Asymptotic Models for an Elasto-Acoustic Problem with a Thin Layer

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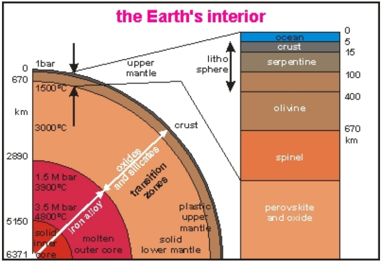
Motivations

HPC-GA Project

- **High Performance Computing for Geophysics Applications** : Earthquakes
- **Numerical Models** : Time dependent Elasto-Acoustic Coupling
  - Elasto-Acoustic Coupling : to reproduce earthquakes
  - Modeling the effects of the ocean on seismic waves

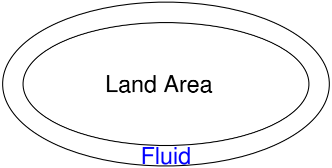


- The medium : a Land Area surrounded by a thin Fluid zone

Motivations

Configuration, Difficulty and Method

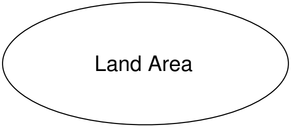
- **Configuration**



- **Difficulty**

  Apply a FEM on a mesh with thin cells in the Fluid and much larger in the solid

- **Method**
  1. Replace the Fluid layer by an Equivalent boundary condition
     Approach : an asymptotic method
  2. Couple this condition with the elastic wave equation
  3. Use a FEM



Motivations

Bibliography and Outline

Bibliography

- Modeling Fluid-Solid Interaction Problems

  JONES 83, LUKE-MARTIN 95, NATROCHVILI ET AL. 00, ... ,
  MONK-SELGAS 09

- Derivation of Equivalent Conditions for thin layer Problems

  LAFITTE 93, ENGQUIST-NÉDÉLEC 93, VOLAKIS-SENIOR 95,
  BENDALI-LEMRABET 96, CALOZ-COSTABEL-DAUGE-VIAL 06 ...

Outline

- Asymptotic Analysis for the time-harmonic **Elasto-acoustic** equations in a domain with a thin acoustic layer.

Outline

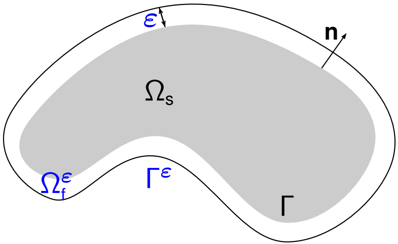
1. Uniform Estimates
2. Equivalent Conditions
3. Numerical Simulations

An Elasto-Acoustic Problem with a Thin Layer

Framework and Issue

The Problem $(\mathbf{P}_\varepsilon)$ set in a smooth bounded domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

$$
\begin{array}{lll}
\Delta p + \kappa^2 p = 0 & \text{in} & \Omega_f^\varepsilon \\
\nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 & \text{in} & \Omega_s \\
\partial_{\mathbf{n}} p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i & \text{on} & \Gamma \\
\mathbf{n}\, \underline{C}(x) \underline{\nabla} \mathbf{u} = -p\mathbf{n} - p_i \mathbf{n} & \text{on} & \Gamma \\
p = 0 & \text{on} & \Gamma^\varepsilon
\end{array}
$$



Assumption

(i) $\underline{C}(x) = (C_{ijkl}(x))$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$.

(ii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ.

(iii) $\exists \alpha > 0, \quad \forall \xi = (\xi_{ij})$ *sym. tensor,* $\displaystyle\sum_{i,j,k,l} C_{ijkl}\, \xi_{ij} \overline{\xi_{kl}} \geqslant \alpha \sum_{i,j} |\xi_{ij}|^2$.

Weak Solutions

Variational Problem $(\mathbf{VP})_\varepsilon$

- $V_\varepsilon = \{(\mathbf{u},\mathrm{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(\Omega_f^\varepsilon) \mid \gamma_0 \mathrm{p} = 0 \quad \text{on } \Gamma^\varepsilon \}$
- For all $\varepsilon > 0$, $(\mathbf{VP})_\varepsilon$ writes : Find $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in V_\varepsilon$ such that

$$\forall (\mathbf{v},\mathrm{q}) \in V_\varepsilon \,, \qquad a_\varepsilon\left((\mathbf{u}_\varepsilon,\mathrm{p}_\varepsilon),(\mathbf{v},\mathrm{q})\right) = \langle F,(\mathbf{v},\mathrm{q})\rangle_{V'_\varepsilon, V_\varepsilon}$$

$$a_\varepsilon\left((\mathbf{u},\mathrm{p}),(\mathbf{v},\mathrm{q})\right) := \int_{\Omega_f^\varepsilon} \left(\nabla \mathrm{p} \cdot \nabla \bar{\mathrm{q}} - \kappa^2 \mathrm{p}\bar{\mathrm{q}}\right)\, \mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_s} \left(\underline{C}(x)\underline{\nabla}\mathbf{u} : \underline{\nabla}\bar{\mathbf{v}} - \omega^2 \rho \mathbf{u}\bar{\mathbf{v}}\right) \mathrm{d}\mathbf{x} + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u}\cdot\mathbf{n}\,\bar{\mathrm{q}} + \mathrm{p}\bar{\mathbf{v}}\cdot\mathbf{n}\right)\, \mathrm{d}s$$

Uniform Estimates

Some resonant frequencies may appear in the solid

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{C}(x)\underline{\nabla}\mathbf{u} + \omega^2 \rho \mathbf{u} = 0 & \text{in} \quad \Omega_s \\ \underline{C}(x)\underline{\nabla}\mathbf{u}\,\mathbf{n} = 0 & \text{on} \quad \Gamma\,. \end{cases}$$

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{VP})_\varepsilon$ with data $F \in V'_\varepsilon$ has a unique solution $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in V_\varepsilon$ and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_f^\varepsilon} \leqslant C\|F\|_{V'_\varepsilon}\,.$$

Application : Convergence of an asymptotic expansion as $\varepsilon \to 0$.

Key for the proof: introduce a ”Scaled Problem”

Scaled Problem $(\mathfrak{P})_\varepsilon$

Formulation in a fixed domain

- For any function p defined in Ω_f^ε, define $\mathfrak{p}$ in $\Omega_f := \Gamma \times (0,1)$:

$$\mathrm{p}(\mathbf{x}) = \mathfrak{p}(\Psi(y_\alpha), S)\,, \quad S = \frac{s}{\varepsilon} \in (0,1)\,.$$

- $V = \{(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(\Omega_f) \,|\, \mathfrak{p}(.\,,1) = 0 \quad \text{on}\, \Gamma\}$
- $(\mathfrak{P})_\varepsilon$ writes : Find $(\mathbf{u}, \mathfrak{p}) \in V$ such that $\forall (\mathbf{v}, \mathfrak{q}) \in V$

$$\varepsilon \int_0^1 \int_\Gamma \left[(\mathbb{I} + \varepsilon S \mathcal{R})^{-2} \nabla_\Gamma \mathfrak{p} \nabla_\Gamma \bar{\mathfrak{q}} + \varepsilon^{-2} \partial_S \mathfrak{p} \partial_S \bar{\mathfrak{q}} - \kappa^2 \mathfrak{p} \bar{\mathfrak{q}} \right] \det(\mathbb{I} + \varepsilon S \mathcal{R}) \, \mathrm{d}\Omega_f$$

$$+ a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \, \bar{\mathfrak{q}} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n} \right) \, \mathrm{d}\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V', V}$$

Scaled Problem $(\mathfrak{P})_\varepsilon$

Uniform estimates

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ of problem $(\mathfrak{P})_\varepsilon$ with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon)\|_{0,\Omega_s \times \Omega_\mathfrak{f}} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_\varepsilon\|_{0,\Gamma} \leqslant C \|\mathfrak{F}_\varepsilon\|_{V'} \,.$$

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$, the problem $(\mathfrak{P})_\varepsilon$ with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon}\|\nabla_\Gamma \mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \sqrt{\varepsilon}^{-1}\|\partial_S \mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \|\mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \|\mathbf{u}_\varepsilon\|_{1,\Omega_s} \leqslant C \|\mathfrak{F}_\varepsilon\|_{V'} \,.$$

Sketch of the proof of the Lemma

Proof by contradiction.
Assume : $\exists (\mathbf{u}_m, \mathfrak{p}_m) \in V$, $m \in \mathbb{N}$, satisfying $(\mathfrak{P})_{\varepsilon_m}$ associated with $\varepsilon_m \to 0$ and $\mathfrak{F}_m \in V'$ such that

$$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_{0,\Omega_s \times \Omega_\mathfrak{f}} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_m\|_{0,\Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \to 0$$

1. Prove that $\|(\mathbf{u}_m, \mathfrak{p}_m)\|_W \leqslant C$
   with $W = \{(\mathbf{u}, \mathfrak{p}) \in V \,|\, \mathfrak{p} \in \mathrm{H}^1(0,1;\mathrm{L}^2(\Gamma)), \mathfrak{p}(.\,,1) = 0 \quad \text{on } \Gamma\}$
2. Compactness argument :
   $\mathbf{u}_m \to \mathbf{u}$ in $\mathbf{L}^2(\Omega_s)$ and $\mathfrak{p}_m \to \mathfrak{p} = 0$ in $\mathrm{L}^2(\Omega_\mathfrak{f})$.
3. Prove that $\|\mathbf{u}\|_{0,\Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1$
4. Using (SA), we prove $\mathbf{u} = 0$: contradiction

Outline

Methodology

Framework : Γ, Γ^ε smooth surfaces in $\mathbb{R}^3$. Smooth data

- Step 1 : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ when $\varepsilon \to 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon\mathbf{u}_1(x) + \varepsilon^2\mathbf{u}_2(x) + \cdots ,$$

$$\mathrm{p}_\varepsilon(x) = \mathrm{p}_0(x;\varepsilon) + \varepsilon\mathrm{p}_1(x;\varepsilon) + \varepsilon^2\mathrm{p}_2(x;\varepsilon) + \cdots , \quad \mathrm{p}_j(x;\varepsilon) = \mathfrak{p}_j(y_\alpha, \frac{S}{\varepsilon}).$$

- Step 2 : Derive Equivalent Conditions of order $k \in \mathbb{N}$:
  Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1})$$

- Step 3 : Prove the Stability of Equivalent models $\mathbf{u}^k_\varepsilon$ and

$$\|\mathbf{u}_\varepsilon - \mathbf{u}^k_\varepsilon\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1}$$

Step 1 : Multiscale Expansion

First terms

- At step $n = 0$, we obtain

$$\mathfrak{p}_0 = 0 \,,$$

and then $\mathbf{u}_0$ solves the problem

$$\begin{cases} \nabla \cdot \underline{C}(x)\underline{\nabla}\mathbf{u}_0 + \omega^2 \rho \mathbf{u}_0 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -\mathrm{p}_i \mathbf{n} & \text{on} \quad \Gamma \,. \end{cases}$$

- At step $n = 1$, we find successively

$$\mathfrak{p}_1(., S) = (S-1)\left(\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i\right) \,, \quad S \in (0,1)$$

and that $\mathbf{u}_1$ solves

$$\begin{cases} \nabla \cdot \underline{C}(x)\underline{\nabla}\mathbf{u}_1 + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\, \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i \mathbf{n} & \text{on} \quad \Gamma \,. \end{cases}$$

Step 1 : Multiscale Expansion

First terms

- At step $n = 2$,

$$\mathfrak{p}_2(., Y_3) = (S^2 - 1)\mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i] + (S - 1)\rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

and then, $\mathbf{u}_2$ solves

$$\begin{cases} \nabla \cdot \underline{C}(x)\underline{\nabla}\mathbf{u}_2 + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} + \mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i]\mathbf{n} & \text{on} \quad \Gamma\,. \end{cases}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\varepsilon^{N+1} := \mathbf{u}_\varepsilon - \sum_{n=0}^{N} \varepsilon^n \mathbf{u}_n \quad \text{in} \quad \Omega_s \,, \quad \text{and} \quad r_\varepsilon^{N+1} := p_\varepsilon - \sum_{n=0}^{N} \varepsilon^n p_n \quad \text{in} \quad \Omega_f^\varepsilon$$

Evaluation of the right hand sides in

$$\begin{cases} \kappa^2 r_\varepsilon^{N+1} + \Delta r_\varepsilon^{N+1} & = & f_\varepsilon & \text{in} & \Omega_f^\varepsilon \\ \omega^2 \rho \mathbf{r}_\varepsilon^{N+1} + \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{r}_\varepsilon^{N+1} & = & 0 & \text{in} & \Omega_s \\ \partial_\mathbf{n} r_\varepsilon^{N+1} - \rho_f \omega^2 \mathbf{r}_\varepsilon^{N+1} \cdot \mathbf{n} & = & g_\varepsilon & \text{on} & \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^{N+1}) + r_\varepsilon^{N+1} \mathbf{n} & = & 0 & \text{on} & \Gamma \\ r_\varepsilon^{N+1} & = & 0 & \text{on} & \Gamma^\varepsilon \,. \end{cases}$$

The RHS are explicit: $f_\varepsilon = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$ in Ω_f^ε ; $g_\varepsilon = \mathcal{O}(\varepsilon^N)$ on Γ

Optimal Estimates : $\|\mathbf{r}_\varepsilon^{N+1}\|_{1,\Omega_s} + \sqrt{\varepsilon} \|r_\varepsilon^{N+1}\|_{1,\Omega_f^\varepsilon} \leqslant C_N \varepsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

Identify a simpler problem

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \omega^2 \rho \mathbf{u}_{\varepsilon}^{k} + \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}_{\varepsilon}^{k} = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k}) = \mathbf{h}_{k,\varepsilon} & \text{on} \quad \Gamma , \end{cases}$$

with $\mathbf{B}_{k,\varepsilon}$ a surfacic differential operator.

- $k \in \{0, 1, 2\}$ for a Dirichlet b.c. on Γ^{ε}
- $k \in \{0, 1\}$ for a Fourier-Robin b.c. on Γ^{ε}

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = -\mathrm{p}_i \mathbf{n} \quad \text{on} \quad \Gamma\,.$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon\omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{1,\varepsilon} \quad \text{on} \quad \Gamma\,.$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon\omega^2 \rho_f \left(1 + \varepsilon\mathcal{H}\right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{2,\varepsilon} \quad \text{on} \quad \Gamma\,.$$

Step 2 : Equivalent Conditions

Example : Order 1

$\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$ satisfies the elastic equation in Ω_s and

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - \varepsilon\omega^2\rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n}\,\mathbf{n} = -\mathrm{p}_i \mathbf{n} - \varepsilon \partial_{\mathbf{n}} \mathrm{p}_i \mathbf{n} - \varepsilon^2 \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} \quad \text{on} \quad \Gamma$$

We infer a simpler problem

$$(\mathbf{P}^1_\varepsilon) \quad \begin{cases} \omega^2 \rho \mathbf{u}^1_\varepsilon + \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}^1_\varepsilon = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}^1_\varepsilon) - \varepsilon\omega^2 \rho_f \mathbf{u}^1_\varepsilon \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{1,\varepsilon} & \text{on} \quad \Gamma\,, \end{cases}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

Fourier-Robin b.c. : $\partial_{\mathbf{n}} \mathrm{p}_\varepsilon - i\kappa \mathrm{p}_\varepsilon = 0$ on Γ^ε

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^0) - i\omega c \rho_f \mathbf{u}_\varepsilon^0 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{g}_{0,\varepsilon} \quad \text{on} \quad \Gamma$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} + \varepsilon c^2 \rho_f \left(\Delta_\Gamma + \kappa^2 \mathbb{I} - 2i\kappa \mathcal{H} \mathbb{I}\right)\left(\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\right)\mathbf{n} = \mathbf{g}_{1,\varepsilon} \quad \text{on} \quad \Gamma$$

Step 3 : Stability and Convergence of Equivalent Conditions

Step 3.1 : Stability

Step 3.1 : Prove that problems $\left(\mathbf{P}_{\varepsilon}^{k}\right)$ are well-posed

Proposition

Under Assumption (SA), $\exists \varepsilon_0 > 0$ *s.t.* $\forall \varepsilon \in (0, \varepsilon_0)$, $\forall k \in \{0, 1, 2\}$, *the problem* $\left(\mathbf{P}_{\varepsilon}^{k}\right)$ *with data* $\mathrm{h}_k \in \mathrm{L}^2(\Gamma)$ *has a unique solution* $\mathbf{u}_{\varepsilon}^{k} \in \mathrm{H}^1(\Omega_{\mathrm{s}})$

$$\|\mathbf{u}_{\varepsilon}^{k}\|_{1,\Omega_{\mathrm{s}}} \leqslant C \|\mathrm{h}_k\|_{0,\Gamma}$$

Application : Convergence of an asymptotic expansion of $\mathbf{u}_{\varepsilon}^{k}$ as $\varepsilon \to 0$.

Step 3.2 : Convergence of Equivalent Conditions

Step 3.2 : Prove that the solution $\mathbf{u}_\varepsilon^k$ satisfies an error estimate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1} .$$

Proof :

(i) Derive an expansion of $\mathbf{u}_\varepsilon^k$ and show that :

$$\mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^{k+1} .$$

Recall :

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k + \mathbf{r}_\varepsilon^{k+1}$$

hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^{k+1} - \tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s} .$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on} \quad \Gamma .$$

According to the step 3.1, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1} .$$

Outline

Finite Element Method

Joint work with : J. Diaz (INRIA).

We use a Discontinuous Galerkin Method (IPDGM).

- 2D Computational domain : Ω_s is a disk of radius 0.01
- $\omega = 1.5 \times 10^6$
- Source : $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$-finite elements (Lagrange) available in the Library Hou10ni
- Navier equation :

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 \quad \text{in} \quad \Omega_s$$

with Lamé coefficients : $\mu = 26.32 \times 10^9$ and $\lambda = 51.08 \times 10^9$.

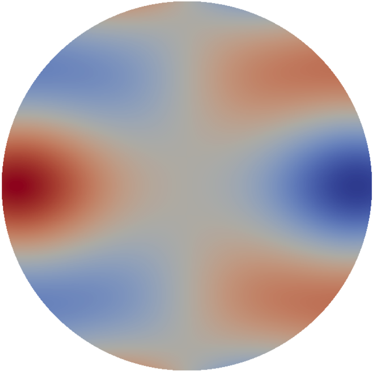
- $c = 1500$ m.s^{-1}, $\rho_f = 1000$ kg.m^{-3} , $\rho = 2700$ kg.m^{-3}.

Numerical Simulations

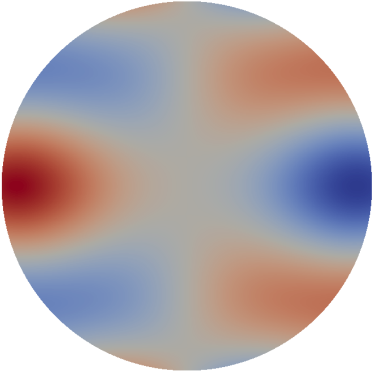
$\varepsilon = 0.0001$;

$\mathbf{u}_\varepsilon$: analytical solution;

$\mathbf{u}_\varepsilon^2$: FE solution (equivalent model of order 2).



$|\operatorname{Re}\mathbf{u}_\varepsilon|$



$|\operatorname{Re}\mathbf{u}_\varepsilon^2|$

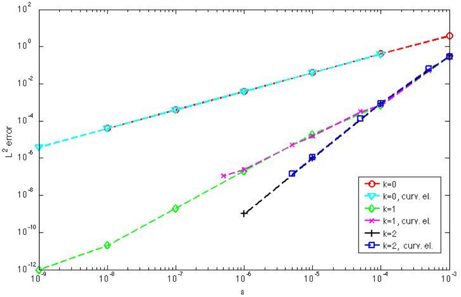
Convergence of the models

Dirichlet b.c.

For $k \in \{0, 1, 2\}$, we plot the L^2-errors $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε:

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution / FE solutions with Curved elements



The convergence rate coincide with the theory $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s} = \mathcal{O}(\varepsilon^{k+1})$

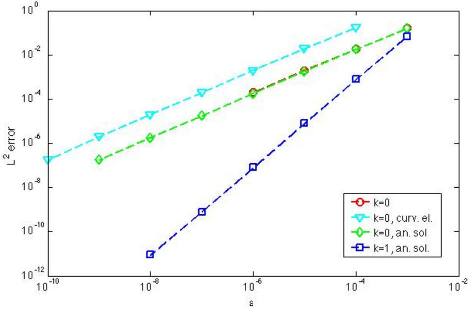
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2-errors $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε,

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution / FE solutions



Conclusion

1. Time dependent Equivalent Conditions
2. Higher Order Equivalent Conditions
3. Thin layer with variable thickness

Thank you for your attention!