Nomographic Gossiping for f-Consensus

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***Abstract*—In this paper, we present a novel class of iterative gossip algorithms called *nomographic gossiping* that partly allow to efficiently achieve a rapid global consensus among nodes/agents in a clustered wireless network with respect to an *arbitrary function* of the initial states. The algorithms are based on the surprising fact that *every* real-valued multivariate function has a nomographic representation, which is simply a function of a superposition of a finite number of univariate functions. Since superpositions can be effectively generated via the wireless channel by letting nodes in a cluster transmit simultaneously their pre-processed states to a cluster head, the convergence speed can be significantly increased provided that some connectivity condition between clusters is fulfilled.**

I. Introduction

Efficiently reaching a rapid consensus among nodes or agents in decentralized wireless networks constitutes one of the fundamental problems in distributed signal processing and optimization [1], [2], where efficiency refers to the economical use of wireless resources (e.g., bandwidth, energy). A distinction is made between unconstrained consensus problems in which the state of all nodes should asymptotically be the same, and constrained consensus problems, where the nodes have to achieve a global agreement with respect to some function f of the initial states. Therefore, a constrained consensus problem is often referred to as an *f-consensus problem* [1]. A popular special case of this distributed computation problem is the average consensus problem in which f is simply chosen to be the arithmetic mean.

In this context, gossip algorithms have received a great deal of attention in recent years because they allow nodes to distributively achieve a global consensus without any complicated routing protocol by letting nodes locally exchange data with their nearest neighbors (see [2]–[5] and references therein). Consequently, each node has only a local view on the network dynamics and therefore does not have any information about the remaining nodes apart from its neighbors.

To enable gossip algorithms the operation in a fully distributed manner, randomized approaches are of particular interest. For example, Boyd et al. presented in [4] an extensive framework for the design and the analysis of randomized gossip algorithms for average consensus in arbitrary connected networks, where pairs of nodes are chosen randomly to exchange their current states. Since then, the algorithms and results of [4] are extended in many different ways to improve the relatively moderate convergence rates. Aysal et al. realized for example in this context in [6] that at a cost of a single transmission, more than a single nearby node can benefit due to the inherent broadcast nature of the wireless channel. On the other hand, since averaging is in the core superposition, Nazer at el. designed in [7] a randomized gossip algorithm that exploits the superposition property of wireless multiple-access channels to allow instantaneous averages over a larger neighborhood, rather than between pairs of nodes. But what if a consensus with respect to a nonlinear function is desired?

To solve the problem of efficiently reaching a global rapid consensus with respect to an *arbitrary* function of the initial states in a wireless network that is organized into overlapping clusters, we introduce in this paper a novel class of iterative gossip algorithms summarized as *nomographic gossiping*. The algorithms rely on the fact that *every* real-valued multivariate function has a nomographic representation (see Definition 6). Since a nomographic representation is simply a function of a superposition of a finite number of univariate functions, such functions can be effectively computed exploiting the natural superposition property of wireless multiple-access channels by letting nodes in a cluster transmit simultaneously their pre-processed initial states to a designated cluster head [8], [9]. This constitutes a kind of a paradigm shift, since in classical communication theory channel collisions are generally avoided by costly coordination.

Under the assumption of error-free local computations, to focus in this paper on presenting the general idea of nomographic gossiping, the proposed class of algorithms consists of randomized as well as of deterministic protocols. While the described randomized algorithm, which is a generalization of superposition gossiping [10] to any f-consensus problem, requires infinitely many steps to converge to the function of interest, at the cost of some coordination the deterministic approach always converges to the exact consensus in a finite number of iterations.

A. Related Work

To the best of our knowledge, the work of Kirti et al. [11] and Nazer et al. [7] are the first in which the multiple-access property of the physical layer in wireless networks is profitably

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be exploited to efficiently achieve a network-wide average consensus. Besides the fact that the work in the present paper is primarily focused on achieving a consensus with respect to an arbitrary function by performing local analog computations in the sense of [8], [9], the authors in [7] consider reliable local averaging over a noisy multiple-access channel through the use of structured computation codes.

Other publications in which the authors consider distributed computations of nonlinear functions are for example [12] and [13]. In [12], Aoyama and Shah propose an iterative distributed algorithm for rapidly computing separable functions of the initial states. Separable functions are multivariate functions that can be represented as sums of univariate functions, why the corresponding function space constitutes only a small subset of the space of nomographic functions, considered in this paper. In contrast, Sundaram and Hadjicostis use in [13] a dynamical system approach to develop a linear iterative strategy that enables nodes to compute arbitrary functions.

B. Contributions

The contributions of the paper are summarized as follows:

- We recap the notion and the basic properties of nomographic functions from [9] and explain why the structure of these functions can be useful to efficiently solve arbitrary f-consensus problems.
- Then, we propose a deterministic and a randomized nomographic gossip algorithm for solving f-consensus problems in wireless networks in which nodes are organized into overlapping clusters.
- Theorem 3 states that deterministic nomographic gossiping always reaches a global consensus in a finite number of iterations, where the consensus can be chosen to be any multivariate function.
- Theorem 4 states that randomized nomographic gossiping convergence almost surely to any continuous nomographic function (i.e., to nomographic functions that consist of continuous components).

C. Notational Remarks

The k-times cartesian product $\mathbb{A} \times \cdots \times \mathbb{A}$ of a space $\mathbb{A}$ is written as $\mathbb{A}^k$. The natural, nonnegative integer and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R} = (-\infty, \infty)$, respectively, and $\mathbb{E} := [0, 1] \subset \mathbb{R}$ defines the closed unit interval. Let $\mathbb{A}^k$, $k \in \mathbb{N}$, be a compact metric space, then $\mathcal{C}[\mathbb{A}^k]$ denotes with the infinity norm $\|\cdot\|_\infty$ the Banach space of real-valued continuous functions of $k \in \mathbb{N}$ variables, defined on $\mathbb{A}^k$. Finally, $\mathcal{F}[\mathbb{A}^k]$ denotes the space of any function $g : \mathbb{A}^k \subseteq \mathbb{R}^k \to \mathbb{R}$.

II. System Model and Problem Statement

A. Network Model

Consider a wireless network (WN) consisting of $N \in \mathbb{N}$ spatially distributed nodes that are organized into a finite set $C := \{C_1, \dots, C_{|C|}\}$ of time-invariant single-hop clusters, where C_i denotes the set of nodes belonging to cluster i, $i = 1, \dots, |C|$, and $|C_i| = N_i \in \mathbb{N}$ the corresponding number of nodes, respectively. The nodes in a cluster are arbitrarily numbered such that $C_i = \{0_i, \dots, N_i - 1\}$, with 0_i the label of the i^{th} designated cluster head (see Fig. 1).

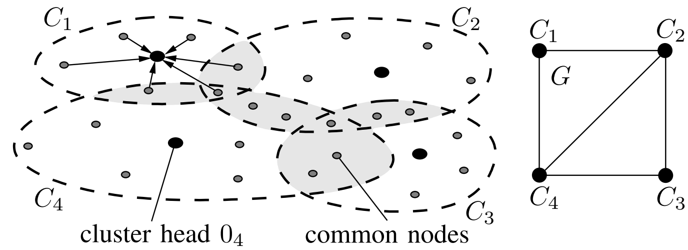

Fig. 1. A qualitative example of a connected clustered wireless network, and its associated graph G, consisting of $|C| = 4$ clusters and $N = 25$ nodes.

Definition 1 (Connected Clusters). Two clusters in a clustered WN, say C_i and C_j, $i, j = 1, 2, ..., |C|$ and $i \neq j$, are called *connected* if C_i and C_j share at least one common node (i.e., $C_i \bigcap C_j \neq \varnothing$).

Definition 2 (Connected Clustered WN). A clustered WN is called *connected* if for any two clusters C_i and C_j, $i \neq j$, there exists a sequence of connected clusters from C_i to C_j.

Definition 3 (Associated Graph). We call the finite undirected graph $G = (C, E)$ with vertex set C (i.e., the set of clusters) and edge set E as the graph *associated* to the clustered WN. Clusters C_i and C_j are connected if and only if $(i, j) \in E$, and if the clustered WN is connected, then G is connected.

Let the nodes possess any initial states $x_n(0) \in \mathbb{E}$, $n = 1, \dots, N$, such that the *initial state* of the network can be summarized in the vector $\boldsymbol{x}(0) := [x_1(0), \dots, x_N(0)]^T \in \mathbb{E}^N$. Then, the intra-cluster communication can usually be described by the standard affine model of a wireless multiple-access channel (MAC) [14]. More precisely, if cluster C_i, $i = 1, \dots, |C|$, is active at time $t \in \mathbb{Z}_+$, the real-valued signal received by cluster head 0_i can be written as

$$y_{0_i}(t) = \sum_{n \in C_i \setminus \{0_i\}} h_{in}(t) x_n(t) + v_i(t) \ , \tag{1}$$

where $h_{in} \in \mathbb{R}$ denotes a flat-fading coefficient between node n and cluster head 0_i and $v_i \in \mathbb{R}$ is receiver noise at 0_i, respectively. If we ignore fading and noise, (1) reduces to an *ideal MAC*

$$y_{0_i}(t) = \sum_{n \in C_i \setminus \{0_i\}} x_n(t) \ , \tag{2}$$

which reveals that the natural mathematical operation of a wireless MAC is simply *superposition*.

B. The f-Consensus Problem

Definition 4 (Desired Consensus). Any function $f : \mathbb{E}^N \to \mathbb{R}$ of the initial state $(x_1(0), \dots, x_N(0)) \in \mathbb{E}^N$, (i.e., $f(x_1(0), \dots, x_N(0))$), is called *desired consensus*.

Definition 5 (f-Consensus Problem). Let f be the desired consensus. Then, the *f-Consensus Problem* is defined as the requirement that the states of all nodes in the network asymptotically become equal to f (i.e., $\forall n \in C : \lim_{t\to\infty} \big\| x_n(t) - f(x_1(0), \ldots, x_N(0)) \big\| = 0$, with $\| \cdot \|$ a certain distance).

A naive approach to solve any f-consensus problem is to disseminate all initial states $x_n(0)$, $n \in C$, throughout the entire network such that each node knows $\boldsymbol{x}(0)$. Since such a flooding-like solution would be very inefficient regarding the amount of interference avoiding coordination, we introduce in the following section a class of gossip algorithms that exploit the superposition property (2) as well as the broadcast property of wireless channels to solve f-consensus problems much more efficiently.

Remark 1. Although practical gossip algorithms suffer from fading, receiver noise, synchronization issues, we focus in this paper on the principal of nomographic gossiping and consider therefore ideal superpositions (2) only. Extensions to realistic MACs (1) will be part of future work.

Remark 2. Note that considering initial states that are drawn from the canonical unit interval $\mathbb{E}$ generates no loss in generality since all results in this paper are also valid if states are from an arbitrary compact metric space.

III. Nomographic Gossiping

As already mentioned, most of the work on gossip algorithms are focused on the average consensus problem, which means with respect to Definition 5 that the desired consensus is chosen to be $f(x_1(0), \ldots, x_N(0)) = \frac{1}{N}\sum_{n=1}^{N} x_n(0)$. Let $\{\varphi_n\}_{n=1}^N$ be certain *pre-processing functions* $\varphi_n : \mathbb{E} \to \mathbb{R}$, $n = 1, \ldots, N$, operating on the initial states (i.e., $\varphi_n(x_n(0))$). Then, it is obvious that gossip algorithms that are proven to achieve an average consensus can also be used to distributively solve every f-consensus problem with f chosen from the function space

$$\mathcal{S}[\mathbb{E}^N] := \left\{ f \in \mathcal{F}[\mathbb{E}^N] \,\middle|\, \exists(\varphi_1, \ldots, \varphi_N) \in \mathcal{F}[\mathbb{E}] \times \cdots \right.$$
$$\left. \cdots \times \mathcal{F}[\mathbb{E}] : f(x_1, \ldots, x_N) = \sum_{n=1}^{N} \varphi_n(x_n) \right\}. \quad (3)$$

Elements of $\mathcal{S}[\mathbb{E}^N]$ are often referred to as separable functions (or sum separable functions) [12] and although $\mathcal{S}[\mathbb{E}^N]$ contains many functions of high practical relevance [2], to the best of our knowledge, a rigorous characterization of (3) is still missing in gossiping literature.[1] However, (3) is merely a subset of the space of so called nomographic functions, defined as follows.

Definition 6 (Nomographic Functions). Let $\mathbb{A}^k$, $k \geq 2$, be a metric space. Then, every $f \in \mathcal{F}[\mathbb{A}^k]$ for which a representation

$$f(x_1, \ldots, x_k) = \psi\left(\sum_{n=1}^{k} \varphi_n(x_n)\right) \quad (4)$$

exists, with $\psi \in \mathcal{F}[\mathbb{R}]$ and $\varphi_n \in \mathcal{F}[\mathbb{A}]$, for all $n = 1, \ldots, k$, is called *nomographic function* and we denote the corresponding function space as $\mathcal{N}[\mathbb{A}^k]$.

Example 1. (i) *Arithmetic Mean:* $f(x_1, \ldots, x_N) = \frac{1}{N}\sum_n x_n$, with $\varphi_n(x) = x$, for all $n = 1, \ldots, N$, and $\psi(y) = y/N$. (ii) *Euclidean Norm:* $f(x_1, \ldots, x_N) = \sqrt{x_1^2 + \cdots + x_N^2}$, with $\varphi_n(x) = x^2$, for all $n = 1, \ldots, N$, and $\psi(y) = \sqrt{y}$. (iii) *Number of Active Nodes:* $f(x_1, \ldots, x_N) = N$, with $\varphi_n(x) = 1$, for all $x \in \mathbb{E}$ and $n = 1, \ldots, N$, and $\psi(y) = y$.

Remark 3. Nomographic functions owe their name from nomographs, which are graphical representations often used in engineering to solve certain types of equations [15].

The following surprising theorem provides a complete characterization of the space of nomographic functions and introduces the meaning of *universal pre-processing functions* (i.e., pre-processing functions which are independent of f).

Theorem 1 (Buck'79 [16]). Every $f \in \mathcal{F}[\mathbb{E}^k]$, $k \geq 2$, is nomographic (i.e., $\mathcal{N}[\mathbb{E}^k] \equiv \mathcal{F}[\mathbb{E}^k]$), and the pre-processing functions $\{\varphi_n\}_{n=1}^N$ can be chosen to be independent of f.

Since nomographic functions are in the core *superpositions* of univariate functions, the basic idea behind *nomographic gossiping* is to exploit besides Theorem 1 the analog superposition property (2) of the wireless channel in each cluster to significantly improve the efficiency compared to standard interference avoiding protocols (e.g., time-division multiple access (TDMA)). This approach merges the processes of local computation and communication (i.e., within clusters) such that for nomographic gossiping, the nodes belonging to an active cluster transmit *simultaneously* their current states to the cluster head, which subsequently *broadcasts* the received superposition back to update the entire cluster. In other words, a network-wide rapid consensus with respect to an $f \in \mathcal{F}[\mathbb{E}^N]$ can be achieved by a certain number of superposition and broadcast phases as long as the clustered network is connected according to Definition 2. Finally, if the state of all nodes converged to $\sum_{n=1}^{N} \varphi_n(x_n(0))$, the application of the so called *post-processing function* ψ results in the desired consensus.

In what follows, we interpret and denote a single superposition in combination with the subsequent broadcast phase as an *elementary step*.

It should be emphasized that the universality of pre-processing functions (see Theorem 1) is one of the great advantages of nomographic gossiping. Precisely, the universality property means that there exist $(\varphi_1, \ldots, \varphi_N) \in \mathcal{F}[\mathbb{E}] \times \cdots \times \mathcal{F}[\mathbb{E}]$ such that for every $f \in \mathcal{F}[\mathbb{E}^N]$ there is a $\psi \in \mathcal{F}[\mathbb{R}]$ with $f(x_1(0), \ldots, x_N(0)) = \psi\big(\sum_{n\in C} \varphi_n(x_n(0))\big)$. Consequently, an update of the pre-processing functions $\{\varphi_n\}_{n=1}^N$ is not necessary if f changes due to for example a change of the application.

This universality is not shared by algorithms that solve f-consensus problems with $f \in \mathcal{S}[\mathbb{E}^N]$ since in this case, $\{\varphi_n\}_{n=1}^N$ can not be universally chosen. Efficiently coordinating updates of $\{\varphi_n\}_{n=1}^N$ if f changes would indeed be a

[1] A rigorous characterization would answer the question which multivariate functions are sum separable and thus which f-consensus problems are solvable by algorithms designed for achieving an average consensus.

problem in itself for corresponding algorithms.

So far, no demands on the properties of pre- and post-processing functions were posed, such that they could be chosen arbitrarily. Since continuity is a property that can be beneficial for practical implementations, we denote in the following in contrast to $\mathcal{N}[\mathbb{E}^N]$ the space of nomographic functions with continuous pre- and post-processing functions by $\mathcal{N}_0[\mathbb{E}^N]$. The next theorem makes immediately clear how this property affects the statement of Theorem 1.

Theorem 2 (Buck'82 [17]). The space of nomographic functions is nowhere dense in the space of continuous functions (i.e., $\mathcal{N}_0[\mathbb{E}^N]$ nowhere dense in $\mathcal{C}[\mathbb{E}^N]$).

Even if, according to Theorem 2, continuity reduces the amount of functions representable in the form (4), there are some *nomographic approximations* to functions of practical relevance that are not in the space $\mathcal{N}_0[\mathbb{E}^N]$. Here, a nomographic approximation to a function $f \in \mathcal{C}[\mathbb{E}^N]$, with regard to the infinity norm, means that there exist continuous pre- and post-processing functions such that [9]

$$\Big\| f - \psi\Big(\sum_n \varphi_n(x_n)\Big)\Big\|_\infty \le \varepsilon\ . \tag{5}$$

Example 2. Let $\varepsilon > 0$ be arbitrary but fixed and let $p_0(\varepsilon)$ be chosen such that (5) is fulfilled for all $p \ge p_0(\varepsilon)$. (i) *Geometric Mean:* $f(x_1,\dots,x_N) = \big(\prod_n x_n\big)^{1/N} \approx \psi\big(\sum_n \varphi_n(x_n)\big)$, with $\varphi_n(x) = \log_{\mathrm{e}}(x + 1/p_0(\varepsilon))$, for all $n = 1,\dots,N$, and $\psi(y) = \exp(y/N)$. (ii) *Maximum Value:* $f(x_1,\dots,x_N) = \max_{1\le n\le N}\{x_n\} \approx \psi\big(\sum_n \varphi_n(x_n)\big)$, with $\varphi_n(x) = x^{p_0(\varepsilon)}$, for all $n = 1,\dots,N$, and $\psi(y) = y^{\frac{1}{p_0(\varepsilon)}}$. (iii) *Minimum Value:* $f(x_1,\dots,x_N) = \min_{1\le n\le N}\{x_n\} \approx \psi\big(\sum_n \varphi_n(x_n)\big)$, with $\varphi_n(x) = \frac{1}{x^{p_0(\varepsilon)}}$, for all $n = 1,\dots,N$, and $\psi(y) = y^{-\frac{1}{p_0(\varepsilon)}}$.

After we discussed some preliminaries on nomographic functions, we develop in the next two subsections concrete nomographic gossip algorithms that essentially differ in the way clusters are activated (e.g., deterministic or random).

A. Deterministic Nomographic Gossiping

1) Algorithm: Let there any connected clustered WN be given and let $f \in \mathcal{N}[\mathbb{E}^N]$ be the desired consensus with $(\varphi_1,\dots,\varphi_N,\psi) \in \mathcal{F}[\mathbb{E}]\times\cdots\times\mathcal{F}[\mathbb{E}]\times\mathcal{F}[\mathbb{R}]$ such that $f(x_1(0),\dots,x_N(0)) = \psi(\sum_{n\in C}\varphi_n(x_n(0)))$ (Theorem 1 ensures the existence). For deterministic nomographic gossiping, we assume that there is a fixed activation sequence $\pi(t) \in \{1,\dots,|C|\}$ be given that uniquely determines which cluster has to be active at time $t \in \mathbb{Z}_+$ and we suppose that $\pi(t)$ defines a repeated closed walk on the associated graph G.[2] In what follows, the length of the closed walk measured in the number of elementary steps is denoted by $\Pi \in \mathbb{N}$ such that $\pi(k\Pi + 1) = \pi((k+1)\Pi)$, for all $k \in \mathbb{Z}_+$.

Each node in the network is equipped with a transmission counter $c_n(t)$, $n = 1,\dots,N$, which is set to zero prior to

[2]Activation of a cluster at time $t \in \mathbb{Z}_+$ means for instance that cluster head $0_{\pi(t)}$ becomes active and wakes up all remaining nodes in $C_{\pi(t)}$.

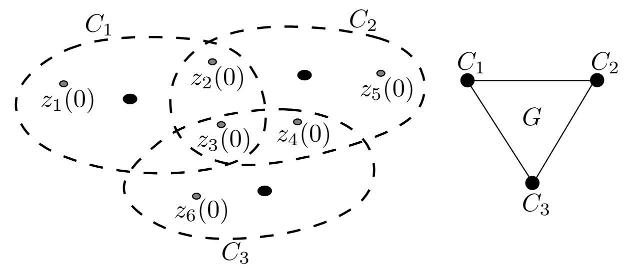

Fig. 2. A simple example of a connected clustered network and the associated graph.

running the algorithm (i.e., $\forall n \in C : c_n(0) = 0$). If for example the i^{th} node transmits at $t \in \mathbb{Z}_+$, the corresponding transmission counter increases by one such that $c_i(t) = c_i(t-1) + 1$. Furthermore, all nodes in the network know if they are common nodes or not (see Fig. 1) and we assume that common nodes belonging to clusters $C_{\pi(t-1)}$ and $C_{\pi(t)}$ are aware of the cardinalities $|C_{\pi(t-1)} \cap C_{\pi(t)}|$, for all $t \in \mathbb{Z}_+$. Having these assumptions in mind, the corresponding iterative deterministic gossiping procedure is described in Algorithm 1.

Remark 4. Note that in Algorithm 1, all standard nodes (i.e., nodes that never belong to more than one cluster) transmit only once. This means, if the transmission counter of a standard node indicates 1, the node does not participates in any further simultaneous transmission, which can be beneficial in terms of energy consumption.

2) A Simple Example: To illustrate the functioning of deterministic nomographic gossiping, we describe in the following Algorithm 1 by using the simple network example depicted in Fig. 2. Therefore, let $f \in \mathcal{F}[\mathbb{E}^6]$ be the desired consensus and let $(\varphi_1,\dots,\varphi_6,\psi) \in \mathcal{F}[\mathbb{E}]\times\cdots\times\mathcal{F}[\mathbb{E}]\times\mathcal{F}[\mathbb{R}]$ be such that $f\big(x_1(0),\dots,x_6(0)\big) = \psi\big(z_1(0)+\cdots+z_6(0)\big)$, with $z_n(0) := \varphi_n\big(x_n(0)\big)$ the pre-processed initial state stored at node n, $n = 1,\dots,6$.

Suppose that the activation sequence is given to be $(\pi(1),\pi(2),\dots) = (1,2,3,1,2,3,1,2,3,\dots)$, forming a repeated closed walk on the associated graph with $\Pi = 3$. Then, at $t = 1$, cluster $C_{\pi(1)} = C_1$ is activated and since any transmission counter is still zero, all nodes belonging to C_1 transmit simultaneously to cluster head 0_1, resulting in the receive signal $y_{0_{\pi(1)}}(1) = \varphi_1\big(x_1(0)\big) + \varphi_2\big(x_2(0)\big) + \varphi_3\big(x_3(0)\big)$. The subsequent broadcast updates all states in C_1 to $z_1(1) = z_2(1) = z_3(1) = y_{0_{\pi(1)}}(1)$. Finally, all nodes belonging to C_1 increase their transmission counter by 1.

At $t = 2$, cluster $C_{\pi(2)} = C_2$ is activated and it is obvious that C_1 and C_2 share $|C_1 \cap C_2| = 2$ common nodes, both with current state $y_{0_{\pi(1)}}(1)$. In order not to falsify the forthcoming superposition, they have to appropriately weight their current states. Hence, the simultaneous transmission of all nodes in the cluster leads to receive signal $y_{0_{\pi(2)}}(2) = \frac{1}{|C_1\cap C_2|}\,[z_2(1) + z_3(1)] + z_4(0) + z_5(0) = \sum_{n=1}^{5}\varphi_n\big(x_n(0)\big)$, and the subsequent broadcast to the updated states $z_n(2) =$

Algorithm 1 Deterministic Nomographic Gossiping

Initialization: Let $t = 0$ and let $\boldsymbol{x}(0) \in \mathbb{E}^N$ be any initial state of the network. Then, all nodes set their transmission counter to zero (i.e., $\forall n \in C : c_n(0) = 0$) and pre-process their initial states such that $\boldsymbol{z}(0) := [z_1(0) = \varphi_1(x_1(0)), \ldots, z_N(0) = \varphi_N(x_N(0))]^\mathsf{T} \in \mathbb{R}^N$ is the initial state of the algorithm.
while $t \leq 2\Pi - 3$ **do**
  Cluster head $0_{\pi(t)}$ wakes up all nodes in cluster $C_{\pi(t)}$
  if $t = 1$ **then**
    All nodes $n \in C_{\pi(1)}$ simultaneously transmit their initial states $z_n(0)$ to cluster head $0_{\pi(1)}$, resulting in receive signal

$$y_{0_{\pi(1)}}(1) = \sum_{n \in C_{\pi(1)}} z_n(0) \tag{6}$$

  else
    Prior to transmission, each node $n \in C_{\pi(t-1)} \cap C_{\pi(t)}$ weights their current state by $\frac{1}{|C_{\pi(t-1)} \cap C_{\pi(t)}|}$
    All nodes $n \in C_{\pi(t-1)} \cap C_{\pi(t)}$, as well as all remaining nodes in cluster $C_{\pi(t)}$ for which $c_n(t) = 0$ holds, simultaneously transmit their current state to cluster head $0_{\pi(t)}$, resulting in receive signal

$$y_{0_{\pi(t)}}(t) = \frac{1}{|C_{\pi(t-1)} \cap C_{\pi(t)}|} \sum_{n \in C_{\pi(t-1)} \cap C_{\pi(t)}} z_n(t-1) + \sum_{n \in C_{\pi(t)} \setminus C_{\pi(t-1)} \cap C_{\pi(t)} : c_n(t) = 0} z_n(t-1) \tag{7}$$

  end if
  $c_n(t) = c_n(t-1) + 1$, for all $n \in C_{\pi(t)}$ that contributed to (6) or (7)
  if $t < \Pi - 1$ **then**
    Cluster head $0_{\pi(t)}$ broadcasts receive signal (6) or (7), resulting in state updates

$$z_n(t) = \begin{cases} y_{0_{\pi(t)}}(t) & , n \in C_{\pi(t)} \\ z_n(t-1) & , n \notin C_{\pi(t)} \end{cases} \tag{8}$$

  else
    Cluster head $0_{\pi(t)}$ applies the post-processing function to receive-signal (7) and broadcasts the result such that

$$z_n(t) = \begin{cases} \psi\big(y_{0_{\pi(t)}}(t)\big) = f\big(\boldsymbol{x}(0)\big) & , n \in C_{\pi(t)} \\ z_n(t-1) & , n \notin C_{\pi(t)} \end{cases} \tag{9}$$

  end if
  $t = t + 1$
end while

$y_{0_{\pi(2)}}(2)$, for all $n \in C_2$.

At $t = 3 = \Pi - 1$, because of the corresponding superimposed signal $y_{0_{\pi(3)}}(3) = \frac{1}{|C_2 \cap C_3|}[z_3(2) + z_4(2)] + z_6(0) = \sum_{i=1}^{6} \varphi_n\big(x_n(0)\big)$, the application of the post-processing function ψ at cluster head 0_3 in conjunction with the subsequent broadcast results in state updates $z_n(3) = \psi\big(y_{0_{\pi(3)}}(3)\big) = f\big(x_1(0), \ldots, x_6(0)\big)$, $n \in C_3$. Hence, the nodes in cluster C_3 already reached the desired consensus.

Now, the common nodes spread the information back to the remaining clusters by the receive signals $y_{0_{\pi(4)}}(4) = \frac{1}{|C_1 \cap C_3|} z_3(3) = z_3(3) = f\big(x_1(0), \ldots, x_6(0)\big)$ for $t = 4$ and $y_{0_{\pi(5)}}(5) = \frac{1}{|C_1 \cap C_2|}[z_2(4) + z_3(4)] = f\big(x_1(0), \ldots, x_6(0)\big)$ for $t = 5$, respectively. The final broadcast leads then to the global f-consensus $z_n(5) = f\big(x_1(0), \ldots, x_6(0)\big)$, for all $n \in C$, in merely $2\Pi - 3 = 5$ elementary steps.

Remark 5. Algorithm 1 can be varied in different ways. For example, with respect to saving energy, common nodes can be prioritized in the sense that in each overlap $C_{\pi(t-1)} \cap C_{\pi(t)}$, $t \in \mathbb{Z}_+$, a unique node $n^\star$ is declared the leader that participates in (7) while the other common nodes are silent. In this case, (7) has the form

$$y_{0_{\pi(t)}}(t) = z_{n^\star}(t-1) + \sum_{n \in C_{\pi(t)} \setminus C_{\pi(t-1)} \cap C_{\pi(t)} : c_n(t) = 0} z_n(t-1).$$

B. Randomized Nomographic Gossiping

As for deterministic nomographic gossiping, let there any connected clustered WN be given and let $f \in \mathcal{N}[\mathbb{E}^N]$ be the desired consensus with pre- and post-processing functions such that $f(x_1(0), \ldots, x_N(0)) = \psi(\sum_{n \in C} \varphi_n(x_n(0)))$. We assume that each cluster head has a clock that ticks independently (over time and across clusters) at a rate $\mu_i \in \mathbb{R}_+$ Poisson process, $i = 1, \ldots, |C|$. The μ_i are chosen such that in a sufficiently small time interval and with high probability two cluster heads do not wake up simultaneously [10].

Prior to running the algorithm, all nodes compute their initial pre-processed state $z_n(0) = \varphi_n(x_n(0))$, $n = 1, \ldots, N$. Then, if cluster C_i, $i \in \{1, \ldots, |C|\}$, randomly wakes up at time t, the corresponding nodes transmit simultaneously their current state values to the cluster head, which subsequently computes the local average by dividing the receive signal by $N_i - 1$. Finally, the broadcast of this intermediate result updates the entire cluster. If the state at each node converged to $\bar{\varphi} := \frac{1}{N}\sum_{k=1}^{N} \varphi_k(x_k(0))$, applying $\psi(N\bar{\varphi})$ results in the desired global consensus f.

A formal description of the iterative nomographic gossiping procedure is given in Algorithm 2.

IV. Convergence Analysis

In this section we study the convergence properties of Algorithms 1 and 2 for connected clustered wireless networks and discuss some advantages and disadvantages.

A. Deterministic Nomographic Gossiping

Intuitively, the rate of convergence of nomographic gossip algorithms will significantly depend on the clustering (i.e., on the properties of the associated graph). Therefore, to obtain explicit convergence results for deterministic nomographic gossiping, which would be valid for arbitrary connected associated graphs, is difficult. In order to still make a useful

Algorithm 2 Randomized Nomographic Gossiping

Initialization: Let $t = 0$, $\varepsilon > 0$ be a desired accuracy and let $\boldsymbol{x}(0) \in \mathbb{E}^N$ be any initial state of the network. Let $f \in \mathcal{N}[\mathbb{E}^N]$ be the desired consensus and let $(\varphi_1, \ldots, \varphi_N, \psi) \in \mathcal{C}[\mathbb{E}] \times \cdots \times \mathcal{C}[\mathbb{E}] \times \mathcal{C}[\mathbb{R}]$ such that $f(\boldsymbol{x}(0)) = \psi\left(\sum_{n \in C} \varphi_n(x_n(0))\right)$. Then, all nodes pre-process their initial states such that $\boldsymbol{z}(0) := [z_1(0) = \varphi_1(x_1(0)), \ldots, z_N(0) = \varphi_N(x_N(0))]^\mathsf{T} \in \mathbb{R}^N$ is the initial state of the algorithm.

while $\exists n \in C : \|\psi(Nz_n(t)) - \psi(Nz_n(t-1))\| \geq \varepsilon$ **do**

  The clock of cluster head $i \in \{1, \ldots, |C|\}$ ticks at random

  Cluster head 0_i wakes up all nodes in cluster $C_i \in C$

  Nodes $1_i, \ldots, N_i$ of C_i transmit their pre-processed current states simultaneously to cluster head 0_i, resulting in receive signal

$$y_{0_i}(t) = \sum_{n \in C_i \setminus \{0_i\}} z_n(t-1) \tag{10}$$

  Cluster head 0_i computes the local average

$$z_{0_i}(t) = \frac{1}{N_i - 1} y_{0_i}(t) \tag{11}$$

  Cluster head 0_i broadcasts the local average $z_{0_i}(t)$, resulting in state updates

$$z_n(t) = \begin{cases} z_{0_i}(t) & , \ n \in C_i \\ z_n(t-1) & , \ n \notin C_i \end{cases}, \tag{12}$$

  $n = 1, \ldots, N$

  $t = t + 1$

end while

statement, we consider connected clustered networks with associated graphs that have a Hamilton cycle, defined as follows [18].

Definition 7 (Hamilton Cycle). A cycle in a graph is called *Hamilton cycle* if it visits each vertex exactly once.

Definition 8 (Hamilton Graph). A graph is called Hamilton graph, if it has a Hamilton cycle.

Theorem 3. Consider any connected clustered wireless network consisting of $|C| < \infty$ clusters and let $f \in \mathcal{F}[\mathbb{E}^N]$ be any desired consensus. Then, Algorithm 1 always converges to the exact desired consensus in a finite number of elementary steps. If the associated graph G is a Hamilton graph, then the convergence can be achieved in at most $2|C| - 1$ elementary steps.

Proof: Since any connected clustered WN with $|C| < \infty$ has a finite connected associated graph, there exists at least one closed walk on G that is finite as well. Hence, let the activation sequence π be chosen such that it describes such closed walk on G and let Π denote the corresponding finite length (i.e., the number of elementary steps) of the walk in the sense that $C_{\pi(1)} = C_{\pi(\Pi)}$, with $C_{\pi(1)}$ the starting vertex. Then, from the description of Algorithm 1 it is easy to see that deterministic nomographic gossiping converges to the desired consensus in at most $2(\Pi - 1) - 1 < \infty$ elementary steps. If the associated graph is hamiltonian, it has a Hamilton cycle and the length of a Hamilton cycle in a Hamilton graph is equal to the number of vertices such that we conclude $\Pi - 1 = |C|$ and $2|C| - 1$ for the number of elementary steps until convergence. ■

Remark 6. The example of a chain of $|C|$ connected clusters shows that the hamiltonian property of the associated graph is sufficient to achieve convergence in at most $2|C| - 1$ elementary steps but not necessary. It is well known that determining whether or not a given graph with $|C| \geq 3$ is hamiltonian constitutes a serious problem. However, there exist several sufficient conditions for the existence of a Hamilton cycle [18], which can be helpful for designing a clustering that enables convergence in $2|C| - 1$ elementary steps. The associated graphs depicted in Figures 1 and 2 are Hamilton graphs, possessing multiple Hamilton cycles.

Remark 7. It should be emphasized, that the rate of convergence in Theorem 3 is independent of the chosen desired consensus (i.e., independent of f).

The convergence of deterministic nomographic gossiping to the exact desired consensus in a finite number of iterations comes at the cost of the fact that all nodes in the network require a certain amount of knowledge about the network topology, the clustering and the activation sequence. In practice, a system designer has therefore to trade off between rate of convergence and amount of coordination.

B. Randomized Nomographic Gossiping

In Algorithm 2, clusters randomly wake up due to an asynchronous time model so that in contrast to the deterministic case, a coordinated activation procedure is superfluous. On the other hand, the resulting distributed nature of randomized nomographic gossiping implies that statements regarding convergence can only be made in some probabilistic sense so that the distance in Definition 5 has to be chosen appropriately.

Theorem 4. Let f be any desired consensus from $\mathcal{N}_0[\mathbb{E}^N]$. Then, for any connected clustered wireless network, Algorithm 2 converges to the desired consensus almost surely.

Proof: Let $\boldsymbol{z}(0) \in \mathbb{R}^N$ be the vector of pre-processed initial states $z_n(0) := \varphi_n(x_n(0))$ and let $\bar{z} := \frac{1}{N}\sum_{n=1}^{N} \varphi_n(x_n(0))$ be the corresponding average. Then, with $(\varphi_1, \ldots, \varphi_N) \in \mathcal{C}[\mathbb{E}] \times \cdots \times \mathcal{C}[\mathbb{E}]$ we can conclude from [10, Theorem 2] that $\boldsymbol{z}(t)$ converges to $\bar{z}\mathbf{1}_N$ almost surely as t tends to infinity (i.e., $\mathbb{P}(\lim_{t\to\infty} \boldsymbol{z}(t) = \bar{z}\mathbf{1}_N) = 1$). Now, if ψ is continuous on $\mathbb{R}$, we have with the Mann-Wald Theorem [19] that $\boldsymbol{z}(t) \overset{\text{a.s.}}{\to} \bar{z}\mathbf{1}_N \Rightarrow \left(\psi(Nz_1(t)), \ldots, \psi(Nz_N(t))\right) \overset{\text{a.s.}}{\to} \psi\left(\sum_{n \in C} \varphi_n(x_n(0))\right)\mathbf{1}_N = f(x_1(0), \ldots, x_N(0))\mathbf{1}_N$, which proves the result. ■

Remark 8. Note that compared with Theorem 3, the convergence of randomized nomographic gossiping essentially

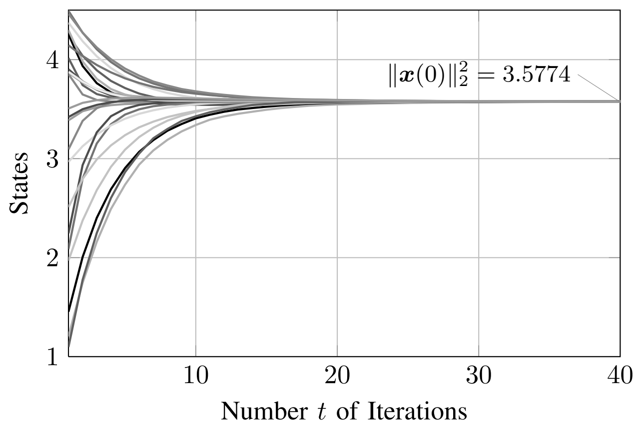

Fig. 3. The mean state evolutions (averaged over 10^3 independent realizations) $z_n(t)$, $n = 1, \ldots, 25$, of the network depicted in Fig. 1 using *randomized nomographic gossiping*. The desired consensus is chosen to be the "Euclidean norm" of the initial state $\boldsymbol{x}(0)$, where $\boldsymbol{x}(0)$ uniformly drawn from $\mathbb{E}^N$.

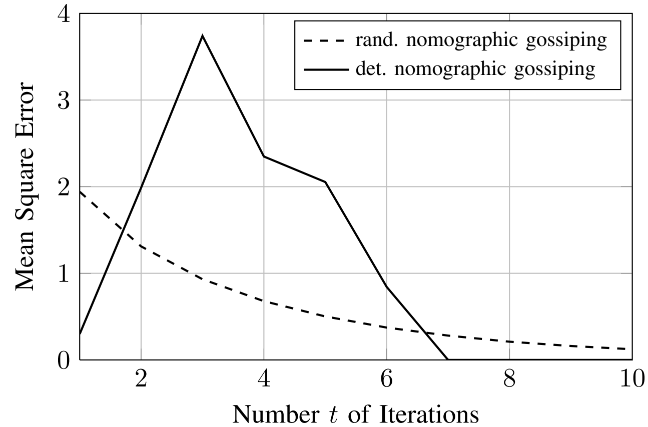

Fig. 4. Mean square error over number of iterations (i.e., elementary steps) for deterministic and randomized nomographic gossiping, using the network example depicted in Fig. 1. The desired consensus is chosen to be "geometric mean" and the initial states are independent and uniformly drawn from $(0, 1]$.

requires continuous pre- and post-processing functions, which means a significant limitation according to Theorem 2 even if there are some nomographic approximations of functions that are not in $\mathcal{N}_0[\mathbb{E}^N]$ (see Example 2).

V. Numerical Examples

In the following, we present some numerical examples to demonstrate the huge potential of nomographic gossiping. Starting with an f-consensus example that illustrates the convergence behavior of Algorithms 1 and 2, we subsequently compare the algorithms with the well established pairwise gossiping from [4] and the broadcast gossiping protocol proposed in [6], respectively.

Consider the connected clustered network from Fig. 1, consisting of $N = 25$ nodes and $|C| = 4$ clusters. According to Definitions 7 and 8 we conclude that the depicted associated graph is hamiltonian, possessing multiple Hamilton cycles. Corresponding examples of activation sequences that form repeated Hamilton cycles of length $\Pi = 5$ on G are $\pi_1(t) = 1, 2, 3, 4, 1, 2, 3, 4, 1, \ldots$ and $\pi_2(t) = 1, 3, 2, 4, 1, 3, 2, 4, 1, \ldots$, respectively. In what follows, we choose without loss in generality $\pi_1(t)$, $t \in \mathbb{Z}_+$, for deterministic nomographic gossiping.

Now, consider an arbitrary but fixed initial state $\boldsymbol{x}(0) \in \mathbb{E}^N$. Fig. 3 shows the corresponding mean state evolutions of all nodes in the network (averaged over 10^3 independent realizations), with the desired consensus chosen to be the "Euclidean norm" from Example 1 (ii). Even after 20 iterations/elementary steps, all states are on average within ± 0.03 of the actual Euclidean norm $\|\boldsymbol{x}(0)\|_2^2 = 3.5774$.

Fig. 4 depicts a comparison of the mean square error performance of deterministic nomographic gossiping and randomized nomographic gossiping, where in contrast to the previous example, the desired consensus is now chosen to be the "geometric mean" from Example 2 (i) with an arbitrary but fixed initial state $\boldsymbol{x}(0) \in (0, 1]^N$. From Theorem 3 we conclude a convergence to the exact desired consensus in $2|C| - 1 = 7$ elementary steps, which is confirmed by the solid plot. Even if the convergence of randomized nomographic gossiping requires infinitely many steps to achieve the exact desired consensus, the mean square error is in the depicted example significantly smaller within iterations $2 - 6$.

Remark 9. Note that the choice of the activation sequence does not impact the convergence speed as long as the sequence describes a repeated closed walk on the associated graph. However, the choice may impact the magnitude of the mean square error during state evolutions such that given a connected clustered WN, the Algorithm could be optimized over all existing activation sequences that form repeated closed walks on G.

As mentioned at the beginning of this section, we finally compare Algorithms 1 and 2 with the standard benchmarks from [4] and [6]. The corresponding numerical data is depicted in Fig. 5, where the desired consensus is chosen to be the arithmetic mean from Example 1 (i). The plots show the huge improvements concerning the convergence rate.

VI. Conclusions

In this paper we presented a class of iterative gossip algorithms for achieving a network-wide consensus in a clustered wireless network with respect to functions of the initial state values. The algorithms rely on the fact that every real-valued multivariate function has a nomographic representation, which is simply a function of a superposition of univariate functions. Since the natural mathematical operation of a wireless multiple-access channel is superposition, nomographic functions can be efficiently computed in each cluster in merely a single step by letting nodes simultaneously transmit pre-processed analog state values to a cluster head that appropriately post-processes the receive signal. Analog systems become again more important for specific applications

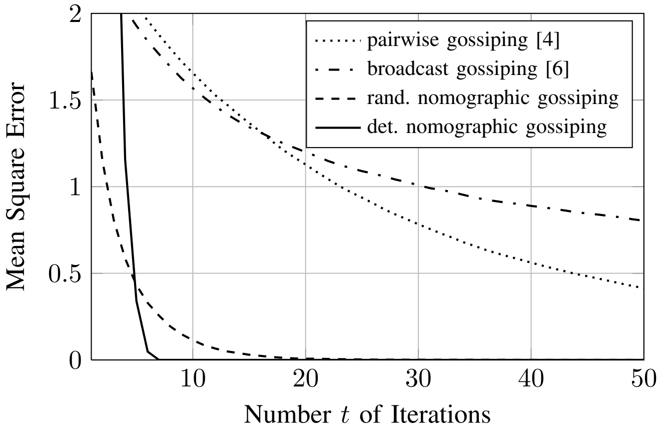

Fig. 5. Performance comparisons of nomographic gossiping, pairwise gossiping (Boyd et al. [4]) and broadcast gossiping (Aysal et al. [6]), using the network Fig. 1 and "arithmetic mean" as the desired consensus.

since recently it has been shown that digital signal processing has several fundamental limits, some of which can not be overcome by oversampling [20], [21].

The class of proposed nomographic gossip algorithms consists of deterministic as well as randomized approaches that differ in the way clusters are activated. The described deterministic algorithm activates clusters due to a fixed activation sequence, while in the randomized counterpart clusters wake up randomly, leading to a distributed gossip procedure.

At the cost of some coordination, we have shown that using deterministic nomographic gossiping, all states in the network converge to the exact desired consensus (i.e., an arbitrary function of the states) in a finite number of iterations. On the other hand, randomized nomographic gossiping converges almost surely to the desired consensus as long as the consensus function consists of continuous component functions. A numerical comparison of the algorithms with standard approaches known from the literature demonstrates that huge performance gains are possible with respect to the speed of convergence.

Finally, we mention some further useful properties of nomographic gossiping. The pre-processing functions, operating on the initial states, can be chosen to be universal in the sense that they are independent of the desired consensus. An update of the post-processing at nodes is therefore superfluous if the desired consensus changes. The universality is even preserved if clusters change due to nodes that drop out or due to additional nodes that join a cluster [22].

Part of our future work will be an extension of the proposed nomographic gossip algorithms to a more realistic intra-cluster communication (i.e., using model (1) instead of (2)) by exploiting the analog computation scheme published in [8].

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