# Extension without Cut

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**Abstract**

In proof theory one distinguishes sequent proofs with cut and cut-free sequent proofs, while for proof complexity one distinguishes Frege-systems and extended Frege-systems. In this paper we show how deep inference can provide a uniform treatment for both classifications, such that we can define cut-free systems with extension, which is neither possible with Frege-systems, nor with the sequent calculus. We show that the propositional pigeon-hole principle admits polynomial-size proofs in a cut-free system with extension. We also define cut-free systems with substitution and show that the cut-free system with extension p-simulates the cut-free system with substitution.

## 1. Introduction

For studying proof complexity (for propositional classical logic) one essentially distinguishes between two kinds of proof systems: *Frege systems* and *extended Frege systems* [1]. Roughly speaking, a Frege-system consists of a set of axioms and *modus ponens*, and in an extended Frege-system one can also use "abbreviations", i.e., fresh propositional variables abbreviating arbitrary formulas appearing in the proof. Clearly, any extended Frege-proof can be converted into a Frege-proof by systematically replacing the abbreviations by the formulas they abbreviate, at the cost of an exponential increase of the size of the proof. Surprisingly, this distinction is not investigated from the proof theoretic viewpoint.

On the other hand, in proof theory one also distinguishes between two kinds of proof systems: those *with cut* and those *without cut*. In a well-designed proof system, it is always possible to convert a proof with cuts into a cut-free proof, at the cost of an exponential increase of the size of the proof (see, e.g., [2]). The cuts are usually understood as "the use of auxiliary lemmas inside the proof". The main tool for investigating the cut and its elimination from a proof is Gentzen's sequent calculus [3].

The two proof classifications are usually not studied together. In fact, every Frege-system contains cut because of the presence of modus ponens. Hence, there is no such thing as a "cut-free Frege system", or a "cut-free extended Frege-system". Similarly, there are no "extended Gentzen systems", because it does not make sense to speak of abbreviations in the sequent calculus, where formulas are decomposed along their main connectives during proof search.[1] This can be summarized by the classification of proof systems shown in Figure 1, where $S_1 \subseteq S_2$ means that $S_2$ includes $S_1$ (and therefore $S_2$ p-simulates $S_1$).

| cut-free sequent systems | $\subseteq$ | sequent systems with cut | $=$ | Frege systems | $\subseteq$ | extended Frege systems |
|---|---|---|---|---|---|---|

Figure 1: Classification of proof systems

There are classes of tautologies that admit no polynomial size proofs in cut-free sequent calculus [5] (and related systems, like resolution [6] and tableaux). But no such class is known for systems with cut

[1] The extension discussed in this paper should not be confused with the notion of "definition" in the sequent calculus LKDe [4], in which the abbreviation may occur in the endsequent of the proof.

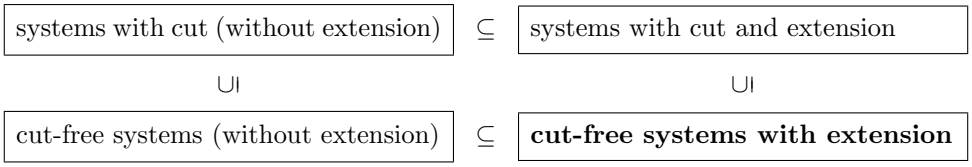


Figure 2: Refined classification of proof systems

or for extended Frege systems. The question whether there is a short, i.e., polynomial size, proof of every tautology $A$ is equivalent to the question whether **NP** is equal to **coNP**.

The contributions of this paper can be summarized as follows:

(i) I provide a deductive system in which extension is independent from the cut, i.e., we can now study cut-free systems with extension.[2] Figure 2 shows the refined classification of proof systems. I use as formalism the *calculus of structures* [9, 10, 11]. Thus, this paper is a continuation of the work by Bruscoli and Guglielmi [12], who observed that by using *deep inference* one can bring the extension rule to a deductive formalism which has originally been designed to study cut-elimination.

(ii) I also present a cut-free system with substitution and investigate the relation between substitution and extension in a cut-free setting.

(iii) In order to provide evidence that it indeed makes sense to study extension (or substitution) independently from cut, I will show polynomial-size proofs for the propositional pigeon-hole principle (PHP) without cut. At the same time, I propose a new class of tautologies (called QHQ), that have similar properties as the PHP wrt. proof complexity, but have the additional property of being *balanced*, i.e., every atom occurs exactly once positive and exactly once negative.

Sections 2–4 of this paper contain preliminaries on proof systems in general and the calculus of structures in particular. Then, Sections 5–7 are dedicated to points (i)–(iii) above.

## 2. Preliminaries on Proof Systems

Following [1], we define a *proof system* to be a surjective **PTIME**-function $S\colon \Sigma^* \to \mathscr{T}$ where $\Sigma$ is some finite alphabet (and $\Sigma^*$ the set of all finite words over $\Sigma$) and $\mathscr{T}$ is the set of all Boolean tautologies. An element of $\pi \in \Sigma^*$ is called a *proof* and $S(\pi)$ its *conclusion*. We denote by $|\pi|$ the size of $\pi$, i.e., the number of symbols in $\pi$. Given two proof systems $S_1\colon \Sigma_1^* \to \mathscr{T}$ and $S_2\colon \Sigma_2^* \to \mathscr{T}$, we say that $S_2$ *p-simulates* $S_1$ iff there is a polynomial $p$ such that for every proof $\pi_1 \in \Sigma_1^*$ there is a proof $\pi_2 \in \Sigma_2^*$ of the same conclusion (i.e., $S_2(\pi_2) = S_1(\pi_1)$) such that $|\pi_2| \leq |p(\pi_1)|$. We say that $S_1$ and $S_2$ are *p-equivalent* iff they p-simulate each other. A proof system $S\colon \Sigma^* \to \mathscr{T}$ is *polynomially bounded* iff there is a polynomial $p$ such that for every tautology $T \in \mathscr{T}$ there is a proof $\pi \in \Sigma^*$ with $S(\pi) = T$ and $|\pi| \leq p(|T|)$. If $|\pi| \leq p(|T|)$ for some $\pi \in \Sigma^*$ with $S(\pi) = T$, we also say that $\pi$ is a *short* proof of $T$. Thus, a polynomially bounded proof system is one in which every tautology has a short proof wrt. some fixed bounding polynomial $p$. The question whether there is a polynomially bounded proof system is equivalent to the question whether the complexity class **NP** is closed under complement:

**Theorem 2.1.** *There exists a polynomially bounded proof system if and only if* **NP** = **coNP**. [1]

[2] Technically speaking, Haken's extended resolution [6] is a cut-free system with extension, but his system is not suited to study cut elimination. And it is not clear how to incorporate Haken's extension into the recent work on resolution and cut elimination [7, 8].

For the sake of simplicity, let us consider for the rest of the paper only formulas in negation normal form. More precisely, formulas are generated from a countable set $\mathscr{A} = \{a, b, c, \ldots\}$ of propositional variables and their negations $\bar{\mathscr{A}} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$ via the binary connectives $\wedge$ and $\vee$, called *and* and *or*, respectively.[3] I denote formulas by capital Latin letters $(A, B, C, \ldots)$. Negation is defined for all formulas via the de Morgan laws: $\bar{\bar{a}} = a$ and $\overline{A \wedge B} = \bar{B} \vee \bar{A}$ and $\overline{A \vee B} = \bar{B} \wedge \bar{A}$. It follows immediately that $\bar{\bar{A}} = A$ for all formulas $A$. The elements of the set $\mathscr{A} \cup \bar{\mathscr{A}}$ are also called *literals*. We can write $A \Rightarrow B$ for $\bar{A} \vee B$ and $A \Leftrightarrow B$ for $[\bar{A} \vee B] \wedge [\bar{B} \vee A]$.

## 3. Preliminaries on the Calculus of Structures

I assume the reader to be familiar with sequent calculus or natural deduction systems [3], in which inference rules decompose formulas along their main connectives. On the other hand, in the *calculus of structures* [9, 10, 11], inference rules are allowed to do arbitrary rewriting deep inside formulas. In this paper, I use the following rule schemes (to be applied on formulas in negation normal form):

$$\mathsf{ai}{\downarrow}\ \frac{F\{B\}}{F\{B \wedge [\bar{a} \vee a]\}} \qquad \mathsf{s}\ \frac{F\{A \wedge [B \vee C]\}}{F\{(A \wedge B) \vee C\}}$$
$$\mathsf{w}{\downarrow}\ \frac{F\{B\}}{F\{B \vee A\}} \qquad \mathsf{ac}{\downarrow}\ \frac{F\{a \vee a\}}{F\{a\}} \qquad \mathsf{m}\ \frac{F\{(A \wedge B) \vee (C \wedge D)\}}{F\{[A \vee C] \wedge [B \vee D]\}} \tag{1}$$

where $A$, $B$, $C$, and $D$ must be seen as formula variables, and $a$ is a propositional variable or its negation. These rules are called *(atomic) identity*, *switch*, *weakening*, *(atomic) contraction*, and *medial*, respectively. The rules in (1) are written in the style of inference rule schemes in proof theory but they behave as rewrite rules in term rewriting, i.e., they can be applied *deep* inside any (positive) formula context $F\{\ \}$. To ease readability of large formulas, I will use [ ] for brackets around disjunctions and ( ) for brackets around conjunctions. The rewriting rules in (1) are applied modulo associativity and commutativity for $\wedge$ and $\vee$. More precisely, we will do rewriting modulo the equational theory generated by

$$\begin{array}{ll} A \wedge (B \wedge C) = (A \wedge B) \wedge C & A \wedge B = B \wedge A \\ A \vee [B \vee C] = [A \vee B] \vee C & A \vee B = B \vee A \end{array} \tag{2}$$

Because of this, we can systematically omit superfluous parentheses in order to ease readability; e.g., instead of $A \wedge ((B \wedge C) \wedge D)$ we write $A \wedge B \wedge C \wedge D$. A *derivation* is a rewrite path via (1) modulo (2). Here is an example:

$$\begin{array}{rl} & (a \wedge [a \vee a]) \vee [\bar{b} \vee \bar{b}] \wedge [\bar{b} \vee \bar{b}]) \\ 2{*}\mathsf{ac}{\downarrow} & \hline \\ & (a \wedge [a \vee a]) \vee [\bar{b} \vee \bar{b}] \wedge \bar{b}) \\ \mathsf{ai}{\downarrow} & \hline \\ & (a \wedge [a \vee a]) \vee [\bar{b} \vee \bar{b}] \wedge (\bar{b} \wedge [\bar{a} \vee a])) \\ \mathsf{s} & \hline \\ & (a \wedge [a \vee a]) \vee [\bar{b} \vee \bar{b}] \wedge [(\bar{b} \wedge \bar{a}) \vee a]) \\ = & \hline \\ & (a \wedge [a \vee a]) \vee [\bar{b} \vee \bar{b}] \wedge [a \vee (\bar{a} \wedge \bar{b})]) \\ \mathsf{ac}{\downarrow} & \hline \\ & (a \wedge [a \vee a]) \vee (\bar{b} \wedge [a \vee (\bar{a} \wedge \bar{b})]) \\ \mathsf{s} & \hline \\ & [a \vee a) \vee (\bar{a} \wedge \bar{b})] \vee (\bar{b} \wedge a) \end{array} \tag{3}$$

The notation $n{*}\mathsf{r}$ is used to indicate that there are $n$ applications of the rule $\mathsf{r}$. In the hope of helping the reader, I sometimes use a "fake inference rule"

$$= \frac{A}{B} \tag{4}$$

governed by the side condition that $A = B$ under the equivalence relation generated by (2).

[3] For simplicity, I do not introduce special symbols for the units *truth* and *falsum*. Note that these units can be recovered by the formulas $p_0 \vee \bar{p}_0$ and $p_0 \wedge \bar{p}_0$, respectively, where $p_0$ is a fresh propositional variable.

**Remark 3.1.** Instead of doing rewriting modulo, one could equivalently add four inference rules

$$=_1 \frac{F\{[A \vee B] \vee C\}}{F\{A \vee [B \vee C]\}} \qquad =_2 \frac{F\{(A \wedge B) \wedge C\}}{F\{A \wedge (B \wedge C)\}} \qquad =_3 \frac{F\{A \vee B\}}{F\{B \vee A\}} \qquad =_4 \frac{F\{A \wedge B\}}{F\{B \wedge A\}} \tag{5}$$

Computationally there is no difference between the two approaches since the the equivalence modulo $=$ can be checked in time $O(n \log n)$.

In order to obtain proofs without hypotheses, we need an axiom, which is in our case just a variant of the rule $\mathsf{ai}{\downarrow}$:

$$\mathsf{ai}{\downarrow} \frac{}{\bar{a} \vee a} \tag{6}$$

The rule in (6) cannot be applied inside a context $F\{\ \}$, but it is in spirit the same rule as $\mathsf{ai}{\downarrow}$ in (1), and I use therefore the same name. Given a system $\mathsf{S}$, we write

$$\begin{array}{c} A \\ \mathsf{S}\big\|\pi_1 \\ B \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{S}\big\|\pi_2 \\ B \end{array}$$

to denote a derivation $\pi_1$ in the system $\mathsf{S}$ from premise $A$ to conclusion $B$, and a proof $\pi_2$ in the system $\mathsf{S}$ without premise and with conclusion $B$, respectively. I write $\mathsf{KS}$ to denote the system shown in (1), together with the rule in (6). A proof in $\mathsf{KS}$ uses the axiom (6) exactly once.

**Remark 3.2.** The original formulation of $\mathsf{KS}$ in [11] uses explicit units $\mathbf{t}$ and $\mathbf{f}$ for *truth* and *falsum*, respectively, and thus contains more rules and equations to deal with them. If we denote by $\mathsf{KS}^+$ the system with units, and by $\mathsf{KS}^-$ the system without units, then we have that $\mathsf{KS}^+$ and $\mathsf{KS}^-$ are p-equivalent under the translation mentioned in Footnote 3. This can easily be shown by using the equations $B \wedge \mathbf{t} = B = B \vee \mathbf{f}$ and $B \wedge \mathbf{f} = \mathbf{f}$ and $B \vee \mathbf{t} = \mathbf{t}$ (see also [13]). Since many Frege systems are given without explicit units, and Gentzen's original $\mathsf{LK}$ comes without units, it might be helpful to see a presentation of $\mathsf{KS}$ that does not rely on the presence of units. Anyhow, everything that is said in this paper does also hold for the variant of $\mathsf{KS}$ with units.

The following two propositions have first been proved in [11]:

**Proposition 3.3.** *The rules*

$$\mathsf{i}{\downarrow} \frac{}{\bar{A} \vee A} \qquad \mathsf{i}{\downarrow} \frac{F\{B\}}{F\{B \wedge [\bar{A} \vee A]\}} \qquad \mathsf{c}{\downarrow} \frac{F\{A \vee A\}}{F\{A\}} \qquad \mathsf{d}{\downarrow} \frac{F\{[A \vee C] \wedge [B \vee C]\}}{F\{(A \wedge B) \vee C\}}$$

*are derivable in* $\mathsf{KS}$. *More precisely,* $\mathsf{KS}$ *p-simulates* $\mathsf{KS} \cup \{\mathsf{i}{\downarrow}, \mathsf{c}{\downarrow}, \mathsf{d}{\downarrow}\}$.

The rules $\mathsf{i}{\downarrow}$ and $\mathsf{c}{\downarrow}$ are the general (non-atomic) versions of $\mathsf{ai}{\downarrow}$ and $\mathsf{ac}{\downarrow}$, respectively.

**Proposition 3.4.** *The system* $\mathsf{KS}$ *p-simulates cut-free sequent calculus.*

The converse is not true, i.e., cut-free sequent calculus cannot p-simulate $\mathsf{KS}$. A counter-example can be found in [12], where Bruscoli and Guglielmi show that the example used by Statman [5] to prove an exponential lower bound for cut-free sequent calculus admits polynomial size proofs in $\mathsf{KS}$. This situation changes when we add the cut rule, which is dual to the identity rule

$$\mathsf{ai}{\uparrow} \frac{F\{(a \wedge \bar{a}) \vee B\}}{F\{B\}} \quad . \tag{7}$$

The system $\mathsf{KS} \cup \{\mathsf{ai}{\uparrow}\}$ will in the following be denoted by $\mathsf{SKS}$. The following two propositions are also due to [11]:

**Proposition 3.5.** *The rules*

$$\mathsf{i}\!\uparrow \frac{F\{(A \wedge \bar{A}) \vee B\}}{F\{B\}} \qquad \mathsf{c}\!\uparrow \frac{F\{A\}}{F\{A \wedge A\}} \qquad \mathsf{w}\!\uparrow \frac{F\{A \wedge B\}}{F\{B\}} \tag{8}$$

*are derivable in* $\mathsf{SKS}$*. More precisely,* $\mathsf{SKS}$ *p-simulates* $\mathsf{SKS} \cup \{\mathsf{i}\!\uparrow, \mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}$*.*

**Proposition 3.6.** $\mathsf{SKS}$ *is p-equivalent to every sequent system with cut.*

Finally, let us mention the following theorem, stating soundness, completeness, cut elimination, and the deduction theorem for $\mathsf{KS}$.

**Theorem 3.7.** *For any formulas* $A$ *and* $B$*, we have:*

$$\begin{array}{l}\textit{The formula } A \Rightarrow B \textit{ is}\\ \textit{a valid implication.}\end{array} \qquad \textit{iff} \qquad \mathsf{KS}\,\frac{\|}{\bar{A} \vee B} \qquad \textit{iff} \qquad \mathsf{SKS}\,\frac{\|}{\begin{array}{c}A\\ B\end{array}}$$

This does not only hold for classical logic, but also for linear logic and modal logic (for a proof see [14, 15]).

## 4. Relation between Calculus of Structures and Frege Systems

*Frege systems* (also known as *Hilbert systems* or *Hilbert-Frege-systems* or *Hilbert-Ackermann-systems* [16, 17]), consist of a set of axioms (more precisely, axiom schemes) and a set of inference rules, which in the case of classical propositional logic only contains *modus ponens*:

$$\mathsf{modus\ ponens}\ \frac{A \qquad A \Rightarrow B}{B}$$

I assume the reader to be familiar with Frege systems, and I will not go into further details. The important facts are that the set of axioms in a Frege system has to be sound and complete, and that all Frege systems p-simulate each other [1]. We also immediately have the following theorem:

**Theorem 4.1.** $\mathsf{SKS}$ *is p-equivalent to every Frege-system.*

This follows immediately from Proposition 3.6 and a result by [1]. In [12] one can find a direct proof. Because it will be needed later, I sketch here the basic idea. For p-simulating a Frege system $\mathsf{F}$ with $\mathsf{SKS}$, we first exhibit an $\mathsf{SKS}$ proof for every axiom in $\mathsf{F}$. Then we proceed by induction on the length of the proof $\pi$ in $\mathsf{F}$ and keep all formulas appearing in $\pi$ in a conjunction $F_1 \wedge F_2 \wedge \cdots \wedge F_n$. Now we can simulate modus ponens:

$$\mathsf{modus\ ponens}\ \frac{A \qquad \bar{A} \vee B}{B} \qquad \rightsquigarrow \qquad \mathsf{i}\!\uparrow \frac{\mathsf{s}\,\dfrac{A \wedge [\bar{A} \vee B]}{(A \wedge \bar{A}) \vee B}}{B}$$

Note that we might need to duplicate a formula $F_i$ by using $\mathsf{c}\!\uparrow$. Finally we remove the superfluous copies by using $\mathsf{w}\!\uparrow$. Conversely, we show that a Frege system can p-simulate $\mathsf{SKS}$ by exhibiting for every rule

$$\mathsf{r}\,\frac{A}{B}$$

a Frege-proof of $\bar{A} \vee B$. Then we show by induction that for every context $F\{\ \}$ also $\overline{F\{A\}} \vee F\{B\}$ has a Frege proof. Then the application of an inference rule in $\mathsf{SKS}$ can be simulated by modus ponens.

## 5. Extension

Let us now turn to the actual interest of this paper, the extension rule (first formulated by Tseitin [18]), which allows us to use abbreviations in the proof. I.e., there is a finite set of fresh and mutually distinct propositional variables $a_1, \ldots, a_n$ which can abbreviate formulas $A_1, \ldots, A_n$, that obey the side condition that for all $1 \leq i \leq n$, the variable $a_i$ does not appear in $A_1, \ldots, A_i$. Extension can easily be integrated in a Frege-system by simply adding the formulas $a_i \Leftrightarrow A_i$, for $1 \leq i \leq n$, to the set of axioms. In that case we speak of an *extended Frege-system* [1]. In the sequent calculus one could add these formulas as non-logical axioms, with the consequence that cut-elimination would not hold anymore. This very idea is used by Bruscoli and Guglielmi in [12] for adding extension to a system in the calculus of structures: instead of starting a proof from no premises, they use the conjunction

$$[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \tag{9}$$

of all extension formulas as premise. Let us write $\mathsf{xSKS}$ to denote the system $\mathsf{SKS}$ with the extension incorporated this way, i.e., a proof of a formula $B$ in $\mathsf{xSKS}$ is a derivation

$$\begin{array}{c} [\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\ \mathsf{SKS}\Big\| \pi \\ B \end{array} \tag{10}$$

where

$$\begin{array}{l} \text{the propositional variables } a_1, \ldots, a_n \text{ are mutually distinct, and for} \\ \text{all } 1 \leq i \leq n, \text{ the variable } a_i \text{ does not appear in } A_1, \ldots, A_i \text{ nor in } B. \end{array} \tag{11}$$

**Theorem 5.1.** $\mathsf{xSKS}$ *is p-equivalent to every extended Frege-system.*

The proof can be found in [12], and is almost literally the same as for Theorem 4.1.

It should be clear that $\mathsf{xSKS}$ crucially relies on the presence of cut, in the same way as extended Frege-system rely on the presence of modus ponens: The premise of (10) contains the variables $a_1, \ldots, a_n$, which do not appear in the conclusion $B$. Thus, the derivation in (10) must contain cuts. This raises the question whether the virtues of extension can also be used in a cut-free system.[4] For this, let us for every extension axiom $a_i \Leftrightarrow A_i$ add the following two rules (we use the same name for both of them):

$$\mathsf{ext}{\downarrow}\,\frac{F\{a_i\}}{F\{A_i\}} \qquad \text{and} \qquad \mathsf{ext}{\downarrow}\,\frac{F\{\bar{a}_i\}}{F\{\bar{A}_i\}} \tag{12}$$

We write $\mathsf{eKS}$ to denote the system $\mathsf{KS} \cup \{\mathsf{ext}{\downarrow}\}$ and we write $\mathsf{eSKS}$ for $\mathsf{SKS} \cup \{\mathsf{ext}{\downarrow}\}$. Note that the rule $\mathsf{ext}{\downarrow}$ is not sound. Consider for example the extension axiom $a \Leftrightarrow b \wedge c$ where $a$ abbreviates $b \wedge c$. Applying it to $a \vee \bar{a}$ (which is a tautology) yields $(b \wedge c) \vee \bar{a}$ (which is *not* a tautology). Nonetheless, we allow to apply (12) in an arbitrary context $F\{\ \}$, provided that condition (11) is satisfied. Then we have the following:

**Theorem 5.2.** *The systems* $\mathsf{eKS}$ *and* $\mathsf{eSKS}$ *are sound and complete for classical propositional logic.*

*Proof.* Completeness of both systems follows from completeness of $\mathsf{KS}$, and soundness of $\mathsf{eSKS}$ follows from Theorem 5.3 below and Theorem 5.1 above. This entails soundness of $\mathsf{eKS}$. □

**Theorem 5.3.** *The systems* $\mathsf{eSKS}$ *and* $\mathsf{xSKS}$ *are p-equivalent.*

[4] One could allow to add a disjunction of formulas $a_i \wedge \bar{a}_i$ to the conclusion, in the same way as we add a conjunction of $[\bar{a}_i \vee A_i] \wedge [\bar{A}_i \vee a_i]$ to the premise [12]. Some readers might consider this to be cut-free, but the question remains whether we can obtain cut-freeness without changing the notions of derivation and proof.

*Proof.* Given a proof $\pi$ of a formula $B$ in $\mathsf{xSKS}$, we can construct

$$\begin{array}{c} \\ [\bar{a}_1 \vee a_1] \wedge [\bar{a}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee a_n] \wedge [\bar{a}_n \vee a_n] \\ [\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\ B \end{array} \quad \begin{array}{l} \{\mathsf{ai}{\downarrow}\}\Big\|\pi_2 \\ \{\mathsf{ext}{\downarrow}\}\Big\|\pi_1 \\ \mathsf{SKS}\Big\|\pi \end{array} \tag{13}$$

where $\pi_1$ consists of $2n$ instances of $\mathsf{ext}{\downarrow}$ and $\pi_2$ of $2n$ instances of $\mathsf{ai}{\downarrow}$. Hence, $\mathsf{eSKS}$ p-simulates $\mathsf{xSKS}$. For the converse, assume we have an $\mathsf{eSKS}$ proof $\pi$ of a formula $B$. We can put every line of $\pi$ in conjunction with the formula (9), and add a coweakening $\mathsf{w}{\uparrow}$ (see Proposition 3.5) at the bottom:

$$\mathsf{eSKS}\Big\|\pi \quad B \qquad \rightsquigarrow \qquad \mathsf{w}{\uparrow}\dfrac{\begin{array}{c}[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\ \mathsf{eSKS}\Big\|\pi' \\ [\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \wedge B\end{array}}{B}$$

The instances of $\mathsf{ext}{\downarrow}$ in $\pi'$ can now be removed as follows:

$$\mathsf{ext}{\downarrow}\dfrac{\cdots \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{a_i\}}{\cdots \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{A_i\}} \quad \rightsquigarrow \quad \mathsf{ai}{\uparrow}\dfrac{\mathsf{s}\dfrac{\mathsf{c}{\uparrow}\dfrac{\cdots \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{a_i\}}{\cdots \wedge [\bar{a}_i \vee A_i] \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{a_i\}}}{\begin{array}{c}\{\mathsf{s}\}\Big\|\pi_{\mathsf{s}} \\ \cdots \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{a_i \wedge [\bar{a}_i \vee A_i]\}\end{array}}}{\cdots \wedge [\bar{a}_i \vee A_i] \wedge \cdots \wedge F\{(a_i \wedge \bar{a}_i) \vee A_i\}} \tag{14}$$

where $F\{\ \}$ is an arbitrary (positive) context, and the existence of $\pi_{\mathsf{s}}$ (which contains only instances of the rule $\mathsf{s}$) can be shown by an easy induction on $F\{\ \}$ (see e.g, Lemma 4.3.20 in [15]). The length of $\pi_{\mathsf{s}}$ is bound by the depth of $F\{\ \}$. Note the crucial use of the cut rule in (14). □

System $\mathsf{eKS}$ gives us a way of adding extension to a deductive system independently from cut. To show that extension without cut is potentially useful for giving short proofs for some of the standard benchmark tautologies, we give in Section 7 polynomial size proofs of the propositional pigeon hole principle in $\mathsf{eKS}$.

**Remark 5.4.** When we say "independent from cut", we have to clarify what "cut" means. The way we added extension to $\mathsf{KS}$ to get $\mathsf{eKS}$ is clearly independent from the chosen language. E.g., it does not matter whether we chose a presentation with or without units: if we define $\mathsf{eKS}^-$ and $\mathsf{eKS}^+$ by adding $\mathsf{ext}{\downarrow}$ to $\mathsf{KS}^-$ and $\mathsf{KS}^+$, respectively (see Remark 3.2), then one can easily show that $\mathsf{eKS}^-$ and $\mathsf{eKS}^+$ p-simulate each other (if the units appear inside the extension formulas $A_i$ then they can be removed using $B \wedge \mathbf{t} = B = B \vee \mathbf{f}$ and $B \wedge \mathbf{f} = \mathbf{f}$ and $B \vee \mathbf{t} = \mathbf{t}$, and if $A_i = \mathbf{t}$ or $A_i = \mathbf{f}$ then this extension axiom can be eliminated without increasing the size of the proof). However, the power of the rule $\mathsf{ext}{\downarrow}$ depends on the other rules that are present. For example, in order to make Theorem 5.3 work, the rules $\mathsf{ai}{\downarrow}$ and $\mathsf{s}$ should be derivable in the chosen system. Furthermore, for the polynomial size proofs of the propositional pigeon hole principle in Section 7 we will need associativity and commutativity of conjunction. More precisely, for completeness of $\mathsf{KS}$ only the rules $=_1$ and $=_3$ in (5) would be necessary [19], but for Section 7 we also need rules $=_2$ and $=_4$ (see Remark 3.1).

## 6. Substitution

Let us next consider systems with substitution. A *substitution* is a function $\sigma$ from the set $\mathscr{A}$ of propositional variables to the set $\mathscr{F}$ of formulas, such that $\sigma(a) = a$ for almost all $a \in \mathscr{A}$. We can define

$\sigma(A)$ inductively for all formulas via $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$ and $\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$ and $\sigma(\bar{A}) = \overline{\sigma(A)}$. Following the tradition, we write $A\sigma$ for $\sigma(A)$. For example, if $A = a \vee \bar{b} \vee b$ and $\sigma = \{a \mapsto a \wedge b, b \mapsto a \vee \bar{c}\}$ then $A\sigma = (a \wedge b) \vee (\bar{a} \wedge c) \vee a \vee \bar{c}$. We can define the inference rule for substitution

$$\mathsf{sub}{\downarrow}\,\frac{A}{A\sigma} \tag{15}$$

Note that the rule $\mathsf{sub}{\downarrow}$ cannot be applied inside a context $F\{\ \}$. Thus, it is exactly the same rule as in Frege systems and in strong contrast to all other rules in deep inference. Let us define $\mathsf{sSKS} = \mathsf{SKS} \cup \{\mathsf{sub}{\downarrow}\}$ and $\mathsf{sKS} = \mathsf{KS} \cup \{\mathsf{sub}{\downarrow}\}$. The following has been proved in [12]:

**Theorem 6.1.** sSKS *is p-equivalent to any Frege-system with substitution.*

However, contrary to the previous cases, there is no immediate easy proof of Theorem 6.1, because the substitution rule is stronger in Frege systems than in SKS. The reason is that in Frege systems one can, after a substitution $\sigma$ has been applied to a formula $A$, reuse the original $A$ as well as the substituted version $A\sigma$.

Thus, to prove Theorem 6.1, Bruscoli and Guglielmi use in [12] the result by Krajícek and Pudlák [20] on the p-equivalence of Frege-system with extension and Frege-system with substitution.

We give here a direct proof of the p-equivalence of sSKS and xSKS (and eSKS).

**Theorem 6.2.** sSKS *p-simulates* xSKS.

*Proof.* This proof can already be found in [12]. For a given xSKS proof $\pi$ of a formula $B$, we construct

$$\begin{array}{c}
\mathsf{i}{\downarrow}\,\dfrac{}{(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee ([\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n])} \\
\mathsf{sKS}\,\Big\|\,\pi' \\
(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B \\
= \quad \dfrac{}{(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee (\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
\mathsf{sub}{\downarrow}\,\dfrac{}{(\bar{A}_n \wedge A_n) \vee (\bar{A}_n \wedge A_n) \vee (\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
{}_{2*}\mathsf{i}{\uparrow}\,\dfrac{}{(\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
\mathsf{sub}{\downarrow}\,\dfrac{}{\vdots} \\
{}_{2*}\mathsf{i}{\uparrow}\,\dfrac{}{(\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
\mathsf{sub}{\downarrow}\,\dfrac{}{(\bar{A}_1 \wedge A_1) \vee (\bar{A}_1 \wedge A_1) \vee B} \\
{}_{2*}\mathsf{i}{\uparrow}\,\dfrac{}{B}
\end{array} \tag{16}$$

where $\pi'$ is obtained from $\pi$ by putting every formula in disjunction with

$$(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1)$$

The derivation (16) is a valid derivation in sSKS because of condition (11). Note that we proceed backwards in eliminating the $a_i$ in order to keep the size of the proof polynomial. □

For the other direction, the basic idea is to simulate the substitution inference step from $A$ to $A\sigma$ by many extension inference steps, one for each occurrence of a variable $a$ with $\sigma(a) \neq a$ in $A$. Consider for example:

$$\mathsf{sub}{\downarrow}\,\frac{\begin{array}{c}\Big\|\,\pi_2\\ F\{a \vee (b \wedge c) \vee \bar{a}\}\end{array}}{\begin{array}{c}F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}\\ \Big\|\,\pi_1\\ B\end{array}} \quad \rightsquigarrow \quad \begin{array}{c}\Big\|\,\pi_2\\ \mathsf{ext}{\downarrow}\,\dfrac{F\{a \vee (b \wedge c) \vee \bar{a}\}}{F\{(a \wedge c) \vee (b \wedge c) \vee \bar{a}\}}\\ \mathsf{ext}{\downarrow}\,\dfrac{}{F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a}\}}\\ \mathsf{ext}{\downarrow}\,\dfrac{}{F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}}\\ \Big\|\,\pi_1\\ B\end{array} \tag{17}$$

where the used substitution is $\{a \mapsto a \wedge c, c \mapsto a \vee c\}$ and the context $F\{\ \}$ does not contain any occurrences of $a$ or $c$. The problem with this is that the result will, in general, not be a valid proof because both conditions in (11) might be violated. For this reason we first have to rename the variables $a$ and $c$ in $\pi_2$:

$$\mathsf{sub}\downarrow \dfrac{\begin{array}{c}\Vdots\pi_2' \\ F\{a' \vee (b \wedge c') \vee \bar{a}'\}\end{array}}{\begin{array}{c}F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\} \\ \Vdots\pi_1 \\ B\end{array}} \quad \rightsquigarrow \quad \begin{array}{c}\Vdots\pi_2' \\ \mathsf{ext}\downarrow \dfrac{F\{a' \vee (b \wedge c') \vee \bar{a}'\}}{F\{(a \wedge c) \vee (b \wedge c') \vee \bar{a}'\}} \\ \mathsf{ext}\downarrow \dfrac{F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a}'\}}{F\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}} \\ \Vdots\pi_1 \\ B\end{array} \tag{18}$$

Here $a$ and $c$ have been replaced everywhere in $\pi_2$ by fresh variables $a'$ and $c'$, respectively. The new substitution is $\{a' \mapsto a \wedge c, c' \mapsto a \vee c\}$, which can be replaced by instances of extension, without violating (11).

**Theorem 6.3.** eSKS *p-simulates* sSKS.

*Proof.* Let $\pi$ be an sSKS proof of a formula $B$. Suppose $\pi$ contains $k$ instances of $\mathsf{sub}\downarrow$, and let $\sigma_{1,1}, \ldots, \sigma_{k,1}$ be the $k$ substitutions used in them. Then $\pi$ is of the shape as shown in the left-most derivation in Figure 3. In the following, we use $\mathscr{A}_{i,j}$ to denote the set of variables $a$ with $\sigma_{i,j}(a) \neq a$. As explained above, we can now iteratively rename the propositional variables in $\mathscr{A}_{1,1}, \ldots, \mathscr{A}_{k,1}$, starting from the bottommost instance of $\mathsf{sub}\downarrow$, as indicated in Figure 3. The result of this renaming is shown in the rightmost derivation in Figure 3 and has the property that

$$\begin{array}{l}\text{for all } i \text{ with } 1 \leq i \leq k, \text{ we have that no variable in } \mathscr{A}_{i,i+1} \\ \text{appears in any of } \pi_{1,1}, \pi_{2,2}, \ldots, \pi_{i,i}.\end{array} \tag{19}$$

Let $\mathscr{A}_{i,i+1} = \{a_{i,1}, \ldots, a_{i,m_i}\}$, and let $A_{i,h} = \sigma_{i,i+1}(a_{i,h})$. We now have $n = m_1 + m_2 + \cdots + m_k$ extension variables, defined via

$$\mathsf{ext}\downarrow \frac{a_{i,h}}{A_{i,h}} \qquad \text{and} \qquad \mathsf{ext}\downarrow \frac{\bar{a}_{i,h}}{\bar{A}_{i,h}}$$

If we give the index pair $(i, h)$ the lexicographic ordering, it immediately follows from (19) that condition (11) is fulfilled. Hence, we can trivially replace each instance of $\mathsf{sub}\downarrow$ by a sequence of instances of $\mathsf{ext}\downarrow$, whose number is bound by the size of the $B_{i,i+1}$. Hence, the size of the resulting eSKS proof is at most quadratic in the size of $\pi$. □

Note that the transformation in the proof of Theorem 6.3 does not involve any cuts. Hence, we have also proved the following:

**Theorem 6.4.** eKS *p-simulates* sKS.

**Remark 6.5.** The possible interest of sKS has already been mentioned in [12], but up to now it is unknown whether sKS can p-simulate eKS, although I conjecture this to be the case.

## 7. Pigeonhole Principle and Balanced Tautologies

In this section we see two classes of tautologies which both admit polynomial-size proofs in eKS and sKS. The first one is the propositional pigeon-hole principle. The second one is a variation which has the property that every member of the class is a balanced tautology. A formula $A$ is *balanced* if every propositional variable occurring in $A$ occurs exactly twice, once positive and once negated. For example,

$$([a \vee b] \wedge [d \vee e]) \vee \left([\bar{a} \vee c] \wedge \left[\bar{d} \vee f\right]\right) \vee \left(\left[\bar{b} \vee \bar{c}\right] \wedge \left[\bar{e} \vee \bar{f}\right]\right)$$

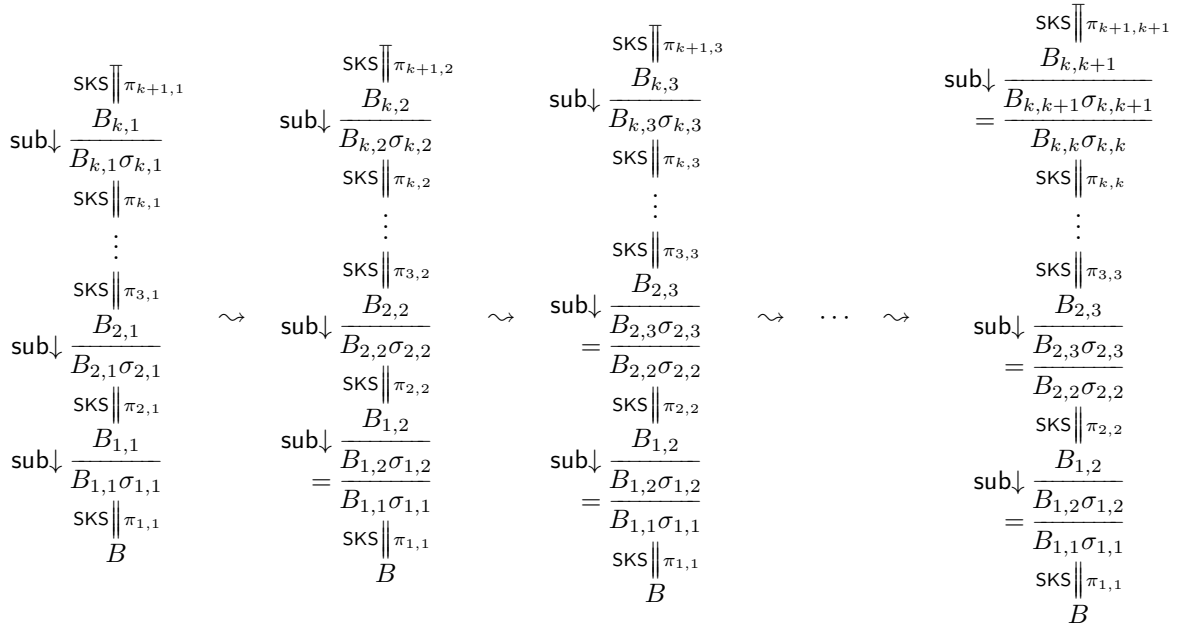

Figure 3: Renaming propositional variables in an sSKS proof

is balanced (and a tautology), whereas

$$a \vee a \vee (\bar{a} \wedge \bar{a}) \qquad \text{and} \qquad a \wedge \bar{a} \wedge b$$

are not balanced. I use the notation $\bigwedge_{0\leq i\leq n} F_i$ as abbreviation for $F_0 \wedge \cdots \wedge F_n$, and similarly for $\bigvee$. Furthermore, for a literal $a$, I abbreviate the formula $a \vee \cdots \vee a$ by $a^n$, if there are $n$ copies of $a$. Consider now

$$\mathrm{PHP}_n = \bigwedge_{0\leq i\leq n} \bigvee_{1\leq j\leq n} p_{i,j} \;\Rightarrow\; \bigvee_{0\leq i<m\leq n} \bigvee_{1\leq j\leq n} (p_{i,j} \wedge p_{m,j}) \tag{20}$$

This formula is called the propositional pigeon hole principle because it expresses the fact that if there are $n+1$ pigeons and only $n$ holes and every pigeon is in a hole then at least one hole contains two pigeons, provided one reads the propositional variable $p_{i,j}$ as "pigeon $i$ sits in hole $j$".

The formulas (20) have been well investigated from the viewpoint of proof complexity. In [1] they where presented as a candidate for separating Frege systems and extended Frege systems (wrt. p-simulation). But Buss [21] has shown that $\mathrm{PHP}_n$ admits a polynomial-size proof in a Frege system (and therefore in SKS) for every $n$.

I will here show that in eKS as well as in sKS we have cut-free polynomial-size proofs for (20). For this I use a new class of tautologies which also admit polynomial-size proofs in eKS, and which are defined as follows:

$$\mathrm{QHQ}_n = \bigvee_{0\leq i\leq n} \bigwedge_{1\leq j\leq n} \left[ \bigvee_{1\leq k\leq i} \bar{q}_{i,j,k} \vee \bigvee_{i<k\leq n} q_{k,j,i+1} \right] \tag{21}$$

Here are the first three examples:

$$\begin{array}{lcl}
\mathrm{QHQ}_1 & = & q_{1,1,1} \vee \bar{q}_{1,1,1} \\
\mathrm{QHQ}_2 & = & ([q_{1,1,1} \vee q_{2,1,1}] \wedge [q_{1,2,1} \vee q_{2,2,1}]) \vee ([\bar{q}_{1,1,1} \vee q_{2,1,2}] \wedge [\bar{q}_{1,2,1} \vee q_{2,2,2}]) \vee \\
& & ([\bar{q}_{2,1,1} \vee \bar{q}_{2,1,2}] \wedge [\bar{q}_{2,2,1} \vee \bar{q}_{2,2,2}]) \\
\mathrm{QHQ}_3 & = & ([q_{1,1,1} \vee q_{2,1,1} \vee q_{3,1,1}] \wedge [q_{1,2,1} \vee q_{2,2,1} \vee q_{3,2,1}] \wedge [q_{1,3,1} \vee q_{2,3,1} \vee q_{3,3,1}]) \vee \\
& & ([\bar{q}_{1,1,1} \vee q_{2,1,2} \vee q_{3,1,2}] \wedge [\bar{q}_{1,2,1} \vee q_{2,2,2} \vee q_{3,2,2}] \wedge [\bar{q}_{1,3,1} \vee q_{2,3,2} \vee q_{3,3,2}]) \vee \\
& & ([\bar{q}_{2,1,1} \vee \bar{q}_{2,1,2} \vee q_{3,1,3}] \wedge [\bar{q}_{2,2,1} \vee \bar{q}_{2,2,2} \vee q_{3,2,3}] \wedge [\bar{q}_{2,3,1} \vee \bar{q}_{2,3,2} \vee q_{3,3,3}]) \vee \\
& & ([\bar{q}_{3,1,1} \vee \bar{q}_{3,1,2} \vee \bar{q}_{3,1,3}] \wedge [\bar{q}_{3,2,1} \vee \bar{q}_{3,2,2} \vee \bar{q}_{3,2,3}] \wedge [\bar{q}_{3,3,1} \vee \bar{q}_{3,3,2} \vee \bar{q}_{3,3,3}])
\end{array}$$

The tautologies $\mathrm{QHQ}_n$ are balanced. This means that the size of a proof in $\mathsf{KS}$ (or related systems) of such a tautology is directly related to the number of applications of $\mathsf{ac}{\downarrow}$. Furthermore, all proofs that we show here do not contain any weakening. This makes this class interesting for investigating the gap between linear logic and classical logic [22, 23].

The formulas $\mathrm{QHQ}_1$ and $\mathrm{QHQ}_2$ are easily provable in $\mathsf{KS} \setminus \{\mathsf{ac}{\downarrow}\}$. One might be tempted to conjecture that $\mathsf{KS} \setminus \{\mathsf{ac}{\downarrow}\}$ or $\mathsf{eKS} \setminus \{\mathsf{ac}{\downarrow}\}$ is already complete for the class of balanced tautologies. But unfortunately, this is not the case. The smallest counterexample known to me is $\mathrm{QHQ}_3$. Every possible application of $\mathsf{ai}{\downarrow}$, $\mathsf{s}$, $\mathsf{m}$, or $\mathsf{w}{\downarrow}$ leads to a non-tautologous formula. Thus also the extension rule is of no use. (The same is true for all formulas $\mathrm{QHQ}_n$ with $n \geq 3$.)

This is not surprising under the view of the following theorem, which says that balanced tautologies are not easier to prove than other tautologies.

**Theorem 7.1.** *The set of balanced tautologies is* **coNP***-complete.*

*Proof.* We can reduce provability of general tautologies to provability of balanced tautologies. For a formula $B$, we let $B'$ be the formula obtained from $B$ by doing the following replacement for every propositional variable $a$ occurring in $B$: Let $n$ be the number of occurrences of $a$ in positive form in $B$, and let $m$ be the number of occurrences of $\bar{a}$ in $B$. If $n \geq 1$ and $m \geq 1$, then introduce $n \cdot m$ fresh propositional variables $a_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Now replace for every $1 \leq i \leq n$ the $i$th occurrence of $a$ by $a_{i,1} \vee \cdots \vee a_{i,m}$, and replace for every $1 \leq j \leq m$ the $j$th occurrence of $\bar{a}$ by $\bar{a}_{1,j} \vee \cdots \vee \bar{a}_{n,j}$. If $n = 0$, then introduce $m$ fresh variables $a_1, \ldots a_m$ and replace the $j$th $\bar{a}$ by $\bar{a}_j \wedge a_j$. If $m = 0$, proceed similarly (cf. Footnote 3). Then $B'$ is balanced, and its size is quadratic in the size of $B$. Furthermore, $B'$ is a tautology if and only if $B$ is a tautology. This can be seen as follows: Any $\mathsf{KS}$-proof of $B$ can be transformed into a $\mathsf{KS}$-proof of $B'$ by propagating the replacements of literals through the proof; atomic contractions $\mathsf{ac}{\downarrow}$ are replaced by general contractions $\mathsf{c}{\downarrow}$, and identities $\mathsf{ai}{\downarrow}$ by weakenings and identities:

$$\mathsf{ai}{\downarrow}\,\frac{F\{B\}}{F\{B \wedge [\bar{a} \vee a]\}} \qquad \rightsquigarrow \qquad \frac{\mathsf{ai}{\downarrow}\,\dfrac{F\{B\}}{F\{B \wedge [\bar{a}_{i,j} \vee a_{i,j}]\}}}{(m+n-2)*\mathsf{w}{\downarrow}\,F\{B \wedge [\bar{a}_{1,j} \vee \cdots \vee \bar{a}_{n,j} \vee a_{i,1} \vee \cdots \vee a_{i,m}]\}}$$

Conversely, an $\mathsf{SKS}$-proof of $B'$ can be transformed into a $\mathsf{SKS}$-proof of $B$ by forgetting the indices and adding a sufficient number of contractions $\mathsf{c}{\downarrow}$ and coweakenings $\mathsf{w}{\uparrow}$

$$\begin{array}{ccc}
F\{a \vee \cdots \vee a\} & F\{\bar{a} \vee \cdots \vee \bar{a}\} & \mathsf{w}{\uparrow}\,\dfrac{F\{\bar{a} \wedge a\}}{F\{a\}} \qquad \mathsf{w}{\uparrow}\,\dfrac{F\{\bar{a} \wedge a\}}{F\{\bar{a}\}} \\
\mathsf{c}{\downarrow}\,\| & \mathsf{c}{\downarrow}\,\| & \\
F\{a\} & F\{\bar{a}\} &
\end{array}$$

at the bottom of the derivation. (Thus, by coweakening-elimination, also a $\mathsf{KS}$-proof of $B'$ can be transformed into a $\mathsf{KS}$-proof of $B$.) □

Let us now reduce $\mathrm{PHP}_n$ to $\mathrm{QHQ}_n$. We first replace the implication by disjunction and negation, and

then apply associativity and commutativity of $\vee$:

$$\begin{aligned}
\mathrm{PHP}_n &= \bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}\bar{p}_{i,j} \vee \bigvee_{0\le i<n}\bigvee_{i<m\le n}\bigvee_{1\le j\le n}(p_{i,j}\wedge p_{m,j}) \\
&= \bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}\bar{p}_{i,j} \vee \bigvee_{0\le i\le n}\bigvee_{1\le j\le n}\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j}) \\
&= \bigvee_{0\le i\le n}\left[\bigwedge_{1\le j\le n}\bar{p}_{i,j} \vee \bigvee_{1\le j\le n}\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j})\right]
\end{aligned}$$

Now consider the following class of formulas (where $\bar{p}^i_{i,j}$ abbreviates $\bar{p}_{i,j}\vee\cdots\vee\bar{p}_{i,j}$ with $i$ copies of $\bar{p}_{i,j}$):

$$\mathrm{PHP}'_n = \bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}\left[\bar{p}^i_{i,j} \vee \bigvee_{i<m\le n} p_{m,j}\right]$$

We have for each $n$ a derivation from $\mathrm{PHP}'_n$ to $\mathrm{PHP}_n$ of length $O(n^3)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{PHP}'_n}{=}}{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}[\bar{p}^i_{i,j}\vee\bigvee_{i<m\le n}p_{m,j}]}}{n(n+1)/2*\mathsf{ai}\downarrow}}{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}[\bar{p}^i_{i,j}\vee\bigvee_{i<m\le n}([\bar{p}_{i,j}\vee p_{i,j}]\wedge p_{m,j})]}}{n(n+1)/2*\mathsf{s}}}{=}}{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}[\bar{p}^i_{i,j}\vee\bar{p}^{n-i}_{i,j}\vee\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j})]}}{=}}{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}[\bar{p}^n_{i,j}\vee\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j})]}}{n(n+1)(n-1)*\mathsf{ac}\downarrow}}{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}[\bar{p}_{i,j}\vee\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j})]}}{n(n+1)*\mathsf{s}}}{=}}{\bigvee_{0\le i\le n}[\bigwedge_{1\le j\le n}\bar{p}_{i,j}\vee\bigvee_{1\le j\le n}\bigvee_{i<m\le n}(p_{i,j}\wedge p_{m,j})]}}{\mathrm{PHP}_n}$$

**Remark 7.2.** Since $\mathrm{PHP}'_n$ is just an instance of $\mathrm{QHQ}_n$ with $q_{i,j,k}=p_{i,j}$, every polynomial-size proof of $\mathrm{QHQ}_n$ yields also a polynomial-size proof of $\mathrm{PHP}_n$. On the other hand, with the substitution (found by an anonymous referee)

$$p_{i,j}\mapsto\bigwedge_{1\le k\le i}q_{i,j,k}\wedge\bigwedge_{i<k\le n}\bar{q}_{k,j,i+1}$$

a polynomial-size proof of $\mathrm{PHP}_n$ can be transformed into a polynomial-size proof of $\mathrm{QHQ}_n$. Thus the result by Buss [21] can be used to give a polynomial-size proof of $\mathrm{QHQ}_n$ in $\mathsf{SKS}$.

For a given number $n$, we define for all $0\le i\le n$ and $1\le j\le n$ the formula

$$\begin{aligned}
Q_{i,j} &= \bigvee_{1\le k\le i}\bar{q}_{i,j,k}\vee\bigvee_{i<k\le n}q_{k,j,i+1} \\
&= \bar{q}_{i,j,1}\vee\bar{q}_{i,j,2}\vee\cdots\vee\bar{q}_{i,j,i}\vee q_{i+1,j,i+1}\vee q_{i+2,j,i+1}\vee\cdots\vee q_{n,j,i+1}
\end{aligned}\tag{22}$$

Then $\mathrm{QHQ}_n=(Q_{0,1}\wedge\cdots\wedge Q_{0,n})\vee(Q_{1,1}\wedge\cdots\wedge Q_{1,n})\vee\cdots\vee(Q_{n,1}\wedge\cdots\wedge Q_{n,n})$. The formula $Q_{i,j}$ consists of $n$ disjuncts. Let $Q^{\vee m}_{i,j}$ denote the formula obtained from $Q_{i,j}$ by removing the $m$th disjunct. Then for all $m\le i$ we have $Q_{i,j}=Q^{\vee m}_{i,j}\vee\bar{q}_{i,j,m}$ and for all $m>i$ we have $Q_{i,j}=Q^{\vee m}_{i,j}\vee q_{m,j,i+1}$. Figure 4 shows a derivation in $\mathsf{sKS}$ from $\mathrm{QHQ}_{n-1}$ to $\mathrm{QHQ}_n$ of length $O(n^3)$. In that figure, the number $z_1$ is $n\cdot(n-1)\cdot(n-2)/2$, and $z_2$ is $n\cdot(n-1)\cdot(n-1)$. The used substitution is defined as follows:

$$q_{i,j,k}\mapsto[q_{i,j,k}\vee q_{n,j,k}]\wedge[\bar{q}_{n,j,i+1}\vee q_{i,n,k}]\quad .$$

Since the proof of $\mathrm{QHQ}_1$ is trivial, we exhibited a cut-free polynomial-size proof of $\mathrm{QHQ}_n$ and $\mathrm{PHP}_n$. We can transform the complete proof of $\mathrm{QHQ}_n$ into an $\mathsf{eKS}$ proof by renaming the variables $q_{i,j,k}$ at each stage

$$
\begin{array}{rl}
 & = \dfrac{\mathrm{QHQ}_{n-1}}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}[\bigvee_{1\le k\le i}\bar{q}_{i,j,k}\vee\bigvee_{i<k<n}q_{k,j,i+1}]} \\
\mathsf{sub}{\downarrow} & \dfrac{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}[\bigvee_{1\le k\le i}[(\bar{q}_{i,j,k}\wedge\bar{q}_{n,j,k})\vee(q_{n,j,i+1}\wedge\bar{q}_{i,n,k})]\vee\bigvee_{i<k<n}([q_{k,j,i+1}\vee q_{n,j,i+1}]\wedge[\bar{q}_{n,j,k+1}\vee q_{k,n,i+1}])]}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}[\bigvee_{1\le k\le i}([\bar{q}_{i,j,k}\vee q_{n,j,i+1}]\wedge[\bar{q}_{n,j,k}\vee\bar{q}_{i,n,k}])\vee\bigvee_{i<k<n}([q_{k,j,i+1}\vee q_{n,j,i+1}]\wedge[\bar{q}_{n,j,k+1}\vee q_{k,n,i+1}])]} \\
z_1{*}\mathsf{m} & \dfrac{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}[\bigvee_{1\le k\le i}([\bar{q}_{i,j,k}\vee q_{n,j,i+1}]\wedge[\bar{q}_{n,j,k}\vee\bar{q}_{i,n,k}])\vee\bigvee_{i<k<n}([q_{k,j,i+1}\vee q_{n,j,i+1}]\wedge[\bar{q}_{n,j,k+1}\vee q_{k,n,i+1}])]}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}([\bigvee_{1\le k\le i}[\bar{q}_{i,j,k}\vee q_{n,j,i+1}]\vee\bigvee_{i<k<n}[q_{k,j,i+1}\vee q_{n,j,i+1}]]\wedge} \\
z_2{*}\mathsf{m} & \dfrac{}{[\bigvee_{1\le k\le i}[\bar{q}_{n,j,k}\vee\bar{q}_{i,n,k}]\vee\bigvee_{i<k<n}[\bar{q}_{n,j,k+1}\vee q_{k,n,i+1}]]} \\
 & = \dfrac{}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}([\bar{q}_{i,j,1}\vee\cdots\vee\bar{q}_{i,j,i}\vee q_{i+1,j,i+1}\vee\cdots\vee q_{n-1,j,i+1}\vee q^{n-1}_{n,j,i+1}]\wedge} \\
 & = \dfrac{}{[\bar{q}_{n,j,1}\vee\cdots\vee\bar{q}_{n,j,i}\vee\bar{q}_{n,j,i+2}\vee\cdots\vee\bar{q}_{n,j,n}\vee\bar{q}_{i,n,1}\vee\cdots\vee\bar{q}_{i,n,i}\vee q_{i+1,n,i+1}\vee\cdots\vee q_{n-1,n,i+1}]} \\
z_2{*}\mathsf{ac}{\downarrow} & \dfrac{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}([Q^{\vee n}_{i,j}\vee q^{n-1}_{n,j,i+1}]\wedge[Q^{\vee i+1}_{n,j}\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}([Q^{\vee n}_{i,j}\vee q_{n,j,i+1}]\wedge[Q^{\vee i+1}_{n,j}\vee Q^{\vee n}_{i,n}])} \\
 & = \dfrac{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}([Q^{\vee n}_{i,j}\vee q_{n,j,i+1}]\wedge[Q^{\vee i+1}_{n,j}\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}(Q_{i,j}\wedge[Q^{\vee i+1}_{n,j}\vee Q^{\vee n}_{i,n}])} \\
 & = \dfrac{\bigvee_{0\le i<n}\bigwedge_{1\le j<n}(Q_{i,j}\wedge[Q^{\vee i+1}_{n,j}\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[Q^{\vee i+1}_{n,1}\vee Q^{\vee n}_{i,n}]\wedge\cdots\wedge[Q^{\vee i+1}_{n,n-1}\vee Q^{\vee n}_{i,n}])} \\
n(n-2){*}\mathsf{d}{\downarrow} & \dfrac{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[Q^{\vee i+1}_{n,1}\vee Q^{\vee n}_{i,n}]\wedge\cdots\wedge[Q^{\vee i+1}_{n,n-1}\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1})\vee Q^{\vee n}_{i,n}])} \\
n{*}\mathsf{ai}{\downarrow} & \dfrac{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge[\bar{q}_{n,n,i+1}\vee q_{n,n,i+1}])\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge[\bar{q}_{n,n,i+1}\vee q_{n,n,i+1}])\vee Q^{\vee n}_{i,n}])} \\
n{*}\mathsf{s} & \dfrac{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge\bar{q}_{n,n,i+1})\vee q_{n,n,i+1}\vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge\bar{q}_{n,n,i+1})\vee q_{n,n,i+1}\vee Q^{\vee n}_{i,n}])} \\
 & = \dfrac{\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge[Q_{i,n}\vee(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge\bar{q}_{n,n,i+1})])}{[\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge Q_{i,n})]\vee[\bigvee_{0\le i<n}(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge\bar{q}_{n,n,i+1})]} \\
n{*}\mathsf{s} & \dfrac{}{[\bigvee_{0\le i<n}(Q_{i,1}\wedge\cdots\wedge Q_{i,n-1}\wedge Q_{i,n})]\vee[\bigvee_{0\le i<n}(Q^{\vee i+1}_{n,1}\wedge\cdots\wedge Q^{\vee i+1}_{n,n-1}\wedge\bar{q}_{n,n,i+1})]} \\
 & = \dfrac{}{[\bigvee_{0\le i<n}\bigwedge_{1\le j\le n}Q_{i,j}]\vee[(Q^{\vee 1}_{n,1}\wedge\cdots\wedge Q^{\vee 1}_{n,n-1}\wedge\bar{q}_{n,n,1})\vee\cdots\vee(Q^{\vee n}_{n,1}\wedge\cdots\wedge Q^{\vee n}_{n,n-1}\wedge\bar{q}_{n,n,n})]} \\
(n-1)(n-1){*}\mathsf{m} & \dfrac{}{[\bigvee_{0\le i<n}\bigwedge_{1\le j\le n}Q_{i,j}]\vee([Q^{\vee 1}_{n,1}\vee\cdots\vee Q^{\vee n}_{n,1}]\wedge\cdots\wedge[Q^{\vee 1}_{n,n-1}\vee\cdots\vee Q^{\vee n}_{n,n-1}]\wedge[\bar{q}_{n,n,1}\vee\cdots\vee\bar{q}_{n,n,n}])} \\
 & = \dfrac{}{[\bigvee_{0\le i<n}\bigwedge_{1\le j\le n}Q_{i,j}]\vee([\bar{q}^{n-1}_{n,1,1}\vee\cdots\vee\bar{q}^{n-1}_{n,1,n}]\wedge\cdots\wedge[\bar{q}^{n-1}_{n,n-1,1}\vee\cdots\vee\bar{q}^{n-1}_{n,n-1,n}]\wedge[\bar{q}_{n,n,1}\vee\cdots\vee\bar{q}_{n,n,n}])} \\
z_2{*}\mathsf{ac}{\downarrow} & \dfrac{}{[\bigvee_{0\le i<n}\bigwedge_{1\le j\le n}Q_{i,j}]\vee([\bar{q}_{n,1,1}\vee\cdots\vee\bar{q}_{n,1,n}]\wedge\cdots\wedge[\bar{q}_{n,n-1,1}\vee\cdots\vee\bar{q}_{n,n-1,n}]\wedge[\bar{q}_{n,n,1}\vee\cdots\vee\bar{q}_{n,n,n}])} \\
 & = \dfrac{}{[\bigvee_{0\le i<n}\bigwedge_{1\le j\le n}Q_{i,j}]\vee(Q_{n,1}\wedge\cdots\wedge Q_{n,n-1}\wedge Q_{n,n})} \\
 & = \dfrac{\bigvee_{0\le i\le n}\bigwedge_{1\le j\le n}Q_{i,j}}{\mathrm{QHQ}_n}
\end{array}
$$

Figure 4: Derivation from $\mathrm{QHQ}_{n-1}$ to $\mathrm{QHQ}_n$

(see proof of Theorem 6.3) and use the extension formulas[5]

$$q'_{i,j,k} \Leftrightarrow [q_{i,j,k}\vee q_{n,j,k}]\wedge[\bar{q}_{n,j,i+1}\vee q_{i,n,k}]$$

as extension axioms, i.e., the rules

$$\mathsf{ext}{\downarrow}\ \frac{q'_{i,j,k}}{[q_{i,j,k}\vee q_{n,j,k}]\wedge[\bar{q}_{n,j,i+1}\vee q_{i,n,k}]} \qquad \mathsf{ext}{\downarrow}\ \frac{\bar{q}'_{i,j,k}}{(\bar{q}_{i,j,k}\wedge\bar{q}_{n,j,k})\vee(q_{n,j,i+1}\wedge\bar{q}_{i,n,k})} \tag{23}$$

In [24], Japaridze provides another cut-free polynomial size proof of $\mathrm{PHP}_n$. His system of *deep cirquents* uses a form of sharing instead of extension or substitution.

## 8. Conclusions and future work

This paper provides more new open problems than it provides answers. Figure 5 shows a refined version of Figure 2 (see also [12]). A solid arrow $A \longrightarrow B$ means that $A$ p-simulates $B$, the notation $A \not\longrightarrow B$ means that $A$ does not p-simulate $B$, and a dotted arrow $A \dashrightarrow B$ means that it is not known whether $A$ p-simulates $B$ or not. The open problems indicated by these dotted arrows are surprisingly difficult:

[5]To distinguish between the propositional variable occurrences in $\mathrm{QHQ}_n$ and the occurrences $\mathrm{QHQ}_{n-1}$, we use $q'$ for those in $\mathrm{QHQ}_{n-1}$. This is more legible than adding yet another index to the $q$.

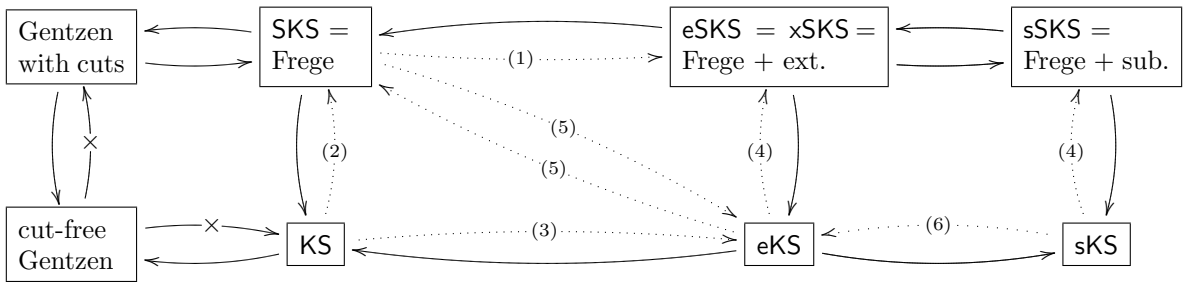


Figure 5: Classification of propositional proof systems

(1) The question whether SKS p-simulates eSKS is equivalent to the question whether Frege systems p-simulate extended Frege systems. This question has already been asked in [1], and is one of the most important open problems in the area of proof complexity.

(2) I conjecture that KS does not p-simulate SKS (see also [12] and [25]),

(3) I also conjecture that KS does not p-simulate eKS. More precisely, it is conjectured that KS cannot provide polynomial size proofs of the formulas $\mathrm{PHP}_n$ (or $\mathrm{QHQ}_n$), whereas this is possible in SKS (as shown in [21]) as well as in eKS (as shown in Section 7). However, so far, no technique has been developed for showing that something *cannot be done* in KS.

(4) This is the question whether extension or substitution can simulate the behavior of the cut. It is one of the contributions of this paper that this question can now be asked. I conjecture that the answer is positive, but it is not clear how to prove it. Note that the naive cut elimination procedures fail in the presence of extension. Even if we manage to modify the technicalities such that we get a cut elimination procedure for eSKS, it is not clear how to avoid the exponential blow-up usually caused by cut elimination.

(5) The questions whether extension without cut is as powerful as the cut without extension, and vice-versa, can be seen as the little brothers of (1).

(6) It has already been shown in [1] that under the presence of cut substitution p-simulates extension, but without cut, this question is not trivial.

**Remark 8.1.** The general cocontraction rule c↑ (see Proposition 3.5) can be reduced to its atomic version

$$\mathsf{ac}\uparrow \frac{F\{a\}}{F\{a \wedge a\}}$$

if the medial rule m (see (1)) is present [11]. It has recently been shown [26, 25] that the system KS + ac↑ quasi-polynomially simulates SKS, and it is conjectured that this result can be improved to a polynomial simulation. Furthermore, one can show that sKS (and therefore also eKS) p-simulates KS + ac↑. The two results together could provide an answer for one direction of (5). This also raises the question whether KS + ac↑ can p-simulate eKS. In any case, we have here four ways of proof compression—cocontraction, cut, extension, and substitution—and they can all be studied independently in KS.

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