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ASYMPTOTIC BEHAVIOR OF A DIFFUSIVE SCHEME SOLVING THE INVISCID ONE-DIMENSIONAL PRESSURELESS GASES SYSTEM

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Abstract. In this work, we discuss some numerical properties of the viscous numerical scheme introduced in [8] to solve the one-dimensional pressureless gases system, and study in particular, from a computational viewpoint, its asymptotic behavior when the viscosity parameter $\varepsilon > 0$ used in the scheme becomes smaller.

1. **Introduction.** In this work, we focus on the one-dimensional system describing a pressureless gas, which writes, for any $T > 0$,

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1}$$

$$\partial_t q + \partial_x(q u) = 0, \tag{2}$$

in $(0, T] \times \mathbb{R}$, where $\rho(t, x) \geq 0$ is the gas density and $q(t, x) \in \mathbb{R}$ is the momentum at time $t \in [0, T]$ and location $x \in \mathbb{R}$. The gas velocity $u(t, x) \in \mathbb{R}$ must be somehow defined as a quotient of q by ρ, but there is trouble when ρ is 0. That is why one needs the notion of duality solutions [5], previously introduced for conservation laws [4]. Clearly, (1) and (2) can be seen as conservation laws, for mass and momentum respectively. System (1)–(2) is supplemented with initial conditions

$$\rho(0, \cdot) = \rho^{\text{in}}, \qquad q(0, \cdot) = q^{\text{in}}. \tag{3}$$

This system arises from very various physical situations (cold plasmas [3], astrophysics [17, 9], traffic models [1, 13]...) and has been mathematically studied in numerous articles, such as [14, 10, 16, 5].

We here choose a periodic framework: we focus on $[0, 1]$ and impose that mass density, velocity and momentum have the same values at both $x = 0$ and $x = 1$, so that the solutions are 1-periodic.

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When ρ and u are smooth, and if ρ remains non zero, (2) can be modified into the standard Burgers equation thank to (1):

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \partial_t u + u \partial_x u = 0. \tag{4}$$

The mass density ρ then solves a basic transport equation, where u is given: it does not depend on ρ because of (4). But it is common knowledge that, in finite time, a mass concentration phenomenon can happen, for instance, when u does not increase. Consequently, the regularity properties of u and ρ are lost, and u does not solve (4) anymore.

From the numerical viewpoint, one can think about several ways to discretize (1)–(3). Kinetic schemes [3, 6] or particle methods [12] allow to use the kinetic framework underlying the pressureless gas dynamics. It is also natural to try numerical schemes related to hyperbolic conservation laws [15] or relaxation schemes [2].

In [8], we prove that upwind schemes were not an option, since they failed to ensure the discrete one-sided Lipschitz (OSL) condition introduced [11] for convex scalar conservation laws. Then, following the strategy of [7] at the continuous level, we add an artificial viscosity. The new diffusive scheme we obtain is proven to be L^∞-stable and consistent, hence converging towards the solution of the viscous pressureless gases system. In particular, it satisfies the discrete OSL condition. We here investigate, at the numerical level, the asymptotic behavior of the same numerical scheme when the artificial viscosity vanishes.

2. **Diffusive numerical scheme.** Let us recall the diffusive scheme we presented in [8]. Consider Δt, $\Delta x > 0$ such that $N = T/\Delta t \in \mathbb{N}$ and $I = 1/\Delta x \in \mathbb{N}$, and set $\lambda = \Delta t/\Delta x$. Denote ρ_i^n, q_i^n and u_i^n the approximate values of ρ, q and u at time $n\Delta t \in [0,T]$ and location $(i+1/2)\Delta x \in [0,1]$, for $0 \le n \le N$ and $0 \le i < I$. Of course, thanks to the periodicity property, ρ_i^n, q_i^n and u_i^n can be extended for any $i \in \mathbb{Z}$. For the sake of readability, in the previous notations, we may drop the time iteration index n and replace $n+1$ by a prime symbol “ $'$ ”.

Note that the discrete OSL condition can be written as $n\lambda(u_{i+1}^n - u_i^n) \le 1$, for any i and $n > 0$.

Let us describe step by step the strategy for our scheme.

2.1. **Periodic initial data.** We choose arbitrary 1-periodic initial data $\rho^{\mathrm{in}} \ge 0$, $u^{\mathrm{in}} \ge 0$. Indeed, the viscous problem in [7] deals with mass density and velocity, and not momentum.

2.2. **Regularizing initial data.** We take a fixed $\varepsilon > 0$, small enough. We may have to regularize both u^{in} and ρ^{in} so that u^{in}, $\rho^{\mathrm{in}} \in C^1(\mathbb{R};\mathbb{R}_+^*)$ satisfy the assumptions of Theorem 2 in [8], i.e.

$$\rho^{\mathrm{in}}(x) \ge C\varepsilon^{1/4}, \qquad u^{\mathrm{in}}(x) \le C, \qquad (u^{\mathrm{in}})'(x) \le \frac{C}{\sqrt{\varepsilon}}, \qquad \forall x \in [0,1], \tag{5}$$

where $C \ge 0$ is a constant not depending on ε. Note that ρ^{in} must lie in $\mathbb{R}_+^*$, since the continuous viscous model involves a division by ρ.

The following quantities

$$U = \max_{[0,1]} u^{\text{in}} > 0, \qquad V = \min_{[0,1]} u^{\text{in}} > 0,$$
$$A = \max(0, \max_{[0,1]} (u^{\text{in}})') \geq 0, \qquad R = \min_{[0,1]} \rho^{\text{in}} > 0,$$

may depend on ε. More precisely, they must satisfy properties inherited from (5), i.e.

$$R \geq C\varepsilon^{1/4}, \qquad V \leq U \leq C, \qquad A \leq \frac{C}{\sqrt{\varepsilon}}, \tag{6}$$

where $C \geq 0$ does not depend on ε.

2.3. **Choosing the time and space steps.** The steps Δt and $\Delta x > 0$ are then chosen such that

$$0 < \quad \Delta x \quad \leq \frac{2V}{1+A}, \tag{7}$$
$$0 < \quad \Delta t \quad \leq \min\left(\frac{1}{4A+1}, \frac{1}{4U}\Delta x, \frac{R}{4\varepsilon(1+AT)}\Delta x^2\right), \tag{8}$$

where we set $\lambda = \Delta t/\Delta x$ and $\sigma = \Delta t/\Delta x^2$.

2.4. **Writing the scheme.** We eventually write the discretization of the viscous pressureless gases system as

$$u_i' = u_i - \lambda\left(\frac{{u_i}^2}{2} - \frac{{u_{i-1}}^2}{2}\right) + \frac{\varepsilon\sigma}{\rho_i}(u_{i-1} + u_{i+1} - 2u_i), \tag{9}$$
$$\rho_i' = (1 - \lambda u_i')\rho_i + \lambda u_{i-1}' \rho_{i-1}. \tag{10}$$

2.5. **Properties of the scheme.** In [8], we prove the following

Theorem 2.1. *We assume that (7)–(8) hold. Then we have, for any i and $n \geq 0$,*

$$V \leq u_i^n \leq U, \qquad u_i^n - u_{i-1}^n \leq \frac{A\Delta x}{1 + An\Delta t}, \qquad \rho_i^n \geq \frac{R}{1 + An\Delta t} \geq \frac{R}{1+AT} > 0.$$

Moreover, the discrete total mass is conserved. Finally, when $\varepsilon > 0$ is fixed, scheme (9)–(10) is first-order consistent (in both t and x) with the viscous pressureless gases system, and is monotonic.

In other words, both discrete unknown functions satisfy maximum principles, and the discrete velocity satisfies the OSL condition. Assumptions (8) on Δt are crucial to ensure the stability of the scheme. They are related to the standard assumptions to get stability for explicit schemes on transport or diffusion equations.

3. **Numerical study.** The viscosity parameter $\varepsilon > 0$ is chosen in the first place. We use several values of ε in the following computations: 10^{-4}, 10^{-5}, 10^{-6}, 5×10^{-7}, 10^{-7} and the smallest value $\varepsilon_0 = 10^{-6}/14$, which will be considered as the reference situation. We take $T = 1$ s as the final time. Moreover, we focus on one case with (almost, see the discussion below) smooth initial data. It allows to point out all the problems arising the pressureless gases system, in particular the regularity loss phenomenon, already explained in Section 1.

We set, for any $x \in [0,1]$,

$$\rho^{\text{in}}(x) = 1 + \frac{1}{2}\cos(4\pi x), \qquad u^{\text{in}}(x) = 2x(1-x) + \frac{1}{2}.$$

We clearly have $U = 1$, $V = 1/2$, $A = 2$ and $R = 1/2$. When ε belongs to the set of values above, (6) is clearly satisfied. Nevertheless, u^{in} is not $C^1(\mathbb{R};\mathbb{R}_+^*)$. Indeed, there is an issue at the bounds of $[0,1]$. Therefore, we should need to regularize u^{in}. Fortunately, it is not necessary: the regularization near 0 and 1 can happen on intervals whose length may be chosen smaller than $2\Delta x$. We can choose it so that $(u^{\mathrm{in}})'$ remains in $[-2,2]$, and consequently, A still equals 2.

Let us now choose $\Delta x = 20\varepsilon$. Again, if ε belongs to our same set of values, (7) obviously holds. Then Δt can be taken as the optimal possible value given by (8), i.e. $\Delta t = \varepsilon \min\left(5, 50/(1+2T)\right) = 5\varepsilon$.

We want to study the behavior of our numerical solution with respect to ε. In the following, since ϱ has a measure meaning when ε goes to zero, it is really more convenient to focus on the cumulative mass M, defined by

$$M(t,x) = \int_0^x \varrho(t,y)\,\mathrm{d}y.$$

We first draw the evolution of M and u with respect to x, at two different times, before and after approximately $t = 0.5$. Indeed, until $t = 0.5$, at the continuous level, u solves the Burgers equation, and there is no regularity loss. But, near time $t = 0.5$, a regularity loss happens because of mass concentration and subsequent vacuum appearance.

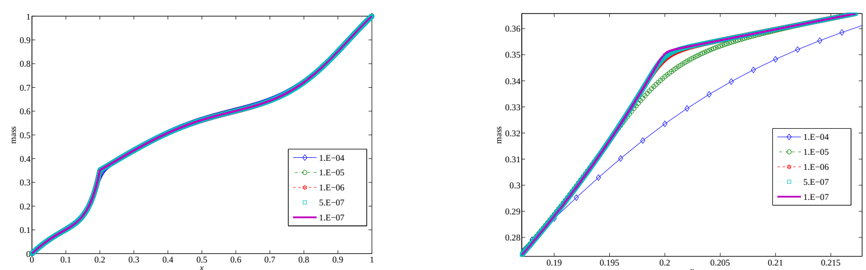

Figure 1. Behavior, at time 0.4, of M (a) on $[0,1]$, (b) focused around $x = 0.2$

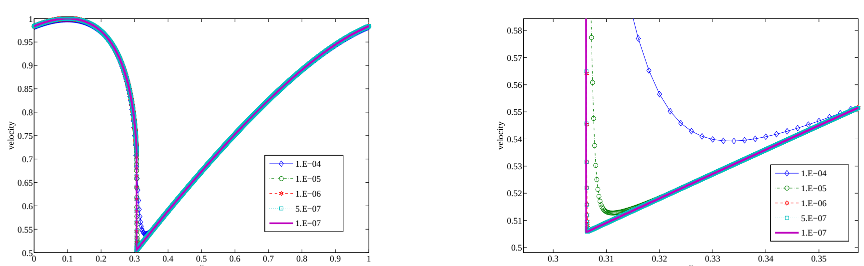

Figure 2. Behavior, at time 0.4, of u (a) on $[0,1]$, (b) focused around $x = 0.2$

The solutions are then displayed for various values of ε. More precisely, Fig. 1–2 respectively give the behavior of M and u at time $t = 0.4$, i.e. before the regularity loss. Of course, the curves are sharpened with smaller values of ε, but there is still no jump at all for either ρ or u.

Fig. 3–4 then present the typical profiles of both M and u after the regularity loss, here given at final time $T = 1$. The jump is now visible at $x = 0.59$ on both figures, and u of course still decreases at the jump abscissa.

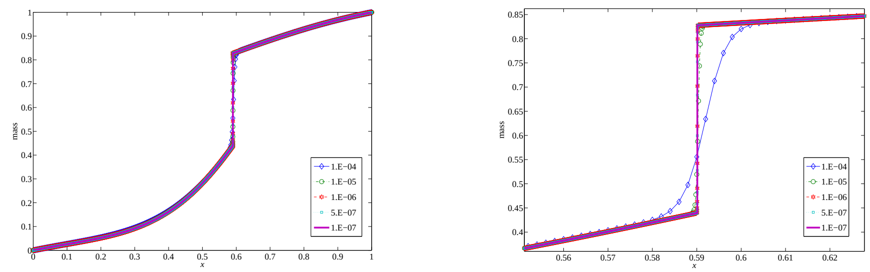

FIGURE 3. Behavior, at final time, of M (a) on $[0, 1]$, (b) focused around $x = 0.59$

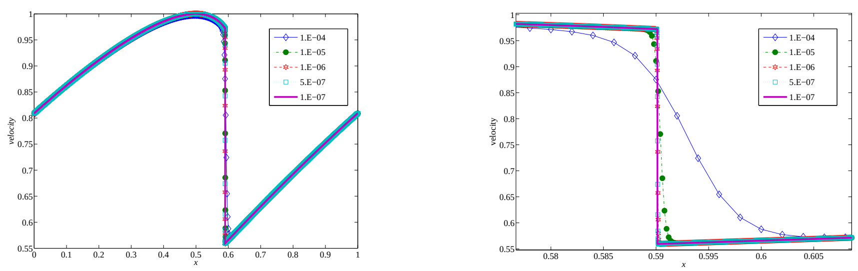

FIGURE 4. Behavior, at final time, of u (a) on $[0, 1]$, (b) focused around $x = 0.59$

Let us now study the evolution of the total momentum

$$Q(t) = \int_0^1 \rho(t, x)\, u(t, x)\, \mathrm{d}x$$

with respect to time.

As the reader can see on Fig. 5, till the regularity loss, the numerical conservation of Q is quite satisfactorily ensured. The property is not recovered after the loss. This drawback of our scheme (9)–(10) was already pointed out in [8]: we had to choose between the total momentum conservation (if we write a scheme for ρ and q) and the OSL condition (for which we wrote a scheme for ρ and u). The latter is in fact crucial to ensure that we select the right solution to the pressureless gases system. Nevertheless, we can at least state that the order of magnitude of Q is conserved.

Eventually, we focus on the behavior with respect to ε. We focus on the following quantities

$$\mathcal{E}_M(t) = \max_{x \in [0,1]} |M(t, x) - M_0(t, x)|, \tag{11}$$

$$\mathcal{E}_u(t) = \max_{x \in [0,1]} |u(t, x) - u_0(t, x)|, \tag{12}$$

for the values ε under study. Quantities indexed by 0 are of course the ones related to the reference value of ε, ε_0. Quantities on coarser grids are projected on the

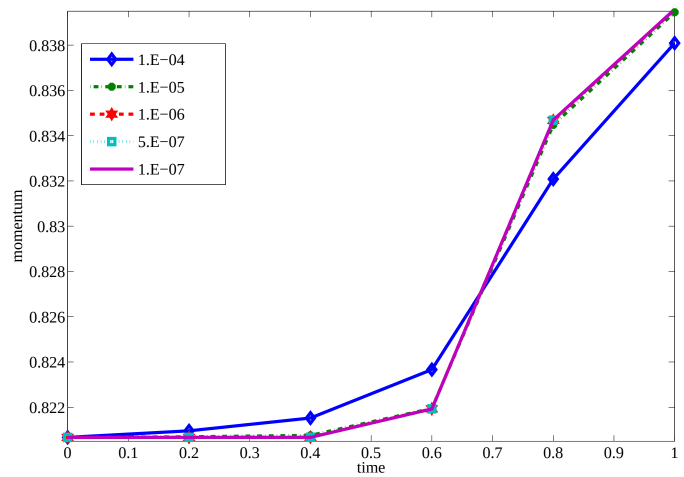

FIGURE 5. Checking the momentum for several values of ε

(fine) grid of the reference value (ε_0 corresponds to 700,000 space cells) in order to compute the discretized versions of (11)–(12).

Fig. 6–7 respectively show the behavior of $\mathcal{E}_M$ and $\mathcal{E}_u$ with respect to time.

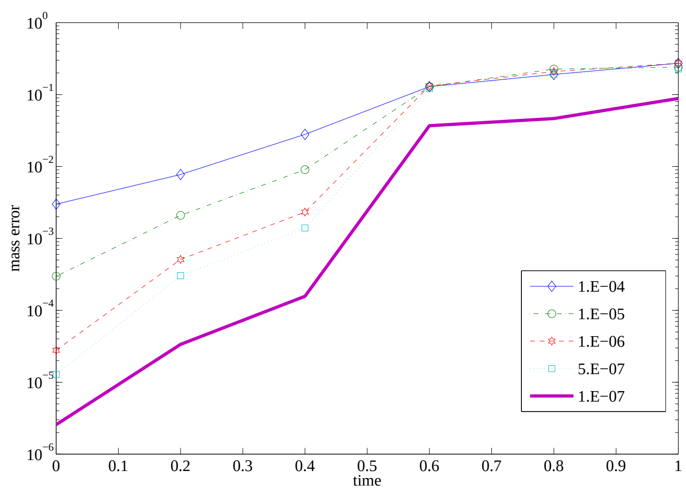

FIGURE 6. Behavior of $\mathcal{E}_M$ w.r.t. t for several values of ε

These results give us hints of what should be the asymptotic behavior of our solutions, in particular when we loose regularity. The convergence seems very slow in ε but it was expected since the smallest power of ε in the main hypotheses of the scheme is equal to one fourth, see Eqn. 6. That partially explains that the errors are quite large, even for quite small values of ε. In fact, we were not able to understand why $\varepsilon = 10^{-7}$ gives such a different behavior (in order of magnitude) compared to not so larger values of ε, whereas, from 10^{-4} to 10^{-6}, the results are close after the regularity loss.

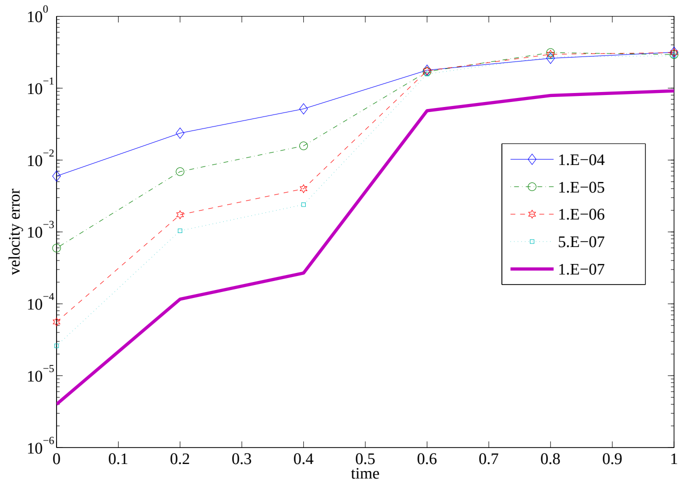

FIGURE 7. Behavior of $\mathcal{E}_u$ w.r.t. t for several values of ε

4. **Perspectives.** As expected, the crucial point in the asymptotic behavior of our scheme is strongly linked to the regularity loss phenomenon which occurs for the pressureless gases. We should be able to prove reasonable convergence estimates with respect to ε as long as the solutions remain smooth. Beyond the regularity loss, the situation remains quite unclear.

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