Long-Time Stability Analysis of Acoustic Absorbing Boundary Conditions for Regular-Shaped Surfaces

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This work deals with the stability analysis of a one-parameter family of Absorbing Boundary Conditions (ABC) that have been derived for the acoustic wave equation. We tackle the problem of long-term stability of the wave field both at the continuous and the numerical levels. We first define a function of energy and we show that it is decreasing in time. Its discrete form is also decreasing under a Courant-Friedrichs-Lewy (CFL) condition that does not depend on the ABC. Moreover, the decay rate of the continuous energy can be determined: it is exponential if the computational domain is star-shaped and this property can be illustrated numerically.

Keywords: Absorbing boundary conditions; Acoustic wave equation; Exponential decay; Discontinuous Galerkin method

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1. Introduction

In a previous work[5], we have constructed a one-parameter family of Absorbing Boundary Conditions (ABC) for the acoustic wave equation. We have obtained second-order boundary conditions that can be applied on regular arbitrarily-shaped surfaces. They read as:

$$\partial_t \left(\partial_n u + \partial_t u\right) = \left(\frac{\kappa}{4} - \gamma\right) \partial_n u - \left(\frac{\kappa}{4} + \gamma\right) \partial_t u \text{ on } \Sigma. \tag{1.1}$$

The function κ stands for the curvature of Σ and γ is a regular parameter defined on Σ. Condition (1.1) states the normal derivative of the wave field u via the pseudo-differential operator $\left(\partial_t + \gamma - \frac{\kappa}{4}\right)^{-1}$. It thus interferes with the sparse structure of the finite element matrices that may be used for the numerical computations. That is why we proposed[7] to introduce an auxiliary unknown ψ defined on the absorbing surface. By combining ψ with u, we thus avoid discretizing the pseudo-differential operator and the sparsity of the discrete matrices is preserved.
The problem reads then as:

$$\begin{cases} \partial_t^2 u - \Delta u = F \text{ in } \Omega \times (0, +\infty)\,; \\ \left(\partial_t - \dfrac{\kappa}{4} + \gamma\right) \psi = \partial_t u \text{ on } \Sigma \times (0, +\infty)\,; \\ \partial_n u + \partial_t u + \dfrac{\kappa}{2} \psi = 0 \text{ on } \Sigma \times (0, +\infty)\,; \\ u = 0 \text{ on } \Gamma \times (0, +\infty)\,; \\ u(0, x) = 0,\ \partial_t u(0, x) = 0 \text{ in } \Omega; \\ \psi(0, x) = 0 \text{ on } \Sigma; \end{cases} \tag{1.2}$$

where Ω is a bounded domain and its boundary $\partial\Omega = \Gamma \cup \Sigma$ is assumed to be regular, with $\Gamma \cap \Sigma = \varnothing$. F is a given source that is compactly supported in time t. In Ref. 7, we have performed a collection of numerical experiments that show that if $\gamma \geq \frac{\kappa}{4}$, the condition (1.1) performs well, with the same degree of accuracy.
This work aims at refining the analysis carried out in Ref. 5 where we obtained a weak stability result by proving that, if F is compactly supported in time, the solution to (1.2) converges to 0 as t tends to infinity. In this paper, we focus on the study of energy by considering its long-term behavior. To this end, we perform a stability analysis describing the time variations of the energy. This way, we can obtain a strong stability result that depends on the geometrical properties of the computational domain.
An outline of the rest of the paper is as follows. First, we show that the time behavior of the solution to the continuous problem (1.2) can be represented by an energy $\mathcal{E} := \mathcal{E}(t)$, defined as a Lyapunov function. Next, we study the variations of $\mathcal{E}$ when t tends to infinity. We show that $\mathcal{E}$ is decreasing and we establish that if the domain Ω is star-shaped, the decay rate is exponential. Thus we obtain a strong stability result. Regarding the discrete problem, we show that there also exists a discrete

energy defined as a suitable approximation of the continuous energy. The discrete energy is also decreasing, provided that the time step satisfies a CFL condition that does not depend on the ABC. Numerical experiments are then performed to illustrate that the discrete energy decays very fast like an exponential function whose coefficients can be computed in the case of a circular boundary.
In this paper, we focus on the 2D case because it is easier to perform large series of numerical experiments. Nevertheless, regarding theoretical results, there is no difference between 2D and 3D.

2. Strong long-term stability of the continuous problem

In Ref. 5, we have established that if $F \in C^1([0\,;\,T]; L^2(\Omega))$, where $T > 0$ is given, the solution u to (1.2) satisfies

$$u \in C^0([0\,;\,+\infty[\,;\,H^1(\Omega)) \cap C^1([0\,;\,T])\,;\,L^2(\Omega)) \text{ and } \psi \in C^1([0\,;\,T])\,;\,L^2(\Sigma)).$$

Then, assuming that Σ is convex, κ is positive and the function

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right) dx + \frac{1}{2}\int_\Sigma \frac{\kappa}{2}|\psi|^2 d\sigma$$

is positive and decreasing. Moreover, since u is defined through a contraction semi-group, we have proved that $\lim_{t\to\infty} \mathcal{E}(t) = 0$. This is what we call a weak stability result because this result does not give the decay rate of $\mathcal{E}(t)$.
Our purpose is thus to establish a strong stability result by proving that the decay rate of the function $\mathcal{E}$ is exponential. In practice, the source term in (1.2) is timely compactly supported and, just as was formerly observed in Ref. 5, the long-term behavior of the energy can be characterized by considering the Cauchy problem only. In this section, we thus consider the system

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0, +\infty)\,; \\ \left(\partial_t - \dfrac{\kappa}{4} + \gamma\right)\psi = \partial_t u & \text{on } \Sigma \times (0, +\infty)\,; \\ \partial_n u + \partial_t u + \dfrac{\kappa}{2}\psi = 0 & \text{on } \Sigma \times (0, +\infty)\,; \\ u = 0 & \text{on } \Gamma \times (0, +\infty)\,; \\ u(0,x) = u_0,\ \partial_t u(0,x) = u_1 & \text{in } \Omega; \\ \psi(0,x) = \psi_0 = \psi(t_0, x) & \text{on } \Sigma; \end{cases} \tag{2.1}$$

The mathematical framework of this section has been previously set in papers dealing with the boundary stabilization of the wave equation. In particular, in Refs. 14, 13, it has been proved that it is sufficient to show that there exists a positive constant C such that

$$\int_S^T \mathcal{E}(t)dt \le C\mathcal{E}(S) \tag{2.2}$$

with $0 \le S < T < +\infty$ to have the exponential decay of the energy.

2.1. *Preliminary results*

We first introduce the function space H defined by $H = H^1_\Gamma(\Omega) \times L^2(\Omega) \times L^2(\Sigma)$, where $H^1_\Gamma(\Omega) = \{h_1 \in H^1(\Omega), h_1 = 0 \text{ on } \Gamma\}$. Let V be the product space defined by

$$V = \{(v_1, v_2, \psi) \in H, \, \Delta v_1 \in L^2(\Omega), \, v_2 \in H^1_\Gamma(\Omega), \, \partial_n v_{1|_\Sigma} \in L^2(\Sigma),$$
$$\partial_n v_1 + v_2 + \frac{\kappa}{2}\psi = 0 \text{ on } \Sigma\}.$$

Lemma 2.1. *Let* $(u_0, u_1, \psi_0) \in V$ *with* $\psi_0 \in H^{1/2}(\Sigma)$. *Then* $u \in C^0\left([0, +\infty[; H^2(\Omega)\right)$.

Proof. In Ref. 5, the Hille-Yosida theorem allowed us to prove that for any $(u_0, u_1, \psi_0) \in V$,

$$(u, \partial_t u, \psi) \in C^0\left([0, +\infty[; V\right) \cap C^1\left([0, +\infty[; H\right).$$

By definition of V, we thus have :

$$u \in C^0\left([0, +\infty[; H^1(\Omega)\right) \text{ and } \Delta u \in C^0\left([0, +\infty[; L^2(\Omega)\right). \tag{2.3}$$

Moreover, if $\psi_0 \in H^{1/2}(\Sigma)$, the auxiliary unknown satisfies $\psi \in C^0\left([0, +\infty[; H^{1/2}(\Sigma)\right)$.

Indeed, according to the Duhamel formula, ψ reads as :

$$\psi(t, x) = \int_0^t e^{(\gamma - \frac{\kappa}{4})(s-t)} \partial_t u(s, x) ds, \text{ for any } x \in \Sigma.$$

We know that $\partial_t u \in C^0\left([0, +\infty[; H^1(\Omega)\right)$, which implies that
$\partial_t u_{|\Sigma} \in C^0\left([0, +\infty[; H^{1/2}(\Sigma)\right)$. We thus obtain that $\psi \in C^0\left([0, +\infty[; H^{1/2}(\Sigma)\right)$. Now, u, $\partial_t u$ and ψ satisfy the boundary condition

$$\partial_n u + \partial_t u + \frac{\kappa}{2}\psi = 0 \text{ on } \Sigma.$$

Since κ belongs to $L^\infty(\Sigma)$, we thus have

$$\partial_n u_{|\Sigma} \in C^0\left([0, +\infty[; H^{1/2}(\Sigma)\right)$$

and the previous regularity result combined with (2.3) implies that $u \in C^0\left([0, +\infty[; H^2(\Omega)\right)$. □

Lemma 2.1 generalizes a result that has been glimpsed in Ref. 7 where we assumed that $\psi_0 = 0$ to show that the solution to (2.1) converges as $t \to +\infty$.

Lemma 2.2. *Let* $(u_0, u_1, \psi_0) \in V$. *Then,*

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right) |\psi|^2 d\sigma dt, \tag{2.4}$$

Proof. Following the Hille-Yosida theory, when $(u_0, u_1, \psi_0) \in V$, $u \in C^1\left([0,+\infty[;\, H^1(\Omega)\right)$, $\partial u \in C^1\left([0,+\infty[;\, H^1(\Omega)\right)$, $\partial_t^2 u \in\in C^0\left([0,+\infty[;\, L^2(\Omega)\right)$ and $\psi \in C^1\left([0,+\infty[;\, L^2(\Sigma)\right)$.

The function $\mathcal{E}$ is thus differentiable and it derivative is given by

$$\frac{d\mathcal{E}}{dt}(u,\partial_t u,\psi) = \int_\Omega \partial_t u \partial_t^2 u\, dx + \int_\Omega \nabla u \cdot \nabla \partial_t u\, dx + \int_\Sigma \frac{\kappa}{2}\psi \partial_t \psi d\Sigma$$

Then, using the Green formula, we get

$$\frac{d\mathcal{E}}{dt}(u,\partial_t u,\psi) = -\int_\Sigma |\partial_t u|^2 d\sigma + \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma.$$

and by integrating on $[S,T]$, we have

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma dt.$$

□

Lemma 2.3. *Let $(u_0, u_1, \psi_0) \in V$ with $\psi_0 \in H^{1/2}(\Sigma)$. If*

$$\gamma(x) > \frac{\kappa(x)}{4},\ \forall x \in \Sigma \text{ and } \gamma - \frac{\kappa(x)}{4} \in L^\infty(\Sigma) \tag{2.5}$$

and if Σ is strictly convex so that $\kappa(x) > 0$, $\forall x \in \Sigma$, then, we have

$$\begin{cases} \displaystyle\int_S^T \int_\Sigma |\partial_t u|^2\, d\sigma\, dt \le \mathcal{E}(S) \\ \displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2\, d\sigma\, dt \le \mathcal{E}(S) \end{cases} \tag{2.6}$$

and if $\alpha_{min} = \min_{x\in\Sigma}\left(\gamma - \frac{\kappa}{4}\right)$, we have

$$\int_S^T \int_\Sigma \frac{\kappa}{2}|\psi|^2\, d\sigma\, dt \le \frac{1}{\alpha_{min}}\mathcal{E}(S) \tag{2.7}$$

Proof. We know that

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma dt,$$

which is equivalent to

$$\mathcal{E}(S) = \int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2 d\sigma dt + \mathcal{E}(T).$$

Since each term is positive, we obviously get

$$\begin{cases} \displaystyle\int_S^T \int_\Sigma |\partial_t u|^2\, d\sigma\, dt \le \mathcal{E}(S); \\ \displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2\, d\sigma\, dt \le \mathcal{E}(S). \end{cases}$$

Moreover $\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2 \, d\sigma \, dt \leq \mathcal{E}(S)$ implies that, if $\alpha_{\min} = \min_{x\in\Sigma}\left(\gamma - \frac{\kappa}{4}\right)$,

$$\alpha_{\min} \int_S^T \int_\Sigma \frac{\kappa}{2}|\psi|^2 \, d\sigma \, dt \leq \mathcal{E}(S)$$

which ends the proof of Lemma 2.3. □

Lemma 2.4. *There exists a constant $C > 0$ such that, for any $0 \leq S \leq t$,*

$$\int_\Sigma |u(t,x)|^2 d\sigma \leq C\mathcal{E}(S). \tag{2.8}$$

Proof. The trace map from $H^1(\Omega)$ into $H^{1/2}(\Sigma)$ is continuous and there exist $C_1 > 0$ and $C_2 > 0$ such that for any $u \in H^1(\Omega)$,

$$\|u\|^2_{L^2(\Sigma)} \leq C_1 \|u\|^2_{H^{1/2}(\Sigma)} \leq C_2 \|u\|^2_{H^1(\Omega)}.$$

Moreover, u satisfies the Poincaré inequality: there exists a positive constant C_3 such that

$$\|u\|_{L^2(\Omega)} \leq C_3 \|\nabla u\|_{L^2(\Omega)} \leq C_3 \mathcal{E}(t). \tag{2.9}$$

This implies that $\|u\|^2_{L^2(\Sigma)} \leq C\mathcal{E}(t)$, and we conclude easily since $t \longmapsto \mathcal{E}(t)$ is decreasing. □

Lemma 2.5. *Let $m(x)$ be a function in $\mathcal{C}^1(\bar{\Omega})^3$ and u a regular solution to the wave equation (see Lemma 2.1). Then, we have*

$$\left[\int_\Omega \partial_t u \,(m \cdot \nabla u)\, dx\right]_S^T - \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\, |\partial_t u|^2 d\sigma \, dt + \frac{1}{2}\int_S^T \int_\Omega \operatorname{div} m |\partial_t u|^2 dx \, dt +$$
$$\int_S^T \int_\Omega \nabla u \cdot \nabla (m \cdot \nabla u)\, dx \, dt - \int_S^T \int_{\partial\Omega} \partial_n u \,(m\cdot \nabla u)\, d\sigma \, dt = 0. \tag{2.10}$$

Proof. For a proof of this identity, we refer for instance to Refs. 13, 14. We just mention that it is based on the identity

$$\int_S^T \int_\Omega \left(\partial_t^2 u - \Delta u\right)(m \cdot \nabla u)\, dx \, dt = 0$$

Let us point out that the condition ψ_0 in $H^{1/2}(\Sigma)$ is necessary to have $u(t,.)$ in $H^2(\Omega)$ and to have (2.10) well-defined. □

2.2. *Proof of the exponential energy decay*

In this section, we set $m(x) = x - x_0$ where $x_0 \in \mathbb{R}^d$ ($d = 2, 3$ denotes the space dimension). We suppose that x_0 is chosen such that

$$\Gamma = \{x \in \partial\Omega, m.n \leq 0\} \tag{2.11}$$

and

$$\Sigma = \{x \in \partial\Omega, m.n \geq 0\}. \tag{2.12}$$

Remark 2.1. The existence of x_0 is guaranteed if Γ is the exterior boundary of a star-shaped domain ω. Indeed, we can then choose x_0 inside ω such that ω is star-shaped with respect to x_0 and (2.11) is satisfied (remark that n is the interior normal with respect to ω). Moreover, since Σ surrounds Γ, x_0 is inside the domain delimited by Σ. Since Σ is convex, this domain is star-shaped with respect to all its point and in particular to x_0, so that (2.12) is satisfied.

In that case, we know that div $m = d$. For the sake of simplicity, we suppose that $d = 3$ but there is no difficulty to obtain the same result for $d = 2$.

To prove that there exists a positive constant C that satisfies (2.2), we only have to find an upper bound of

$$\frac{1}{2}\int_S^T \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\, dx\, dt$$

since we already know from Lemma 2.3 that

$$\int_S^T \int_\Sigma \frac{\kappa}{2}|\psi|^2\, d\sigma\, dt \leq \frac{1}{\alpha_{\min}}\mathcal{E}(S).$$

Lemma 2.6. *Let $(u_0, u_1, \psi_0) \in V$. Then, we have*

$$\begin{aligned}
&\frac{1}{2}\int_S^T \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\, dx\, dt = -\left[\int_\Omega \partial_t u\left((m\cdot\nabla u) + u\right) dx\right]_S^T \\
&+\frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T \int_{\partial\Omega} \partial_n u u\, d\sigma\, dt \\
&+\int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt.
\end{aligned} \tag{2.13}$$

Proof. From Lemma 2.5, we know that

$$\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T - \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\partial_t u|^2 d\sigma\, dt + \frac{1}{2}\int_S^T \int_\Omega \text{div } m|\partial_t u|^2 dx\, dt +$$
$$\int_S^T \int_\Omega \nabla u\cdot\nabla\left(m\cdot\nabla u\right) dx\, dt - \int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt = 0.$$

Moreover, we can check that

$$\partial_j (m \cdot \nabla u) = (m \cdot \nabla) \partial_j u + \nabla u \cdot \partial_j m \text{ for } j = 1, 2, 3;$$

and since $m(x) = x - x_0$, we get

$$\nabla (m \cdot \nabla u) = (m \cdot \nabla) \nabla u + \nabla u.$$

Consequently, we have

$$\begin{aligned}\int_S^T \int_\Omega \nabla u \cdot \nabla (m \cdot \nabla u)\, dx\, dt &= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \int_S^T \int_\Omega ((m \cdot \nabla) \nabla u) \cdot \nabla u\, dx\, dt \\ &= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \frac{1}{2} \int_S^T \int_\Omega m \cdot \nabla |\nabla u|^2\, dx\, dt \\ &= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) |\nabla u|^2\, d\sigma\, dt \\ &-\frac{1}{2} \int_S^T \int_\Omega \text{div } m |\nabla u|^2\, dx\, dt.\end{aligned}$$

Therefore, since div $m = 3$, we get

$$\begin{aligned}&\left[\int_\Omega \partial_t u\, (m \cdot \nabla u)\, dx\right]_S^T - \frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) |\partial_t u|^2 d\sigma\, dt + \frac{3}{2} \int_S^T \int_\Omega |\partial_t u|^2 dx\, dt \\ &-\frac{1}{2} \int_S^T \int_\Omega |\nabla u|^2 dx\, dt + \frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) |\nabla u|^2 d\sigma\, dt \\ &-\int_S^T \int_{\partial\Omega} \partial_n u\, (m \cdot \nabla u)\, d\sigma\, dt = 0\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{3}{2} \int_S^T \int_\Omega |\partial_t u|^2 dx\, dt - \frac{1}{2} \int_S^T \int_\Omega |\nabla u|^2 dx\, dt = \\ &-\left[\int_\Omega \partial_t u\, (m \cdot \nabla u)\, dx\right]_S^T + \frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) |\partial_t u|^2 d\sigma\, dt \\ &-\frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) |\nabla u|^2 d\sigma\, dt + \int_S^T \int_{\partial\Omega} \partial_n u\, (m \cdot \nabla u)\, d\sigma\, dt.\end{aligned} \tag{2.14}$$

Moreover, we have

$$\int_S^T \int_\Omega \left(\partial_t^2 u - \Delta u\right) u = 0$$

which implies that

$$\left[\int_\Omega \partial_t u u\, dx\right]_S^T - \int_S^T \int_\Omega |\partial_t u|^2\, dx\, dt + \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt - \int_S^T \int_{\partial\Omega} \partial_n u u\, d\sigma\, dt = 0.$$

Then, adding this equation to (2.14), we obtain

$$\left[\int_\Omega \partial_t u u\, dx\right]_S^T + \frac{1}{2}\int_S^T \int_\Omega |\partial_t u|^2\, dx\, dt + \frac{1}{2}\int_S^T \int_\Omega |\nabla u|^2\, dx\, dt - \int_S^T \int_{\partial\Omega} \partial_n u u\, d\sigma\, dt =$$

$$-\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T + \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\partial_t u|^2 d\sigma\, dt$$

$$-\frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\nabla u|^2 d\sigma\, dt + \int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt.$$

Finally, we arrive at

$$\frac{1}{2}\int_S^T \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\, dx\, dt = -\left[\int_\Omega \partial_t u\,((m\cdot\nabla u)+u)\, dx\right]_S^T$$

$$+\frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T \int_{\partial\Omega} \partial_n u u\, d\sigma\, dt$$

$$+\int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt,$$

which completes the proof of Lemma 2.6. □

In the following, the letter C will be used to denote any positive constant.

Lemma 2.7. *We have*

$$I = \frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt$$

$$+ \int_S^T \int_\Gamma \partial_n u u\, d\sigma\, dt + \int_S^T \int_\Gamma \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt \le 0.$$

Proof. We know that $u = 0$ on Γ, so that $\nabla_\Gamma u = 0$ and $\partial_t u = 0$ on Γ. Moreover, since $\nabla u = \nabla_\Gamma u + (\nabla u\cdot n)\, n$, $\nabla u = (\nabla u \cdot n)\, n$ on Γ. Then we get

$$I = -\frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2\, d\sigma\, dt + \int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2\, d\sigma\, dt$$

$$= \frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2\, d\sigma\, dt.$$

By hypothesis, $m\cdot n$ is negative on Γ. Therefore, we obtain $I \le 0$. □

Lemma 2.8. *There exists $C > 0$ such that*

$$\int_S^T \int_\Sigma \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt - \frac{1}{2}\int_S^T \int_\Sigma (m\cdot n)\,|\nabla u|^2\, d\sigma\, dt \le C\mathcal{E}(S).$$

Proof. Since $m \cdot n > 0$ on Σ, we have

$$\partial_n u\,(m \cdot \nabla u) = \sqrt{m \cdot n}\,(m \cdot \nabla u)\,\frac{\partial_n u}{\sqrt{m \cdot n}}.$$

Therefore, if we set $R = \max_{x \in \Sigma} |m(x)|$, we get

$$\begin{aligned}|\partial_n u\,(m \cdot \nabla u)\,| &\leq 2R\frac{|\partial_n u|}{\sqrt{m \cdot n}}\frac{\sqrt{m \cdot n}}{2}|\nabla u| \\ &\leq \frac{R^2}{m \cdot n}|\partial_n u|^2 + \frac{m \cdot n}{4}|\nabla u|^2 \\ &\leq C|\partial_n u|^2 + \frac{m \cdot n}{4}|\nabla u|^2\end{aligned}$$

From this inequality, we deduce that

$$\begin{aligned}&\int_S^T \int_\Sigma \partial_n u\,(m \cdot \nabla u)\;d\sigma\,dt - \frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\,|\nabla u|^2\,d\sigma\,dt \\ &\leq -\frac{1}{4}\int_S^T \int_\Sigma (m \cdot n)\,|\nabla u|^2\,d\sigma\,dt + C\int_S^T \int_\Sigma |\partial_n u|^2\,d\sigma\,dt,\end{aligned}$$

which implies that

$$\int_S^T \int_\Sigma \partial_n u\,(m \cdot \nabla u)\;d\sigma\,dt - \frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\,|\nabla u|^2\,d\sigma\,dt \leq C\int_S^T \int_\Sigma |\partial_n u|^2\,d\sigma\,dt, \tag{2.15}$$

since $m.n \geq 0$ on Σ.

The proof of the lemma is thus completed if we prove that

$$\int_S^T \int_\Sigma |\partial_n u|^2\,d\sigma\,dt \leq C\mathcal{E}(S). \tag{2.16}$$

We know that $\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi$ on Σ. Since $\kappa > 0$ on Σ, we can write

$$|\partial_n u|^2 \leq \kappa\left(\frac{\kappa}{2}|\psi|^2 + \frac{2}{\kappa}|\partial_t u|^2\right).$$

Then, if $\kappa_{\max}$ denotes the maximum of κ on Σ, we get

$$\int_S^T \int_\Sigma |\partial_n u|^2\,d\sigma\,dt \leq \frac{\kappa_{\max}}{\alpha_{\min}}\mathcal{E}(S) + 2\int_S^T \int_\Sigma |\partial_t u|^2\,d\sigma\,dt.$$

But, according to Lemma 2.3, we have

$$\int_S^T \int_\Sigma |\partial_t u|^2\,d\sigma\,dt \leq C\mathcal{E}(S)$$

which ends the proof of (2.16). Therefore, plugging (2.16) in (2.15), we obtain

$$\int_S^T \int_\Sigma \partial_n u\,(m \cdot \nabla u)\;d\sigma\,dt - \frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\,|\nabla u|^2\,d\sigma\,dt \leq C\mathcal{E}(S),$$

which ends the proof of Lemma 2.8. □

Lemma 2.9. *There exists a constant $C > 0$ such that*

$$\frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\, |\partial_t u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S).$$

Proof. By definition of R, we have $|m \cdot n| \leq R$. Therefore,

$$\frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\, |\partial_t u|^2 \, d\sigma \, dt \leq \frac{R}{2}\int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt,$$

and by applying (2.6), we get

$$\frac{1}{2}\int_S^T \int_\Sigma (m \cdot n)\, |\partial_t u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S).$$

□

Lemma 2.10. *There exists a constant $C > 0$ such that*

$$\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt \leq C\mathcal{E}(S).$$

Proof. On Σ, $\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi$. Therefore,

$$\begin{aligned}
\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt &= \int_S^T \int_\Sigma \Big(-\partial_t u - \frac{\kappa}{2}\psi\Big)\, u \, d\sigma \, dt; \\
&= -\frac{1}{2}\left[\int_\Sigma |u|^2 \, d\sigma\right]_S^T - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt; \\
&= -\frac{1}{2}\int_\Sigma |u(T)|^2 \, d\sigma + \frac{1}{2}\int_\Sigma |u(S)|^2 \, d\sigma - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt; \\
&\leq \frac{1}{2}\int_\Sigma |u(S)|^2 \, d\sigma - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt.
\end{aligned}$$

Using the continuity of the trace operator, we know that there exists a constant $C > 0$ such that

$$\int_\Sigma |u(S)|^2 \, d\sigma \leq C\|u(S)\|^2_{H^1(\Omega)}.$$

From the Poincaré inequality (2.9), we thus get that there exists a positive constant C such that

$$\|u(S)\|^2_{H^1} \leq C\int_\Omega |\nabla u(S)|^2 \, dx.$$

Finally, we obtain

$$\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt \leq C\mathcal{E}(S) - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt.$$

Now, there is still to control $-\int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt$. Since $\gamma > \frac{\kappa}{4}$, ψ satisfies

$$\psi = \left(\frac{\kappa}{4} - \gamma\right)^{-1} (\partial_t \psi - \partial_t u) \text{ on } \Sigma.$$

We thus get

$$\begin{aligned}
-\int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt &= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} (\partial_t \psi - \partial_t u) \, u \, d\sigma \, dt \\
&= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \partial_t \psi u \, d\sigma \, dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \partial_t u u \, d\sigma \, dt \\
&= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \partial_t \psi u \, d\sigma \, dt + \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \left[|u|^2\right]_S^T \, d\sigma
\end{aligned}$$

which gives rise to

$$\begin{aligned}
-\int_S^T\int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt =& \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \left[|u|^2\right]_S^T \, d\sigma \\
&+ \int_S^T\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \psi \partial_t u \, d\sigma \, dt - \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} [\psi u]_S^T \, d\sigma.
\end{aligned} \tag{2.17}$$

First, we know that

$$\begin{aligned}
\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \left[|u|^2\right]_S^T \, d\sigma = & \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} |u(T)|^2 \, d\sigma \\
& - \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} |u(S)|^2 \, d\sigma,
\end{aligned}$$

and since $\gamma > \frac{\kappa}{4}$ and $\kappa > 0$,

$$\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} |u(T)|^2 \, d\sigma \leq 0,$$

which implies that

$$\begin{aligned}
\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \left[|u|^2\right]_S^T \, d\sigma &\leq \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)^{-1} |u(S)|^2 \, d\sigma \\
&\leq \frac{\kappa_{\max}}{4\alpha_{\min}} \int_\Sigma |u(S)|^2 \, d\sigma
\end{aligned}$$

Then, according to Lemma 2.4, we get that there exists a positive constant C such that

$$\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \left[|u|^2\right]_S^T \, d\sigma \leq C\mathcal{E}(S). \tag{2.18}$$

Moreover, using Lemma 2.3, we know that

$$\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1} \psi \partial_t u \, d\sigma \, dt \leq C\mathcal{E}(S). \tag{2.19}$$

Regarding the last term in (2.17), we have

$$\begin{aligned}
&-\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}[\psi u]_S^T\,d\sigma\\
&=\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}\psi(T)u(T)\,d\sigma-\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}\psi(S)u(S)\,d\sigma\\
&\le \frac{1}{\alpha_{\min}}\left|\int_\Sigma \frac{\kappa}{2}\psi(T)u(T)\,d\sigma+\int_\Sigma \frac{\kappa}{2}\psi(S)u(S)\,d\sigma\right|.
\end{aligned}$$

Now, using the Cauchy inequality, we get

$$|\psi(T)u(T)+\psi(S)u(S)|\le \frac{1}{2}\left(|\psi(T)|^2+|u(T)|^2+|\psi(S)|^2+|u(S)|^2\right).$$

Therefore,

$$\begin{aligned}
&-\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}[\psi u]_S^T\,d\sigma\\
&\le \frac{1}{2\alpha_{\min}}\int_\Sigma\left(\frac{\kappa}{2}|\psi(T)|^2+\frac{\kappa}{2}|u(T)|^2+\frac{\kappa}{2}|\psi(S)|^2+\frac{\kappa}{2}|u(S)|^2\right)d\sigma.
\end{aligned}\tag{2.20}$$

By definition of $\mathcal{E}$, we immediately have

$$\int_\Sigma\left(\frac{\kappa}{2}|\psi(T)|^2+\frac{\kappa}{2}|\psi(S)|^2\right)d\sigma\le(\mathcal{E}(T)+\mathcal{E}(S))\tag{2.21}$$

and, according to Lemma 2.4, there exists a constant $C>0$ such that

$$\int_\Sigma\left(\frac{\kappa}{2}|u(T)|^2+\frac{\kappa}{2}|u(S)|^2\right)d\sigma\le C\left(\mathcal{E}(T)+\mathcal{E}(S)\right).\tag{2.22}$$

Thus, since $t\longmapsto\mathcal{E}(t)$ is decreasing, $\mathcal{E}(T)\ge\mathcal{E}(S)$ and from (2.20), (2.21) and (2.22) we infer that

$$-\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}[\psi u]_S^T\,d\sigma\le C\mathcal{E}(S)\tag{2.23}$$

Finally, according to (2.17) and the successive estimates (2.18), (2.19), (2.23), we obtain

$$-\int_S^T\int_\Sigma\frac{\kappa}{2}\psi u\,d\sigma\,dt\le C\mathcal{E}(S)$$

which proves that

$$\int_S^T\int_\Sigma\partial_n u u\,d\sigma\,dt\le C\mathcal{E}(S)$$

and completes the proof of Lemma 2.10. □

Lemma 2.11. *We have*

$$-\left[\int_\Omega\partial_t u\left((m\cdot\nabla u)+u\right)dx\right]_S^T\le C\mathcal{E}(S).$$

Proof. We know that

$$-\left[\int_\Omega \partial_t u\,((m\cdot\nabla u)+u)\,dx\right]_S^T = -\int_\Omega \partial_t u\,(m\cdot\nabla u)_{|t=T}\;dx - \int_\Omega \partial_t u u_{|t=T}\,dx$$
$$+\int_\Omega \partial_t u\,(m\cdot\nabla u)_{|t=S}\;dx + \int_\Omega \partial_t u u_{|t=S}\,dx.$$

Moreover, we have

$$\begin{cases} -\displaystyle\int_\Omega \partial_t u\,(m\cdot\nabla u)_{|t=T}\;dx \le C\left(\int_\Omega |\partial_t u_{|t=T}|^2\,dx + \int_\Omega |\nabla u_{|t=T}|^2\,dx\right), \\ \displaystyle\int_\Omega \partial_t u\,(m\cdot\nabla u)_{|t=S}\;dx \le C\left(\int_\Omega |\partial_t u_{|t=S}|^2\,dx + \int_\Omega |\nabla u_{|t=S}|^2\,dx\right), \end{cases}$$

and

$$\begin{cases} -\displaystyle\int_\Omega \partial_t u u_{|t=T}\,dx \le C\left(\int_\Omega |\partial_t u_{|t=T}|^2\,dx + \int_\Omega |u_{|t=T}|^2\,dx\right), \\ \displaystyle\int_\Omega \partial_t u u_{|t=S}\,dx \le C\left(\int_\Omega |\partial_t u_{|t=S}|^2\,dx + \int_\Omega |u_{|t=S}|^2\,dx\right). \end{cases}$$

Using the Poincaré inequality (2.9) and that $\mathcal{E}$ decreases, we obviously get

$$-\left[\int_\Omega \partial_t u\,((m\cdot\nabla u)+u)\,dx\right]_S^T \le C\mathcal{E}(S),$$

which ends the proof. □

Theorem 2.1. *There exists a positive constant C such that for all $0 \le S < T < +\infty$,*

$$\int_S^T \mathcal{E}(u,\partial_t u,\psi)dt \le C\mathcal{E}(S). \tag{2.24}$$

Proof. From Lemma 2.6 to Lemma 2.11, we get

$$\frac{1}{2}\int_S^T\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\,dx\,dt \le C\mathcal{E}(S). \tag{2.25}$$

Combining (2.25) with the result of Lemma 2.3, we get

$$\int_S^T\int_\Sigma \frac{\kappa}{2}|\psi|^2\,d\sigma\,dt \le C\mathcal{E}(S). \tag{2.26}$$

We then obtain

$$\frac{1}{2}\int_S^T\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\,dx\,dt + \frac{1}{2}\int_S^T\int_\Sigma \frac{\kappa}{2}|\psi|^2\,d\sigma\,dt \le C\mathcal{E}(S) \tag{2.27}$$

and the proof of Theorem 2.1 is completed. □

Theorem 2.2. *There exists a positive constant C such that for all initial data in V with $\psi_0 \in H^{1/2}(\Sigma)$,*

$$\mathcal{E}(u, \partial_t u, \psi) \leq e^{-(t-C)/C} \mathcal{E}(u, \partial_t u, \psi)_{|t=0}. \tag{2.28}$$

Proof. In Theorem 2.1 we have shown that there exists a positive constant C such that for all $0 \leq S < T < +\infty$,

$$\int_S^T \mathcal{E}(u, \partial_t u, \psi) dt \leq C\mathcal{E}(u, \partial_t u, \psi)_{|t=S}.$$

Letting T to $+\infty$, we get

$$\int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \leq C\mathcal{E}(u, \partial_t u, \psi)_{|t=S}, \tag{2.29}$$

which implies that

$$\frac{d}{ds}\left(e^{S/C} \int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt\right) \leq 0.$$

The map $S \longmapsto e^{S/C} \int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt$ is thus decreasing and, using Gronwall lemma, we get

$$e^{S/C} \int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \leq \int_0^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt.$$

Besides, when we apply (2.29) for $S = 0$, we get

$$\int_0^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \leq C\mathcal{E}(u, \partial_t u, \psi)_{|t=0}.$$

Therefore,

$$e^{S/C} \int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \leq C\mathcal{E}(u, \partial_t u, \psi)_{|t=0}. \tag{2.30}$$

Moreover, since $\mathcal{E}$ is positive

$$\int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \geq \int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi) dt,$$

and since $\mathcal{E}$ decreases

$$\int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi) dt \geq \int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi)_{|t=S+C} = C\mathcal{E}(S + C). \tag{2.31}$$

Consequently, by plugging (2.30) into (2.31), we obtain

$$e^{S/C} \mathcal{E}(u, \partial_t u, \psi)_{|t=S+C} \leq \mathcal{E}(u, \partial_t u, \psi)_{|t=0},$$

which implies that for all $t > 0$

$$\mathcal{E}(u, \partial_t u, \psi) \leq e^{-(t-C)/C} \mathcal{E}(u, \partial_t u, \psi)_{|t=0}.$$

□

We have thus established that the energy of the continuous problem is exponentially decreasing under two conditions. The first one requires more regularity on ψ_0. Indeed, ψ_0 must be in $H^{1/2}(\Sigma)$ while it is sufficient to have ψ_0 in $L^2(\Sigma)$ to prove that the problem is well-posed[7]. From a practical point of view, that is not annoying because we choose $\psi_0 = 0$ on Σ. Indeed, the initial data u_0 and u_1 are compactly supported inside Ω. Their traces are thus vanishing on Σ, which implies that $\psi_0 = 0$ on Σ, according to the compatibility condition $\partial_n u_0 + u_1 + \frac{\kappa}{2}\psi_0 = 0$ on Σ that must be satisfied in V. The condition $\psi_0 \in H^{1/2}(\Sigma)$ is thus satisfied. The second condition concerns the geometrical form of Ω. The energy decay has been proved under the condition : there exists $x_0 \in \mathbb{R}^2$ such that $(x-x_0)\cdot n \leq 0$ on Γ and $(x-x_0)\cdot n > 0$ on Σ and Σ is convex. It is satisfied if the obstacle is star-shaped with respect to x_0. The exponential decay should be obtained for more general cases where the obstacle is non-trapping, by using micro-local analysis arguments like in Ref. 4.

3. Numerical analysis of stability

In Ref. 7, we have presented numerical results showing the exponential decay of a discrete energy in the case of a circular boundary (i.e. when the curvature is constant). These results confirmed that the condition $\gamma \geq \kappa/4$ is a necessary condition for the decay of the energy. We also showed that the decay rate of the energy is exponential.

Herein our goals are **a)** to prove the decay of the discrete energy by a stability analysis (Subsection 3.1); **b)** to illustrate that condition $\gamma \geq \kappa/4$ is also necessary when the curvature is not constant (Subsection 3.2); and **c)** to estimate the decay rate of the energy as a function of the curvature of the boundary (Subsection 3.3).

To perform the experiments, we have implemented a numerical scheme coupling the Interior Penalty Discontinuous Galerkin (IPDG) method for the space discretization with a Leap-Frog scheme for the time discretization. The scheme is detailed in Ref. 7.

3.1. *Discrete stability analysis*

In this subsection, we recall the discrete energy presented in Ref. 7 and we show that it is decreasing under a CFL condition, provided that $\gamma \geq \frac{\kappa}{4}$ and that the curvature is assumed constant by edges.

As explained in Ref. 7, the fully discretized scheme we obtain reads as

$$\begin{cases} M\dfrac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\Delta t^2} + B\dfrac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t} + B_\kappa\dfrac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2} + K\mathbf{U}^n = 0, \\ C\dfrac{\mathbf{\Psi}^{n+1} - \mathbf{\Psi}^{n-1}}{2\Delta t} + C_{\kappa,\gamma}\dfrac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2} - D\dfrac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t} = 0, \end{cases} \tag{3.1}$$

and we will suppose that κ is constant by edges. We recall that the mass matrix M

and the matrices B, B_κ and C are symmetric positive definite and that the stiffness matrices D and K are positive.

For $n \in \mathbb{N}$, we set

$$\begin{aligned} E^{n+1/2} = & \left(M \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} \right) + \left(K \mathbf{U}^{n+1}, \mathbf{U}^n \right) \\ & + \frac{1}{2} \left[\left(C_\kappa \mathbf{\Psi}^{n+1}, \mathbf{\Psi}^{n+1} \right) + \left(C_\kappa \mathbf{\Psi}^n, \mathbf{\Psi}^n \right), \right] \end{aligned} \tag{3.2}$$

with

$$C_\kappa = \left(\sum_{\sigma \in \Sigma_{\text{abs}}} \int_\sigma \frac{\kappa}{2} v_i v_j \right)_{1 \leq i,j \leq M}.$$

Since M and K are symmetric positive matrices and M is definite, the eigenvalues of $M^{-1}K$ are real and non-negative.
Let λ_{max} be the maximum of these eigenvalues.

Proposition 3.1. *Under the Courant-Friedrichs-Lewy condition (CFL)*

$$\Delta t < \frac{2}{\sqrt{\lambda_{max}}}, \tag{3.3}$$

$E^{n+1/2}$ *defines a discrete energy.*

Proof. To show that $E^{n+1/2}$ defines a discrete energy, we only have to prove that $E^{n+1/2}$ is positive.
We easily check that

$$\begin{aligned} E^{n+1/2} = & \left(\left(M - \frac{\Delta t^2}{4} K \right) \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} \right) \\ & + \left(K \frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2}, \frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2} \right) + \frac{1}{2} \left[\left(C_\kappa \mathbf{\Psi}^{n+1}, \mathbf{\Psi}^{n+1} \right) + \left(C_\kappa \mathbf{\Psi}^n, \mathbf{\Psi}^n \right) \right] \end{aligned}$$

since

$$\left(-\frac{\Delta t^2}{4} K \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} \right) + \left(K \frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2}, \frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2} \right) = \left(K \mathbf{U}^{n+1}, \mathbf{U}^n \right).$$

It is obvious that $E^{n+1/2}$ is positive if $M - \dfrac{\Delta t^2}{2} K$, K and C_κ are positive. We know that K and C_κ are positive matrices by construction. Moreover since M is a symmetric positive definite matrix, the positivity of the first matrix is equivalent to the positivity of $I - \dfrac{\Delta t^2}{4} M^{-1} K$. Hence, if λ_{max} denotes the largest eigenvalue of $M^{-1}K$, $I - \dfrac{\Delta t^2}{4} M^{-1} K$ is positive if

$$\lambda_{max} \leq \frac{4}{\Delta t^2}.$$

□

Remark 3.1. The eigenvalue λ_{max} depends on the space discretization and satisfies $\lambda_{max} \simeq \frac{C}{h^2}$ when the space step h (i.e. the diameter of the smallest cell) is small enough. Therefore, the CFL condition (3.3) can be written as

$$\Delta t < \beta h, \tag{3.4}$$

where β is a constant depending on the geometry of the mesh, on the degree of the polynomial approximation used for the space discretization, and on the penalization parameter introduced to stabilize the bilinear form.

The determination of the analytic expression of β for IPDG is still an open problem. In a recent work[1] it has been shown that, on cartesian meshes and when the penalization parameter is small enough, $\beta \approx 0.58/\sqrt{d}$ (resp. $0.26/\sqrt{d}$, $0.15/\sqrt{d}$, $0.10/\sqrt{d}$, $0.07/\sqrt{d}$). for $Q1$ (resp. $Q2$, $Q3$, $Q4$, $Q5$ elements), where d denotes the dimension of the problem. When the penalization parameter α increases, it has been shown that $\beta \approx \frac{1}{d}\sqrt{\frac{2}{\alpha(p+1)(p+2)}}$. In the numerical experiments we present below, we consider triangular cells with P_1 elements and we have set empirically $\alpha = 3$ and $\beta = 0.25$.

Remark 3.2. The CFL condition only depends on the matrices M and K and not on the boundary matrices. This shows that the ABCs do not penalize the CFL.

Proposition 3.2. *Under the CFL condition (3.4) and if $\gamma > \frac{\kappa}{4}$ on Σ, the energy $E^{n+1/2}$ is decreasing.*

Proof. We first write the restriction of the second equation of (3.1) on an external edge σ_i :

$$C_{\sigma_i}\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}-\mathbf{\Psi}^{n-1}_{\sigma_i}}{2\Delta t}+C_{\kappa,\gamma_{\sigma_i}}\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}-D_{\sigma_i}\frac{\mathbf{U}^{n+1}_{K_{j_i}}-\mathbf{U}^{n-1}_{K_{j_i}}}{2\Delta t}=0$$

where K_{j_i} is the element containing the edge σ_i. We multiply this equation by $\dfrac{\kappa_i}{2}\dfrac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}$:

$$\left(\frac{\kappa_i}{2}C_{\sigma_i}\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}-\mathbf{\Psi}^{n-1}_{\sigma_i}}{2\Delta t},\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}\right)+\frac{\kappa_i}{2}\left(C_{\kappa,\gamma_{\sigma_i}}\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2},\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}\right)$$
$$-\frac{\kappa_i}{2}\left(D_{\sigma_i}\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2},\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}\right)=0.$$

We sum on all the exterior edges to obtain

$$\begin{aligned}&\left(C_\kappa\frac{\mathbf{\Psi}^{n+1}-\mathbf{\Psi}^{n-1}}{2\Delta t},\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right)=\\&\sum_{i=1}^{n_f}\frac{\kappa_i}{2}\left(\left(D_{\sigma_i}-C_{\kappa,\gamma_{\sigma_i}}\right)\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2},\frac{\mathbf{\Psi}^{n+1}_{\sigma_i}+\mathbf{\Psi}^{n-1}_{\sigma_i}}{2}\right).\end{aligned}\tag{3.5}$$

Let us now multiply the first equation of (3.1) by $\dfrac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}$:

$$\left(M\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\Delta t^2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)+\left(B\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)$$
$$+\left(B_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)+\left(K\mathbf{U}^{n},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=0. \tag{3.6}$$

Remark that the term $\left(B_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)$ can be written as

$$\sum_{i=1}^{n_f}\left(B_{\kappa\sigma_i}\frac{\mathbf{\Psi}_{\sigma_i}^{n+1}+\mathbf{\Psi}_{\sigma_i}^{n-1}}{2},\frac{\mathbf{U}_{K_{j_i}}^{n+1}-\mathbf{U}_{K_{j_i}}^{n-1}}{2\Delta t}\right).$$

But

$$\begin{aligned}(B_{\kappa\sigma_i})_{k,l}&=\int_{\sigma_i}\frac{\kappa_i}{2}w_{n_{d-1}*(i-1)+k}v_{n_d*(j_i-1)+l}\\&=\frac{\kappa_i}{2}\int_{\sigma_i}w_{n_{d-1}*(i-1)+k}v_{n_d*(j_i-1)+l}\\&=\frac{\kappa_i}{2}\,(D_{\sigma_i})_{l,k}\end{aligned}$$

since we have supposed that κ is constant by edge.
We then have

$$\left(B_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=\sum_{i=1}^{n_f}\frac{\kappa_i}{2}\left(\frac{\mathbf{\Psi}_{\sigma_i}^{n+1}+\mathbf{\Psi}_{\sigma_i}^{n-1}}{2},D_{\sigma_i}\frac{\mathbf{\Psi}_{\sigma_i}^{n+1}+\mathbf{\Psi}_{\sigma_i}^{n-1}}{2}\right).$$

By adding (3.5) and (3.6), we get

$$\left(M\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\Delta t^2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)+\left(K\mathbf{U}^{n},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)+$$
$$\left(C_\kappa\frac{\mathbf{\Psi}^{n+1}-\mathbf{\Psi}^{n-1}}{2\Delta t},\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right)=$$
$$-\left(B\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)-\sum_{i=1}^{n_f}\frac{\kappa_i}{2}\left(C_{\kappa,\gamma_{\sigma_i}}\frac{\mathbf{\Psi}_{\sigma_i}^{n+1}+\mathbf{\Psi}_{\sigma_i}^{n-1}}{2},\frac{\mathbf{\Psi}_{\sigma_i}^{n+1}+\mathbf{\Psi}_{\sigma_i}^{n-1}}{2}\right). \tag{3.7}$$

Since M is a symmetric matrix, we check that

$$\left(M\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\Delta t^2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=\frac{1}{2\Delta t}\left[\left(M\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t}\right)\right.$$
$$\left.-\left(M\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t},\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t}\right)\right].$$

In the same way, since K and C are both symmetric matrices, we obtain

$$\left(K\mathbf{U}^{n},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=\frac{1}{2\Delta t}\left[\left(K\mathbf{U}^{n+1},\mathbf{U}^{n}\right)-\left(K\mathbf{U}^{n},\mathbf{U}^{n-1}\right)\right]$$

and

$$\left(C_\kappa \frac{\mathbf{\Psi}^{n+1} - \mathbf{\Psi}^{n-1}}{2\Delta t}, \frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}\right) = \frac{1}{4\Delta t}\left[\left(C_\kappa \mathbf{\Psi}^{n+1}, \mathbf{\Psi}^{n+1}\right) - \left(C_\kappa \mathbf{\Psi}^{n-1}, \mathbf{\Psi}^{n-1}\right)\right]$$

$$= \frac{1}{2\Delta t}\left[\frac{\left(C_\kappa \mathbf{\Psi}^{n+1}, \mathbf{\Psi}^{n+1}\right) + \left(C_\kappa \mathbf{\Psi}^{n}, \mathbf{\Psi}^{n}\right)}{2} - \frac{\left(C_\kappa \mathbf{\Psi}^{n}, \mathbf{\Psi}^{n}\right) + \left(C_\kappa \mathbf{\Psi}^{n-1}, \mathbf{\Psi}^{n-1}\right)}{2}\right]$$

so that (3.7) reads as

$$\frac{1}{2\Delta t}\left(E^{n+1/2} - E^{n-1/2}\right) = -\left(B\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right)$$
$$-\sum_{i=1}^{n_f} \frac{\kappa_i}{2}\left(C_{\kappa,\gamma_{\sigma_i}} \frac{\mathbf{\Psi}_{\sigma_i}^{n+1} + \mathbf{\Psi}_{\sigma_i}^{n-1}}{2}, \frac{\mathbf{\Psi}_{\sigma_i}^{n+1} + \mathbf{\Psi}_{\sigma_i}^{n-1}}{2}\right).$$

Since B and $C_{\kappa,\gamma_{\sigma_i}}$ are positive definite matrices if $\gamma > \frac{\kappa}{4}$ and κ is positive, we get

$$\frac{1}{2\Delta t}\left(E^{n+1/2} - E^{n-1/2}\right) < 0,$$

which ends the proof of the proposition. □

3.2. *Behavior of the discrete energy in the case of an elliptical boundary*

In this part, we present numerical results that extend the results presented in Ref. 7 to the case of a boundary with a variable curvature. We consider a two-dimensional domain Ω_1, delimited by an exterior boundary Σ_1 and by an interior boundary Γ_1. Γ_1 is the boundary of an elliptical obstacle of semi-major axis a =2m and semi-minor axis b =1m centered at the origin. Σ_1 is an ellipse of semi-major axis $a_{\text{ext}} = \delta a$ and semi-minor axis $b_{\text{ext}} = \delta b$ centered at the origin. The source term F represents a point source in space set at (0m, 1.3m). Its time variations are represented by a second-derivative of a Gaussian with a dominant frequency f_0 of 1Hz:

$$F = \delta_{x_0} 2\lambda \left(\lambda\left(t - t_0\right)^2 - 1\right) e^{-\lambda(t-t_0)^2},$$

with $x_0 = (0\text{m}, 1.3\text{m})$, $\lambda = \pi^2 f_0^2$ and $t_0 = 1/f_0$.

First, we set $\gamma = \kappa$ in order to analyze the behavior of the discrete energy. In Fig. 2, we depict the evolution of these energy with respect to the time. Obviously, the energy is first increasing until the source is switched off. Then, it is constant until the waves reach the external boundary. Finally, the energy is decreasing, which conforms to Proposition 3.2.

Now, we focus on the sufficient condition $\gamma \geq \kappa/4$ to see if it may be necessary to ensure the decay of the energy. To this aim, we have performed two experiments, one for the critical case $\gamma = 0.25\kappa$ and the other one for the case $\gamma = 0.249\kappa$. In Fig.

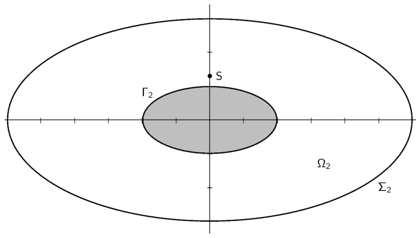

Fig. 1. Computational domain

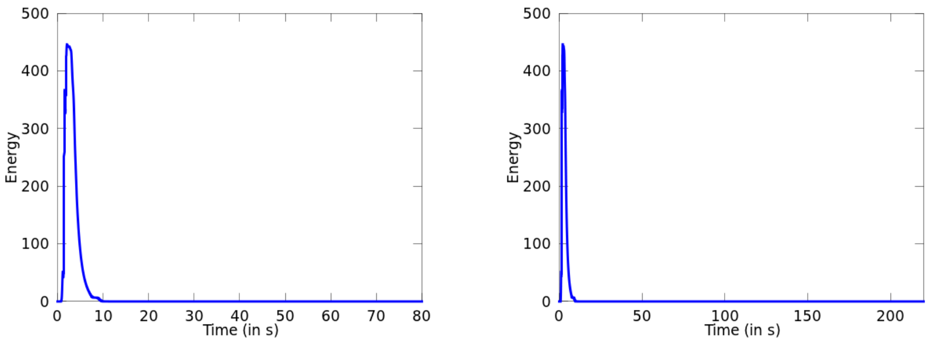

Fig. 2. Energy vs time for $\gamma = \kappa$

3 (resp. in Fig. 4), we represent the evolution of the discrete energy until $T = 1000s$ (which represents approximately 150 000 iterations) when $\gamma = 0.249\kappa$ (resp. when $\gamma = 0.25\kappa$). Considering only these figures, it seems that both systems are stable, even when $\gamma < \frac{\kappa}{4}$. The y-scale has been magnified by a factor 10^{10} to show there is no instability. In order to investigate more carefully the stability of the energy, we have performed the same experiments during a much longer time : $T = 35000s$ (which represents 4 000 000 iterations). The results are presented in Fig. 5 for $\gamma = 0.249\kappa$ and in Fig. 6), for $\gamma = 0.25\kappa$. It is then clear that the boundary condition is unstable when $\gamma < \frac{\kappa}{4}$. Note that the number of iterations we have performed is much larger than the one needed for practical applications. Nevertheless, it was necessary to achieve this number in order to exhibit the instabilities in the case $\gamma = 0.249\kappa$. Moreover, this emphasizes the long-term stability both of the ABCs and of the numerical scheme.

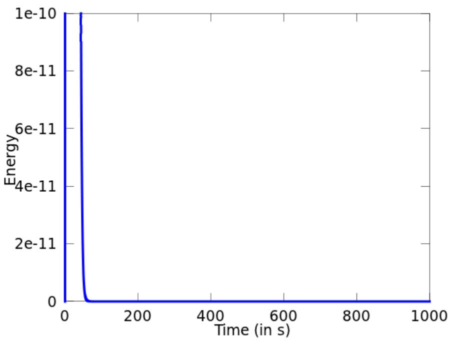

Fig. 3. Energy for $\gamma = 0.249\kappa$ - y-scale magnified by 10^{10}

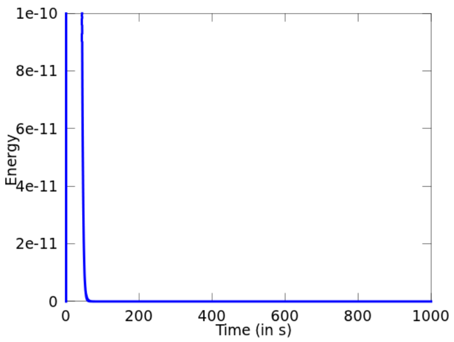

Fig. 4. Energy for $\gamma = 0.25\kappa$ - y-scale magnified by 10^{10}

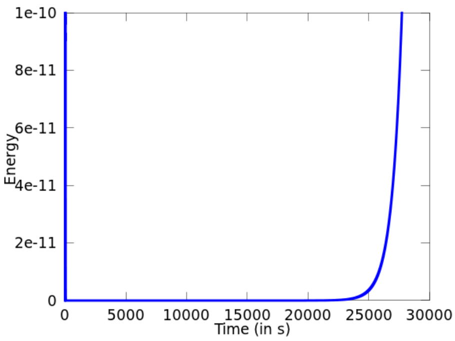

Fig. 5. Energy for $\gamma = 0.249\kappa$ - y-scale magnified by 10^{10}

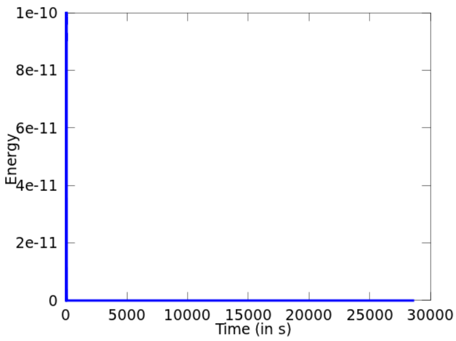

Fig. 6. Energy for $\gamma = 0.25\kappa$ - y-scale magnified by 10^{10}

3.3. *Analysis of the decay rate of the energy*

In Section 2, we have seen that the continuous energy can be controlled by an exponential decreasing function $g(t)$ which read as

$$g(t) = \exp(-\alpha_1 t + \alpha_2)).$$

To check this property, we have computed the evolution of the logarithm of the discrete energy for $\gamma = \kappa$ (see Fig. 7). Note that the energy stops decreasing after $T = 100s$, which is due to the fact that it becomes smaller than the round off error of 10^{-16}. In order to evaluate the rate of the decay α_1, we have used a linear regression method. We have focused on the time interval $[0\,;\,10]$, after which the energy is divided by 1000. In Fig. 8, we compare the evolution of the discrete energy obtained with $\gamma = \kappa$ (blue curve) with the function g (red curve). The two curves are perfectly superimposed, which illustrates the exponential decay of the energy. Now, we wish to analyze the dependence of the decay rate with respect to the curvature

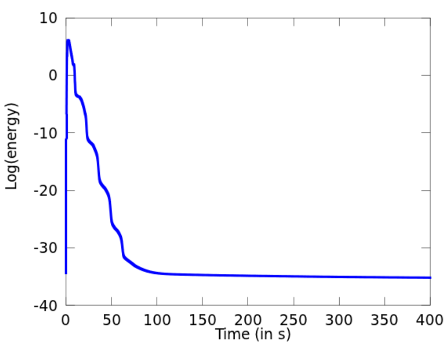

Fig. 7. Evolution of the logarithm of the energy

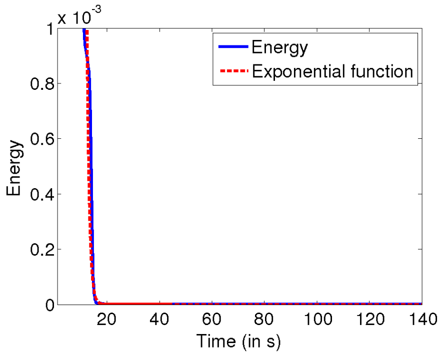

Fig. 8. Exponential decay of the energy

of the boundary. The matter is to show that it is possible to impact on the decay rate of the solution by changing the location and/or the geometry of the artificial boundary. This is an interesting property of the ABCs because the solution of time dependent problem can be used to compute quickly and efficiently the solution to the Helmholtz equation[4]. This point deserves attention in particular at high frequency where the solution to the Helmholtz equation is difficult to compute.

We have computed the decay rate of the energy for six values of δ, 1.5, 2, 3, 4, 5, 6 which represents the distance from the boundary to the obstacle. The results are presented in Fig. 9. Except the particular cases $\delta = 1.5$ and 2, where the boundary is very close to the obstacle, the decay rate grows linearly with δ. In Fig. 10, we compare the previous curve with the function $s(\delta) = 0.12\delta - 1$. However, since the

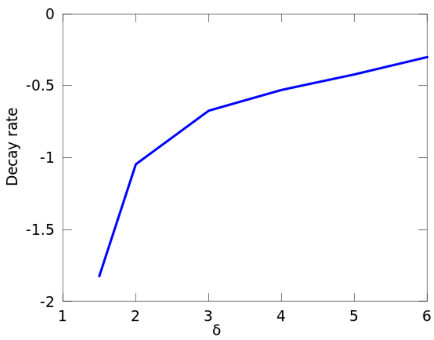

Fig. 9. Decay rate of the energy with respect to δ

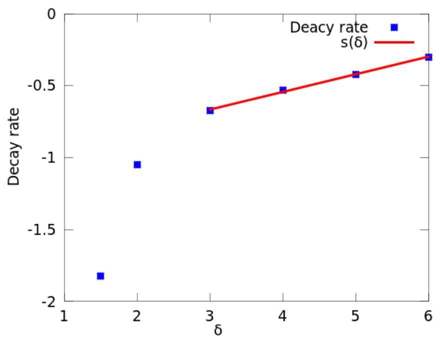

Fig. 10. Comparison with the function s

curvature is not constant, it is not possible to deduce a relation between the decay rate and the curvature. It is however possible to express this relation as function of the minimal curvature $\kappa_{\min}$ or of the maximal curvature $\kappa_{\max}$ of the external boundary. In this configuration, we have $\kappa_{\min} = 1/(2\delta)$ and $\kappa_{\max} = 1/\delta$ and two possible relations for the decay rate α_1 are $\alpha_1 = 0.24/\kappa_{\min} - 1$ or $\alpha_1 = 0.12/\kappa_{\max} - 1$. In order to analyze this relation more precisely, we have reproduced in Ref. 7 the same experiments by replacing the elliptical obstacle by a circle of radius 1 and the elliptical artificial boundary by a circle of radius δ. In this configuration, the curvature κ is constant and $\kappa = 1/\delta$. We have obtained that this decay could be approximated by $s(\delta) = 0.15\delta - 1.5$ for $\delta \geq 3$. Then, the relation between the decay rate and the curvature could be $s(\delta) = 0.15/\kappa - 1.5$. This relation, which differs from the two previous ones indicates that it is not possible to express the decay rate only as a function of $\kappa_{\min}$ or of $\kappa_{\max}$. Moreover, it not clear that the curvature is the only parameter impacting on the decay rate. Therefore, it might be interesting to use the same approach than Komornik[12] in order to get an explicit decay rate, at least in the case of a circle.

4. Conclusion

By combining the results of this work with Refs. 5, 7, we are now in position to consider that the curvature condition is a good candidate for modeling propagating acoustic waves. The next step should be to combine the curvature condition with a condition modeling the evanescent waves. This idea has already been proposed by Hagstrom *et al*[9,10,11] for plane boundary but has not been generalized to arbitrary regular convex boundary.

We have already addressed this issue[6] but by performing numerical experiments only. The conclusion is that combining the curvature condition with a Fourier condition improves the accuracy of the solution. The Fourier condition represents the evanescent waves near the absorbing surface. It depends on a parameter and a mathematical study is still needed to determine its optimal value. Following this paper, the corresponding boundary value problem remains to be studied mathematically but numerical experiments that have been carried out in Ref. 6 indicate that the enriched boundary condition is long-term stable. This is an on-going work.

But this condition could also be used as a second-order condition in the harmonic case. Then, it should be compared with existing condition like the popular BGT condition[8] or the complete second-order condition[2]. Besides, the new conditions could be used as On-Surface-Radiation-Conditions. Indeed, by introducing the auxiliary unknown, we are able to write the normal derivative of the scattered field as a function of the scattered field on the surface of the obstacle. The same conclusion applies in the time dependent case because the ABC gives a relation that can be plugged into the retarded potential easily (see Ref. 3).

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