Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem

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Motivations

A usual model for electrical engineering applications

- Dielectric relaxation vs magnetic diffusion:

$$\tau_{\mathrm{e}} = \frac{\epsilon}{\sigma} \ll \tau_{\mathrm{m}} = \mu\sigma L^2$$

L, ϵ, σ, μ: characteristic length, permittivity, conductivity and permeability.

$\Longrightarrow$ magneto-quasistatic (eddy-current) model.

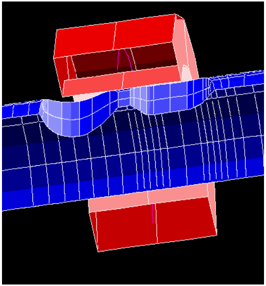
$$\mathbf{curl}\,\mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \qquad \mathrm{div}(\mu\mathbf{H}) = 0,$$

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \cancel{\frac{\partial(\epsilon\mathbf{E})}{\partial t}} \qquad \mathrm{div}(\epsilon\mathbf{E}) = 0.$$

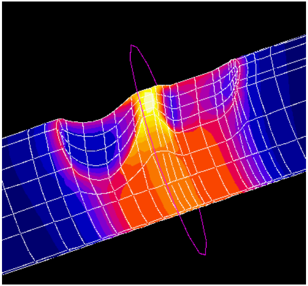
Motivations

Usage of the magneto-quasistatic model

- Electrothermics



(a) Position of the inductor.



(b) Induced current density.

Figure : Surface steel hardening of a shift fork. PhD thesis Sven Wanser ('93 - Renault+EDF/ECL).

- Others: non-destructive testing, most of electromechanical systems.

A computational simplification

Skin Effect

- Skin effect: "exponential decrease" of the electromagnetic field in a volume conductor, *i.e.*

$$\delta = \sqrt{\frac{1}{\mu\pi f\sigma}} \ll L.$$

δ: penetration (skin) depth.

- Instead of strongly meshing close to the surface of the volume conductors: surface impedance conditions. Most usual:

$$Z_\delta\big(\mathbf{H} \times \mathbf{n}\big) = \mathbf{E}_t, \text{ with } Z_\delta = \frac{1+\mathrm{i}}{\sigma\delta}.$$

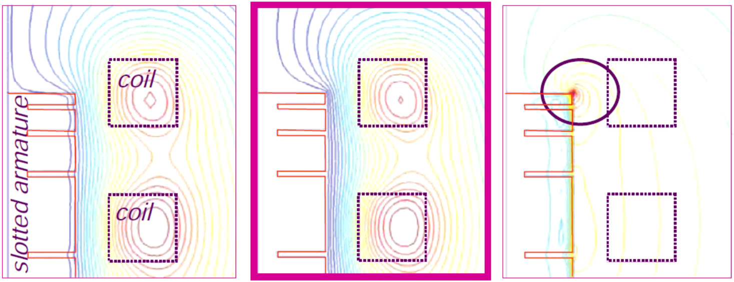
n: unitary normal to the conductor, $\mathbf{E}_t$: tangential component of the electric field.

- For smooth enough surface, the impedance can even be "more accurate".

A computational simplification

Difficulties in the corner

- Example in a domain decomposition process.



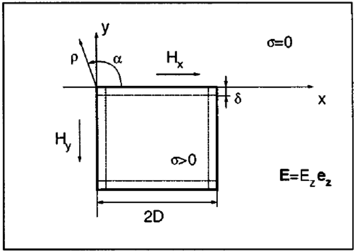
global solution = surf. imp. solution + iterative corrections

P. Dular et al. (Compel, 2007)

A computational simplification

Difficulties in the corner

- Few authors have considered impedance modifications :
  "Surface impedance boundary conditions near corners and edges: rigorous consideration". [IEEE Trans. on Mag., 2002]
- Application of the same asymptotic expansion as in the smooth surface case. Impedance obtained:



$$Z_x = -\frac{\tan(\pi/6)}{\rho\sigma}, \; Z_y = \frac{\tan(\pi/6)}{\rho\sigma}.$$

$\Longrightarrow$ blows up in the corner for any $\sigma < +\infty$.

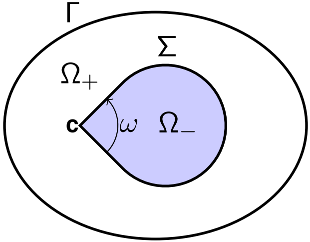
It is not valid for non-magnetic materials with $\sigma < +\infty$ [Buret *et al*, IEEE Trans. on Mag., 2012]

Statement of the Problem

The magnetic vector potential $\mathcal{A}$ satisfies

$$\begin{cases} -\Delta \mathcal{A}^+ = \mu_0 J \text{ in } \Omega_+, \\ -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ \mathcal{A}^+ = 0 \text{ on } \Gamma, \end{cases} \quad \begin{array}{r} [\mathcal{A}]_\Sigma = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_\Sigma = 0, \text{ on } \Sigma. \end{array} \tag{1}$$

with $\zeta^2 = \kappa\mu_0\sigma/4$, and a smooth data J.



Aim : Description of the magnetic potential $\mathcal{A}$ near the corner **c** in 2D

Statement of the Problem

Weak solutions

Variational Formulation

Find $\mathcal{A} \in H_0^1(\Omega)$ *such that* $\forall \mathcal{A}_* \in H_0^1(\Omega)$

$$\int_{\Omega_+} \nabla \mathcal{A}^+ \cdot \nabla \overline{\mathcal{A}_*^+}\, \mathrm{d}x + \int_{\Omega_-} \nabla \mathcal{A}^- \cdot \nabla \overline{\mathcal{A}_*^-} + 4\mathrm{i}\zeta^2 \mathcal{A}^-\, \overline{\mathcal{A}_*^-}\, \mathrm{d}x = \mu_0 \int_\Omega J\, \overline{\mathcal{A}_*}$$

Proposition

There exists a unique solution $\mathcal{A}$ *in* $H_0^1(\Omega)$ *to problem* (1). *Moreover,* $\mathcal{A}$ *belongs to* $H^{\frac{5}{2}-\varepsilon}(\Omega)$ *for any* $\varepsilon > 0$ *.*

In particular, by Sobolev embedding, we obtain that $\mathcal{A}$ belongs to $\mathcal{C}^1(\overline{\Omega})$.

Corner asymptotics

- In general $\mathcal{A}$ does not belong to $\mathcal{C}^2(\overline{\Omega})$ but $\mathcal{A}$ possesses a corner asymptotic expansion as the distance r to **c** goes to zero

$$\mathcal{A}(r,\theta) \underset{r\to 0}{\sim} \Lambda^{0,0}\mathfrak{S}^{0,0}(r,\theta) + \sum_{k\geqslant 1}\sum_{p\in\{0,1\}} \Lambda^{k,p}\mathfrak{S}^{k,p}(r,\theta)$$

- Key points for the behavior of $\mathcal{A}$ in the vicinity of **c** :
  - Derivation of the singular functions $\mathfrak{S}^{k,p}$
  - Determination of the singular coefficients $\Lambda^{k,p}$
- We can provide a constructive procedure to determine $\mathfrak{S}^{k,p}$

$$\mathfrak{S}^{k,p}(r,\theta) = \underbrace{r^k\cos(k\theta - p\pi/2)}_{\textit{kernel functions of } \Delta} \underbrace{+\, r^k \sum_{j\geqslant 1}(\mathrm{i}\zeta^2 r^2)^j \sum_{n=0}^{j} \log^n r\, \Phi_{j,n}^{k,p}(\theta)}_{\textit{shadow terms}}$$

Outline

1. **The case** $\zeta = 0$: For the determination of coefficients $\Lambda^{k,p}$ we introduce the method of dual singular functions
2. **The case** $\zeta \neq 0$: we introduce the method of quasi-dual singular functions. This method is an approximate method. We prove estimates for its convergence.
3. Numerical simulations illustrate the theoretical results.

Laplace Operator ($\zeta = 0$)

- We consider the solution $\mathcal{A}$ to

$$\begin{cases} -\Delta \mathcal{A} = \mu_0 J \text{ in } \Omega, \\ \mathcal{A} = 0 \text{ on } \Gamma, \end{cases}$$

with a smooth right hand side with support outside the interior point **c**.

- $\mathcal{A}$ admits the Taylor expansion at **c** :

$$\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} + \sum_{k \geqslant 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} r^k \cos(k\theta - p\pi/2)$$

Methods to extract the coefficients $\Lambda^{k,p}$

- The method of moments :
  It uses the orthogonality of the angular parts $\cos(k\theta - p\pi/2)$
- The dual function method :
  It is the application to the present smooth case of a general method [Mazya-Plamenevskii, '78] valid for corner coefficient extraction

The Dual Function Method

- Dual harmonic functions singular at $r = 0$:

$$\mathfrak{k}_0^{k,p}(r,\theta) = \begin{cases} -\dfrac{1}{2\pi} \log r, & \text{if } k = 0,\ p = 0, \\ \dfrac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \geqslant 1,\ p = 0, 1. \end{cases}$$

- For $R > 0$, let us introduce the bilinear form

$$\mathcal{J}_R(K, A) = \int_{r=R} \left(K \partial_r A - \partial_r K\, A\right)\, R \mathrm{d}\theta.$$

Proposition

Let $\mathcal{A}$ be the solution to the Laplace equation with a smooth right hand side with support outside the ball $\mathcal{B}(\mathbf{c}, R)$. Then

$$\mathcal{J}_R(\mathfrak{k}_0^{0,0}, \mathcal{A}) = \Lambda^{0,0} \quad \textit{and} \quad \mathcal{J}_R(\mathfrak{k}_0^{k,p}, \mathcal{A}) = \Lambda^{k,p}, \ \ k \geqslant 1,\ p = 0, 1.$$

The Quasi-Dual Function Method ($\zeta \neq 0$)

- [Costabel *et al*, '04; '12] : The QDFM for straight edges, circular edges and homogeneous operators with constant coefficients.
- We use quasi-dual functions $\mathfrak{K}_m^{k,p}$:

$$\mathfrak{K}_m^{k,p}(r,\theta) = \mathfrak{k}_0^{k,p}(r,\theta) + \sum_{j=1}^{m} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{k}_j^{k,p}(r,\theta)}_{\textit{shadow term}}$$

where $\mathfrak{k}_j^{k,p}$ are solution to

$$\begin{cases} \Delta \mathfrak{k}_j^{k,p\,+} = 0, & \text{in } \mathcal{S}_+ , \\ \Delta \mathfrak{k}_j^{k,p\,-} = 4\mathfrak{k}_{j-1}^{k,p\,-}, & \text{in } \mathcal{S}_- \end{cases}$$

in the space :

$$r^{-k+2j}\mathrm{Span}\left\{\log^q r\,\Phi(\theta),\ \ q \leqslant j+1\ ,\ \Phi \in \mathcal{C}^1(\mathbb{T}),\ \Phi^{\pm} \in \mathcal{C}^{\infty}(\overline{\mathbb{T}}_{\pm})\right\}$$

Extraction of coefficients

The extraction of $\Lambda^{k,p}$ is performed through the evaluation of quantities

$$\mathcal{J}_R(\mathfrak{K}_m^{k,p}, \mathcal{A}), \quad k = 0, 1, 2, \ldots$$

Theorem

Let $\mathcal{A}$ be the solution to problem (1)*, under the assumptions of the introduction. Let $k \in \mathbb{N}$ and $p \in \{0, 1\}$ ($p = 0$ if $k = 0$). Let m such that $2m + 2 > k$. There exist coefficients $\mathcal{J}^{k,p;k',p'}$ independent of R and $\mathcal{A}$ such that*

$$\mathcal{J}_R(\mathfrak{K}_m^{k,p}, \mathcal{A}) \underset{R \to 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{[k/2]} \mathcal{J}^{k,p\,;\,k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}\big(R^{-k} R_0^{2m+2} \log R\big)\,,$$

where $\boxed{R_0 = \zeta R \left(1 + \sqrt{|\log R|}\right)}$.

Extraction of coefficients

Example

- For $k = 0$

$$\Lambda^{0,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$$

- For $k = 1$

$$\Lambda^{1,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$$

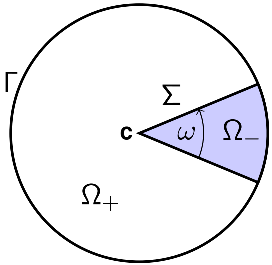
- For $k = 2$, we need $m \geqslant 1$

$$\Lambda^{2,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0\,;\,0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2} R_0^4 \log R)$$

The Problem

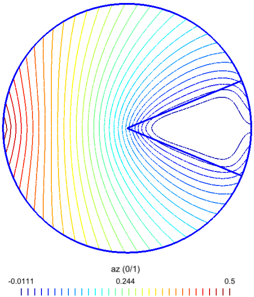
$$\begin{cases} -\Delta \mathcal{A}^{+} = 0 \text{ in } \Omega_{+}, & \\ -\Delta \mathcal{A}^{-} + 4i\zeta^2 \mathcal{A}^{-} = 0 \text{ in } \Omega_{-}, & [\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma, \\ \mathcal{A}^{+} = \dfrac{|\theta|}{\pi} \text{ on } \Gamma, & [\partial_n \mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{cases}$$

- Ω : disk of radius 50 mm.
- Conducting sector : $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2})$ mm^{-1} corresponds to a physical skin depth of 5 mm.

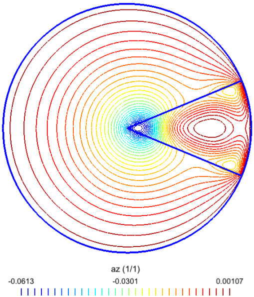


Finite Element Solution

We use P_2 finite elementss



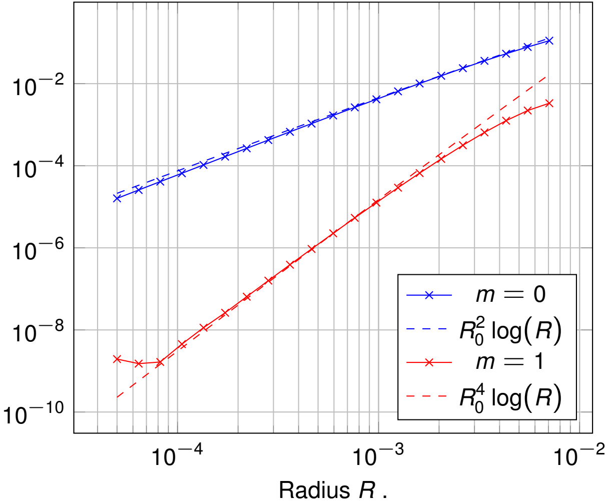
(b) Real part of the solution $\mathcal{A}$.



(c) Imaginary part of the solution $\mathcal{A}$.

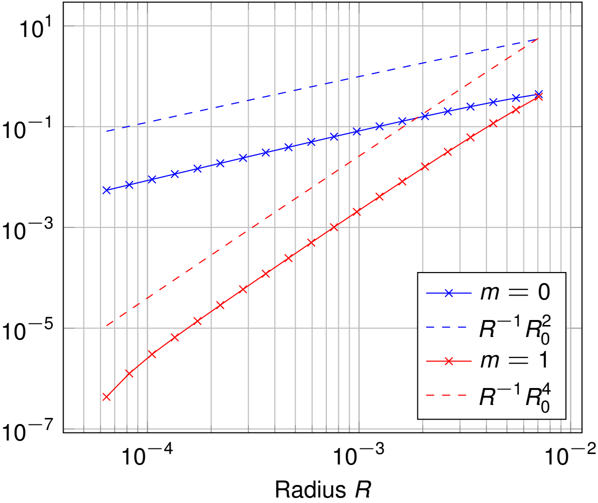
Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

Recall $\Lambda^{0,0} = \mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$.
We plot $\left|\mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) - \mathcal{A}|_{\mathbf{c}}\right|$ w.r.t. the radius R, for $m = 0, 1$



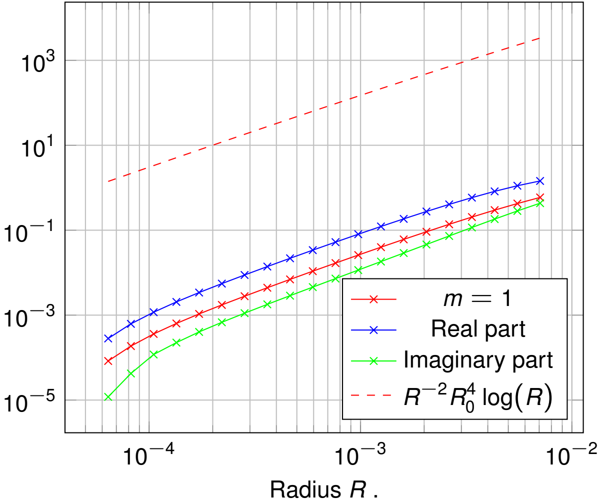
Computation of the coefficient $\Lambda^{1,0}$

Recall $\Lambda^{1,0} = \mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$.
We plot $\left|\mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}_m^{1,0}, \mathcal{A})\right|$.



Computation of the coefficient $\Lambda^{2,0}$

Recall $\Lambda^{2,0} = \mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0\,;\,0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2}R_0^4 \log R)$.
We plot $\left|\mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{2,0}, \mathcal{A})\right|$.



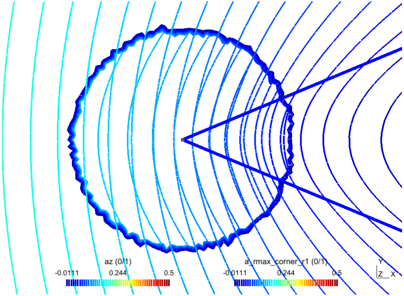
Comparison of the FE solution and of the local expansion

Expansion restricted to order 1:

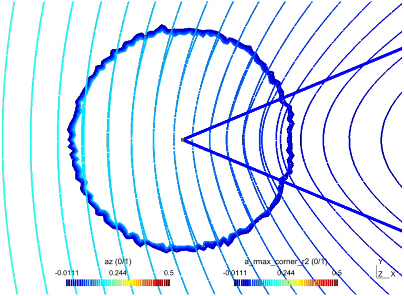
$$\mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{0,0}, \mathcal{A})\left(1 + \mathrm{i}\zeta^2 \mathfrak{s}_1^{0,0}\right) + \mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{1,0}, \mathcal{A})\mathfrak{s}_0^{1,0},$$

Expansion restricted to order 2: adding to the above expression the term

$$\left(\mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0}\mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{0,0}, \mathcal{A})\right)\mathfrak{s}_0^{2,0}.$$



(d) Expansion restricted to order 1. Real part.



(e) Expansion restricted to order 2. Real part.

Thank you for your attention!