

# Equivalent Conditions for an Elasto-Acoustic Problem with a Thin Layer

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# Equivalent Conditions for an Elasto-Acoustic Problem with a Thin Layer

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First Russian-French Conference on Mathematical Geophysics,  
Mathematical Modeling in Continuum Mechanics and Inverse Problems  
Biarritz, June 18-22, 2012

# Motivations

## HPC-GA Project

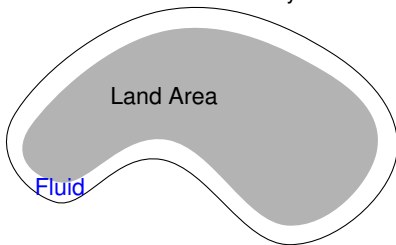
- **High Performance Computing for Geophysics Applications**

FP7 Project : INRIA, BCAM, UFRGS, UNAM

Earthquakes

- **Numerical Models** (Work Package 2)

- Elasto-Acoustic Coupling : to reproduce earthquakes
- Modeling the effects of the ocean on seismic waves
- The medium : Land Areas surrounded by **thin Fluid zones**



# Motivations

## Difficulty and Method

- **Difficulty**

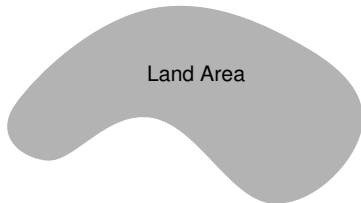
Applying a FEM on a mesh with **fine cells** in the **Fluid** and much larger in the **Solid**

- **Method**

- 1 Replace the **Fluid layer** by a boundary condition

**Approach** : an asymptotic method to derive Equivalent Conditions

- 2 Couple this condition with the elastic wave equation
- 3 Use a FEM



# Motivations

## Bibliography and Outline

### Bibliography

- Modeling Fluid-Solid Interaction Problems  
JONES 1983, LUKE-MARTIN 1995, NATROCHVILI ET AL. 2000, ... ,  
MONK-SELGAS 2009
- Derivation of Equivalent Conditions for **thin layer** Problems: Laplace,  
Helmholtz, Maxwell equations  
LAFITTE 1993, ENGQUIST-NÉDÉLEC 1993, VOLAKIS-SENIOR 1995,  
BENDALI-LEMRAËT 1996, ... , CALOZ ET AL. 2006 ...

### Outline

- Asymptotic Analysis for the time-harmonic **Elasto-acoustic** equations in  
a domain with a **thin acoustic layer**.

# Outline

- 1 **Uniform Estimates**
- 2 Equivalent Conditions
- 3 Numerical Simulations

# An Elasto-Acoustic Problem with a Thin Layer

## Framework and Issue

The Problem ( $\mathbf{P}_\epsilon$ ) set in a smooth bounded domain  $\Omega^\epsilon = \Omega_s \cup \Gamma \cup \Omega_f^\epsilon$ :

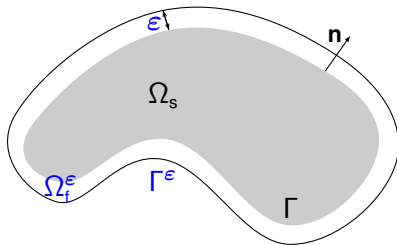
$$\Delta p + \kappa^2 p = 0 \quad \text{in } \Omega_f^\epsilon$$

$$\nabla \cdot \underline{C}(x) \nabla \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 \quad \text{in } \Omega_s$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \partial_n p_i \quad \text{on } \Gamma$$

$$\underline{C}(x) \nabla \mathbf{u} \mathbf{n} = -p \mathbf{n} - p_i \mathbf{n} \quad \text{on } \Gamma$$

$$p = 0 \quad \text{on } \Gamma^\epsilon$$



### Assumption

(i)  $\underline{C}(x) = (C_{ijkl}(x))$  is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$ .

(ii)  $C_{ijkl}(x)$  are real valued smooth functions, up to  $\Gamma$ .

(iii)  $\exists \alpha > 0, \quad \forall \xi = (\xi_{ij})$  sym. tensor,  $\sum_{i,j,k,l} C_{ijkl} \xi_{ij} \bar{\xi}_{kl} \geq \alpha \sum_{i,j} |\xi_{ij}|^2$ .

## Weak Solutions

### Variational Problem $(\mathbf{VP})_\varepsilon$

- $V_\varepsilon = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\varepsilon\}$
- For all  $\varepsilon > 0$ ,  $(\mathbf{VP})_\varepsilon$  writes : Find  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$  such that

$$\forall (\mathbf{v}, q) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, p_\varepsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V_\varepsilon', V_\varepsilon}$$

$$a_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) := \int_{\Omega_f^\varepsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, dx$$

$$+ \int_{\Omega_s} (\underline{C}(x) \underline{\nabla} \mathbf{u} : \underline{\nabla} \bar{\mathbf{v}} - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, dx + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma$$



## Uniform Estimates

Some resonant frequencies may appear in the solid

### Assumption (SA)

*The angular frequency  $\omega$  is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{C}(x) \underline{\nabla} \mathbf{u} \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

### Theorem

*Under Assumption (SA),  $\exists \varepsilon_0 > 0$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathbf{VP})_\varepsilon$  with data  $F \in V'_\varepsilon$  has a unique solution  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$  and*

$$\|\mathbf{u}_\varepsilon\|_{1, \Omega_s} + \|p_\varepsilon\|_{1, \Omega_\varepsilon} \leq C \|F\|_{V'_\varepsilon}.$$

**Application** : Convergence of an asymptotic expansion as  $\varepsilon \rightarrow 0$ .

**Key for the proof**: introduce a "Scaled Problem"

## Scaled Problem $(\mathfrak{P})_\epsilon$

### Formulation in a fixed domain

- For any function  $p$  defined in  $\Omega_f^\epsilon$ , define  $\mathfrak{p}$  in  $\Omega_f := \Gamma \times (0, 1)$  :

$$\mathfrak{p}(\mathbf{x}) = p(\mathbf{x}(t), S), \quad S = \frac{s}{\epsilon} \in (0, 1).$$

- $V = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f) \mid p(\cdot, 1) = 0 \text{ on } \Gamma\}$
- $(\mathfrak{P})_\epsilon$  writes : Find  $(\mathbf{u}, p) \in V$  such that  $\forall (\mathbf{v}, q) \in V$

$$\begin{aligned} \epsilon \int_0^1 \int_\Gamma [(1 + \epsilon S c(t))^{-2} \partial_t p \partial_t q + \epsilon^{-2} \partial_S p \partial_S q - \kappa^2 p \bar{q}] (1 + \epsilon S c(t)) dt dS \\ + a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) dt = \langle \mathfrak{F}_\epsilon, (\mathbf{v}, q) \rangle_{V', V} \end{aligned}$$

## Scaled Problem $(\mathfrak{P})_\varepsilon$

Uniform estimates

### Lemma

Under Assumption (SA),  $\exists \varepsilon_0 > 0$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , any solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  of problem  $(\mathfrak{P})_\varepsilon$  with a data  $\mathfrak{F}_\varepsilon \in V'$  satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

### Theorem

Under Assumption (SA),  $\exists \varepsilon_0 > 0$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathfrak{P})_\varepsilon$  with data  $\mathfrak{F}_\varepsilon \in V'$  has a unique solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  which satisfies

$$\sqrt{\varepsilon} \|\partial_t \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

## Scaled Problem $(\mathfrak{P})_\varepsilon$

### Proof of Lemma

Proof by contradiction.

Assume :  $\exists(\mathbf{u}_m, \mathbf{p}_m) \in V$ ,  $m \in \mathbb{N}$ , satisfying  $(\mathfrak{P})_{\varepsilon_m}$  associated with  $\varepsilon_m \rightarrow 0$  and  $\mathfrak{F}_m \in V'$  such that

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

- 1 Prove that  $\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C$   
 with  $W = \{(\mathbf{u}, \mathbf{p}) \in V \mid \mathbf{p} \in \mathbf{H}^1(0, 1; L^2(\Gamma)), \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}$
- 2 Compactness argument :  
 $\mathbf{u}_m \rightarrow \mathbf{u}$  in  $L^2(\Omega_s)$  and  $\mathbf{p}_m \rightarrow \mathbf{p} = 0$  in  $L^2(\Omega_f)$ .
- 3 Prove that  $\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} = 1$
- 4 Using (SA), we prove  $\mathbf{u} = 0$  : contradiction

# Outline

- 1 Uniform Estimates
- 2 Equivalent Conditions**
- 3 Numerical Simulations

# Methodology

**Framework** :  $\Gamma$ ,  $\Gamma^\varepsilon$  smooth curves in  $\mathbb{R}^2$ . Smooth data

- Step 1 : Derive an Asymptotic Expansion for  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  when  $\varepsilon \rightarrow 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots,$$

$$p_\varepsilon(x) = p_0(x; \varepsilon) + \varepsilon p_1(x; \varepsilon) + \varepsilon^2 p_2(x; \varepsilon) + \dots, \quad p_j(x; \varepsilon) = p_j\left(t, \frac{s}{\varepsilon}\right).$$

- Step 2 : Derive Equivalent Conditions of order  $k \in \mathbb{N}$  :  
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- Step 3 : Stability of Equivalent models  $\mathbf{u}_\varepsilon^k$ 
  - (i) Prove well-posedness and estimates
  - (ii) Show that  $\mathbf{u}_\varepsilon^k$  satisfies

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1}$$

## Step 1 : Multiscale Expansion

### First terms

At step  $n = 0$ , we obtain

$$p_0 = 0 ,$$

and then  $\mathbf{u}_0$  solves the problem

$$\begin{cases} \nabla \cdot \underline{\mathbf{C}}(x) \underline{\nabla} \mathbf{u}_0 + \omega^2 \rho \mathbf{u}_0 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -p_i \mathbf{n} & \text{on } \Gamma . \end{cases}$$

At step  $n = 1$ , we find successively

$$p_1(\cdot, S) = (S - 1) (\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i)$$

and that  $\mathbf{u}_1$  solves

$$\begin{cases} \nabla \cdot \underline{\mathbf{C}}(x) \underline{\nabla} \mathbf{u}_1 + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} - \partial_n p_i \mathbf{n} & \text{on } \Gamma . \end{cases}$$

## Step 1 : Multiscale Expansion

### First terms

At step  $n = 2$ ,

$$p_2(\cdot, S) = -(S^2 - 1) \frac{c(t)}{2} [\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i] + (S - 1) \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

and then,  $\mathbf{u}_2$  solves

$$\begin{cases} \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}_2 + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} - \frac{1}{2} c(t) [\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i] \mathbf{n} & \text{on } \Gamma. \end{cases}$$



## Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^{N+1} := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^{N+1} := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon$$

Evaluation of the right hand sides in

$$\begin{cases} \kappa^2 r_\epsilon^{N+1} + \Delta r_\epsilon^{N+1} & = f_\epsilon & \text{in } \Omega_f^\epsilon \\ \omega^2 \rho \mathbf{r}_\epsilon^{N+1} + \nabla \cdot \underline{C}(x) \nabla \mathbf{r}_\epsilon^{N+1} & = 0 & \text{in } \Omega_s \\ \partial_n r_\epsilon^{N+1} - \rho_f \omega^2 \mathbf{r}_\epsilon^{N+1} \cdot \mathbf{n} & = g_\epsilon & \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\epsilon^{N+1}) + r_\epsilon^{N+1} \mathbf{n} & = 0 & \text{on } \Gamma \\ r_\epsilon^{N+1} & = 0 & \text{on } \Gamma^\epsilon. \end{cases}$$

The RHS are explicit:  $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$  in  $\Omega_f^\epsilon$ ;  $g_\epsilon = \mathcal{O}(\epsilon^N)$  on  $\Gamma$

**Optimal Estimates :**  $\|\mathbf{r}_\epsilon^{N+1}\|_{1, \Omega_s} + \sqrt{\epsilon} \|r_\epsilon^{N+1}\|_{1, \Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

## Step 2 : Equivalent Conditions on $\Gamma$

Identify a simpler problem

$$(\mathbf{P}_\varepsilon^k) \quad \begin{cases} \omega^2 \rho \mathbf{u}_\varepsilon^k + \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}_\varepsilon^k = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k) = \mathbf{h}_{k,\varepsilon} & \text{on } \Gamma, \end{cases}$$

with  $\mathbf{B}_{k,\varepsilon}$  a surfacic differential operator.

- $k \in \{0, 1, 2\}$  for a Dirichlet b.c. on  $\Gamma^\varepsilon$
- $k \in \{0, 1\}$  for a Fourier-Robin b.c. on  $\Gamma^\varepsilon$

## Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$ :

$$\mathbf{T}(\mathbf{u}_0) = -p_j \mathbf{n} \quad \text{on } \Gamma .$$

- $k = 1$ :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - \epsilon \omega^2 \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\epsilon} \quad \text{on } \Gamma .$$

- $k = 2$ :

$$\mathbf{T}(\mathbf{u}_\epsilon^2) - \epsilon \omega^2 \rho_f \left( 1 - \frac{\epsilon}{2} c(t) \right) \mathbf{u}_\epsilon^2 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{2,\epsilon} \quad \text{on } \Gamma .$$

## Step 2 : Equivalent Conditions

Example : Order 1

$\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$  satisfies the elastic equation in  $\Omega_s$  and

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - \varepsilon \omega^2 \rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n} \mathbf{n} = -\mathbf{p}_i \mathbf{n} - \varepsilon \partial_n \mathbf{p}_i \mathbf{n} - \varepsilon^2 \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma$$

We infer a simpler problem

$$(\mathbf{P}_\varepsilon^1) \quad \begin{cases} \omega^2 \rho \mathbf{u}_\varepsilon^1 + \nabla \cdot \underline{\mathbf{C}}(x) \underline{\nabla} \mathbf{u}_\varepsilon^1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon \omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\varepsilon} & \text{on } \Gamma, \end{cases}$$

## Step 2 : Equivalent Conditions

### Fourier-Robin b.c.

Fourier-Robin b.c. :  $\partial_n p_\epsilon - i\kappa p_\epsilon = 0$  on  $\Gamma^\epsilon$

- $k = 0$  :

$$\mathbf{T}(\mathbf{u}_\epsilon^0) - i\omega c \rho_f \mathbf{u}_\epsilon^0 \cdot \mathbf{n} \mathbf{n} = \mathbf{g}_{0,\epsilon} \quad \text{on } \Gamma$$

- $k = 1$  :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c \rho_f [1 - \epsilon c(t)] \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} + \epsilon c^2 \rho_f \partial_t^2 (\mathbf{u}_\epsilon^1 \cdot \mathbf{n}) \mathbf{n} = \mathbf{g}_{1,\epsilon} \quad \text{on } \Gamma$$

## Step 3 : Stability of Equivalent Conditions

**Step 3.1** : Prove that problems  $(\mathbf{P}_\varepsilon^k)$  are well-posed, and uniform estimates.

### Proposition

*Under Assumption (SA),  $\exists \varepsilon_0 > 0$  s.t.  $\forall \varepsilon \in (0, \varepsilon_0)$ ,  $\forall k \in \{0, 1, 2\}$ , the problem  $(\mathbf{P}_\varepsilon^k)$  with data  $h_k \in L^2(\Gamma)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in \mathbf{H}^1(\Omega_s)$*

$$\|\mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C \|h_k\|_{0, \Gamma}$$

**Application** : Convergence of an asymptotic expansion of  $\mathbf{u}_\varepsilon^k$  as  $\varepsilon \rightarrow 0$ .

## Step 3 : Order of Equivalent Conditions

**Step 3.2 :** Prove that the solution  $\mathbf{u}_\epsilon^k$  satisfies an error estimate

$$\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{1,\Omega_s} \leq C\epsilon^{k+1}.$$

**Proof :**

(i) Derive an expansion of  $\mathbf{u}_\epsilon^k$  and show that :

$$\begin{aligned}\mathbf{u}_\epsilon &= \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots + \epsilon^k\mathbf{u}_k + \mathbf{r}_\epsilon^{k+1}, \\ \mathbf{u}_\epsilon^k &= \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots + \epsilon^k\mathbf{u}_k + \tilde{\mathbf{r}}_\epsilon^{k+1}.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\epsilon^{k+1} - \tilde{\mathbf{r}}_\epsilon^{k+1}\|_{1,\Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\epsilon^{k+1}) + \mathbf{B}_{k,\epsilon}(\tilde{\mathbf{r}}_\epsilon^{k+1}) = \mathcal{O}(\epsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the step 3.1, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\epsilon^{k+1}\|_{1,\Omega_s} \leq C\epsilon^{k+1}.$$

# Outline

- 1 Uniform Estimates
- 2 Equivalent Conditions
- 3 Numerical Simulations**



# Finite Element Method

**Joint work with : J. Diaz (INRIA).**

We use a Discontinuous Galerkin Method (IPDGM).

- $\omega = 1.5 \times 10^6$
- $\Omega_s$  is a disk of radius 0.01
- Source :  $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$  with  $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$ -finite elements (Lagrange) available in the Library Hou10ni
- Navier equation :  $\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0$  in  $\Omega_s$ .

$$\text{Lamé coefficients : } \mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}$$

Young's modulus :  $E = 70 \times 10^9$  Pa (aluminum)

Poisson ratio :  $\nu = 0.33$

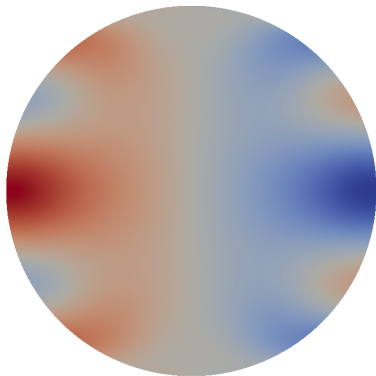
- $c = 1500 \text{ m.s}^{-1}$   
 $\rho_f = 1000 \text{ kg.m}^{-3}$   
 $\rho = 2700 \text{ kg.m}^{-3}$ .

## Numerical Tests

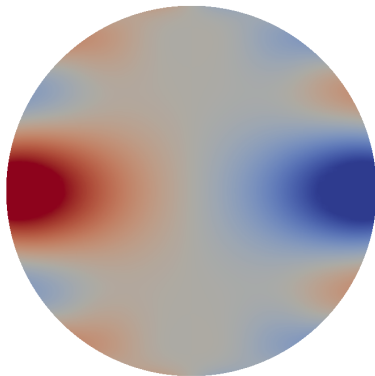
$\varepsilon = 0.001$ ;

$\mathbf{u}_\varepsilon$ : analytical solution;

$\mathbf{u}_\varepsilon^2$ : FE solution, equivalent model of order 2.



$|\operatorname{Re} \mathbf{u}_\varepsilon|$



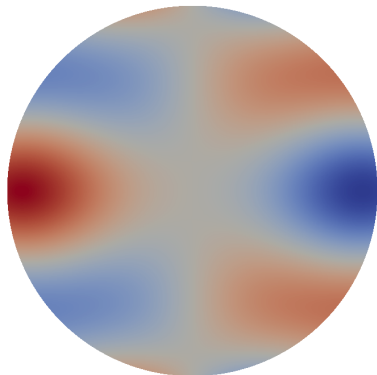
$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

## Numerical Simulations

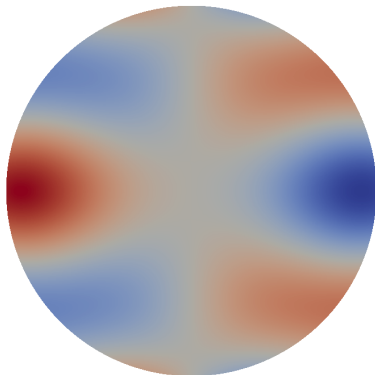
$\varepsilon = 0.0001$ ;

$\mathbf{u}_\varepsilon$ : analytical solution;

$\mathbf{u}_\varepsilon^2$ : FE solution, equivalent model of order 2.



$|\operatorname{Re} \mathbf{u}_\varepsilon|$



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

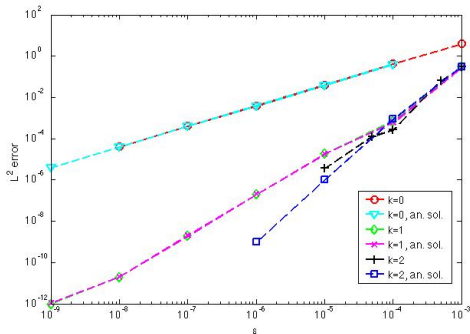
# Convergence of the models

Dirichlet b.c.

For  $k \in \{0, 1, 2\}$ , we plot the  $L^2$ -errors  $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s}$  w.r.t.  $\varepsilon$ :

$\mathbf{u}_\varepsilon$ : analytical solution

$\mathbf{u}_\varepsilon^k$ : FE solution / analytical solution



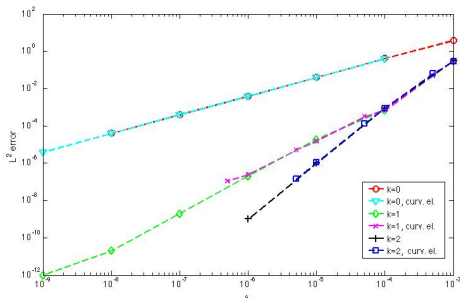
# Convergence of the models

## Dirichlet b.c.

For  $k \in \{0, 1, 2\}$ , we plot the  $L^2$ -errors  $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$  w.r.t.  $\epsilon$ :

$\mathbf{u}_\epsilon$ : analytical solution

$\mathbf{u}_\epsilon^k$ : analytical solution / FE solutions with Curved elements



The convergence rate coincide with the theory  $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s} = \mathcal{O}(\epsilon^{k+1})$

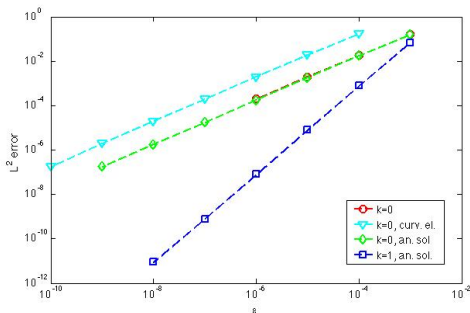
# Convergence of the models

## Fourier-Robin b.c.

For  $k \in \{0, 1\}$ , we plot the  $L^2$ -errors  $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$  w.r.t.  $\epsilon$ ,

$\mathbf{u}_\epsilon$ : analytical solution

$\mathbf{u}_\epsilon^k$ : analytical solution / FE solutions



# Prospects

- 1 3D Modeling
- 2 Higher Order Equivalent Conditions
- 3 Thin layer with varying thickness ...

Thank you for attention!