Collusion of Operators in Wireless Spectrum Markets

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***Abstract*—The liberalization of wireless spectrum markets has been envisioned as a method for improving mobile communication services and accommodating the increasing traffic volumes of wireless users. It has been assumed that competition among operators will foster optimal utilization of wireless spectrum and ensure the provision of cost-efficient wireless services to users. However, spectrum markets often function inefficiently due to collusion of the operators. Although it is illegal and detrimental to users, such phenomena are often observed in real life as, for example, in the form of implicit price fixing. This results in a de-facto monopoly in the spectrum market to the detriment of the users. In this paper, we consider a general wireless spectrum market where a set of operators sell bandwidth to a large population of users. We use an evolutionary game model to capture the user dynamics in the presence of limited market information and analyze the interaction of the operators using coalitional game theory. We define a partition formation game in order to rigorously study the conditions that render the grand coalition (emergence of monopoly) stable under various stability notions. The results obtained provide a foundation for effective measures against operator collusion by altering the underlying motivations rather than fighting against the symptoms through law enforcement.**

Index Terms:Evolutionary Game Theory, Pricing, Coalitional Game Theory, Resource Allocation, Collusion.

I. Introduction

Competition of sellers in markets has been traditional envisioned as an ideal method for ensuring that clients will enjoy modest prices for products and services of high quality. In agreement with this view, the full liberalization of spectrum markets has been proposed as a method for managing the ever increasing volume of wireless data traffic, [1], [2]. It was expected that competition both for prices and quality of services would foster the effective allocation of spectrum and spectrum-enabled services in a cost-efficient fashion for the wireless users. Although to a large extent this expectation has been realized, notable cases of malfunctioning wireless spectrum markets have been observed due to collusion of the service providers (operators). These phenomena, despite being illegal and detrimental to users, are often observed in real life in the form of price fixing which yields an effective monopoly. Our goal is to model such markets and understand when and under what conditions do the operators collide.

We consider a **general wireless spectrum market** where a set of operators provide bandwidth to a large common pool of users, as it is depicted in Figure 1. The users select operators according to their net utilities, bounded below by a *reservation utility*. The latter represents the users' minimum requirements or, equivalently, an alternative method that satisfies their communication needs. For example, in a 3G wireless Internet market, the alternative choice may be to use a WiFi connection. We model this option through the *neutral* operator. The net utility users receive is the difference between the *valuation function*, which is monotonically increasing in the amount of the specific operator's total effective resource (bandwidth) divided by the number of users sharing them, and the price they pay to that operator. The operators obtain their revenue by selling bandwidth for the prices they choose, both of which are fixed in the time scale of user dynamics. Based on these prices -ideally- they compete with each other to attract users as customers.

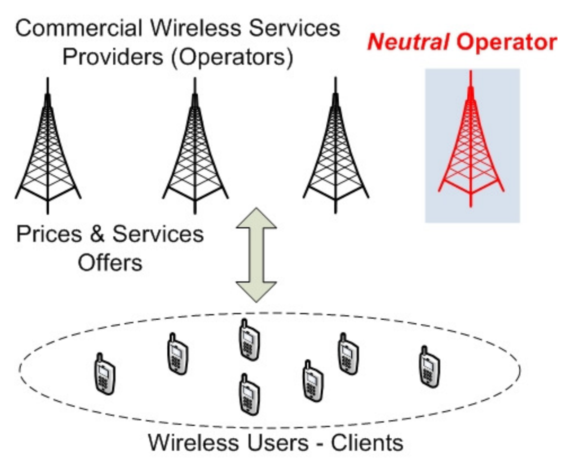

Fig. 1. Operators provide communication services of certain quality at different prices to a common pool of users. Neutral operator models the alternative choice that users have to satisfy their communication needs.

Given the complexity of this market, we adopt the model presented in [3] where the interaction between users and operators is captured through an evolutionary game. Instead of modeling individual users, which leads to an intractable complexity in analysis, we study dynamics of user populations distributed over the operators. These dynamics are a result of users selecting operators and updating their selection according to a specific hybrid revision protocol. This protocol is based partially on imitation of other users who receive a better net utility and partially on direct selection of the *neutral* operator. This alternative choice is actually equivalent to abstaining from the market and can be used as a criterion for quantifying the successful operation of the market.

The competitive nature of this market has been inherently assumed in [3]. However, the operators may have an incentive to collude and form *coalitions*, increasing the prices in secret agreement with each other in order to maximize their revenue, or even act as a de-facto monopoly by creating a *grand coalition*. While such price-fixing practices are against the law in many countries, history shows that law enforcement measures against them are often quite ineffective and hence not beneficial for the users. Investigating the motivation for building such coalitions may lead to solutions that remove them, and therefore address the problem at its source rather than fighting the symptoms. This paper aims to rigorously investigate the properties of potential collusion between operators and the conditions that motivate them.

A. Related Work and Contribution

The competition of sellers for attracting buyers has been studied extensively in the context of network economics, [4], [5], and recently for wireless services markets where operators fight to attract subscribers/clients, [6], [7], [8]. In many cases, the competition yields an undesirable outcome for the sellers. For example, in [9] the authors have shown that selfish pricing strategies of ISPs may decrease their accrued revenue. Similarly, in our previous work, [3], we have established that under certain conditions, the price competition of wireless operators may yield decreased revenue for them and, even worse, counterbalance any further CAPEX investments they make, such as buying more spectrum.

The analysis of collusion phenomena in such markets remains a quite unexplored area although our every day experience manifests that operators very often collude and provide non-differentiable services. The few existing studies either adopt quite abstract network models, e.g. [10], or analyze collusion from the perspective of the operators without assessing the impact of this phenomenon on the users, [6], [11]. In part, this shortage of works is due to the complexity and intractability of coalitional game theoretic models, [12]. In this work, we restrict our analysis to the case where operators collusion is realized by adopting identical prices, partly due to the covert nature of this activity. This way, we are able to use a rigorous model, based on coalitional game theory, while decreasing the complexity of the analysis and maintain its tractability. Moreover, we do not restrict our study to the stability investigation of the grand coalition, as it it the case in the vast majority of related works (e.g. [11]). On the contrary we analyze the non-cooperative interaction (competition) between different coalitions by using the theory of partition formation games.

In particular, **the main contributions of this work are** the following: **(i)** we develop and analyze a system model which accounts for the realistic aspects of large user population and limited information about user demand and operator's capacity, **(ii)** we model the operators collusion as a *partition formation game* with externalities which captures the competition among the different coalitions and their interdependency, **(iii)** we analyze the conditions that render operator collusion beneficial in terms of revenue, and **(iv)** we study the stability of the grand coalition under different *stability criteria*. **The latter two provide a foundation for creating measures against operator collusion, e.g. by a regulator on behalf of their customers.**

The rest of the paper is organized as follows. The next Section introduces the system model and presents the necessary background on the non-cooperative interaction of operators in this market. Section III analyzes the cooperative game among the operators and explores the stability of the grand coalition under different stability criteria. In Section IV we present representative numerical results that support our theoretical analysis, which is followed by the concluding remarks of Section V.

II. System Model and Background

We consider a wireless spectrum market with a very large set of users $\mathcal{N} = (1, 2, \ldots, N)$ and a set of operators $\mathcal{I} = (1, 2, \ldots, I)$ as depicted in Figure 2. Each user may either select one of the I operators or opt not to purchase services from any of them. The net utility perceived by each user who is served by operator i, is:

$$U_i(W_i, n_i, \lambda_i) = \log \frac{W_i}{n_i} - \lambda_i \tag{1}$$

where n_i is the number of the users served by this specific operator, W_i the total spectrum at his disposal, and λ_i the charged price. We assume that the effective resource of the operator is *on average* equally shared among his subscribers due to network management and load balancing. That is, the level of service users receive is the same for a certain type of service when averaged over time and location. Finally, we model each user's utility as a logarithmic function of the allocated resource in order to represent his perceived satisfaction saturation as the allocated resource increases. Notice that this type of concave functions are compliant with the economics' standard principle of diminishing marginal returns, [4], and has been extensively used to model the benefit of users in best-effort wired or wireless networks, [6].

The system operation is discrete and time slotted. In each slot $t = 1, 2, \ldots$, users have the opportunity to change their association and select another operator or decide not to purchase services from any of them. Due to the large number of users, the limited information about the market (e.g. unknown W_i) and their bounded rationality, each user updates his choice through

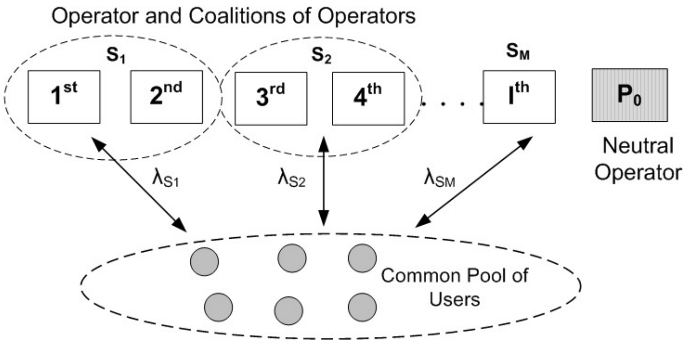

Fig. 2. The market consists of I operators and N users. Each user is associated with one operator and each operator can serve concurrently more than one users. Operators may form coalitions, $S_1, S_2, \ldots, S_M$ and fix their prices $\lambda_{S_1}, \lambda_{S_2}, \ldots, \lambda_{S_M}$. Users that fail to satisfy their minimum requirements U_0, abstain from the market and select the neutral operator P_0.

an evolutionary process which is mainly based on imitation of other users. Namely, the probability that a user associated with operator i will move to operator j, is [13]:

$$p_{ij}(t) = x_j(t)[U_j(t) - U_i(t)]_+ \tag{2}$$

where $x_j(t) = n_j/N$ is the portion of users already associated with the j^{th} operator at time slot t. Notice that, for simplicity, we express the user utilities as a function with a single argument, the time t.

At the same time, we assume that each user has a reservation utility of U_0 units that must be satisfied in order to agree to pay for the service. In case it is $U_i < U_0$, $\forall i \in \mathcal{I}$, the user abstains from the market. We model this option by using the *neutral* operator P_0, which is shown in Figure 2. The probability with which a user switches from operator i to P_0, is:

$$p_{i0}(t) = \gamma[U_0 - U_i(t)]_+ \tag{3}$$

P_0 captures many different aspects in this kind of markets. For example, in a WiFi Internet market, P_0 may represent the municipal WiFi provider that serves the citizens with a minimum service rate at no cost. Similarly, in a 3G wireless market, P_0 may represent the alternative choice of using a WiFi connection. From the technical point of view, incorporating P_0 in our model, allows us to calculate exactly how many users are not served by the I operators and hence we can assess the efficacy of the market.

In our previous work, [3], we proved that in this system the evolution of the user population that is associated with each operator reaches a stable point which depends on the vector of the prices set by the operators $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_I)$. Namely, according to the evolutionary game theoretic framework, [13], the users' strategy revision protocol described by the switching probabilities, $p_{ij}(t)$ and $p_{i0}(t)$, yields the following population dynamics:

$$\begin{aligned}\frac{\mathrm{dx_i(t)}}{\mathrm{dt}} &= x_i(t)[U_i(t) - U_{avg}(t) - x_0(t)(U_i(t) - U_0) \quad (4)\\ &- \gamma(U_0 - U_i(t))_+ + x_0(t)(U_i(t) - U_0)_+],\ \forall i \in \mathcal{I}\end{aligned}$$

where $U_{avg}(t) = \sum_{i\in\mathcal{I}} x_i(t)U_i(t)$ is the average utility of the market in each slot t. The user population associated with P_0 is:

$$\frac{\mathrm{dx_0(t)}}{\mathrm{dt}} = x_0 \sum_{i\in\mathcal{I}^+} x_i(U_0 - U_i) + \gamma \sum_{j\in\mathcal{I}^-} x_j(U_0 - U_j) \tag{5}$$

where $\mathcal{I}^+$ is the subset of operators offering utility $U_i(t) > U_0$, and $\mathcal{I}^-$ is the subset of operators offering utility $U_i(t) < U_0$, at slot t.

Solving these equations, we obtain the stationary state of the system $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_I^*)$ of the users distribution to the I operators. The dependency of $\mathbf{x}^*$ on vector λ gives rise to a non-cooperative price competition game where each operator $i \in \mathcal{I}$ selects the price that maximizes his revenue, $R_i = n_i^*\lambda_i = x_i^* N\lambda_i$. In [3] we expressed the optimal revenue of each operator $i \in \mathcal{I}$ as a function of his price λ_i and the prices set by the other operators $\lambda_{-i} = (\lambda_j : j \in \mathcal{I}\setminus\{i\})$:

$$R_i(\lambda_i, \lambda_{-i}) = \begin{cases} \dfrac{\alpha_i\lambda_i N}{e^{\lambda_i}\sum_{j=1}^{I}(\alpha_j/e^{\lambda_j})} & \text{if } \lambda_i < l_0, \\ \dfrac{\alpha_i\lambda_i N}{e^{\lambda_i}} & \text{if } \lambda_i \geq l_0. \end{cases} \tag{6}$$

where we have defined the scalar parameter $\alpha_i = W_i/Ne^{U_0}$, the respective vector $\alpha = (\alpha_i : i \in \mathcal{I})$ and parameter $l_0 = \log(\alpha_i/(1 - \sum_{j\neq i}\alpha_j/e^{\lambda_j}))$.

Apparently, the best response price λ_i^* depends both on vector λ_{-i} and on vector α:

$$\lambda_i^* = arg\max_{\lambda_i} R_i(\lambda_i, \lambda_{-i}, \alpha) \tag{7}$$

$$\lambda_i^*(\lambda_{-i}, \alpha) = \begin{cases} 1 & \text{if } (1, \lambda_{-i}) \in \Lambda_A, \\ \mu_i^* & \text{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ l_0 & \text{otherwise.} \end{cases} \tag{8}$$

where μ_i^* is the unconstrained optimal point of the upper function case in (6). The price sets Λ_A and Λ_B are

$$\Lambda_A = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} < 1\right\} \tag{9}$$

and

$$\Lambda_B = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} > 1\right\} \tag{10}$$

Under certain conditions, the competition may yield decreased revenue for the competing operators, [3]. Additionally, in case of fierce competition, the revenue of operators does not increase even if they make further CAPEX investments and increase - for example - their spectrum W. This fact suggests that operators may decide to collude and jointly determine their pricing strategy in the market, in order to increase their revenue. In the next section, we employ coalitional game theory and study collusion of operators in the context of the wireless services market of Figure 2.

III. Collusion of Operators

Assume that a group of operators form a coalition which we denote S_k, $S_k \subseteq \mathcal{I}$, and agree to set the same price λ_{s_k}, Figure 2. The potential of each coalition S_k is quantified by using the following two metrics: (i) the number $|S_k|$ of the participating operators and, (ii) the sum of their α parameters, $A_k = \sum_{i \in S_k} \alpha_i$. A coalition acts as a single operator with $\alpha = A_k$. According to (6), the revenue of operator i that participates in coalition S_k, is:

$$R_i(\lambda_{s_k}) = \begin{cases} \dfrac{\alpha_i \lambda_{s_k} N}{A_k + e^{\lambda_{s_k}} \sum_{j \in \mathcal{I} \setminus S_k} \alpha_j / e^{\lambda_j}} & \text{if } \lambda_{s_k} < l_0(S_k), \\ \dfrac{\alpha_i \lambda_{s_k} N}{e^{\lambda_{s_k}}} & \text{if } \lambda_{s_k} \geq l_0(S_k). \end{cases} \tag{11}$$

where $l_0(S_k) = \log(A_k/(1 - \sum_{j \in \mathcal{I} \setminus S_k} \alpha_j / e^{\lambda_j}))$ is a parameter that depends on the specific coalition S_k. Apparently, this revenue depends on the prices set by the operators that do not belong in S_k, $\lambda_{\bar{s_k}}$.

In the case there exist more than one coalitions, $\{S_1, S_2, ..., S_K\}$, $K > 1$, they compete with each other in a similar fashion that the independent operators were competing with each other before their cooperation emerges. In particular, the non-cooperative game among the coalitions is identical to the price competition game defined in [3] with K operators where $\alpha_k = A_k$ for $k = 1, 2, ..., K$. Therefore, the optimal price $\lambda^*_{s_k}$ for each coalition S_k can be found by using equation (8), after substituting i with S_k, α_i with A_k and λ_{-i} with $\lambda_{\bar{s_k}}$.

A price vector $\lambda = (\lambda^*_{s_1}, \lambda^*_{s_2}, \ldots, \lambda^*_{s_K})$ is a Nash Equilibrium (NE) when operators in all coalitions set their best response prices $(\lambda^*_{s_k})$ simultaneously. In [3], we proved the existence of pure NE, and the convergence to it by proving that the competition game is a potential game. The revenue of each operator $i \in S_k$ in the NE, $R^*_{i \in S_k}$, can be computed by substituting NE prices to the function $R_{i \in S_k}$ given in eq. (11). Here we want to stress that, according to the previous analysis, the price and the revenue of each operator at the NE, depends not only on the size of his coalition, but also on the structure of the coalitions that are formed by the rest operators. This *externality* affects the strategy of the operators.

On the other hand, when all operators collude and form one single coalition, the so-called *grand coalition*, i.e. $S_1 = \mathcal{I}$, they set the same price $\lambda_{\mathcal{I}}$, and each one of them accrues revenue, (11):

$$R_i(\lambda_{\mathcal{I}}) = \begin{cases} \dfrac{\alpha_i \lambda_{\mathcal{I}} N}{\sum_{j=1}^{I} \alpha_j} & \text{if } \lambda_{\mathcal{I}} < \log(\sum_{j=1}^{I} \alpha_j), \\ \dfrac{\alpha_i \lambda_{\mathcal{I}} N}{e^{\lambda_{\mathcal{I}}}} & \text{otherwise.} \end{cases} \tag{12}$$

Notice that operators within the same coalition receive different revenue due to their different α parameters. When the *grand coalition* is formed, the operators do not compete with each other, cooperate and act as one single operator. This results in the emergence of a monopolistic market where the optimal price, $\lambda_{\mathcal{I}}$, depends only on the α_i values:

$$\lambda^*_{\mathcal{I}} = \begin{cases} \log(\sum_{i=1}^{I} \alpha_i) & \text{if } \log(\sum_{i=1}^{I} \alpha_i) > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

Now we ask the following questions: **(i)** do operators have an incentive to collude and create an oligopolistic or even a monopolistic market? **(ii)** Under which conditions does collusion increases the revenue of the operators? The answers to these questions are significant from a regulatory perspective since they enable the derivation of methods that promote competition in this kind of markets.

A. Coalitional Game for Colluding Operators

Coalition formation games in economic environments with externalities are conventionally modeled as *Partition Function Games (PFG)* which were first introduced in [14]. Given a partition Π of $\mathcal{I}$ and a coalition $S_k \in \Pi$, the pair $(S_k; \Pi)$ is called an embedded coalition of $\mathcal{I}$. The set of all embedded coalitions is denoted by $EC(\mathcal{I})$. Now, we model the operators coalitional game, $\mathcal{G}_{\mathcal{C}} = (\mathcal{I}, \mathbf{v})$, in PFG format, as follows:

- The set of players is the set of the I operators $\mathcal{I} = (1, 2, ..., I)$.
- $\phi_i(S_k; \Pi) = R^*_{i \in S_k}$ is the equilibrium payoff of any operator i in coalition S_k within a partition Π.
- $\mathbf{v}(\cdot)$ is the function that assigns to every embedded coalition $(S_k; \Pi) \in EC(\mathcal{I})$, a set of values $\{\phi_1(S_k; \Pi), ..., \phi_{|S_k|}(S_k; \Pi)\}$.

We assume that there is no central authority coordinating the interactions of the operators, and also that each operator is a selfish revenue maximizing entity. Therefore, the coalition formation is accomplished in a distributed fashion and hence there is a need to study the respective distributed algorithms. In a distributed coalition formation algorithm, there are three main components as it is discussed in [16] and [12]: (1) well defined orders suitable to compare collections of coalitions, (2) two rules for merging and splitting coalitions, and (3) adequate notions for assessing the stability of a partition. In the sequel, we describe these components for the operators coalition game $\mathcal{G}_{\mathcal{C}}$.

As it is explained below, $\mathcal{G}_{\mathcal{C}}$ is a game with non-transferable utility (NTU). Therefore, rather than the aggregate utility of the players in a coalition, we have to focus and consider the individual utilities. We use Pareto order [16] as a comparison metric between two collections of coalitions. Given two collections S and T, S is preferred over T by Pareto order if at least one player in S improves his payoff without hurting other players. We define merging and splitting rules for forming and breaking coalitions, according to Pareto order, as follows:

- *Merge Rule*: Any collection of disjoint coalitions $S = \{S_1, ..., S_K\}$ can agree to merge into single coalition $T = \cup_{k=1}^{K} S_k$, if the new coalition T is preferred by all the operators over the previous collection of coalitions S according to the Pareto order. In other words, T is preferred over S if at least one operator in S improves his revenue without decreasing the revenue of any other operator.

- *Split Rule*: A single coalition T splits into a collection of disjoint coalitions $S = \{T-R, R\}$ if operators in coalition R prefer S over T according to the Pareto order. Here, we assume that one or more operators can leave the coalition and form a new coalition if it is to their advantage, although this may have an impact on the revenue of other operators in the initial coalition.

The fundamental properties of $\mathcal{G}_{\mathcal{C}}$ are the following:

- **Property 1:** $\mathcal{G}_{\mathcal{C}}$ is a coalitional game with **non-transferable utility (NTU)**. This is due to the fact that utility transfer agreements and revenue sharing are mostly prohibited by laws.
- **Property 2:** If the NE price vector $\lambda^* \notin \Lambda_B$, there is no incentive for operators to collude. Therefore, in this study, we restrict our analysis to the cases where λ^* belongs to Λ_B.
- **Property 3:** $\mathcal{G}_{\mathcal{C}}$ is a coalitional game with **positive** externalities. In other words, a merger between two coalitions always make other coalitions better off.
- **Property 4:** Operators in coalition $(S_i;\Pi)$ get lower revenue per unit spectrum compared to the operators in coalition $(S_k;\Pi)$, if $A_i > A_k$. This property, together with the **Property 3**, may create an incentive for some players to free ride.
- **Property 5:** When two or more coalitions merge, NE prices of all operators increase. There is also a direct proportionality between aggregate revenues of operators in a coalition and their prices. This property suggest that operators have higher incentive to collude when there is no upper-bound on the set prices.
- **Property 6:** $\mathcal{G}_{\mathcal{C}}$ is not always super-additive. This is due to the fact that operators with small α values may obtain lower revenue when merging with a coalition with large A value. Although the aggregate revenue of the coalition increases (after merging), the individual share of the newly merging operator may decrease (due to Property 4).

Detailed proofs of the above properties are provided in the Appendix.

B. Stability Analysis

In games with externalities, when a group of players decides to deviate, they should take into account and anticipate the reaction of the external players. Different assumptions of the behavior of external players give rise to different notions of stability. In the coalitional game theory literature (such as [17] and [18]), there exist various notions of stability depending on the assumed reaction model of the external players. The most restricted stability concept is *core stability*. A partition Π is said to be *core-stable*, if no group of players has an incentive to deviate, even if they consider that external players react in such a way as to maximize the payoff of deviators. On the other extreme, according to the *α-stability* notion, no group of players deviates unless it is guaranteed to obtain higher payoff independently of the reaction of the external players. Between these extremes, *δ-stability* and *γ-stability* are defined. In δ model, players belonging to coalitions that lose some of their members remain together as smaller coalitions. In γ model, coalitions that lose some of their members assumed to dissolve and form singletons. Another and more natural expectation of the deviating players is that external players will take their deviation as given and try to maximize their own payoff through merge and split operations. This is called rational expectations [15] and the respective stability notion can be named as *r-stability*. In this paper, we define deviation of a group of players as simply merging or splitting. We study the stability of the grand coalition, and obtain the following results:

Lemma III.1. *The grand-coalition is α-stable and γ-stable if and only if the following condition holds:*

$$\phi_k(\mathcal{I}) \geq \phi_k(\{k\};[\mathcal{I}]), \text{ where } k = \arg\min_i \alpha_i. \tag{14}$$

$[\mathcal{I}]$ denotes the partition of $\mathcal{I}$ to singletons.

Lemma III.2. *The grand-coalition is core-stable, δ-stable and r-stable if the following condition holds:*

$$\phi_k(\mathcal{I}) \geq \phi_k(\{k\};\{\{k\},\mathcal{I}-\{k\}\}), \ \forall k \in \mathcal{I}. \tag{15}$$

Note that $\phi_k(\mathcal{I}) = R_k(\log(\sum_{i=1}^{I}\alpha_i))$ is the revenue of operator k in grand coalition, given by (12).

Proofs of the above lemmas are given in the Appendix. Intuitively, grand coalition is not α-stable and γ-stable if the operator with smallest α_i value can gain more when all the external operators dissolve and form singletons after his defection. We observed that this situation is very unlikely to happen, except for some highly asymmetric markets, and grand coalition is mostly α-stable and γ-stable.

On the other hand, grand coalition is core-stable, δ-stable and r-stable if none of the operators can gain more by defecting while the other operators do not dissolve. Although we haven't provided a formal proof yet, our observations (also intuition from Property 4) show that the operator with smallest α_i value has the highest willing to defect from a coalition. Therefore, it seems to be sufficient to check the inequality given in (15) for the one with smallest α_i, rather than all the operators.

If the condition (15) does not hold, the grand coalition is neither core-stable nor δ-stable. However, it can be still r-stable, since after defection of an operator, another rational operator may also decide not to stay in the coalition. This may reduce the revenue of the first defecting operator, probably to lower level than its initial revenue. Hence, an operator with rational expectations will not defect from the grand coalition in such cases. One example is given in the next section (see Figure 4).

Similar conditions can be derived for coalition structures other than the grand coalition. There are some cases, where none of the coalition structures are core-stable, δ-stable or r-stable. In the next section, we give an example to illustrate one of these cases. More detailed analysis will be handled in a future work. Existence and structural characteristics of stable coalitions are of interest from the regulatory perspective.

IV. Numerical Study

For the sake of clarity, first we consider a symmetric spectrum market with four operators, where each operator has the same amount of spectrum. In Figure 3, the first two plots show two possible coalition formation steps when $\alpha_i = 2e$. In the both cases, grand coalition is the only stable coalition (in any stability notion), since it provides highest possible revenue for all operators. This result coincides with the fact that condition given in (15) is satisfied.

The third plot in Figure 3 shows coalition formation when $\alpha_i = e$. The output of the coalition game remains same as the previous case, for all coalition structures except the grand coalition. Condition (15) is not satisfied and grand coalition is not core-stable, δ-stable and r-stable. One of the operators can increase his revenue by leaving the coalition if the other operators act rational, and not split after his defection. On the other hand, the grand coalition is α-stable and γ-stable, since the deviating operator cannot increase his revenue if the external operators dissolve and form singletons.

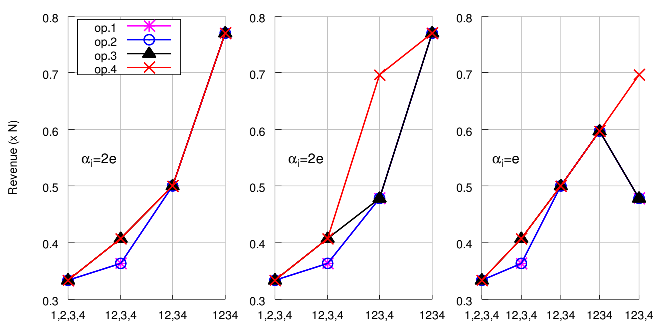

Fig. 3. Revenue for the four operators for changing coalition structures. In the first two figures, $\alpha_i = 2e$ for all operators. In the last figure $\alpha_i = e$. Adjacent numbers in the x-axis stand for coalitions, e.g. 1234 is the grand coalition.

Next, we consider an asymmetric market with three operators, where the first two operators have $\alpha_1 = \alpha_2 = e$ amount of spectrum in their disposal, while the third operator has six times more ($\alpha_3 = 6e$) amount of spectrum. Figure 4 shows possible coalition formation steps. In this example, grand coalition is not core-stable or δ-stable, since operator 1 can gain more by defecting, given that the other operators not split after his defection. However, if operator 1 has rational expectations, he would expect that, after his defection, operator 2 would not stay in the coalition with operator 3 any more in order to increase his revenue, and act alone. In that case, operator 1 would not gain more than his revenue in the grand coalition. Therefore, he would not defect and the grand coalition is r-stable.

Another observation from Figure 4 is that none of the coalition structures are core-stable or δ-stable. Hence, optimistic operators cannot form a stable coalition.

The second plot in Figure 4 illustrates the revenues per unit spectrum, instead of total revenues of operators. One observation is that, operators with smaller α_i values gain more revenue per unit spectrum than the operators with higher α_i values. It is also clearly visible how operators with smaller α_i values are apt to defect from large coalitions and free ride. It can be easily verified that all the examples given in this section support the properties defined in Section III.B.

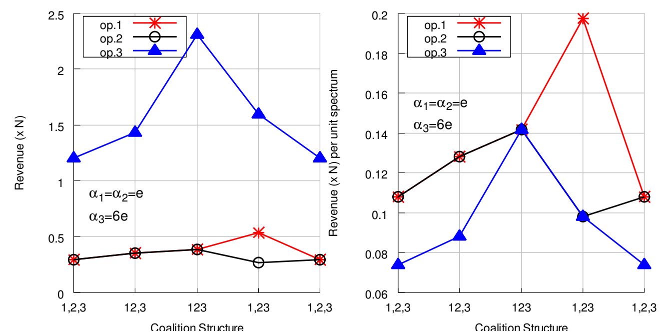

Fig. 4. The first figure illustrates revenues of three operators for changing coalition structures. The second figure illustrates the revenue per unit spectrum for the same case. First two operators have the same amount of spectrum ($\alpha_1 = \alpha_2 = e$), while the third one has six times more spectrum.

V. Conclusions

We have presented a general wireless spectrum market where a set of operators sell bandwidth to a large population of users. Using an evolutionary game model, we have captured the user dynamics in the presence of limited market information and analyzed the interaction of the operators using coalitional game theory. Specifically, we have defined a partition formation game and studied rigorously the conditions that render the grand coalition (emergence of monopoly) stable under various stability notions. Identifying how the spectrum owned by the operators (W_i) and the reservation utility (U_0), are related to the stability of de-facto monopoly, we have obtained a rigorous framework for developing regulation methods to spur competition via adjusting these parameters. Likewise, we have provided mathematical tools to analyze how an upper-bound on the prices may prevent collusion of operators.

Operator collusion is against the law in many countries, yet the history shows that law enforcement measures against them are often quite ineffective. Our results, which quantify the motivation for building such coalitions, may lead to solutions that remove them, and therefore address the problem at its source rather than fighting the symptoms. Such regulatory schemes and their analysis constitutes a promising research direction.

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Appendix

Lemma A.1. *Operators have incentive to collude only when NE price vector* $\lambda \in \Lambda_B$*.*

Proof: If $\lambda \in \Lambda_A$, all the operators already set the same price, i.e. $\lambda_i^* = 1$. On the other hand, if $\lambda \in \Lambda_C$, operators cannot yield a NE in Λ_B by changing their prices, as described in [3]. Hence by colluding, operators may only give rise to another NE in Λ_C which cannot be considered as an improvement.

This lemma proves Property 2 of $\mathcal{G}_\mathcal{C}$. ■

All of the lemmas below are for the case where the NE is in Λ_B.

Lemma A.2. $\mathcal{G}_\mathcal{C}$ *is a coalitional game with positive externalities.*

Proof: We prove this lemma in two steps.

Step 1: First we prove that, higher the NE price, higher the aggregate NE revenue of a coalition.

Let us denote aggregate revenue of coalition S_k with R_{s_k}. When the NE is in Λ_B, it is given as:

$$R_{s_k}^* = \frac{A_k \lambda_{s_k}^* N}{e^{\lambda_{s_k}^*} \sum_{j=1}^{I} (\frac{\alpha_j}{e^{\lambda_j^*}})} = \frac{A_k \lambda_{s_k}^* N}{e^{\lambda_{s_k}^*} \sum_{j \in \mathcal{I} \setminus S_k} (\frac{\alpha_j}{e^{\lambda_j^*}}) + A_k}. \quad (16)$$

NE price of each coalition satisfies the following equation:

$$\frac{\mathrm{dR_{s_k}}(\lambda_{\mathrm{s_k}})}{\mathrm{d}\lambda_{\mathrm{s_k}}} = 0 \Rightarrow e^{\lambda_{s_k}^*}(\lambda_{s_k}^* - 1) = \frac{A_k}{\sum_{j \in \mathcal{I} \setminus S_k} \frac{\alpha_j}{e^{\lambda_j^*}}} \quad (17)$$

Using equations (16) and (17),

$$R_{s_k}^* = \frac{A_k \lambda_{s_k}^* N}{\frac{e^{\lambda_{s_k}^*} A_k}{e^{\lambda_{s_k}^*}(\lambda_{s_k}^* - 1)} + A_k} = N(\lambda_{s_k}^* - 1) \quad (18)$$

Step 2: Second, we prove that, when two operators (or operator coalitions) collude, new NE prices of all operators are greater than their initial NE prices.

Suppose that operators in S_i and operators in S_j collude and form a coalition S_k. Their NE prices before merging are $\lambda_{s_i}^*$ and $\lambda_{s_j}^*$ respectively. Their new NE prices satisfy the equation (17). In [3], we proved that pricing game between competing operators is a potential game. Therefore, the game would converge to NE when operators adopt best response prices sequentially. Newly colluding operators adopt the best response strategy first. Since $A_k > A_i$, and $\sum_{j \in \mathcal{I} \setminus S_k} \alpha_j / e^{\lambda_j} < \sum_{j \in \mathcal{I} \setminus S_i} \alpha_j / e^{\lambda_j}$, best response price of coalition S_i increases due to the following:

$$e^{\lambda_{s_k}^*}(\lambda_{s_k}^* - 1) = \frac{A_k}{\sum_{j \in \mathcal{I} \setminus S_k} \frac{\alpha_j}{e^{\lambda_j}}} > \frac{A_i}{\sum_{j \in \mathcal{I} \setminus S_i} \frac{\alpha_j}{e^{\lambda_j}}} = e^{\lambda_{s_i}^*}(\lambda_{s_i}^* - 1) \quad (19)$$

Similarly,

$$e^{\lambda_{s_k}^*}(\lambda_{s_k}^* - 1) > e^{\lambda_{s_j}^*}(\lambda_{s_j}^* - 1) \quad (20)$$

After coalitions S_i and S_j increase their prices, the other operators (or operator coalitions) also increase their prices sequentially according to (17). Therefore all the operators increase their prices until reaching to a new NE.

This constitutes the final step of our proof. Since, all prices are higher in the new NE, revenues of the external operators increase. This means that $\mathcal{G}_\mathcal{C}$ is a coalitional game with positive externalities.

This lemma proves Property 3 and Property 5 of $\mathcal{G}_\mathcal{C}$. ■

Lemma A.3. *In the NE, operators in coalition* $(S_i;\Pi)$ *get lower revenue per unit spectrum compared to the operators in coalition* $(S_k;\Pi)$*, if* $A_i > A_k$*.*

Proof: Suppose that operator $i \in S_i$ and operator $k \in S_k$. Using equation (11), ratio between revenues per unit spectrum for operators i and k is:

$$\frac{R_i/\alpha_i}{R_k/\alpha_k} = \frac{\lambda^*_{s_i} e^{\lambda^*_{s_k}}}{\lambda^*_{s_k} e^{\lambda^*_{s_i}}} \tag{21}$$

Since NE prices in Λ_B are always greater than one, above ratio is smaller than one (which means operator i has lower revenue per unit spectrum) when $\lambda^*_{s_i} > \lambda^*_{s_k}$.

Now, the lemma will be proven after proving that $\lambda^*_{s_i} > \lambda^*_{s_k}$ if $A_i > A_k$. Suppose that $A_i > A_k$, but $\lambda^*_{s_i} \le \lambda^*_{s_k}$. Then,

$$e^{\lambda^*_{s_i}}(\lambda^*_{s_i} - 1) \le e^{\lambda^*_{s_k}}(\lambda^*_{s_k} - 1) \tag{22}$$

On the other hand,

$$\frac{A_i}{\sum_{j \in \mathcal{I} \setminus S_i} \frac{\alpha_j}{e^{\lambda^*_j}}} > \frac{A_k}{\sum_{j \in \mathcal{I} \setminus S_k} \frac{\alpha_j}{e^{\lambda^*_j}}} \tag{23}$$

However, if both (22) and (23) hold, then eq. (17) cannot be satisfied for $\lambda^*_{s_i}$ and $\lambda^*_{s_k}$ simultaneously. Hence the market cannot be in NE, which is a contradiction.

This lemma proves Property 4 of $\mathcal{G}_{\mathcal{C}}$. ■

Proof of Lemma III.1. First we prove that a deviating operator gets the worst possible revenue if the external players also leave the coalition and form singletons. This is due to Property 3 of $\mathcal{G}_{\mathcal{C}}$, i.e. it is a game with positive externalities. An operator always gains more revenue when the others merge, and it loses when the others split.

Now, according to the Lemma A.3, when all the operators form singletons, the operator with smallest α_i value would get highest revenue per unit spectrum. Therefore, if the operator with lowest α_i value does not have incentive to deviate, none of the operators would deviate (given that the external players form singletons). Therefore γ-stability and α-stability is ensured if and only if the operator with lowest α_i cannot gain more revenue by deviating from the grand coalition, given that the external players form singletons, which proves the lemma.

Proof of Lemma III.2. According to Property 3 of $\mathcal{G}_{\mathcal{C}}$, a deviating operator gets maximum possible revenue if none of the external operators leave the coalition after him. Therefore, if he cannot gain more (compared to his revenue in grand coalition) in that case (where external operators do not deviate), he cannot gain more in any other case. Therefore, core-stability, δ-stability and r-stability is ensured if condition (15) is satisfied.