



## Complexity of greedy edge-colouring

Frédéric Havet, Ana Karolinna Maia, Min-Li Yu

► **To cite this version:**

Frédéric Havet, Ana Karolinna Maia, Min-Li Yu. Complexity of greedy edge-colouring. [Research Report] RR-8171, INRIA. 2012, pp.13. <hal-00762534>

**HAL Id: hal-00762534**

**<https://hal.inria.fr/hal-00762534>**

Submitted on 7 Dec 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# Complexity of greedy edge-colouring

Frédéric Havet, A. Karolinna Maia, Min-Li Yu

**RESEARCH  
REPORT**

**N° 8171**

December 2012

Project-Teams MASCOTTE





## Complexity of greedy edge-colouring

Frédéric Havet<sup>\*†</sup>, A. Karolinna Maia<sup>\*‡§</sup>, Min-Li Yu<sup>¶||</sup>

Project-Teams MASCOTTE

Research Report n° 8171 — December 2012 — 10 pages

**Abstract:** The *Grundy index* of a graph  $G = (V, E)$  is the greatest number of colours that the greedy edge-colouring algorithm can use on  $G$ . We prove that the problem of determining the Grundy index of a graph  $G = (V, E)$  is NP-hard for general graphs. We also show that this problem is polynomial-time solvable for caterpillars. More specifically, we prove that the Grundy index of a caterpillar is  $\Delta(G)$  or  $\Delta(G) + 1$  and present a polynomial-time algorithm to determine it exactly.

**Key-words:** Edge colouring, greedy colouring, greedy algorithm, line graphs, caterpillars, NP-complete.

---

\* Projet Mascotte, I3S (CNRS, UNSA) and INRIA, Sophia Antipolis. Partly supported by ANR Blanc AGAPE.

† email: frederic.havet@inria.fr.

‡ Partly supported by CAPES/Brazil.

§ email: karol.maia@inria.fr.

¶ Department of Maths and Statistics University of the Fraser Valley Abbotsford, BC, Canada.

|| email: joseph.yu@ufv.ca.

**RESEARCH CENTRE  
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

## Complexité de coloration gloutonne d'arêtes

**Résumé :** L'*indice Grundy* d'un graphe  $G$  est le plus grand nombre de couleurs que l'algorithme glouton de coloration d'arêtes peut utiliser pour  $G$ . Nous prouvons que le problème de déterminer l'indice Grundy d'un graphe est NP-dur en général. Nous montrons également que ce problème peut être résolu en temps polynômial pour les chenilles. Plus spécifiquement, nous prouvons que l'indice Grundy d'une chenille est  $\Delta(G)$  ou  $\Delta(G)+1$  et nous présentons un algorithme polynômial pour le déterminer exactement.

**Mots-clés :** Coloration d'arêtes, coloration gloutonne, algorithme glouton, graphe des lignes, chenilles, NP-complet.

## 1 Introduction

All the graphs considered in this paper are loopless.

A  $k$ -colouring of a graph  $G = (V, E)$  is a surjective mapping  $c : V \rightarrow \{1, 2, \dots, k\}$  such that  $c(u) \neq c(v)$  for any edge  $uv \in E$ . The *chromatic number* is  $\chi(G) = \min\{k \mid G \text{ admits a } k\text{-colouring}\}$ . On the algorithmic point of view, finding the chromatic number of a graph is a hard problem. For all  $k \geq 3$  it is NP-complete to decide if a graph admits a  $k$ -colouring (see [2]). Furthermore, it is NP-hard to approximate the chromatic number within  $|V(G)|^{\varepsilon_0}$  for some positive constant  $\varepsilon_0$  as shown by Lund and Yannakakis [5].

Hence lots of heuristics have been developed to colour a graph. The most basic and widespread because it works on-line is the greedy algorithm. Given a vertex ordering  $\sigma = v_1 < \dots < v_n$  of  $V(G)$ , this algorithm colours the vertices in the order  $v_1, \dots, v_n$ , assigning to  $v_i$  the smallest positive integer not used on its lower-indexed neighbours. A colouring resulting of the greedy algorithm is called a *greedy colouring*. The *Grundy number*  $\Gamma(G)$  is the largest  $k$  such that  $G$  has a greedy  $k$ -colouring. Easily,  $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$ .

Zaker [6] showed that for any fixed  $k$ , one can decide in polynomial time if a given graph has Grundy number at most  $k$ . However determining the Grundy number of a graph is NP-hard [6], and given a graph  $G$ , it is even NP-complete to decide if  $\Gamma(G) = \Delta(G) + 1$  as shown by Havet and Sampaio [3]. In addition, Asté et al. [1] showed that for any constant  $c \geq 1$ , it is NP-complete to decide if  $\Gamma(G) \leq c \cdot \chi(G)$ .

Graph colouring of many graph classes has also been studied. One of the classes is the one of line graphs. The *line graph* of a graph  $G$ , denoted  $L(G)$ , is the graph whose vertices are the edges of  $G$ , with  $ef \in E(L(G))$  whenever  $e$  and  $f$  share an endvertex. Colouring line graphs corresponds to edge-colouring. A  $k$ -edge-colouring of a graph  $G$  is a surjective mapping  $\phi : E(G) \rightarrow \{1, \dots, k\}$  such that if two edges  $e$  and  $f$  are adjacent (i.e share an endvertex), then  $\phi(e) \neq \phi(f)$ . A  $k$ -edge colouring may also be seen as a partition of the edge set of  $G$  into  $k$  disjoint *matchings*  $M_i = \{e \mid \phi(e) = i\}$ ,  $1 \leq i \leq k$ . By edge-colouring we mean either the mapping  $\phi$  or the partition.

The *chromatic index*  $\chi'(G)$  of a graph  $G$  is the least  $k$  such that  $G$  admits a  $k$ -edge-colouring. It is easy to see that  $\chi'(G) = \chi(L(G))$ . Obviously,  $\Delta(G) \leq \chi'(G)$  and Shannon's and Vizing's theorems state that  $\chi'(G) \leq \max\{\frac{3}{2}\Delta(G); \Delta(G) + \mu(G)\}$ , where  $\mu(G)$  is the maximum number of edges between two vertices of  $G$ . Holyer [4] showed that for any  $k \geq 3$ , it is NP-complete to decide if a  $k$ -regular graph has chromatic index  $k$ .

One can apply the greedy algorithm to colour a line graph. It corresponds to the following greedy algorithm for edge-colouring. Given a graph  $G = (V, E)$  and an edge ordering  $\theta = e_1 < \dots < e_n$ , assign to  $e_i$  the least positive integer that was not already assigned to lower-indexed edges adjacent to it. An edge-colouring obtained by this process is called a *greedy edge-colouring* and it has the following property:

*For every  $j < i$ , every edge  $e$  in  $M_i$  is adjacent to an edge in  $M_j$ .* (P)

Note that an edge-colouring satisfying (P) is a greedy edge-colouring relative to any edge ordering in which the edges of  $M_i$  precede those of  $M_j$  when  $i < j$ .

The *Grundy index*  $\Gamma'(G)$  of a graph  $G$  is the largest number of colours of a greedy edge-colouring of  $G$ . Notice that  $\Gamma'(G) = \Gamma(L(G))$ . By definition,  $\chi'(G) \leq \Gamma'(G)$ . Furthermore, as an edge is incident to at most  $2\Delta(G) - 2$  other edges ( $\Delta(G) - 1$  at each endvertex), colouring the edges greedily uses at most  $2\Delta(G) - 1$  colours. So  $\Delta(G) \leq \Gamma'(G) \leq 2\Delta(G) - 1$ . There are graphs for which the Grundy index equals the maximum degree: stars for example. On the opposite, for any  $\Delta$  there is a tree with maximum degree  $\Delta$  and Grundy index  $2\Delta(G) - 1$ .

In this paper, we study the complexity of finding the Grundy index of a graph. We prove that it is NP-hard by showing that the following problem is co-NP-complete.

**MINIMUM GREEDY EDGE-COLOURING**Instance: A graph  $G$ .Question:  $\Gamma'(G) = \Delta(G)$ ?

The proof is a reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS which was proved to be NP-complete by Holyer [4]. We recall that a *cubic graph* is a 3-regular graph. The reduction also proves that it is co-NP-complete to decide if  $\Gamma'(G) = \chi'(G)$ .

**3-EDGE-COLOURABILITY OF CUBIC GRAPHS**Instance: A cubic graph  $G$ .Question: Is  $G$  3-edge colourable?

We then extend the result to a more general problem.

 **$f$ -GREEDY EDGE-COLOURING**Instance: A graph  $G$ .Question:  $\Gamma'(G) \leq f(\Delta(G))$ ?

We show that for any function  $f$  such that  $k \leq f(k) \leq 2k - 2$ , the problem  $f$ -GREEDY EDGE-COLOURING is co-NP-Complete.

Since determining the Grundy index is NP-hard, a natural question to ask is for which class of graphs it can be done in polynomial time. In Section 3, we consider *caterpillars* which are trees such that the deletion of all leaves results in a path, called *backbone*. We show that if  $T$  is a caterpillar then  $\Gamma'(T) \leq \Delta(T) + 1$  and then give a linear-time algorithm to compute the Grundy index of a caterpillar.

## 2 Co-NP-completeness results

The aim of this section is to prove that  $f$ -GREEDY EDGE-COLOURING is co-NP-complete for every function  $f$  such that  $k \leq f(k) \leq 2k - 2$  for all  $k$ .

For sake of clarity, we first show that MINIMUM GREEDY EDGE-COLOURING is co-NP-Complete.

**Theorem 1.** MINIMUM GREEDY EDGE-COLOURING is co-NP-Complete.

MINIMUM GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph  $G$  with at least  $\Delta(G) + 1$  colours is a certificate that  $\Gamma'(G) > \Delta(G)$ .

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.

Let  $H$  be a cubic graph on  $n$  vertices  $w_1, \dots, w_n$ . Let  $G$  be the graph defined by  $V(G) = V(H) \cup \{u_1, \dots, u_n\} \cup \{v, a, b, c\}$  and  $E(G) = E(H) \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \{v u_i \mid 1 \leq i \leq n\} \cup \{av, bv, cv\}$ . See Figure 1.

In  $G$ ,  $d(v) = n + 3$ , while the degree of all other vertices is at most 4. Thus,  $\Delta(G) = d(v) = n + 3$  because  $n \geq 4$  has  $H$  is cubic. Moreover, every edge of  $G$  is adjacent to at most  $n + 3$  edges so  $\Gamma'(G) \leq n + 4 = \Delta(G) + 1$ . Hence the Grundy index of  $G$  is either  $\Delta(G)$  or  $\Delta(G) + 1$ . The co-NP-completeness of MINIMUM GREEDY EDGE-COLOURING follows directly from the following claim.

**Claim 1.**  $\chi'(H) = 3$  if and only if  $\Gamma'(G) = \Delta(G) + 1$ .

*Proof.* ( $\Rightarrow$ ) Suppose that there exists a 3-edge-colouring  $\phi$  of  $H$ . Let us extend  $\phi$  into a greedy edge-colouring of  $G$  with  $\Delta(G) + 1 = n + 4$  colours. Set  $\phi(av) = 1$ ,  $\phi(bv) = 2$ ,  $\phi(cv) = 3$ , and for all  $1 \leq i \leq n$ ,  $\phi(u_i w_i) = 4$  and  $\phi(u_i v) = i + 4$ . Notice that every vertex  $w_i$  is incident to an

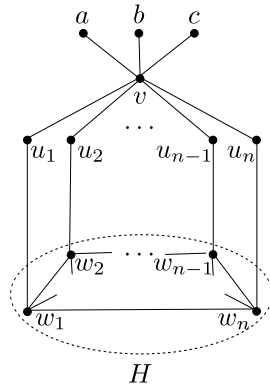


Figure 1: Graph  $G$  obtained from a cubic graph  $H$ .

edge of  $H$  of each colour in  $\{1, 2, 3\}$  since  $H$  is cubic. Then it is straightforward to check that  $\phi$  is a greedy  $(n + 4)$ -edge-colouring of  $G$ .

( $\Leftarrow$ ) Suppose that there is a greedy  $(n + 4)$ -edge-colouring of  $G$ . Some edge is coloured  $n + 4$ . But such an edge has to be adjacent to at least  $n + 3$  edges and thus to be one of the  $vu_i$ , say  $vu_n$ . The edge  $vu_n$  is adjacent to exactly  $n + 3$  edges. So by Property (P), all edges adjacent to  $vu_n$  receive distinct colours in  $\{1, \dots, n + 3\}$ .

Let us first prove by induction on  $1 \leq j \leq n$  that the edge  $e_j$  incident to  $vu_n$  labelled  $n + 5 - j$  is one of the  $vu_i$ , the result holding for  $j = 1$ . Suppose now that  $j \geq 2$ . The edge  $e_j$  must have degree at least  $n + 5 - j$  since it is adjacent to  $vu_n$  and one edge of each colour in  $\{1, \dots, n + 4 - j\}$  by Property (P). Hence  $e_j$  must be incident to  $v$  since  $u_n w_n$  is adjacent to four edges. Then  $e_j$  must have degree at least  $n + 3$  since it is adjacent to the  $j - 1$  edges  $e_l$  for  $1 \leq l < j$  and one edge of each colour in  $\{1, \dots, n + 4 - j\}$ . Hence  $e_j$  is one of the  $vu_i$ .

Hence, without loss of generality, we may assume that  $\phi(vu_i) = i + 4$  for all  $1 \leq i \leq n$ . The edge  $vu_i$  is adjacent to an edge coloured 4. This edge must be  $u_i w_i$  since the edges  $av, bv$  and  $cv$  are adjacent to at most 2 edges coloured in  $\{1, 2, 3\}$ . Thus  $\phi(u_i w_i) = 4$  for all  $1 \leq i \leq n$ .

Now every edge  $u_i w_i$  is adjacent to three edges, one of each colour in  $\{1, 2, 3\}$ . Since  $\phi(vu_i) \geq 5$ , these three edges must be the three edges incident to  $w_i$  in  $H$ . Thus all the edges of  $H$  are coloured in  $\{1, 2, 3\}$ . Hence the restriction to  $\phi$  to  $H$  is a 3-edge-colouring.  $\square$

*Remark 1.* Observe that the graph  $G$  has chromatic index  $\Delta(G)$ . Indeed colour the edges adjacent to  $v$  with the colours  $1, \dots, \Delta(G)$  and then extend greedily this colouring to the other edges. Since all the remaining edges are adjacent to at most 6 edges they will all get a colour less or equal to 7. Since  $\Delta(G) \geq 7$ , we obtain a  $\Delta(G)$ -edge colouring. Hence the above reduction shows that it is co-NP-complete to decide if  $\Gamma'(G) = \chi'(G)$ .

Theorem 1 may be generalized as follows.

**Theorem 2.** *Let  $f$  be a function such that  $k \leq f(k) \leq 2k - 2$  for all  $k \in \mathbb{N}$ .  $f$ -GREEDY EDGE-COLOURING is co-NP-Complete.*

*Proof.*  $f$ -GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph  $G$  with more than  $f(\Delta(G))$  colours is a certificate that  $\Gamma'(G) > f(\Delta(G))$ .

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.



Let  $H$  be a cubic graph on  $n$  vertices  $w_1, \dots, w_n$  and let  $G$  be the graph defined as in the proof of Theorem 1. Set  $p = f(n+3) - (n+3)$ . Then  $0 \leq p \leq n+1$ . For  $1 \leq i \leq p$ , let  $T_i$  be the tree with vertex set  $\{a_i, b_i, c_i, t_i\} \cup \{a_{i,j}, b_{i,j}, c_{i,j}, s_{i,j}, t_{i,j} \mid 1 \leq j \leq n-1\}$  and edge set  $\{a_i t_i, b_i t_i, c_i t_i\} \cup \bigcup_{j=1}^{n-1} \{a_{i,j} t_{i,j}, b_{i,j} t_{i,j}, c_{i,j} t_{i,j}, t_{i,j} s_{i,j}, s_{i,j} t_{i,j}\}$ . Let  $G'$  be a graph obtained from the disjoint union of  $G$  and the  $T_i$  by adding the edge  $u_n t_i$  for all  $1 \leq i \leq p$ . See Figure 2.

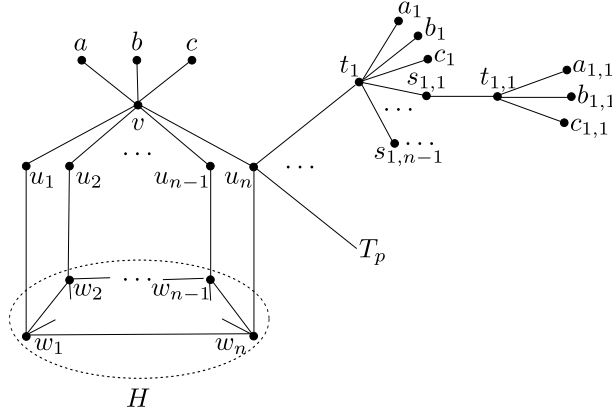


Figure 2: The graph  $G'$  obtained from a cubic graph  $H$ .

Observe that  $\Delta(G') = n+3$  and the vertices of degree  $n+3$  are  $v, t_1, \dots, t_p$  and  $u_n$  when  $p = n+1$ . Moreover every edge is adjacent to at most  $n+3+p$ , so  $\Gamma(G') \leq n+3+p+1 = f(\Delta(G')) + 1$ . The co-NP-completeness of  $f$ -GREEDY EDGE-COLOURING follows directly from the following claim.

**Claim 2.**  $\chi'(H) = 3$  if and only if  $\Gamma(G') = f(\Delta(G')) + 1$ .

( $\Rightarrow$ ) Suppose that there exists a 3-edge-colouring  $\phi$  of  $H$ . Let us extend  $\phi$  into a greedy edge-colouring of  $G'$  with  $f(\Delta(G')) + 1 = n+p+4$  colours. We first extend it into a greedy  $(n+4)$ -colouring of  $G$  as we did in the proof of Theorem 1. In particular, we have  $\phi(u_n w_n) = 4$  and  $\phi(u_n v) = n+4$ . For all  $1 \leq i \leq p$  and all  $1 \leq j \leq n-1$ , we set  $\phi(t_i a_i) = 1$ ,  $\phi(t_i b_i) = 2$ ,  $\phi(t_i c_i) = 3$ ,  $\phi(t_{i,j} a_{i,j}) = 1$ ,  $\phi(t_{i,j} b_{i,j}) = 2$ ,  $\phi(t_{i,j} c_{i,j}) = 3$ ,  $\phi(t_{i,j} s_{i,j}) = j+3$ , and  $\phi(t_i u_n) = n+4+i$ . Then it is straightforward to check that  $\phi$  is a greedy  $(n+p+4)$ -edge-colouring of  $G'$ .

( $\Leftarrow$ ) Suppose that  $G'$  admits a greedy  $(n+p+4)$ -edge-colouring  $\phi$ . For all  $1 \leq i \leq p$ , there is an edge  $e_i$  coloured  $n+4+i$ . This edge must have to be adjacent to at least  $n+3+i$  edges by Property (P). So all the  $e_i$  must be in  $F = \{v u_n\} \cup \{u_n t_i \mid 1 \leq i \leq p\}$ . Now the edge  $e_p$  is adjacent to an edge  $e_0$  coloured  $n+4$ . This edge is adjacent to at least  $n+4$ -edges, one of each colour in  $\{1, \dots, n+3\}$  and  $e_p$ . Hence it also has to be in  $F$ . Since  $|F|$ , all the edges in  $F$  are coloured with distinct labels in  $\{n+4, \dots, n+p+4\}$ .

Now applying the same reasoning as in the proof of Theorem 1, we derive that the restriction to  $\phi$  to  $H$  is a 3-edge-colouring.  $\square$

### 3 Greedy edge-colouring of caterpillars

In this section, we show a polynomial-time algorithm solving GREEDY EDGE-COLOURING for caterpillars. Recall that caterpillars are trees such that the deletion of all leaves results in a path, called backbone.

We first show some properties of a greedy edge-colouring of a caterpillar where we can see that the Grundy index of a caterpillar  $T$  is at most  $\Delta(T) + 1$ . We then give a polynomial-time algorithm that computes its Grundy index.

### 3.1 Grundy index of a caterpillar

**Lemma 3.** *Let  $T$  be a caterpillar and  $v$  a vertex in its backbone. In every greedy edge-colouring of  $T$ , the colours  $1, \dots, d(v) - 2$  appear on the edges incident to  $v$ .*

*Proof.* By the contrapositive. Let  $c$  be a greedy edge-colouring of  $T$ . Suppose that a colour  $\alpha \in \{1, \dots, d(v) - 2\}$  is not assigned to any edge incident to  $v$ . Then, since all the edges incident to  $v$  have different colours, at least three colours strictly greater than  $d(v) - 2$  appear on three edges incident to  $v$ . One of these colours, say  $\beta$  must appear on an edge  $e$  incident with a leaf. But  $e$  is uniquely adjacent to edges incident to  $v$ . So  $e$  is adjacent to no edge coloured  $\alpha$ . Since  $\alpha \leq d(v) - 2 < \beta$ , the edge-colouring  $c$  is not greedy.  $\square$

**Lemma 4.** *Let  $c$  be a greedy edge-colouring of a caterpillar  $T$  and  $v$  a vertex in the backbone of  $T$ . If two edges  $e_1$  and  $e_2$  incident to  $v$  receive colours greater than  $d(v) - 1$ , then  $e_1$  and  $e_2$  are two edges of the backbone and the edges incident to  $v$  and leaves are coloured  $1, \dots, d(v) - 2$ .*

*Proof.* Suppose by way of contradiction that one of these two edges, say  $e_1$ , is incident to a leaf. Then  $e_1$  is adjacent to  $d(v) - 1$  other edges, and one of them, namely  $e_2$ , is assigned a colour greater than  $d(v) - 1$ . Thus  $e_1$  is adjacent to at most  $d(v) - 2$  edges whose colour is less or equal to  $d(v) - 1$ . So, there is a colour  $\alpha$  in  $\{1, \dots, d(v) - 1\}$  such that no edge incident to  $e_1$  is coloured  $\alpha$ . This contradicts the fact that  $c$  is greedy. Hence  $e_1$  and  $e_2$  are edges of the backbone.

Now by Lemma 3, there must be edges incident to  $v$  of each colour in  $\{1, \dots, d(v) - 2\}$ . So the  $d(v) - 2$  edges distinct from  $e_1$  and  $e_2$ , which are the edges linking  $v$  and leaves are coloured in  $\{1, \dots, d(v) - 2\}$ .  $\square$

We use Lemma 4 to prove the following theorem.

**Theorem 5.** *If  $T$  is a caterpillar, then  $\Gamma(T) \leq \Delta(T) + 1$ .*

*Proof.* Set  $\Delta(T) = \Delta$ . Suppose by way of contradiction that it is possible to greedily colour  $T$  with  $\Delta + 2$  colours. Let  $e$  be an edge coloured  $\Delta + 2$ . It must be adjacent to at least  $\Delta + 1$  edges, one of each colour  $1, \dots, \Delta + 1$ . Thus, the edge  $e$  is in the backbone. According to Lemma 4, the edges  $e_1$  and  $e_2$  adjacent to  $e$  with colours  $\Delta$  and  $\Delta + 1$  are in the backbone. Furthermore all the edges adjacent to  $e$  which are neither  $e_1$  nor  $e_2$  are coloured in  $\{1, \dots, \Delta - 2\}$ . Hence  $e$  is adjacent to no edge coloured  $\Delta - 1$ , a contradiction.  $\square$

Theorem 5 is tight. For instance, consider the caterpillar  $T_k$  with backbone  $(t, u, v, w)$  for which the vertex  $t$  has degree  $k - 1$ ,  $u$  has degree 2 and  $v$  and  $w$  degree  $k$ . An edge-colouring in which the  $k - 2$  edges incident to  $t$  and a leaf are coloured with  $1, \dots, k - 2$ , the  $k - 1$  edges incident to  $w$  and a leaf with  $1, \dots, k - 1$ , the  $k - 2$  edges incident to  $v$  and a leaf with  $1, \dots, k - 2$ , the edge  $tu$  with  $k - 1$ , the edge  $vw$  with  $k$  and the edge  $uv$  with  $k + 1$  is greedy.

### 3.2 Finding the Grundy index of a caterpillar

Let us now examine in more details when a caterpillar  $T$  has Grundy index  $\Delta(T) + 1$ .

Let  $T$  be a caterpillar with backbone  $P = (v_1, v_2, \dots, v_n)$ . The first edge of  $P$  is  $v_1v_2$ . For any edge  $e = v_iv_{i+1} \in P$ , removing  $e$  from  $T$  gives two caterpillars  $T_e^-$  and  $T_e^+$ , the first one containing  $v_i$  and the second one containing  $v_{i+1}$ . For convenience, the backbone of  $T_e^-$  is

$P_e^- = (v_i, v_{i-1}, \dots, v_1)$  and the backbone of  $T_e^+$  is  $P_e^+ = (v_{i+1}, \dots, v_n)$ . Hence the first edge of  $T_e^-$  is  $(v_i, v_{i-1})$  and the first edge of  $T_e^+$  is  $(v_{i+1}, v_{i+2})$ .

**Lemma 6.** *Let  $T$  be a caterpillar of maximum degree  $\Delta$  with backbone  $P = (v_1, \dots, v_n)$ . Then  $\Gamma'(T) = \Delta + 1$  if and only if there is an edge  $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$  such that*

1. *one endvertex of  $e$  has degree  $\Delta$ , and*
2. *one of the two caterpillars  $T_e^-$  and  $T_e^+$  has a greedy edge-colouring such that the first edge of its backbone is coloured  $\Delta$  and the other has a greedy edge-colouring such that its first edge of its backbone is coloured  $\Delta - 1$ .*

*Proof.* Assume that  $T$  has a greedy  $(\Delta + 1)$ -edge-colouring. Let  $e$  be an edge coloured  $\Delta + 1$ . By Lemma 4,  $e$  is in the backbone and incident to a vertex of degree  $\Delta$ , proving (i). Moreover, the edge  $e$  is adjacent to an edge coloured  $\Delta$  and another one labelled  $\Delta - 1$ . Again by Lemma 4 these two edges must also be in the backbone. In particular,  $e$  is not  $v_1v_2$  nor  $v_{n-1}v_n$  because these two edges are adjacent to a unique edge of the backbone. Moreover the greedy edge-colourings induced on  $T_e^-$  and  $T_e^+$  clearly satisfy (ii).

Conversely, assume that there is an edge  $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$  satisfying (i) and (ii). Let  $\phi^-$  and  $\phi^+$  be the greedy edge-colourings of  $T_e^-$  and  $T_e^+$  respectively as in (ii). Let  $\phi$  be the edge-colouring of  $T$  defined by  $\phi(e) = \Delta + 1$ ,  $\phi(f) = \phi^-(f)$  for all  $f \in T_e^-$  and  $\phi(f) = \phi^+(f)$  for all  $f \in T_e^+$ . We claim that  $\phi$  is a greedy edge-colouring. Clearly, since  $\phi^-$  and  $\phi^+$  are greedy, it suffices to prove that  $e$  is adjacent to an edge of every colour  $i$  in  $\{1, \dots, \Delta\}$ . Since  $\phi^+$  and  $\phi^-$  satisfy (ii), then  $e$  is adjacent to an edge labelled  $\Delta$  and an edge labelled  $\Delta - 1$ . Now,  $e$  is incident to a vertex  $v$  of degree  $\Delta$ . This vertex is incident to  $e$  and an edge  $f$  in the backbone. The edge  $f$  is the first edge of a tree  $T_f$  in  $\{T_e^+, T_e^-\}$ . In the greedy edge-colouring of  $T_f$ , the edge  $f$  has a colour greater than  $\Delta - 2$ , so the  $\Delta - 2$  edges incident to  $v$  which are not  $e$  nor  $f$  have all one colour in  $1, \dots, \Delta - 2$ . Hence  $e$  is adjacent to an edge of every colour in  $\{1, \dots, \Delta\}$ .  $\square$

In view of this lemma, it is useful to decide when a caterpillar has a greedy edge-colouring with a prescribed colour on the first edge of its backbone.

**Lemma 7.** *Let  $T$  be a caterpillar with backbone  $P$  with first edge is  $e = uv$ . Then  $T$  has a greedy edge-colouring such that  $e$  is coloured  $k$  if and only if one of the following holds:*

1.  $d(u) \geq k$  or  $d(v) \geq k$ ;
2.  $d(u) = k - 1$  and  $T_e^+$  admits a greedy edge-colouring such that the first edge of  $P_e^+$  is coloured  $k - 1$ .

*Proof.* Let  $e = uv$  with  $u$  the first vertex of  $P$ . Assume first that  $T$  has a greedy edge-colouring such that  $e$  is coloured  $k$  and that  $e$  is incident to no vertex of degree  $k$ . Then the edges incident to  $u$  must be coloured by  $1, \dots, d(u) - 1$  and the edges incident to  $v$  and a leaf are coloured by  $1, \dots, d(v) - 2$ . Hence the edge incident to  $e$  and coloured  $k - 1$  must be the first edge of  $P_e^+$  is coloured  $k - 1$  by Property (P). So the edge incident to  $e$  and coloured  $k - 2$  must be incident to  $u$ , and thus  $d(u) - 1 \geq k - 2$ , that is  $d(u) \geq k - 1$ .

Assume now that (i) holds. Let  $x$  be a vertex in  $\{u, v\}$  with degree at least  $k$ . One can colour all the edges incident to  $x$  with  $1, \dots, d(x)$  such that  $e$  is coloured  $k$  and then extend this edge-colouring greedily to obtain the desired greedy edge-colouring of  $T$ .

Finally assume that (ii) holds. Let  $\phi$  be a greedy-edge colouring of  $T_e^+$  such that the first edge of  $P_e^+$  is coloured  $k - 1$ . One can extend it by assigning  $k$  to  $e$ ,  $1, \dots, k - 2$  to the  $k - 2$  edges incident to  $u$  and leaves. It is routine to check that this is a greedy-edge colouring of  $T$ .  $\square$

Lemma 7 translates easily in a linear-time recursive algorithm. Using this, and Lemma 6, one derive a linear-time algorithm for computing the Grundy index of a given caterpillar.

**Theorem 8.** *Determining the Grundy index of a caterpillar  $T$  can be done in  $O(|V(T)|)$ .*

*Proof.* Theorem 5 and Lemma 6 imply that Algorithm 1 return the Grundy index of  $T$  provided that we have a subroutine  $\text{FirstEdge}(T, P, k)$  that returns 'yes' if a caterpillar  $T$  with backbone  $P$  admits a greedy-edge colouring such that the first edge of  $P$  is coloured  $k$ .

**Algorithm 1: GrundyIndex(T)**

**Input:** A caterpillar  $T$ .

**Output:**  $\Gamma'(T)$ .

- (1) Let  $P = (v_1, v_2, \dots, v_n)$  be the backbone of  $T$ . Compute  $d(v_i)$  for all  $i \leq v_n$  and compute  $\Delta = \Delta(T)$ .
- (2) **for**  $i = 2$  to  $n - 2$  **do**
  - (3)  $e := v_i v_{i+1}$ ;
  - (4) **if**  $d(v_i) = \Delta$  or  $d(v_{i+1}) = \Delta$  **then**
    - (5) **if**  $\text{FirstEdge}(T_e^+, P_e^+, \Delta) = \text{TRUE}$  and  $\text{FirstEdge}(T_e^-, P_e^-, \Delta - 1) = \text{TRUE}$  **then**
    - (6) **return**  $\Delta + 1$ ;
    - (7) **if**  $\text{FirstEdge}(T_e^+, P_e^+, \Delta - 1) = \text{TRUE}$  and  $\text{FirstEdge}(T_e^-, P_e^-, \Delta) = \text{TRUE}$  **then**
    - (8) **return**  $\Delta + 1$ ;
- (9) **Return**  $\Delta$ ;

Such a subroutine  $\text{FirstEdge}$  may be obtained by Algorithm 2 according to Lemma 7.

**Algorithm 2: FirstEdge**

**Input:** A caterpillar  $T$  with backbone  $P$  and an integer  $k$ .

**Output:** *TRUE* if there is a greedy  $k$ -edge-colouring of  $T$  with first edge of  $P$  coloured  $k$ , and *FALSE* otherwise.

- (1) Let  $u$  be the first vertex of  $P$  and  $v$  its second. (So  $uv$  is the first edge.)
- (2) **If**  $d(u) \geq k$  or  $d(v) \geq k$  **then**
  - (3) **return** *TRUE*;
- (4) **If**  $d(u) \geq k - 1$  **then**
  - (5) **return**  $\text{FirstEdge}(T - u, P - u, k - 1)$ ;
- (6) **return** *FALSE*;

Let us now examine the complexity of Algorithm 1. Let us first observe that  $\text{FirstEdge}(T, P, k)$  makes a constant number of operations before calling  $\text{FirstEdge}(T - u, P - u, k - 1)$ . Hence an easy induction show that it makes  $O(k)$  operations in total.

Algorithm 1 first compute (line 1) the degrees of all the  $v_i$ , which can be done in time  $O(|V(T)|)$  and then takes the maximum of all this values which can also be done in time  $O(|V(T)|)$ .

In a second phase (line 2 to 8), for each edge  $e \in P$  which is incident to a vertex of degree  $\Delta$ , Algorithm 1 makes at most four calls of FirstEdge with last parameter  $\Delta - 1$  or  $\Delta$ . Hence for each  $e \in P$  it makes  $O(\Delta)$  operations. Let  $S$  be the set of vertices of degree  $\Delta$ , The number of edges of  $P$  incident to a vertex of degree  $\Delta$  is at most  $2|S|$ . But every vertex in  $S$  is adjacent to at least  $\Delta - 2$  leaves. Hence  $|V(T)| \geq |S| + (\Delta - 2)|S|$ , so  $|S| \leq |V(T)|/(\Delta - 1)$ . Hence, in this second phase, the algorithm makes at most  $O\left(2 \times \frac{|V(T)|}{\Delta - 1} \Delta\right) = O(|V(T)|)$  operations.

Thus, in total, Algorithm 1 makes  $O(|V(T)|)$  operations. □

## References

- [1] M. Asté, F. Havet and C. Linhares-Sales. Grundy number and products of graphs. *Discrete Mathematics* 310(9):1482–1490, 2010.
- [2] M. R. Garey and D. S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness*. A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979.
- [3] F. Havet and L. Sampaio. On the Grundy number of a graph. In Proceedings of the International Symposium on Parameterized and Exact Computation(IPEC), *Lecture Notes on Computer science* 6478, December 2010.
- [4] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Computing* 2:225–231, 1981.
- [5] C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. In *Proc. 25th ACM Symposium on Theory of Computing*, pages 286-293, 1993.
- [6] M. Zaker. Results on the Grundy chromatic number of graphs. *Discrete Math.* 306(23):3166–3173, 2006.



**RESEARCH CENTRE  
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-6399