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Mathieu Schmitt*, Monique Teillaud

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Abstract: This short report describes a preliminary study of a method computing meshes of the standard hyperbolic octagon that respects its symmetries. A prototype software was written, using the 2D meshing package of the CGAL library [1, 10] and a software computing hyperbolic triangulations, currently under development [4].

Key-words: Hyperbolic geometry, Poincaré disk, Schwarz triangle, mesh

* This work was done while the first author was working as an intern student at INRIA

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Maillages de l'octogone hyperbolique

Résumé : Ce court rapport décrit une étude préliminaire d'une méthode permettant de calculer des maillages de l'octogone hyperbolique qui respecte ses symétries. Un logiciel prototype a été réalisé, utilisant le module de maillages 2D de la bibliothèque CGAL [1, 10] et un logiciel de calcul de triangulations hyperboliques en cours de développememnt [4].

Mots-clés : Géométrie hyperbolique, disque de Poincaré, triangle de Schwarz, maillage

1 Introduction

Hyperbolic octagon. The hyperbolic octagon [2] and its meshes have been used in very various fields, see eg., [3, 8]. They are often generated in quite a manual way, and they do not fulfill the symmetry properties that actually are required for the application. As far as we know, no available software allows to compute such a mesh in an automatic way.

We consider the unit octagon $\mathcal{O}$ in the Poincaré disk of the hyperbolic plane $\mathbb{H}^2$ (see Figure 1) [6, 5]. The octagon can be seen as the fundamental domain of the action of the finitely presented group

$$\mathcal{G} = \left\langle a, b, c, d \mid ab\overline{a}\overline{b}cd\overline{c}\overline{d} \right\rangle$$

on $\mathbb{H}^2$, where a, b, c, d are hyperbolic translations (see Figure 2).

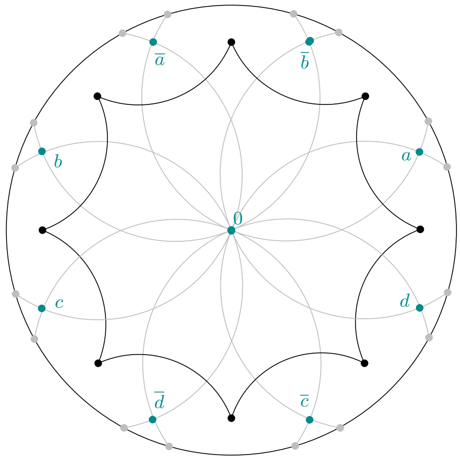

Figure 1: Hyperbolic octagon.

$\mathcal{G}$ can be thought of as the quotient of the free group $\mathcal{F} = \langle a, b, c, d\rangle$ by the normal closure of $\mathcal{R} = \{ab\overline{a}\overline{b}cd\overline{c}\overline{d}\}$ in $\mathcal{F}$. $\mathcal{O}$ is also the Voronoi cell of the origin O in the Voronoi diagram of the set $\mathcal{G}O$ of its images by elements of $\mathcal{G}$. $\mathcal{O}$ tiles $\mathbb{H}^2$, ie., all images of $\mathcal{O}$ by elements of $\mathcal{G}$ form a partition of $\mathbb{H}^2$.

Hyperbolic Delaunay triangulations. We refer the reader to [4].

2 Meshing the octagon

2.1 Basic mesh

The CGAL 2D mesh package implements a Delaunay refinement algorithm [10] to mesh a shape defined by a set of given constrained edges in the Euclidean plane. In a nutshell, a quality criterion can be given by the user, and the algorithm refines triangles and constrained edges as long as they do not all satisfy the criterion. A triangles is refined by inserting its (Euclidean) circumcenter in the Delaunay triangulation, while a constrained edge is refined by inserting its (Euclidean) midpoint.

We propose to reuse the package, adapting it to the hyperbolic case.

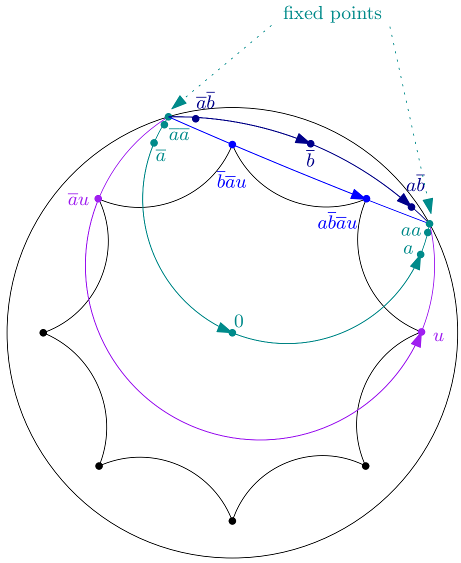

Figure 2: Hyperbolic translation a.

The flexibility of CGAL allows us to do so by changing the so-called *Traits* class, which provides the algorithm with basic geometric predicates and constructions. Instead of the default traits class of the package, which provides Euclidean constructions, we plug a new traits class providing constructions in the hyperbolic plane. The geometric constructions, in the hyperbolic plane, of the bisector of a line segment (represented as a a circular arc in the Poincaré disk), and of the circumcenter of a triangle, are explained in [4]. The midpoint of a line segment can be obtained by intersecting it with its bisecting line.

In a similar way, we can modify the quality criterion that the mesh package uses to stop the refinement. We replace it by a criterion on the hyperbolic size of a triangle. The area of a hyperbolic triangle is equal to $\pi - \Sigma\alpha$, where $\Sigma\alpha$ denotes the sum of its angles [6, 5] (see Figure 3). Such angles can be easily computed using the standard propery that the angle between a chord

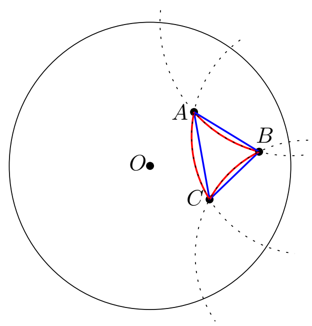

Figure 3: The area of the hyperbolic (red) triangle ABC is the difference between the sum of angles of the Euclidean (blue) triangle and its own sum of angles.

and the tangent at a vertex is the same as half the angle at the center (see Figure 4). Note that these computations make an extensive use of the fact that the Poincaré disk is a conformal

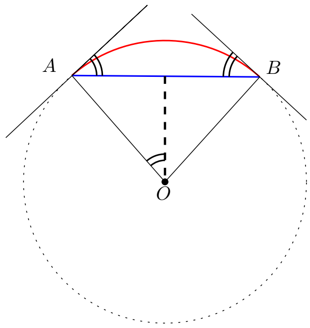

Figure 4: Property of angles.

model of the hyperbolic plane: it preserves angles.

Figure 5 shows an example of a mesh provided in this way. Visually, the mesh looks a bit denser at the boundaries of the octagon than at its centers: all triangles have the same hyperbolic area (up to arithmetic rounding).

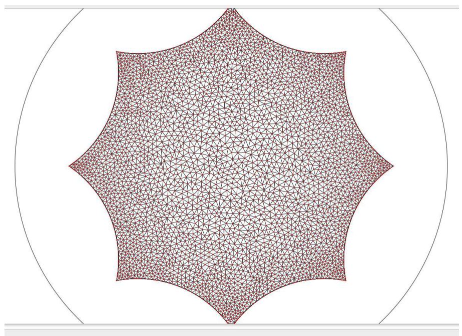

Figure 5: Basic mesh.

2.2 Adding symmetries

The Schwarz triangle $T(k,l,m)$, for

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1,$$

is the hyperbolic triangle with angles π/k, π/l and π/m. Note that in hyperbolic geometry, the angles of a triangle uniquely define it, up to isometry. The group of a Schwarz triangle $T(k,l,m)$ is the group of isometries generated by the reflexions with respect to its edges.

Under the action of its group, the Schwarz triangle $T(8,3,2)$ tiles the octagon (see Figure 6). Hence, from the properties of the octagon mentioned in introduction, it tiles $\mathbb{H}^2$.

Reflexions with respect to a hyperbolic line segment AB are computed in the following way. If AB is supported by a diameter of the Poincaré disk, then the hyperbolic reflexion is a Euclidean reflexion. If AB is supported by a Euclidean circle C of center c and radius r, then the reflexion

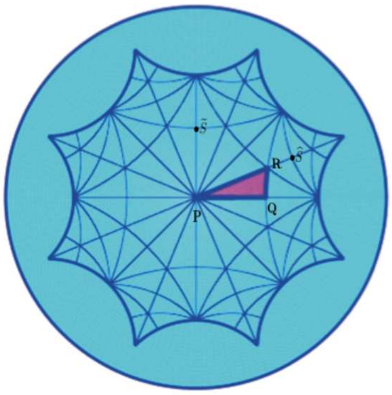

Figure 6: The Schwarz triangle $T(8, 3, 2)$ and the octagon (figure from [9]).

is the inversion defined by C: point M is mapped to M' such that $\| cM \| \cdot \| cM' \| = r^2$ and c, M, M' are collinear.

Ou method consists in

- Computing a mesh of the Schwarz triangle as explained in the previous section,
- Applying reflexions until the octagon is filled (there are 96 such reflexions)

A result can be seen in Figure 7.

In [9, 7], a Euclidean mesh is computed in the Schwarz triangle $T(8, 3, 2)$ using Matlab, by considering that the central triangle is close enough to the center of the Poincaré disk so that the hyperbolic distance can be approximated by the Euclidean distance. This approximation is in fact not precise enough, and the mesh is too sparse close to the boundaries of the octagon. To try to hide this problem, more points are added close to the center, so that more points appear after reflexions close to the boundaries. See Figure 8.

Since we are perfoming the Delaunay refinement using constructions in the hyperbolic plane, our method does not need any manual tuning, and it automatically provides a mesh that respects both the hyperbolic sizing criterion and the symmetries, and which can be as fine as needed.

3 Conclusion and future work

The results obtained by this first study show that the approach goes in the right direction. Still, the current prototype implementation is flawn by arithmetic issues: when computing the midpoint of a hyperbolic line segment, this midpoint is rounded and in fact does not exactly lie on the segment; then it is not a fixed point through the reflexion with respect to this segment. This generates two points that are very close, but different, which must be fixed. Such improvements will allow to release the software in CGAL, maybe as a demo of the package on hyperbolic Delaunay triangulations in preparation [4].

A higher level open question is to generalize the method and make the software generic so that it can handle any Schwarz triangle. This would allow to compute meshes of any fundamental polygon tiling the hyperbolic plane by the action of a Fuchsian group on $\mathbb{H}^2$.

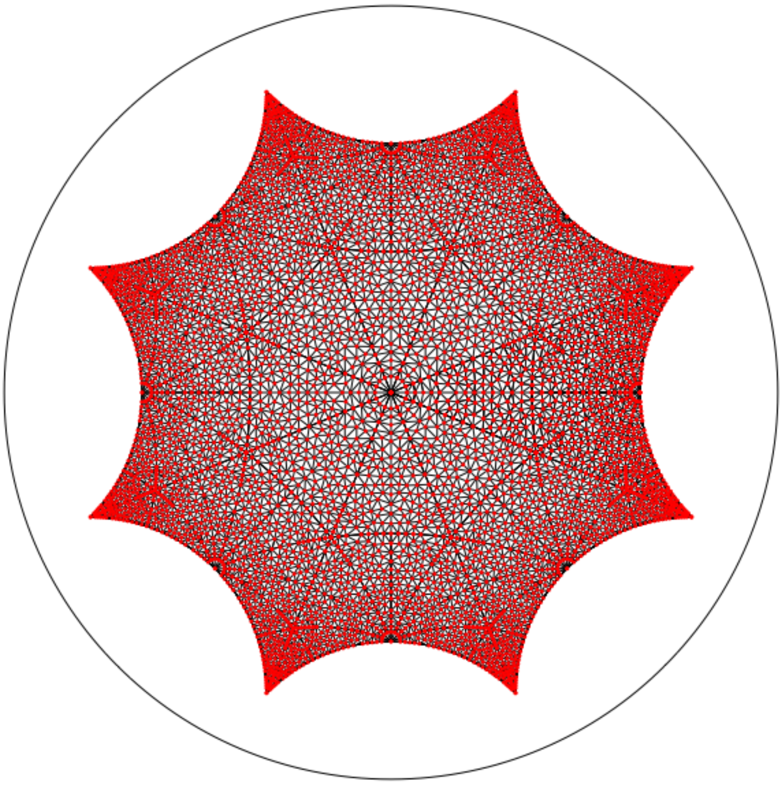

Figure 7: Mesh with symmetries.

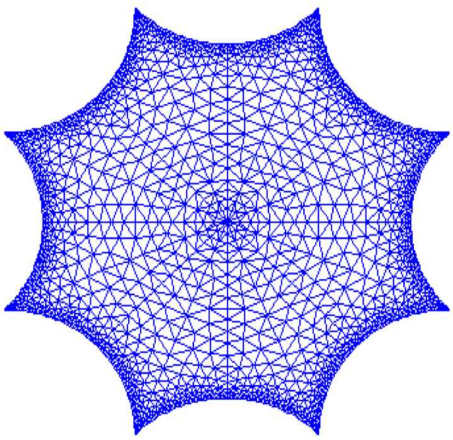

Figure 8: Mesh obtained in [9, 7].

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RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
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