{i-1}(z_1(t^-),...,z_{i-1}(t^-))}(z_i(t^+))$ for all $1 \leq i \leq r^{w\lambda}$, or that $\sigma_{\mu_i}(t) \in -\partial\psi_{T_\Phi^{i-1}(z_1(t^-),...,z_{i-1}(t^-))}(z_i(t^+))$ for all $1 \leq i \leq r^{w\lambda}$, where $\sigma_{\mu_i}(t)$ is the density of μ_i with respect to δ_t. Thus one sees that λ in (22.5) can be given a meaning as

$$\lambda = (CA^{r^{w\lambda}-1}B)^{-1}[\mu_1^{(r^{w\lambda}-1)} + ... + \dot{\mu}_{r^{w\lambda}-1}] + \mu_{r^{w\lambda}} \tag{22.8}$$

provided $CA^{r^{w\lambda}-1}B$ is invertible. Then λ is uniquely defined as in (22.8). The positivity of λ is now understood as the positivity of $\mu_{r^{w\lambda}}$. It is then important to see that the distributional dynamics

$$D^{r^{w\lambda}}z_1 = Dz_{r^{w\lambda}} = CA^{r^{w\lambda}}W^{-1}\{z\} + CA^{r^{w\lambda}-1}B\lambda \tag{22.9}$$

with λ in (22.7) (22.8), is equivalent to (22.5) (22.7). In (22.9) we used the standard notation for distributional derivatives (Ferreira (1997)). Notice that (22.9) (22.8) (22.7) is the same as (22.4) (22.8) (22.7).

Definition 22.1 *The higher order (or extended) sweeping process is the dynamical system represented in (22.4)-(22.5)-(22.7). This is a particular measure differential inclusion.*

Assumption 22.2 *Let the test functions $\phi(\cdot)$ be with compact support and n times differentiable with continuous $(n+1)$th derivative. The solutions of the higher order sweeping process in (22.4)-(22.5)-(22.7) are distributions T such that $\langle T, \phi\rangle = \sum_{i=0}^{n}\int_{supp(\phi)}\phi^{(i)}(t)dh_i$, where h_i, $1 \leq i \leq n$ are RCLSBV (Right Continuous Locally Special Bounded Variation), and $n \leq r^{w\lambda}+1$.*

Thus $dh_i = \dot{h}_i(t)dt + \mu_{h_ia}$, where $\dot{h}_i(\cdot)$ is Lebesgue integrable and μ_{h_ia} is an atomic measure with countable set of atoms. Distributions of the form $f(t) + \sum_{i=0}^{l} a_i\delta_{t_k}^{(i)}$, $l < +\infty$, belong to the above set, with $g_i(t) = a_iH_i(t)$, $H_i(\cdot)$ is the Heaviside function with jump at $t = t_k$. The proposed framework permits atomic distributions with support a set of times $\{t_k\}_{k\geq 0}$, $0 \leq t_k < +\infty$, with possible accumulations, and that may even not be orderable. *We also note that assumption 22.2 implies that outside the state jump times, right and left limits exist so that $z(t^+) = \lim_{s\to t, s>t} z(s)$ and $z(t^-) = \lim_{s\to t, s<t} z(s)$ have a meaning for all t in the interval of existence of solutions, which is a crucial property in the developments which follow.* This framework also allows us to recover the case of Mechanics which involves signed distributions of degree ≤ 2, i.e. measures (Ballard, 2000). The framework proposed in assumption

22.2 is thought to be large enough to encompass the non-autonomous and nonlinear cases as well. One sees that from assumption 22.2, the measures μ_i are the derivatives of some functions $\nu_i(\cdot) \in RCLSBV$. We recall that the continuous part of the derivative of special functions of bounded variation is absolutely continuous.

From assumption 22.2, (22.6) (22.7) and (22.8) the extended sweeping process can be formulated as the evolution Variational Inequality

$$\left\{\begin{array}{l}
\langle D\{z_i\} - \{z_{i+1}\}, v - z_i(t^+)\rangle \geq 0, \\
\forall \;\; v \in T_\Phi^{i-1}(z_1(t^-), ..., z_{i-1}(t^-)), 1 \leq i \leq r^{w\lambda} - 1 \\
\\
\langle (CA^{r^{w\lambda}-1}B)^{-1}[D\{z_{r^{w\lambda}}\} - CA^{r^{w\lambda}}W^{-1}\{z\}], v - z_{r^{w\lambda}}(t^+)\rangle \geq 0, \\
\forall \;\; v \in T_\Phi^{r^{w\lambda}-1}(z_1(t^-), ..., z_{r^{w\lambda}-1}(t^-)) \\
\\
\langle \dot{\xi}(t) - A_\xi \xi(t) - B_\xi z_1(t), v - \xi(t)\rangle \geq 0, \;\; \forall \, v \in I\!R \\
\\
z(0^-) = z_0 \in I\!R^n
\end{array}\right. \tag{22.10}$$

Remark 22.3 *Assumption 22.2 is not too stringent. Especially the fact that solutions admit right and left limits everywhere seems to be a basic requirement. Many of the technical results that follow use it.*

Starting from (22.2) one is tempted to write the inclusion

$$Dz_{r^{w\lambda}} - CA^{r^{w\lambda}}W^{-1}z(t) \in -CA^{r^{w\lambda}-1}B \; \partial\psi_\Phi(z_1(t))$$

which makes sense only if λ is a measure since $\partial\psi_\Phi(z_1(\cdot))$ is a cone. This inclusion is replaced by

$$\mu_{r^{w\lambda}} \in - \; \partial\psi_{T_\Phi^{r^{w\lambda}-1}(z_1(t^-),...,z_{r^{w\lambda}-1}(t^-))}(z_{r^{w\lambda}}(t^+))$$

in (22.7). The following is true

Lemma 22.4 *The inclusion $\partial\psi_{T_\Phi^{r^{w\lambda}-1}(z_1,...,z_{r^{w\lambda}-1})}(z_{r^{w\lambda}}) \subseteq \partial\psi_\Phi(z_1)$ holds for all $z_1, ..., z_{r^{w\lambda}}$. Also from (22.7) one has $z_1 > 0 \Longrightarrow \mu_{r^{w\lambda}} = 0$, and if $z_1 = 0$ then $\mu_{r^{w\lambda}} \geq 0$.*

Proof: If $z_1 > 0$ then $T_\Phi(z_1) = I\!R$ and $\partial\psi_\Phi(z_1) = \{0\}$, and since $T_\Phi^{i-1}(z_1, ..., z_{i-1}) = I\!R$ then $\partial\psi_{T_\Phi^{i-1}(z_1,...,z_{i-1})}(z_i) = \{0\}$ for all $1 \leq i \leq r^{w\lambda}$. In particular from (22.7) $\mu_{r^{w\lambda}} = 0$. If $z_1 = 0$ then $\partial\psi_{I\!R^+}(z_1) = I\!R^-$. Depending on the values of $z_2, ..., z_i$ being positive or non-positive, one may have $\partial\psi_{T^i_\Phi(z_1,\ldots,z_i)}(z_{i+1}) = I\!R^-$ or $\partial\psi_{T^i_\Phi(z_1,\ldots,z_i)}(z_{i+1}) = \{0\}$ for all $1 \le i \le r^{w\lambda}-1$. Indeed assume that $z_1 = z_2 = \ldots = z_j = 0$ and $z_{j+1} > 0$ (this implies that $z \succeq 0$). Then $(\Phi) = T^0_\Phi(z_1) = T_\Phi(z_1) = T^2_\Phi(z_1,z_2) = T^3_\Phi(z_1,z_2,z_3) = \ldots = T^j_\Phi(z_1,\ldots,z_j) = I\!R^+$. And $T^{j+1}_\Phi(z_1,\ldots,z_{j+1}) = T^{j+2}_\Phi(z_1,\ldots,z_{j+2}) = \ldots = T^{r^{r\lambda}-1}_\Phi(z_1,\ldots,z_{r^{w\lambda}-1}) = I\!R$. This can be seen since for instance $T^{j+2}_\Phi(z_1,\ldots,z_{j+2}) = T_{T^{j+1}_\Phi(z_1,\ldots,z_{j+1})}(z_{j+2}) = T_{I\!R}(z_{j+2}) = I\!R$ because also $T^{j+1}_\Phi(z_1,\ldots,z_{j+1}) = T_{I\!R^+}(z_{j+1}) = I\!R$. Consequently $\partial\psi_{T^i_\Phi(z_1,\ldots,z_i)}(z_{i+1}) = I\!R^-$ for all $0 \le i \le j$, whereas $\partial\psi_{T^i_\Phi(z_1,\ldots,z_i)}(z_{i+1}) = \{0\}$ for $j+1 \le i \le r^{w\lambda}-1$. We conclude that under such conditions $\mu_i \ge 0$ for all $1 \le i \le j+1$, and $\mu_i = 0$ for all $j+2 \le i \le r^{w\lambda}$. Consequently $\mu_{r^{w\lambda}} \ge 0$ when $z_1 = 0$. The inclusion is also proved. ■

LEMMA 22.5 *The distribution Dz_1 in (22.5) (22.7) is of degree ≤ 2, so that $z_1(\cdot)$ is a function of time and the zero-dynamics is an ODE (i.e. $D\xi = \frac{d\xi}{dt}(t)$). Also λ has degree $\le r^{w\lambda}+1$.*

Proof: From (22.7) μ_1 has degree ≤ 2 so from (22.5) and (22.8) the result follows. ■

The following lemmas prove that the extended sweeping process inclusion defines a well-posed state jump mapping.

LEMMA 22.6 *From (22.4) (22.5) (22.7) we get for all $1 \le i \le r^{w\lambda}-1$ and all $t \ge 0$,*

$$
\begin{array}{ll}
z_{i+1}(t^+) - z_{i+1}(t^-) \in -\partial\psi_{T^i_\Phi(z_1(t^-),\ldots,z_i(t^-))}(z_{i+1}(t^+)) & (\mathbf{a}) \\
\Updownarrow & \\
z_{i+1}(t^+) = prox\left[T^i_\Phi(z_1(t^-),\ldots,z_i(t^-)); z_{i+1}(t^-)\right] & (\mathbf{b})
\end{array}
\tag{22.11}
$$

Proof: The proof starts by noting that (22.5) is an equality of distributions, and that at state jump times, Dz_i is a distribution of degree strictly larger than $z_{i+1}(\cdot)$ which is a function (Ferreira (1997), Theorem 1.1). Equaling distributions of same degree results in (22.11) (a) (see (Brogliato, 1999, §1.1) for such a reasoning in the case of distributions of degree ≤ 2). The rest of the proof is a direct consequence of the equivalence $-x+y \in \partial\psi_K(x) \iff x = \text{prox}[K;y]$ (Rockafellar and Wets, 1998, Example 10.2) (Brezis, 1973, Example 2.8.2), where K is a nonempty convex set, and $\text{prox}[K;y]$ denotes the closest vector to y in K (i.e. the projection of y on K). ■

REMARK 22.7 *The proximation operation in (22.11) (b) is used here in the sense of (Moreau, 1963), i.e.* $prox_f z$ *is the point where the function* $u \mapsto \frac{1}{2}||z-u||^2 + f(u)$ *attains its minimum value. When* $f(\cdot) = \psi_K(\cdot)$, *then* $prox_f z = prox[K; z]$.

This shows that jumps are automatically taken into account by the dynamics as it is written in (22.4)-(22.5)-(22.7). We notice also that the lower triangular structure of the tangent cones which appear in (22.7) merely reflects the way the measures μ_i appear in (22.5). We also have

LEMMA 22.8 *Let* $m \geq 1$.
Let us assume that $CA^{r^{w\lambda}-1}B = (CA^{r^{w\lambda}-1}B)^T > 0$. *Then for all* $t \geq 0$,

$$-z_{r^{w\lambda}}(t^+) + z_{r^{w\lambda}}(t^-) \in CA^{r^{w\lambda}-1}B\ \partial\psi_{T_\Phi^{r^{w\lambda}-1}(z_1(t^-),...,z_{r^{w\lambda}-1}(t^-))}(z_{r^{w\lambda}}(t^+))$$

$$\Updownarrow$$

$$z_{r^{w\lambda}}(t^+) = prox_{(CA^{r^{w\lambda}-1}B)^{-1}}\left[T_\Phi^{r^{w\lambda}-1}(z_1(t^-),...,z_{r^{w\lambda}-1}(t^-)); z_{r^{w\lambda}}(t^-)\right] \tag{22.12}$$

and these generalized equations possess the same unique solution $z_{r^{w\lambda}}(t^+)$ *for any* $z_{r^{w\lambda}}(t^-)$.

COROLLARY 22.9 *Let the solution of (22.4) (22.5) (22.7) exist on a time interval* $[\tau, \tau+\epsilon]$ *and assumption 22.2 holds. Then* $z_i(t^+) \in T_\Phi^{i-1}(z_1(t^-), ..., z_{i-1}(t^-))$ *for all* $2 \leq i \leq r_{w\lambda}$, *and* $0 \leq z_1(t^+) \perp \mu_{r^{w\lambda}}(t) \geq 0$ *for all* $t \in [\tau, \tau+\epsilon]$.

Proof: From lemmas 22.4, 22.6 and 22.8, and also from assumption 22.2 which implies that solutions have right and left limits everywhere in $[\tau, \tau+\epsilon]$. Indeed from lemma 22.4 we have that $z_1 > 0 \Longrightarrow \mu_{r^{w\lambda}} = 0$, and if $z_1 = 0$ then $\mu_{r^{w\lambda}} \geq 0$. Let us assume now that $\mu_{r^{w\lambda}}(t) > 0$. This implies that $\partial\psi_{T_\Phi^{r^{w\lambda}-1}(z_1(t^-),...,z_{r^{w\lambda}-1}(t^-))}(z_{r^{w\lambda}}(t^+)) = I\!R^+$ which in turn implies that $z_1(t^-) \leq 0$ from the definition of the tangent cones (see also the developments in the proof of lemma 22.4). Now from lemma 22.6 it follows that $z_1(t^+) \geq 0$ for all $t \in [\tau, \tau+\epsilon]$ since $z_1(t^+) = prox[\Phi; z_1(t^-)]$. We deduce that in fact if $\mu_{r^{w\lambda}} > 0$ then $z_1(t^+) = 0$. Thus $0 \leq z_1(t^+) \perp \mu_{r^{w\lambda}}(t) \geq 0$. The assertion $z_i(t^+) \in T_\Phi^{i-1}(z_1(t^-, ..., z_{i-1}(t^-))$ is a consequence of (22.11) and (22.12): right limits, which by assumption exist, belong to the tangent cones. ∎

This result (which is also a consequence of the inclusion in lemma 22.4 and of (22.7) with $i = r^{w\lambda}$) implies that on intervals $[\tau, \tau+\epsilon)$, $\epsilon > 0$, on which $z_1(t) = 0$, the inclusion (22.4)-(22.5)-(22.7) implies the existence of a multiplier $\lambda(t) = \mu_{r^{w\lambda}}(t)$ that belongs to $-\partial\psi_{I\!R^+}(0) = I\!R^+$, equivalently which satisfies $0 \le z_1(t^+) \perp \lambda(t) \ge 0$ and is the solution of the linear complementarity problem (LCP)

$$0 \le \lambda(t) \perp CA^{r^{w\lambda}} W^{-1} z(t) + CA^{r^{w\lambda}-1} B\lambda(t) \ge 0 \qquad (22.13)$$

Equivalently, $\mu_{r^{w\lambda}}$ is the sum of an atomic measure $d\mu_{r^{w\lambda}}$, $\mathrm{supp}(d\mu_{r^{w\lambda}}) \subset \{t|\ z_1(t) = 0\}$, that corresponds to jumps in $z_{r^{w\lambda}}(\cdot)$ and whose magnitude is the solution of the LCP in (22.12), and of a Lebesgue measure $g_{r^{w\lambda}}(t)dt$ where $g_{r^{w\lambda}}(t)$ is the solution of the LCP in (22.13). In corollary 22.9, the complementarity condition $0 \le z_1(t^+) \perp \mu_{r^{w\lambda}}(t) \ge 0$ can equivalently be written as $0 \le z_1(t^+) \perp g_{r^{w\lambda}}(t) \ge 0$, since the complementarity holds at the right limit $z_1(^+)$. Similarly $\mu_i = g_i(t)dt + d\mu_i$ for all $1 \le i \le r^{w\lambda}$, with $\mathrm{supp}(d\mu_i) \subset \{t|\ z_1(t) = 0\}$. On $[\tau, \tau + \epsilon)$ one has $g_i(t) = 0$ for all $1 \le i \le r^{w\lambda} - 1$, as can easily be deduced from $z_1(t) \equiv 0$. We thus have proved the following

LEMMA 22.10 *Consider the extended sweeping process dynamics (22.4) (22.5) (22.7) and let assumption 22.2 hold. Then $g_i(t) = 0$ for all $1 \le i \le r^{w\lambda} - 1$ and almost all $t \ge 0$, whereas $g_{r^{w\lambda}}(t)$ is the solution of the LCP in (22.13).*

LEMMA 22.11 *The following inclusion holds*

$$\partial\psi_{T_\Phi^{i-1}(z_1,\dots,z_{i-1})}(z_i) \subseteq \partial\psi_{T_\Phi^{i-2}(z_1,\dots,z_{i-2})}(z_{i-1}) = N_{T_\Phi^{i-2}(z_1,\dots,z_{i-2})}(z_{i-1}) \qquad (22.14)$$

for all $1 \le i \le r^{w\lambda}$.

Proof: Let $z_1 = z_2 = \dots = z_j = 0$ and $z_{j+1} > 0$. Then as already shown in the proof of lemma 22.4, one has $\partial\psi_{T_\Phi^{i}(z_1,\dots,z_i)}(z_{i+1}) = \mathbb{R}^-$ for all $0 \le i \le j$, and $\partial\psi_{T_\Phi^{i}(z_1,\dots,z_i)}(z_{i+1}) = \{0\}$ for all $j+1 \le i \le r^{w\lambda} - 1$. So one sees that in particular it always holds that $\partial\psi_{T_\Phi^{k}(z_1,\dots,z_k)}(z_{k+1}) \subseteq \partial\psi_{T_\Phi^{k-1}(z_1,\dots,z_{k-1})}(z_k)$ for any $1 \le k \le r^{w\lambda} - 1$. ■

COROLLARY 22.12 *The operators $z_i(t^+) \mapsto -\mu_i$, $1 \le i \le r^{w\lambda}$, in (22.7) are maximal monotone.*

Proof: These operators can be rewritten as the following cone CP

$$T_\Phi^{i-1}(z_1(t^-), \dots, z_{i-1}(t^-)) \ni z_i(t^+) \perp \\ - \mu_i \in \partial\psi_{T_\Phi^{i-1}(z_1(t^-),\dots,z_{i-1}(t^-))}(z_i(t^+)) \qquad (22.15)$$

Since from lemma 22.11 we have

$$\partial\psi_{T_\Phi^{i-1}(z_1(t^-),\dots,z_{i-1}(t^-))}(z_i(t^+)) \subseteq N_{T_\Phi^{i-2}(z_1(t^-),\dots,z_{i-2}(t^-))}(z_{i-1}(t^+)$$

the cones in both sides of (22.15) are closed polar convex cones. Consequently the operators that correspond to these cone CPs are maximal monotone. ■

We notice that from (22.15) we get

$$T_\Phi(z_1(t^-)) \ni z_2(t^+) \perp -\mu_2 \in \partial\psi_{T_\Phi(z_1(t^-))}(z_2(t^+)) \subseteq N_\Phi(z_1(t^+))$$

It is also noteworthy that from corollary 22.9 one gets that the operator $-\mu_{rw\lambda} \mapsto z_1(t^+)$ is also maximal monotone. Hence our framework contains that of the sweeping process for Lagrangian systems (Moreau, 1988).

3. Numerical time integration scheme

This section addresses the problem of the numerical time integration of complementarity dynamical systems with arbitrary relative degree. Particularly, it is shown how one can take advantage of the formalism in (22.4)-(22.7) to design a time-stepping scheme, i.e, a time integration scheme without explicit event handling procedure.

A naive way to design a time-stepping scheme for non smooth systems is to apply a backward Euler method to the dynamical equation and to discretize the complementarity condition in a fully implicit way. One obtains for the system (22.1) the following discretized system:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{h} = Ax_{k+1} + B\lambda_{k+1} \\ w_{k+1} = Cx_{k+1} + D\lambda_{k+1} \\ 0 \leq \lambda_{k+1} \perp w_{k+1} \geq 0 \end{cases} \tag{22.16}$$

where h is the constant time step of a subdivision $\{t_k\}$ of the time interval $[0, T]$ and the subscript k denotes the approximation of a value at time t_k.

A straightforward substitution of w_{k+1} in the complementarity condition leads to solve the following complementarity problem at each step:

$$0 \leq \lambda_{k+1} \perp C(I - hA)^{-1}x_k + hC(I - hA)^{-1}B\lambda_{k+1} \geq 0 \tag{22.17}$$

For the linear complementarity systems, some sufficient conditions for consistency and convergence of this backward Euler scheme have been given in (Camlibel, 2002). They also exhibit several examples for which the scheme does not work at all. Let us consider one of these examples, introduced in (Camlibel, 2001, Example 6.3.3), where the system (22.1) is defined by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; x_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{22.18}$$

In this example, the relative degree $r^{w\lambda}$ is equal to 3 ($D = 0, CB = 0, CAB = 0, CA^2B \neq 0$). Using the time discretization defined above, we can remark that:

$$\lim_{h \longrightarrow 0} hC(I - hA)^{-1}B = 0 \tag{22.19}$$

It is clear that if the time step h vanishes, which may be needed in many practical cases or for the convergence analysis purpose, then the LCP matrix in (22.17) has little chance to be well conditioned due to the fact that $CB = 0$. Furthermore, the numerical solution is given by

$$x_k = \begin{bmatrix} \frac{k(k+1)}{h} \\ k \\ \frac{1}{h} \end{bmatrix}, \forall k \geq 1; \quad \lambda_1 = \frac{1}{h^2}; \quad \lambda_k = 0, \forall k \geq 2 \tag{22.20}$$

which cannot converge towards a solution when h vanishes.

In fact, the backward Euler scheme is only consistent for the systems of relative degree, $r_{w\lambda} \leq 1$. Indeed, we can construct easily many examples of inconsistency with systems of relative degree equal to two.

A similar time-stepping method has been introduced in (Moreau, 1977) for a mathematical analysis purpose of the existence of solution for the first order sweeping process. For systems of relative degree equal to two, such as Lagrangian mechanical systems with unilateral constraints, Moreau (Moreau, 1988) introduces the Contact Dynamics method which extends the simple Euler method when some discontinuities may be encountered in the first time derivative of the state. Although the resulting scheme seems to be very close to a standard Euler scheme, there are several slight, but fundamental differences based on a sound mathematical analysis of Moreau about the nature of the solution. For the Lagrangian systems, majors lessons of this seminal work may be stated as follows:

- the use of a differential measure associated with a function of bounded variation leads to a first order approximation given directly by the integration of the differential measure,

- the use of finite values, as primary unknowns, such as velocity and impulse. This feature allows one to capture the discontinuities when the time step vanishes,

- the reformulation of constraints in terms of velocity associated with a viability lemma to ensure the satisfaction of the position constraint.

In the next section, we propose a time-stepping scheme based on these remarks, which is able to integrate in time linear complementarity problems of any relative degree.

3.1 The proposed numerical scheme

The proposed numerical scheme is based on the ZD dynamical form written in terms of measures (22.6). Let us consider a subdivision $\{t_k\}$ of the interval $[0,T]$. The time integration of the differential measure on $]t_k, t_{k+1}]$ is given by

$$\int_{]t_k,t_{k+1}]} D\{z_i\} = z_i(t_{k+1}^+) - z_i(t_k^+) \tag{22.21}$$

and we pose as a primary unknown the right value of the non singular part of z_i such that $z_{i,k} = z_i(t_k^+)$ (recall that $z(t)$ or $\{z\}$ denote the non singular part of the distribution z). These remarks leads to the following numerical integration rule for a generic line of the system (22.6):

$$\int_{]t_k,t_{k+1}]} D\{z_i\} = z_{i,k+1} - z_{i,k} = \int_{]t_k,t_{k+1}]} \{z_{i+1}\}\,dt + \int_{]t_k,t_{k+1}]} \mu_i \tag{22.22}$$

$$\approx h z_{i+1,k+1} + \int_{]t_k,t_{k+1}]} \mu_i \tag{22.23}$$

In order to manipulate only finite values, this second unknown, which is the multiplier corresponding to μ_i, is defined as:

$$r_{i,k+1} = \int_{]t_k,t_{k+1}]} \mu_i \tag{22.24}$$

and we assume that:

$$r_{i,k+1} \in -\partial\psi_{T_\Phi^{i-1}(z_{1,k},\ldots,z_{i-1,k})}(z_{i,k+1}) \text{ for all } 1 \le i \le r^{w\lambda} \tag{22.25}$$

The approximation of $z_i(t^-)$ in the inclusion (22.7) by $z_{i,k}$ is a basic choice. The operation can be viewed as a prediction of the state before a discontinuity. More accurate prediction may be performed using higher order derivatives, if exist, of $z_i(t^-)$. Finally, the proposed discretization scheme may be summarized as follows:

$$\begin{cases} z_{1,k+1} - z_{1,k} = hz_{2,k+1} + r_{1,k+1} \\ z_{2,k+1} - z_{2,k} = hz_{3,k+1} + r_{2,k+1} \\ \vdots \\ z_{i,k+1} - z_{i,k} = hz_{i+1,k+1} + r_{i,k+1} \\ \vdots \\ z_{r^{w\lambda},k+1} - z_{r^{w\lambda},k} = hCA^{r^{w\lambda}}W^{-1}z_{k+1} + CA^{r^{w\lambda}-1}Br_{r^{w\lambda},k+1} \\ \\ \xi_{k+1} - \xi_k = hA_\xi \xi_{k+1} + hB_\xi z_{1,k+1} \\ \\ r_{i,k+1} \in -\partial\psi_{T_\Phi^{i-1}(z_{1,k},\ldots,z_{i-1,k})}(z_{i,k+1}) \ \text{ for all } \ 1 \leq i \leq r^{w\lambda} \end{cases} \tag{22.26}$$

If $\Phi = I\!R^+$, the inclusion (22.25) implies a sequence of unilateral constraints on $z_{i,k}$ to be satisfied. If the integer j is the first for which $z_{j,k}$ is positive, then the system (22.26) is reduced to a linear complementarity problem involving $z_{i,k+1}, r_{i,k+1}$ for all $i, 1 \leq i \leq j$.

3.2 Numerical examples

We illustrate in this part the ability of the scheme to solve the preliminary example. The numerical solution given by the scheme (22.26) for the initial condition $z_0 = [0; -1; 0]^T$ is given by $z_k = [0; 0; 0]^T, \forall k \geq 1$ and $r_2, 1 = 1, r_2, k = 0, \forall k \geq 2$. The Figure 22.1 depicts a similar result with the initial condition $z_0^T = (1, -1, 0)$.

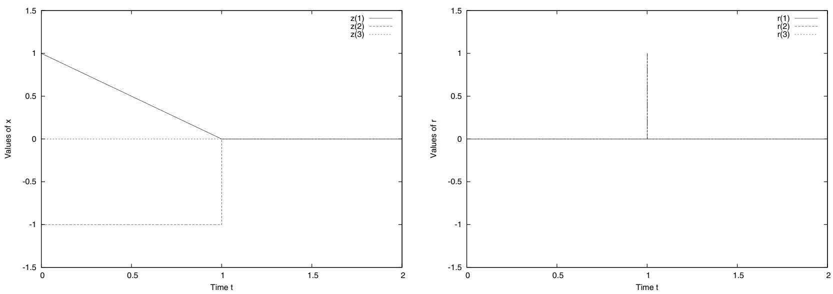

Figure 22.1. EMSP scheme – Initial data $z(0^-)^T = (1, -1, 0)$

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