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# On the decoding of quasi-BCH codes

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## Abstract

In this paper we investigate the structure of quasi-BCH codes. In the first part of this paper we show that quasi-BCH codes can be derived from Reed-Solomon codes over square matrices extending the known relation about classical BCH and Reed-Solomon codes. This allows us to adapt the Welch-Berlekamp algorithm to quasi-BCH codes. In the second part of this paper we show that quasi-BCH codes can be seen as subcodes of interleaved Reed-Solomon codes over finite fields. This provides another approach for decoding quasi-BCH codes.

**keywords:** Quasi-cyclic code, quasi-BCH code, BCH code, Reed-Solomon, interleaved code

## 1 Introduction

Many codes with best known minimum distances are quasi-cyclic codes or derived from them [LS03, Gra07]. This family of codes is therefore very interesting. Quasi-cyclic codes were studied and applied in the context of McEliece's cryptosystem [McE78, BCGO09] and Niederreiter's [Nie86, LDW94]. They permit to reduce the size of keys in opposition to Goppa codes. However, since the decoding of random quasi-cyclic codes is difficult, only quasi-cyclic alternant codes were proposed for the latter cryptosystem. The high structure of alternant codes is actually a weakness and two cryptanalysis were proposed in [FOPT10, UL10]

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## 1.1 Our contributions

In this paper we investigate the structure of quasi-BCH codes. In the first part of this paper we show that quasi-BCH codes can be derived from Reed-Solomon codes over square matrices. It is well known that BCH codes can be obtained from Reed-Solomon codes [MS86, Theorem 2, page 300]. We extend this property to quasi-BCH codes which allows us to adapt the Welch-Berlekamp algorithm to quasi-BCH codes.

**Theorem 1.** *Let  $\Gamma \in M_{\ell \times \ell}(\mathbb{F}_{q^s})$  be a primitive  $m$ -th root of unity and  $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$ . Then there exists a RRS code  $\mathcal{R}$  over the ring  $M_{\ell \times \ell}(\mathbb{F}_{q^s})$  with parameters  $[n, n - \delta + 1]_{M_{\ell \times \ell}(\mathbb{F}_{q^s})}$  and a  $\mathbb{F}_q$ -linear,  $F_q$ -isometric embedding  $\psi : \mathcal{C} \rightarrow \mathcal{R}$ .*

In the second part we show that quasi-BCH codes can be seen as subcodes of interleaved Reed-Solomon codes.

**Theorem 2.** *The quasi-BCH code  $\mathcal{C}$  over  $\mathbb{F}_q$  is an interleaved code of  $\ell$  subcodes of Reed-Solomon codes over  $\mathbb{F}_{q^s}$  in the following sense: there exists  $\ell$  Reed-Solomon codes  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  over  $\mathbb{F}_q$  and an isometric isomorphism from  $\mathcal{C}$ , equipped with the  $\ell$ -block distance, to a subcode of the interleaved code with respect to  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ .*

## 1.2 Related work

In [LF01, LS01],  $\ell$ -quasi-cyclic codes of length  $m\ell$  are seen as  $R$ -submodules of  $R^\ell$  for a certain ring  $R$ . However, in [LF01], Gröbner bases are used in order to describe polynomial generators of quasi-cyclic codes whereas in [LS01], the authors decompose quasi-cyclic codes as direct sums of shorter linear codes over various extensions of  $\mathbb{F}_q$  (when  $\gcd(m, q) = 1$ ). This last work leads to an interesting trace representation of quasi-cyclic codes. In [CCN10], the approach is more analogous to the cyclic case. The authors consider the factorization of  $X^m - 1 \in M_\ell(F_q)[X]$  with reversible polynomials in order to construct  $\ell$ -quasi-cyclic codes canceled by those polynomials and called  $\Omega(P)$ -codes. This leads to the construction of self-dual codes and codes beating known bounds. But the factorization of univariate polynomials over a matrix ring remains difficult. In [Cha11] the author gives an improved method for particular cases of the latter factorization problem.

# 2 Prerequisites

## 2.1 Reed-Solomon codes over rings

We recall some basic definitions of Reed-Solomon codes over rings in this section. We let  $A$  be a ring with identity, we denote by  $A^\times$  the *group of units* of  $A$  and by  $Z(A)$  the *center* of  $A$ , the commutative subring of  $A$  consisting of all the elements of  $A$  which commutes with all the other elements of  $A$ . We denote by

$A[X]$  the ring of polynomials over  $A$  and by  $A[X]_{<k}$  the polynomials over  $A$  of degree at most  $k - 1$ .

**Definition 1.** *Let*

$$f = \sum_{i=0}^d f_i X^i \in A[X]$$

*be a polynomial with coefficients in  $A$  and  $a \in A$ . We call left evaluation of  $f$  at  $a$  the quantity*

$$f(a) := \sum_{i=0}^d f_i a^i \in A$$

*and right evaluation of  $f$  at  $a$  the quantity*

$$(a)f := \sum_{i=0}^d a^i f_i \in A.$$

**Remark 1.** *For  $f, g \in A[X]$  and  $a \in A$ , we obviously have  $f(a) = (a)f$  whenever  $a \in Z(A)$ ,  $(f + g)(a) = f(a) + g(a)$ ,  $(a)(f + g) = (a)f + (a)g$ . If  $a$  commutes with all the coefficients of  $g$  we also have  $(fg)(a) = f(a)g(a)$  and  $(a)(gf) = (a)g(a)f$ .*

**Definition 2.** *Let  $0 < k \leq n$  be two integers. Let  $(x_1, \dots, x_n)$  and  $v = (v_1, \dots, v_n)$  be two vectors of  $A^n$  be such that  $x_i - x_j \in A^\times$  and  $x_i x_j = x_j x_i$  for all  $i \neq j$  and  $v_i \in A^\times$  for all  $i$ .*

*The left submodule of  $A^n$  generated by the vectors*

$$(f(x_1) \cdot v_1, \dots, f(x_n) \cdot v_n) \in A^n \text{ with } f \in A[X]_{<k}$$

*is called a left generalized Reed-Solomon code (LGRS) over  $A$  with parameters  $[v, x, k]_A$  or  $[n, k]$  if there is no confusion on  $x$  and  $v$ .*

*The right submodule of  $A^n$  generated by the vectors*

$$(v_1 \cdot (x_1)f, \dots, v_n \cdot (x_n)f) \in A^n \text{ with } f \in A[X]_{<k}$$

*is called a right generalized Reed-Solomon code (RGRS) over  $A$  with parameters  $[v, x, k]_A$  or  $[n, k]$  if there is no confusion on  $x$  and  $v$ . The vector  $x$  is called the support of the code. If  $v = (1, \dots, 1)$ , the codes constructed above are called left Reed-Solomon (LRS) and right Reed-Solomon (RRS) codes.*

**Definition 3.** *Let  $x = (x_1, \dots, x_n) \in A^n$ . We call the Hamming weight of  $x$  the number of nonzero coordinates.*

$$w(x) := w(x_1, \dots, x_n) = |\{i : x_i \neq 0\}|.$$

*Let  $y = (y_1, \dots, y_n) \in A^n$ . The Hamming distance between  $x$  and  $y$  is*

$$d(x, y) = w(x - y) = |\{i : x_i \neq y_i\}|.$$

*The minimum distance of any subset  $S \subseteq A^n$  is defined as*

$$\min \{d(x, y) : x, y \in S \text{ and } x \neq y\}.$$

**Proposition 1.** *A LGRS (resp. RGRS) code is a free left (resp. right) submodule of  $A^n$ . A LGRS (resp. RGRS) code with parameters  $[n, k]$  has minimum distance  $n - k + 1$ .*

*Proof.* It suffices to see that the maps

$$\begin{aligned} A^n &\longrightarrow A^n \\ (a_1, \dots, a_n) &\longmapsto (a_1 v_1, \dots, a_n v_n) \\ (a_1, \dots, a_n) &\longmapsto (v_1 a_1, \dots, v_n a_n) \end{aligned}$$

are respectively left and right isometric automorphisms of  $A^n$ .  $\square$

## 2.2 Quasi cyclic and quasi BCH codes

Quasi cyclic codes form an important family of codes defined as follow.

**Definition 4.** *Let  $T : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$  to be the left cyclic shift defined by*

$$T(c_1, c_2, \dots, c_n) = (c_2, c_3, \dots, c_1).$$

*We call  $\ell$ -quasi-cyclic code over  $\mathbb{F}_q$  of length  $n$  any code of length  $n$  over  $\mathbb{F}_q$  stable by  $T^\ell$ . If the context is clear we will simply say  $\ell$ -quasi-cyclic code.*

We will focus in this paper on quasi-BCH codes which form a subfamily of quasi-cyclic codes. They can be seen as a generalization of BCH codes in the context of quasi-cyclic codes. For we need primitive roots of unity defined in an extension of  $\mathbb{F}_q$ , say  $\mathbb{F}_{q^s}$  to construct BCH codes over  $\mathbb{F}_q$ .

**Proposition 2.** *Then there exists a primitive  $q^{s\ell} - 1$ -th root of unity in  $M_\ell(\mathbb{F}_{q^s})$ .*

*Proof.* The proof can be found in [BCQ12b, Proposition 16, page 911].  $\square$

**Definition 5.** *Let  $\Gamma$  be a primitive  $m$ -th root of unity in  $M_\ell(\mathbb{F}_{q^s})$  and  $\delta \leq m$ . We define the  $\ell$ -quasi-BCH code of length  $m\ell$ , with respect to  $\Gamma$ , with designed minimum distance  $\delta$ , over  $\mathbb{F}_q$  by*

$$\text{Q-BCH}_q(m, \ell, \delta, \Gamma) := \left\{ (c_1, \dots, c_m) \in (\mathbb{F}_q^\ell)^m : \sum_{j=0}^{m-1} (\Gamma^i)^j (c_{j+1})^T = 0 \text{ for } i = 1, \dots, \delta - 1 \right\}.$$

*Note that  $\text{Q-BCH}_q(m, \ell, \delta, \Gamma)$  is a quasi-cyclic code.*

**Definition 6.** *The  $\ell$ -block weight of  $(x_{11}, \dots, x_{1\ell}, \dots, x_{m1}, \dots, x_{m\ell}) \in \mathbb{F}_q^{m\ell}$  is defined to be*

$$\text{Block-w}_\ell(x) := |\{i : (x_{i1}, \dots, x_{i\ell}) \neq 0\}|.$$

*The  $\ell$ -block distance between  $x, y \in \mathbb{F}_q^{m\ell}$  is defined to be  $\text{Block-w}_\ell(x - y)$ .*

### 3 Reed-Solomon codes and quasi-BCH codes

#### 3.1 The relation between quasi-BCH and Reed-Solomon codes

We show in this section that under certain assumptions on the support of Reed-Solomon codes, the dual of a LRS code is a RRS code. From this fact we show that quasi-BCH can be constructed from Reed-Solomon codes over square matrices rings. In this Subsection we let  $A$  designate a finite ring with identity.

**Definition 7.** Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be two vectors of  $A^n$ . The inner product is defined as

$$\langle x, y \rangle := \sum_{i=0}^n x_i y_i.$$

**Remark 2.** Let  $S$  be a subset of  $A^n$ . Then the set  $\{x \in A^n : \forall s \in S, \langle s, x \rangle = 0\}$  denoted by  $S^\perp$  is called the right dual of  $S$  and is a right submodule of  $A^n$ . Similarly, Let  $S$  be a subset of  $A^n$ . Then the set  $\{x \in A^n : \forall s \in S, \langle x, s \rangle = 0\}$  denoted by  ${}^\perp S$  is called the left dual of  $S$  and is a left submodule of  $A^n$ . Note that for all  $x, y \in A^n$  and  $\mu \in A$  we have  $\mu \langle x, y \rangle = \langle \mu x, y \rangle$  and  $\langle x, y \rangle \mu = \langle x, y \mu \rangle$ .

**Definition 8.** We say that  $a \in A$  is a primitive  $m$ -th root of unity if  $a^m = 1$  and  $\forall 0 \leq i < m, (a^i - 1) \in A^\times$ .

**Remark 3.** Let  $x = (1, \gamma, \gamma^2, \dots, \gamma^{m-1}) \in A^m$  where  $\gamma$  is a primitive  $m$ -th root of unity. Then a RRS or LRS code whose support is  $x$  is cyclic.

**Proposition 3.** Let  $\gamma \in A$  be a primitive  $m$ -th root of unity. Let  $x = (1, \gamma, \gamma^2, \dots, \gamma^{m-1}) \in A^m$ . Then the right (resp. left) dual of the LGRS (resp. RGRS) code with parameters  $[x, x, k]_A$  is the RRS (resp. LRS) code with parameters  $[x, n - k]_A$ .

*Proof.* We denote respectively by  $\mathcal{L}$  and  $\mathcal{R}$  the left generalized Reed-Solomon code with parameters  $[x, x, k]_A$  and the right Reed-Solomon code with parameters  $[x, n - k]_A$ .

First note that  $\mathcal{L}$  is generated by the vectors

$$(1, \gamma^i, \gamma^{2i}, \dots, \gamma^{(m-1)i}) \text{ for } i = 1, \dots, k$$

and that  $\mathcal{R}$  is generated by the vectors

$$(1, \gamma^i, \gamma^{2i}, \dots, \gamma^{(m-1)i}) \text{ for } i = 0, \dots, n - k - 1.$$

And we have for  $0 \leq i + j < n - 1$  in the commutative ring  $Z(A)[\gamma]$

$$\sum_{i=0}^{m-1} \gamma^{(i+1)\ell} \cdot \gamma^{j\ell} = \sum_{i=0}^{m-1} (\gamma^{i+j+1})^\ell = \frac{1 - (\gamma^{i+j+1})^m}{1 - \gamma^{i+j+1}} = 0.$$

Therefore, by Proposition 1 and Remark 2,  $\mathcal{L}^\perp \subseteq \mathcal{R}$  and  ${}^\perp\mathcal{R} \subseteq \mathcal{L}$ .

Again by Proposition 1 and Remark 2 an element  $x \in A^n$  lies in  $\mathcal{L}^\perp$  if and only if

$$\left[ \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \gamma & \gamma^2 & \dots & \gamma^{m-1} \\ 1 & \vdots & \vdots & \dots & \vdots \\ 1 & \gamma^{k-1} & \gamma^{2(k-1)} & \dots & \gamma^{(k-1)(m-1)} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \gamma & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \gamma^{m-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right] = 0. \quad (1)$$

But in the commutative ring  $Z(A)[\gamma]$  the matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \gamma & \gamma^2 & \dots & \gamma^{2(k-1)} \\ 1 & \vdots & \vdots & \dots & \vdots \\ 1 & \gamma^{k-1} & \gamma^{2(k-1)} & \dots & \gamma^{(k-1)(k-1)} \end{pmatrix} \in M_{k \times k}(Z(A)[\gamma])$$

is invertible. Therefore  $H$  is also invertible in  $M_{k \times k}(A)$  and thus induces a group automorphism of  $A^k$ . If we let  $x_H = (x_1, \dots, x_k)$ ,  $x_U = (x_{k+1}, \dots, x_n)$ , we can rewrite equation (1) as

$$\left( H \mid U \right) \begin{pmatrix} x_H \\ x_U \end{pmatrix} = 0 \text{ and } \left( H \mid 0 \right) \begin{pmatrix} x_H \\ 0 \end{pmatrix} = - \left( 0 \mid U \right) \begin{pmatrix} 0 \\ x_U \end{pmatrix}.$$

For each choice of  $x_U$  we have only one possible value for  $x_H$ . Thus  $|\mathcal{L}^\perp| = |A|^{n-k} = |\mathcal{R}|$  by Proposition 1 and therefore  $\mathcal{L}^\perp = \mathcal{R}$ . Similarly, we have  ${}^\perp\mathcal{R} = \mathcal{L}$ .  $\square$

**Theorem 3.** *Let  $\Gamma \in M_{\ell \times \ell}(\mathbb{F}_{q^s})$  be a primitive  $m$ -th root of unity and  $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$ . Then there exists a RRS code  $\mathcal{R}$  over the ring  $M_{\ell \times \ell}(\mathbb{F}_{q^s})$  with parameters  $[n, n - \delta + 1]_{M_{\ell \times \ell}(\mathbb{F}_{q^s})}$  and a  $\mathbb{F}_q$ -linear,  $F_q$ -isometric embedding  $\psi : \mathcal{C} \rightarrow \mathcal{R}$ .*

*Proof.* A parity-check matrix of  $\mathcal{C}$  is

$$H = \begin{pmatrix} I_\ell & \Gamma & \dots & \Gamma^{m-1} \\ I_\ell & \Gamma^2 & \dots & \Gamma^{2(m-1)} \\ \vdots & \vdots & \dots & \vdots \\ I_\ell & \Gamma^{\delta-1} & \dots & \Gamma^{(\delta-1)(m-1)} \end{pmatrix} \in M_{(\delta-1)\ell, m\ell}(\mathbb{F}_{q^s}).$$

Remark that  $H$  is a generator matrix of the LGRS code with parameters  $[x, x, \delta - 1]_{M_{\ell \times \ell}(\mathbb{F}_{q^s})}$  over the ring  $M_{\ell \times \ell}(\mathbb{F}_{q^s})$  and by Proposition 3 its dual is the RRS with parameters  $[x, \delta - 1]_{M_{\ell \times \ell}(\mathbb{F}_{q^s})}$ .

Now let

$$\begin{aligned} \psi : \mathcal{C} &\longrightarrow (M_{\ell \times \ell}(\mathbb{F}_{q^s}))^m \\ (c_{11}, \dots, c_{1\ell}, \dots, c_{m1}, \dots, c_{m\ell}) &\longmapsto \left[ \begin{pmatrix} c_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{1\ell} & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} c_{m1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{m\ell} & 0 & \dots & 0 \end{pmatrix} \right]. \end{aligned}$$

Obviously,  $\psi$  is  $\mathbb{F}_q$ -linear, injective and isometric and by the above remark we have  $\psi(\mathcal{C}) \subseteq \mathcal{R}$ .  $\square$

Theorem 3 generalizes the well-known [MS86, Theorem 2, page 300] relation between BCH codes and Reed-Solomon codes. The above relation will allow us to adapt the unique decoding algorithm from [BCQ12a] to quasi-BCH codes.

### 3.2 The Welch-Berlekamp algorithm for quasi-BCH codes

In this Subsection we let  $A$  designate a finite ring with identity. Before giving the Welch-Berlekamp decoding algorithm, we need to define what the *evaluation* of a bivariate polynomial over  $A$  is. Let  $Q = \sum Q_{i,j} X^i Y^j \in A[X, Y]$  be such a polynomial. We define the *evaluation of  $Q$  at  $(a, b) \in A^2$*  to be

$$(a, b)Q = \sum a^i b^j Q_{i,j} \in A.$$

Be careful of the order of  $a$ ,  $b$  and  $Q_{i,j}$ . This choice will be explained in the proof of Lemma 1. Let  $f \in A[X]$ , we define the *evaluation of  $Q$  at  $f$*  to be

$$(X, f(X))Q = \sum X^j (f(X))^j Q_{i,j} \in A[X].$$

As in the univariate case, the evaluation maps defined above are not ring homomorphisms in general.

**Lemma 1.** *Let  $g \in A[X]$ ,  $Q \in A[X, Y]$  of degree at most 1 in  $Y$  and  $a \in A$ . Then*

$$(a)((X, g(X))Q) = (a, (a)g)Q.$$

*Proof.* We write

$$\begin{aligned} Q(X, Y) &= Q_0(X) + Q_1(X)Y \\ &= Q_0(X) + \left( \sum_i Q_{1i} X^i \right) Y. \end{aligned}$$

The proof is an easy calculation:

$$\begin{aligned} (a)((X, g(X))Q) &= (a) \left( Q_0(X) + \sum_i X^i g(X) Q_{1i} \right) \\ &= (a)Q_0 + \sum_i a^i (a)g Q_{1i} \\ &= (a, (a)g)Q \text{ by definition.} \end{aligned}$$

$\square$



We let  $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$ ,  $\tau = \lfloor \frac{\delta-1}{2} \rfloor$ ,  $n = m$ ,  $k = n - \delta + 1$  and

$$\left[ \begin{array}{c} \text{pr} : (M_{\ell \times \ell}(\mathbb{F}_{q^s}))^m \longrightarrow \mathbb{F}_q^{m\ell} \\ \left( \begin{pmatrix} a_{11}^1 & \cdots & a_{1\ell}^1 \\ \vdots & & \vdots \\ a_{\ell 1}^1 & \cdots & a_{\ell\ell}^1 \end{pmatrix}, \dots, \begin{pmatrix} a_{11}^m & \cdots & a_{1\ell}^m \\ \vdots & & \vdots \\ a_{\ell 1}^m & \cdots & a_{\ell\ell}^m \end{pmatrix} \right) \end{array} \right] \mapsto (a_{11}^1, \dots, a_{\ell 1}^1, \dots, a_{11}^m, \dots, a_{\ell 1}^m).$$

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**Algorithm 1** Welch-Berlekamp for quasi-BCH codes

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**Input:** a received vector  $y \in \mathbb{F}_q^{m\ell}$  with at most  $\tau$  errors.

**Output:** the unique codeword within distance  $\tau$  of  $y$ .

- 1:  $(Z_1, \dots, Z_m) \leftarrow \psi(y)$  where  $\psi$  is the map from Theorem 3.
  - 2: Find  $Q = Q_0(X) + Q_1(X)Y \in (M_{\ell \times \ell}(\mathbb{F}_{q^s})[X])[Y]$  of degree 1 such that
    1.  $(\Gamma^{i-1}, Z_i)Q = 0$  for all  $i = 1, \dots, m-1$ ,
    2.  $\deg Q_0 \leq n - \tau - 1$ ,
    3.  $\deg Q_1 \leq n - \tau - 1 - (k-1)$ .
  - 3:  $f \leftarrow$  the unique root of  $Q$  in  $(M_{\ell \times \ell}(\mathbb{F}_{q^s})[X])_{<k}$  such that  $d((Z_1, \dots, Z_m), ((I_\ell)f, \dots, (\Gamma^{m-1})f)) \leq \tau$ .
  - 4: **return**  $\text{pr}((I_\ell)f, (\Gamma)f, \dots, (\Gamma^{m-1})f)$ .
- 

**Lemma 2.** *Let  $y \in \mathbb{F}_q^{m\ell}$  be a received word containing at most  $\tau$  errors. Then there exists a nonzero bivariate polynomial  $Q = Q_0 + Q_1Y \in (M_{\ell \times \ell}(\mathbb{F}_{q^s})[X, Y])$  satisfying*

1.  $(\Gamma^{i-1}, Z_i)Q = 0$  for  $i = 1, \dots, n$ .
2.  $\deg Q_0 \leq n - \tau - 1$ .
3.  $\deg Q_1 \leq n - \tau - 1 - (k-1)$ .

*Proof.* We solve the problem with linear algebra over  $\mathbb{F}_{q^s}$ . We have, for each column of the solution,  $n\ell$  equations and  $\ell[(n-\tau) + (n-\tau-(k-1))] = \ell(n+1)$  unknowns by Proposition 1.  $\square$

**Lemma 3.** *Let  $Q \in (M_{\ell \times \ell}(\mathbb{F}_{q^s})[X, Y])$  satisfying the three conditions of Lemma 2 and  $f \in (M_{\ell \times \ell}(\mathbb{F}_{q^s})[X])_{<k}$  be such that  $d((Z_1, \dots, Z_m), ((I_\ell)f, \dots, (\Gamma^{m-1})f)) \leq \tau$ . Then  $(X, f(X))Q = 0$ .*

*Proof.* The polynomial  $(X, f(X))Q$  has degree at most  $n - \tau - 1$ . By Lemma 1 we have  $(\Gamma^{i-1})((X, f(X))Q) = (\Gamma^{i-1}, (\Gamma^{i-1})f)Q = (\Gamma^{i-1}, Z_i)Q = 0$  for at least  $n - \tau$  values of  $i \in \{1, \dots, n\}$ . And therefore we must have  $(X, f(X))Q = 0$ .  $\square$

**Proposition 4.** *Algorithm 1 works correctly as expected and can correct up to  $\lfloor \frac{\delta-1}{2} \rfloor$  errors.*

*Proof.* This is a direct consequence of Lemmas 2 and 3.  $\square$

## 4 Quasi-BCH codes as interleaved codes

In this Section we prove that quasi BCH codes can be viewed as an interleaving of classical BCH codes. We fix for this Section  $\Gamma \in M_{\ell \times \ell}(\mathbb{F}_{q^s})$  a primitive  $m$ -th root of unity and  $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$ . We first recall the definition of interleaved codes.

**Definition 9.** Let  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  be error correcting codes over  $\mathbb{F}_q$ . The interleaved code  $\mathcal{C}$  with respect to  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  is a subset of  $M_{\ell \times m}(\mathbb{F}_q)$ , equipped with the  $\ell$ -block distance with respect to the columns, such that  $c \in \mathcal{C}$  if and only if the  $i$ -th row of  $c$  is a codeword of  $\mathcal{C}_i$  for  $i = 1, \dots, \ell$ .

**Lemma 4.** The matrix  $\Gamma$  diagonalizes over an extension of  $\mathbb{F}_{q^s}$  and its eigenvalues are all primitive  $m$ -th roots of unity.

*Proof.* Let  $\mathbb{F}_{q^{s'}} \supseteq \mathbb{F}_{q^s}$  be the splitting field of  $X^m - 1$ . The polynomial  $X^m - 1$  is a multiple of the minimal polynomial  $\mu(X)$  of  $\Gamma$ . Hence the eigenvalues of  $\Gamma$  are  $m$ -roots of unity. Let  $P \in \text{GL}_\ell(\mathbb{F}_{q^{s'}})$  be such that  $P^{-1}\Gamma P$  is diagonal. Now if an eigenvalue  $\lambda_i$  of  $\Gamma$  has order  $d < m$ , then

$$P^{-1}(\Gamma^d - I_\ell)P = \begin{pmatrix} \lambda_1^d & & & & \\ & \ddots & & & \\ & & \lambda_i^d & & \\ & & & \ddots & \\ & & & & \lambda_\ell^d \end{pmatrix} - I_\ell$$

is singular as its  $i$ -th diagonal element would be zero. Consequently  $\Gamma^d - I_\ell \notin \text{GL}_\ell(\mathbb{F}_{q^{s'}})$  which is absurd.  $\square$

**Theorem 4.** The quasi-BCH code  $\mathcal{C}$  over  $\mathbb{F}_q$  is an interleaved code of  $\ell$  subcodes of Reed-Solomon codes over  $\mathbb{F}_{q^{s'}}$  in the following sense: there exists  $\ell$  Reed-Solomon codes  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  over  $\mathbb{F}_q$  and an isometric isomorphism from  $\mathcal{C}$ , equipped with the  $\ell$ -block distance, to a subcode of the interleaved code with respect to  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ .

*Proof.* We take the notation of the proof of Lemma 4. Recall that

$$H = \begin{pmatrix} I_\ell & \Gamma & \dots & \Gamma^{m-1} \\ I_\ell & \Gamma^2 & \dots & \Gamma^{2(m-1)} \\ \vdots & \vdots & & \vdots \\ I_\ell & \Gamma^{\delta-1} & \dots & \Gamma^{(\delta-1)(m-1)} \end{pmatrix} \in M_{(\delta-1)\ell, m\ell}(\mathbb{F}_{q^s})$$

is a parity check matrix for  $\mathcal{C}$  (proof of Theorem 3). By Lemma 4 we have that

$$(c_{11}, \dots, c_{1\ell}, \dots, c_{m1}, \dots, c_{m\ell}) \in \mathcal{C} \iff$$

$$\begin{pmatrix} P^{-1} & & & \\ & \ddots & & \\ & & P^{-1} & \end{pmatrix} \begin{pmatrix} I_\ell & \Gamma & \dots & \Gamma^{m-1} \\ I_\ell & \Gamma^2 & \dots & \Gamma^{2(m-1)} \\ \vdots & \vdots & & \vdots \\ I_\ell & \Gamma^{\delta-1} & \dots & \Gamma^{(\delta-1)(m-1)} \end{pmatrix} \begin{pmatrix} P & & & \\ & \ddots & & \\ & & P & \end{pmatrix} \times$$

$$\left[ \begin{pmatrix} P^{-1} & & & \\ & \ddots & & \\ & & P^{-1} & \end{pmatrix} \begin{pmatrix} c_{11} \\ \vdots \\ c_{1\ell} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{m\ell} \end{pmatrix} \right] = 0$$

and  $(c_{11}, \dots, c_{1\ell}, \dots, c_{m1}, \dots, c_{m\ell}) \in \mathbb{F}_q^{m\ell}$

Let

$$\begin{pmatrix} v_{11} \\ \vdots \\ v_{1\ell} \\ \vdots \\ v_{m1} \\ \vdots \\ v_{m\ell} \end{pmatrix} = \begin{pmatrix} P^{-1} & & & \\ & \ddots & & \\ & & P^{-1} & \end{pmatrix} \begin{pmatrix} c_{11} \\ \vdots \\ c_{1\ell} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{m\ell} \end{pmatrix} \quad (2)$$

Denote by  $\sigma$  the application defined by equation (2). Then

$$(c_{11}, \dots, c_{1\ell}, \dots, c_{m1}, \dots, c_{m\ell}) \in \mathcal{C} \iff$$

$$\sigma^{-1}(v_{11}, \dots, v_{1\ell}, \dots, v_{m1}, \dots, v_{m\ell}) \in \mathbb{F}_q^{m\ell} \text{ and for } i = 1, \dots, \ell$$

$$\begin{pmatrix} 1 & \lambda_i & \dots & \lambda_i^{m-1} \\ 1 & \lambda_i^2 & \dots & \lambda_i^{2(m-1)} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_i^{\delta-1} & \dots & \lambda_i^{(\delta-1)(m-1)} \end{pmatrix} \begin{pmatrix} v_{1i} \\ \vdots \\ v_{mi} \end{pmatrix} = 0. \quad (3)$$

Then it is straightforward that  $\sigma$  is an isometric isomorphism from  $\mathcal{C}$  equipped with the  $\ell$ -block distance and  $\sigma(\mathcal{C})$ , which is by equation (3) a subcode of the interleaved code with respect to  $\ell$  subcodes of Reed-Solomon codes over  $\mathbb{F}_q$ . For  $i = 1, \dots, \ell$  take  $\mathcal{C}_i$  to be the Reed-Solomon code defined by the parity check matrix of equation (3).  $\square$

Note that if the minimal polynomial of  $\Gamma$  has degree one:  $\Gamma = X - \lambda$ , then  $s' = s$  and  $\Gamma$  diagonalizes as  $\lambda I_\ell$ . Consequently the Reed-Solomon codes

$\mathcal{C}_1, \dots, \mathcal{C}_\ell$  are isomorphic, as they are defined by the same control equations in equation (3). In such a case, we can apply the result on the correction capacity for interleaved Reed-Solomon codes [SSB06, BKY07].

**Corollary 1.** *There exists a decoding algorithm that is guaranteed to correct up to  $\frac{\delta-1}{2}$  errors. In particular, if the minimal polynomial of  $\Gamma$  has degree 1 over  $\mathbb{F}_{q^s}$  then it can correct up to  $\frac{\ell}{\ell+1}(\delta-1)$  errors with high probability.*

*Proof.* Taking the notation of Theorem 4 and if  $y = c + e$  is a received word, one can decode  $\sigma(y)$  with the decoding algorithms of  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  obtaining  $c' \in \mathbb{F}_{q^s}^{m\ell}$ . Then  $c = \sigma^{-1}(c')$ .

If the minimal polynomial of  $\Gamma$  has degree 1, then  $\mathcal{C}_1 = \mathcal{C}_2 = \dots = \mathcal{C}_\ell$  and one can apply the algorithm of [BKY07] or [SSB06].  $\square$

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