k-Selection Protocols from Energetic Complexity Perspective

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***Abstract*—In this paper we discuss energetic complexity aspects of k-Selection protocols for the single-hop radio network (that is equivalent to Multiple Access Channel model). The aim is to grant each of k activated stations exclusive access to the communication channel. We consider both deterministic as well as randomized model. Our main goal is to investigate relations between minimal time of execution (time complexity) and energy consumption (energetic complexity). We present lower bound for energetic complexity for some classes of protocols for k-Selection. We also present randomized protocol efficient in terms of both time and energetic complexity.**

I. Introduction

This paper is devoted to energetic efficiency of protocols solving k-Selection problem. Let us recall that the aim of this problem is to grant each of k (out of n) activated stations exclusive access to the communication channel.

It was originally formulated for MAC (Multiple Access Channel). However, this problem can also be stated in an equivalent form for the single-hop radio network. In such a system, for practical reasons, energy consumption is of critical importance. Indeed, while discussing radio networks we often have in mind small battery-supplied sensing devices, that cannot be easily re-charged.

The problem is discussed in various settings. In all of cases we have, however, n stations and some k of them are *activated* and want to broadcast their messages to all other stations. The message is successfully transmitted only if exactly one station transmits at a given time. In the case of simultaneous transmission of two or more stations a *collision* occurs and none of messages is delivered to any recipient. If the collision is distinguishable from the background noise we call the model with *collision detection* (CD). Otherwise, the model is described as *no-collision detection* (noCD). The core of the problem is that the subset of activated stations is unknown in advance (except that its cardinality is constrained) and stations have to communicate via very restricted communication channel.

Remarks about the model of energy: In this paper we concentrate on energy complexity of k-Selection protocols understood as the maximal energetic effort over all stations. The energy usage of a particular station is the number of rounds when the station transmits, whether the message is delivered or not. In many applications it is required that all (or almost all) devices have to be working for proper acting of network. Therefore, a lifespan of the system is determined by the most loaded station, which motivated us to consider maximal energy usage. Such approach is used in literature, however one can also find papers wherein authors consider the average energetic effort of stations instead. It should be noted that in most of the cases in the analysis of the algorithms, finding or even estimating the maximal effort over all stations is technically more challenging.

Similarly, there are two common approaches to energetic expense of station being in listening mode. The first one is to take into account both transmitting and listening rounds. In particular, it is the case when all stations are located close to each other. The second approach assumes that energetic cost of listening is dramatically smaller than transmitting and can be treated as negligible. Note also that in the case of some of considered classes of protocols, both approaches are equivalent. Indeed, for example in oblivious algorithms discussed in Section V receiving any transmission does not influence the execution and stations can be switched-off instead of being in listening mode.

Previous work: The k-Selection problem is a classic issue in distributed computing. In recent years it has gained additional interests motivated by expansions of radio (sensor) networks technologies. It is hard to enumerate all important literature related to this topic, thus we mention only the most fundamental papers we are aware of. Komlos and Greenberg considered the oblivious model with collision detection. They showed in [1] that k-Selection can be deterministically completed in time $O(k\log(\frac{n}{k}))$. This result can be adapted to the model without collision detection. Moreover the lower bound for the time complexity $\Omega(k\log(\frac{n}{k}))$ which was obtained in [2] holds also for model without collision detection. In [2] the superimposed codes method as well as selective families approach were used. Hayes presented in [3] the adaptive solution which satisfies the same time complexity as for the oblivious model. In [4] the lower bound $\Omega(k\log_k n)$ for the family of adaptive deterministic protocols was proved. In a similar model, Martel [5] showed an interesting randomized approach for finding a maximal value among n stations, which succeeds in the expected time $O(k + \log n)$. Kowalski noted in [6] that Martel expected time complexity can be improved to $O(k + \log\log n)$ by using the Willard algorithm as a subprocedure. Martel algorithm can be easily adapted to k-Selection problem. Then the time complexity is $O(k)$, because only active stations transmit a message. Another important, recent paper is [7]. The randomized, adaptive solution presented by Anta and Mosteiro guarantees, that all of k stations successfully transmit a message in time $(e+1+\xi)k+O(\log^2(\frac{1}{\epsilon}))$ with the probability error ϵ from a reasonable interval and a fixed negligible constant ξ. In [8] authors analyzed a problem (connected with k-Selection) of learning a subset of m stations out of k active ones. Work of Nakano and Olariu [9] can be easily adapted to obtain algorithm solving w.h.p. (with high probability) k-Selection problem in $O(k)$ expected time and $O(\log\log k)$ expected energy.

Energetic efficiency of algorithms for radio networks is considered in several papers, devoted to initialization protocols [10], size approximation problem [11], alerts for weak devices [12] or routing aspects [13]. However, to the best of our knowledge, except [9] there are no results about energetic complexity issues of the k-Selection protocols or any other protocol that can be recycled for our problem in a straightforward manner.

Our Results: In the paper we show lower bounds on maximal energy usage for class of so-called *uniform* algorithms. We prove that any uniform algorithm solving k-Selection problem with expected time of execution $O(k \operatorname{polylog}(k))$ has energetic complexity $\Omega(\frac{\log k}{\log\log k})$. We present protocol for solving w.h.p. k-Selection problem in constant energy within $O(k^{1+\epsilon})$ rounds for any $\epsilon > 0$. We also give very general lower bound relating time and energy in a case of deterministic oblivious algorithms.

The paper is organized as follows: in Section II we describe the model in details and formulate the problem we investigate. Section III is devoted to analysis of randomized algorithms, where we provide lower bounds for uniform algorithms and we show algorithm efficient in terms of both energy and time complexity. We also discuss energetic complexity of some known, optimal (in terms of time of execution) protocols. In Section V we present a lower bound for oblivious deterministic algorithms. We conclude in Section VII.

II. Model

We consider a single-hop radio network with n stations. The set of stations is denoted by V. In the case of deterministic algorithms we assume that each station has a unique label from the set $\{1,\ldots,n\}$. Time is assumed to be slotted into rounds. We assume that stations are fully synchronized as if they had access to a global clock. At the beginning of the protocol's execution a subset of k *activated* stations have a message that has to be transmitted. Using terminology from [6] we consider *static* k-Selection — all algorithms are started in the same round.

Stations communicate via a single channel. In our paper we concentrate on the network with collision detection[1], i.e., the background noise that is received if no station transmits is distinguishable from the noise generated by two or more stations transmitting in the same round. Thus, we can have three states of the communication channel — SILENCE, SINGLE transmission and COLLISION.

We consider both deterministic as well as randomized algorithms. In the latter case we assume that stations are indistinguishable and have access to the perfect source of random bits. Moreover, sources of different stations are stochastically independent.

Energetic measures: In radio networks one of the main practical problems is the fact that all devices have limited energy resources and moreover in some realistic cases it is very hard to replace their batteries. Thus, the level of energy usage may really matter. In this paper we use the measure of energetic complexity defined as follows. We define $\mathcal{E}_v$, an *energetic effort* of a station $v \in V$, as the number of rounds wherein v transmitted. Note that both successful as well as unsuccessful (due to collisions) transmissions counts. The energetic complexity of the algorithm is defined as $\mathbb{E}\left[\max_{v\in V}\mathcal{E}_v\right]$ for the worst case over all subsets of activated stations. Note that this value is well defined also for deterministic algorithms. Let us stress that usually $\max_{v\in V}\mathbb{E}\left[\mathcal{E}_v\right] \neq \mathbb{E}\left[\max_{v\in V}\mathcal{E}_v\right]$. That is, we look for the expected energetic effort over all stations. Let us note that such measure has been used among others in [12], [14]. On the other hand in some remarkable papers some different metrics have been used.

III. Energy Efficient Randomized Algorithm

In this section we discuss randomized k-Selection protocols from energetic complexity perspective. First we present a lower bound for so-called uniform algorithms. We also confront the obtained result with other classes of algorithms. Then we present algorithms efficient both in terms of energy and time complexity. The protocol requires $O(k^{1+\epsilon})$ rounds, $\epsilon > 0$, after which w.h.p. all stations successfully transmit their messages. More importantly, the energy usage of each station can be bounded by constant dependent only on ϵ. Therefore, the maximal energetic complexity is $O(1)$, what was the main design goal.

A. Uniform Algorithms

Definition 1: Algorithm $\mathcal{A}$ solving k-Selection is called **uniform** if, and only if, in round i every station that has not yet transmitted successfully, transmits with probability p_i (the same for all active stations). Every other station is not transmitting in round i.

[1] Note however, that all of the results (including analysis of our protocol) remain true in model without collision detection

Note that p_i may depend on the state of the communication channel in previous rounds. In general, p_i can be even chosen randomly from some distribution during the execution of the protocol (all stations have to use, however, the same value p_i). Due to simplicity and robustness, uniform algorithms are commonly used. For example algorithms proposed by Martel in [5] and by Anta and Mosteiro in [7] are uniform ones.

B. Lower Bound for Uniform Algorithms

Before we introduce the key technical lemma let us recall that *selection resolution* (see e.g., [15]) is the problem of obtaining one SINGLE in possibly small number of rounds. More precisely, there are k stations that want to transmit, and the protocol is successfully completed if exactly one station transmits in a round. This problem is in fact equivalent to leader election in a Multiple Access Channel. Let us stress, however, that 1-Selection is a trivial problem that is **not** an instance of a leader election problem.

The lemma below shows some relation between time of execution and expected number of collisions.

Lemma 1: Let $k > 1$. If uniform algorithm $\mathcal{A}$ solves *selection resolution* in expected time t, then the expected number of rounds with COLLISION during the execution of $\mathcal{A}$ is at least $\frac{1}{128t^2}$.

Proof:

Algorithm is uniform, thus in the i-th round each station transmits independently with the same probability p_i. Note however, that in every execution the probabilities $\{p_i\}_i$ may differ and depend for example on the state of the channel in previous rounds. Let P_i be the random variable denoting the probability of transmission used by stations in round i. Finally T denotes run time of the algorithm and $\mathbb{E}[T] = t$. Algorithm works until first SINGLE appears, thus $P_i = 0$ for every $i > T$. Let B denote the random event that there is i such that $P_i \geq \frac{1}{2kt}$ and let $\bar{B}$ be its complement. We want to show, that $P[B] \geq \frac{1}{2}$. Note, that if for some i, $P_i < \frac{1}{2kt}$, then

$$P[\text{SINGLE in round } i] = kP_i\left(1 - P_i\right)^{k-1} \leq kP_i < \frac{1}{2t},$$

$$P[\text{SILENCE or COLLISION in round } i] > 1 - \frac{1}{2t}.$$

We want to bound the conditional expectation $\mathbb{E}[T|\bar{B}]$. The conditional expectation is well defined, if $P[\bar{B}] > 0$. But, if $P[\bar{B}] = 0$, then the statement $P[B] \geq \frac{1}{2}$ holds trivially.

$$\begin{aligned}\mathbb{E}[T|\bar{B}] &= \sum_{t' \geq 1} P[T \geq t'|\bar{B}] \\ &> \sum_{t' \geq 1} \left(1 - \frac{1}{2t}\right)^{t'} = 2t,\end{aligned}$$

$$\mathbb{E}[T] = \mathbb{E}[T|\bar{B}]P[\bar{B}] + \mathbb{E}[T|B]P[B] > 2tP[\bar{B}].$$

But $\mathbb{E}[T] = t$, thus $P[\bar{B}] < \frac{1}{2}$, and $P[B] > \frac{1}{2}$. Therefore, with probability more than $\frac{1}{2}$, during the execution of the algorithm there exists a slot i_0 with probability of transmission $P_i \geq \frac{1}{2kt}$.

Now we want to bound probability P_c of COLLISION in round $i = i_0$. It is clear that $P_c = 1 - (1 - P_i)^{k-1}\left(P_i k + (1 - P_i)\right)$. The following inequality $(1-x)^n \leq 1 - nx + \frac{1}{2}n(n-1)x^2$ works for $0 \leq x \leq 1$, and $n \in \mathbf{N}_+$ and it can be proven using a straightforward induction. Assume, that $P_i = \frac{1}{2kt}$. Then,

$$\begin{aligned}P_c &\geq 1 - \left(1 - \frac{k-1}{2kt} + \frac{(k-1)(k-2)}{8k^2t^2}\right)\left(\frac{k-1}{2kt} + 1\right) \\ &= \frac{(k-1)^2}{4k^2t^2} - \frac{(k-1)(k-2)}{8k^2t^2}\left(\frac{k-1}{2kt} + 1\right) \\ &\geq \frac{(k-1)^2}{8k^2t^2}\left(1 - \frac{k-1}{2kt}\right) \geq \frac{1}{64t^2}.\end{aligned}$$

We use the fact, that $k > 1$, thus $\frac{k-1}{k} \geq \frac{1}{2}$. We also use, that $t \geq 1$, because any algorithm requires at least one step to solve the *selection resolution*. We proved, that if $P_i = \frac{1}{2kt}$, then $P[\text{COLLISION in round } i] \geq \frac{1}{64t^2}$. Obviously, if $P_i \geq \frac{1}{2kt}$, then also $P[\text{COLLISION in round } i] \geq \frac{1}{64t^2}$, because the probability of transmission for each station increases. It follows that with probability at least $\frac{1}{2}$ during any execution of the algorithm there exists a round i_0, where probability of COLLISION is at least $\frac{1}{64t^2}$. This implies that the expected number of COLLISIONs in the algorithm $\mathcal{A}$ is at least $\frac{1}{128t^2}$. ■

Theorem 1: Any uniform k-Selection algorithm with expected time of execution $O(k \operatorname{polylog}(k))$ has energetic complexity $\Omega\left(\frac{\log k}{\log\log k}\right)$.

Proof:

Let us consider any k-Selection algorithm with expected time of execution $O(k \operatorname{polylog}(k))$. We show that the expected number of COLLISIONs during the execution is $\Omega\left(\frac{k}{\operatorname{polylog}(k)}\right)$.

By the i-th *era* we understand the number of rounds between $i-1$ and i-th successful transmissions for $1 < i \leq k$ (including the round with the i-th transmission). The 1st era is just the number of rounds before the first transmission. Let t_i be the expected time of i-th era and T be the expected run time of the algorithm. Moreover, let the station that transmitted successfully in i-th era be called i-th transmitter. Clearly, $\sum_{i=1}^{k} t_i = T$. Since $T \in O(k \operatorname{polylog}(k))$, there has to be $\Omega(k)$ eras, such that $t_i \in O(\operatorname{polylog}(k))$. From Lemma 1, we know that if era has expected run time t, the expected number of COLLISIONs is $\Omega\left(\frac{1}{t^2}\right)$. Finally we have $\Omega(k)$ eras with expected number of COLLISIONs equal $\Omega\left(\frac{1}{\operatorname{polylog}(k)}\right)$. Thus the expected number of COLLISIONs during the execution of the algorithm is $\Omega\left(\frac{k}{\operatorname{polylog}(k)}\right)$. Similarly, during the first $k - \sqrt{k}$ eras the expected number of COLLISIONs is $\Omega\left(\frac{k-\sqrt{k}}{\mathrm{polylog}(k)}\right) = \Omega\left(\frac{k}{\mathrm{polylog}(k)}\right)$.

Since the protocol is uniform, each active station is equally likely to transmit in a round with COLLISION. This can be represented in terms of balls and bins model. More precisely, stations are represented by bins. If COLLISION occurs we throw **one** ball to the bin randomly chosen from bins representing active stations. Clearly the number of balls in the most loaded bin is a lower bound for the number of transmissions of station with maximal number of transmissions[2].

Let us consider a group of the last $\sqrt{k}$ transmitters. All those transmitters are exposed to $\Omega\left(\frac{k}{\mathrm{polylog}(k)}\right)$ COLLISONs (in expectation). If there is $\Omega\left(\frac{k}{\mathrm{polylog}(k)}\right)$ balls than with high probability, $\Omega\left(\frac{\sqrt{k}}{\mathrm{polylog}(k)}\right)$ balls are placed in bins representing the last $\sqrt{k}$ transmitters.

From [16] we have, that in case with $m = \frac{\sqrt{k}}{\mathrm{polylog}(k)}$ balls and $n = \sqrt{k}$ bins, the maximum load is $\Omega\left(\frac{\log k}{\log\log k}\right)$ with high probability. Thus the expected maximum number of transmissions over last $\sqrt{k}$ transmitters is $\Omega\left(\frac{\log k}{\log\log k}\right)$. ∎

C. Non-uniform Algorithms

The result presented in the previous subsection implies that there is no uniform k-Selection algorithm working in linear time with maximum energy usage being $o(\frac{\log k}{\log\log k})$. However, there are non-uniform algorithms that are more efficient in terms of energy consumption. For example, the initialization algorithm by Nakano and Olariu [9] can be modified in a straightforward manner to obtain k-Selection algorithm with linear time of execution and no station being awake for more than $O(\log\log k)$ rounds w.h.p. Thus the number of transmissions of each station is $O(\log\log k)$ as well.

D. Energy Efficient Algorithm Description

In this section we present k-Selection algorithm with extremely small energy consumption and moderate time of execution. Our construction is also based on the protocol described by Nakano and Olariu in [9]. The algorithm consists of $3+\lfloor\log_2(1+\frac{1}{\epsilon})\rfloor$ iterations. In each of iterations, stations that have not transmitted successfully yet, try to transmit its message in one out of $\lceil 2k^{1+\epsilon}\rceil$ rounds. The choice is independent on other stations and uniform over all rounds of a particular iteration. The pseudo code of the protocol is shown in Algorithm 1.

E. Complexity Analysis

It should be clear that the energy usage of any station is at most max_{iter}. Similarly, one can see that the total time of the protocol is $max_{iter} \cdot rounds \in O(k^{1+\epsilon})$. The presented

[2]Note, that each collision affects always more than one station. For simplicity we use however only one ball.

Protocol 1 Energy Efficient k-Selection

$max_{iter} \leftarrow 3 + \lfloor\log_2(1+\frac{1}{\epsilon})\rfloor$ ▷ maximum number of iterations
$rounds \leftarrow \lceil 2k^{1+\epsilon}\rceil$ ▷ number of rounds per iteration
$iter \leftarrow 1$
$status \leftarrow$ COLLISION
while $iter \le max_{iter}$ and $status \ne$ SINGLE **do**
  $iter \leftarrow iter + 1$
  $i \leftarrow \mathrm{uniform}(\{1, 2, \ldots, rounds\})$ ▷ round number to transmit in
  for $round \leftarrow 1$ to $rounds$ **do**
    if $round = i$ **then**
      $status \leftarrow \mathrm{transmit}(packet)$ ▷ try to transmit

algorithm is of Monte Carlo type, which means that with a certain probability, after its execution some stations may fail to transmit. We show that the probability of failure is $O(\frac{1}{k})$.

Theorem 2: For any given $\epsilon > 0$, after execution of Algorithm 1 by k stations, all of them transmit successfully with probability at least $1 - O(\frac{1}{k})$.

Proof: Before we prove the theorem let us prove following lemma.

Lemma 2: Assume that n stations transmits uniformly and independently in one out of m rounds. For $t \ge 1$:

- if $\frac{n(n-1)}{6m} \ge t\log(n)$, then with probability exceeding $1 - \frac{1}{n^t}$, fewer than $\frac{2n(n-1)}{m}$ stations fail to transmit successfully,
- if $\frac{n(n-1)}{6m} < t\log(n)$, then with probability exceeding $1 - \frac{1}{n^t}$, fewer than $20\log(n)$ stations fail to transmit successfully.

Proof: Note that this lemma is a modification of a result of Nakano and Olariu from [9]. Using Corollary 4.2 in [9], we have that if $\frac{n(n-1)}{6m} \ge \log(nf(n))$ for some positive real-valued function $f(n)$, then with probability at least $1 - \frac{1}{nf(n)}$, fewer than $\frac{2n(n-1)}{m}$ stations fail to transmit. Thus, it is sufficient to take $f(n) = n^{t-1}$ to prove the first case. The second case is proved by a simple application of the Lemma 4.3 from [9]. ∎

Now we are ready to prove Theorem 2. Let us consider, what happens after first iteration of Algorithm 1: there are $n = k$ participating stations and $m = 2k^{1+\epsilon}$ rounds. If $\epsilon \ge 1$, then for sufficiently large k we have $\frac{n(n-1)}{6m} < \log(n)$. Therefore, with probability at least $1 - \frac{1}{k}$, after first iteration there are at most $20\log(k)$ remaining stations, for which with probability exceeding $1 - \frac{1}{k}$, two additional rounds are sufficient for successful transmission of all remaining station.

On the other hand, if $0 < \epsilon < 1$, then for sufficiently large k we have $\frac{n(n-1)}{6m} < \log(n)$, thus with probability exceeding $1 - \frac{1}{k}$, first iteration ends with fewer than $\frac{2n(n-1)}{m} < \frac{2k^2}{2k^{1+\epsilon}} = k^{1-\epsilon}$ remaining stations. Inductively, if i-th iteration starts with at most $k^{1-(2^{i-1}-1)\epsilon}$ stations and $(2^i-1)\epsilon < 1$, then by Lemma with probability at least $1-\frac{1}{k}$, after i-th iteration fewer than $k^{1-(2^i-1)\epsilon}$ pass to $(i+1)$-st iteration. Thus, until $i \geq \log_2(1+\frac{1}{\epsilon})$, with probability $1-O(\frac{1}{k})$ after i-th iteration there are fewer than $k^{1-(2^i-1)\epsilon}$ stations that still need to transmit. After iteration $\lfloor \log_2(1+\frac{1}{\epsilon})\rfloor$ we use the second case of the Lemma, thus w.h.p. the next iteration ends with $O(\log k)$ stations. Therefore, again, two additional rounds are sufficient to successful transmission of each station. ■

IV. Energetic Complexity of Martel Algorithm

We present analysis of energy complexity of the Martel algorithm that has optimal expected run time in the assumed model. We also give some general remarks about uniform k-Selection algorithms. The main result of this section expressed in Theorem 3 is also a motivation for constructing more efficient algorithms in terms of energy usage.

Theorem 3: Algorithm solving k-Selection proposed by Martel in [5] has expected energetic complexity $\Omega(\log k)$.

Proof: Let us denote the time between $(i-1)$-th and i-th SINGLE in Martel algorithm, as i-th era. First era is the time until first SINGLE appears. Let X_i be a random variable denoting time of the i-th era. We need to show that in Martel algorithm, for all $i > 1$, $\mathbb{E}[X_i] \in O(1)$. But this fact is proved by Martel in [5], in Lemmas 2.1, 2.2 and 2.3. If for some i, $\mathbb{E}[X_i] \in O(1)$, then from Lemma 1, the expected number of COLLISIONs in i-th era is $\Omega(1)$. Consider energy consumption of the last (i.e., the k-th) station denoted as $\mathcal{E}_{v_{last}}$. Station v_{last} has chance to participate in COLLISION in each era. If i-th era's expected number of COLLISIONs is δ then, since the algorithm is uniform, each active station in this era has equal chance to participate in COLLISION. Expected energy consumption of each active station in this round is at least $\frac{2\delta}{k-i+1}$. Thus

$$\mathbb{E}[\mathcal{E}_{v_{last}}] \in \Omega\left(\sum_{i=2}^{k} \frac{1}{k-i+1}\right).$$

We note that $\sum_{i=2}^{k} \frac{1}{k-i+1} = H_{k-1}$, where H_{k-1} is $(k-1)$-th harmonic number. Let us recall that harmonic number $H_n = \log n + \gamma + O(\frac{1}{n})$ and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant. Finally, $\mathbb{E}[\mathcal{E}_{v_{last}}] \in \Omega(\log k)$. It is clear that $\mathbb{E}[\mathcal{E}_{v_{last}}] \leq \mathbb{E}[\mathcal{E}_{max}]$. ■

V. Lower Bounds for Deterministic Oblivious Algorithms

In this section we investigate oblivious, deterministic k-Selection protocols. This means that schedule of transmissions for each station is defined before execution of the algorithm. That is, each station knows if it shall transmit in each round before the algorithm is started. In particular, decision of transmission does not depend on the state of the communication channel in previous rounds. Thus the algorithm can be viewed as an assignment of binary vectors to stations. More formally, for every station $v \in V$ we denote by $w(v)$ the binary vector, where $w(v)_i$ denotes i-th position in the vector $w(v)$, defined as follows. If station v is transmitting in round i, then $w(v)_i = 1$, otherwise $w(v)_i = 0$.

Below we recall the definition of superimposed codes introduced by Kautz and Singleton in [17]. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ be a set of binary words of length t. The number of vectors n is the size of code. Given k words $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$, we define the sum of vectors $c_{i_1} \vee c_{i_2} \vee \cdots \vee c_{i_k}$ as bitwise Boolean sum. We say that binary vector v covers vector v' if for each coordinate with value 1 in v', the corresponding coordinate in v is also 1.

Definition 2: Let r be a positive integer. We say that set of binary words $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ is r-superimposed code if for any distinct words $c_{i_0}, c_{i_1}, c_{i_2}, \ldots, c_{i_r}$, the word c_{i_0} is **not** covered by $c_{i_1} \vee c_{i_2} \vee \cdots \vee c_{i_r}$.

Algorithm solves the k-Selection problem if and only if the corresponding set of vectors is a $(k-1)$-superimposed code. Indeed, there is $1-1$ correspondence between superimposed codes and oblivious k-Selection algorithms pointed in [18]. In [19] Erdős, Frankl and Füredi proved theorem about families of sets which has direct application in superimposed codes.

Fact 1 (see [19, Proposition 2.1]): Let $f_k(t,\varepsilon)$ be the maximum size of the k-superimposed code of length t, where each codeword has exactly ε ones, then $f_k(t,\varepsilon) \leq \binom{t}{\lceil \frac{\varepsilon}{k} \rceil} / \binom{\varepsilon-1}{\lceil \frac{\varepsilon}{k} \rceil - 1}$.

Lower bound on length of k-superimposed codes implies lower bound on time complexity of any oblivious, deterministic k-Selection algorithms. Using techniques similar as in [19] we can bound the size of any k-superimposed code with restricted number of ones in codewords. In effect we can bound the time complexity of any oblivious k-Selection algorithm with energy complexity $\mathcal{E}_{max}$.

Fact 2: The binomial coefficient $\binom{n}{k}$, satisfies $\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}$.

Theorem 4: Run time t of any deterministic, oblivious algorithm solving k-Selection with energetic complexity $\mathcal{E}_{max}$ satisfies

$$t \in \Omega\left(\mathcal{E}_{max}\left(\frac{n}{(k-1)^2}\right)^{\frac{1}{\lceil \frac{\mathcal{E}_{max}}{k-1} \rceil}}\right).$$

Proof: In the proof we assume, that $k, t, \mathcal{E}_{max}$ depend on n, and n goes to infinity. Firstly we want to prove, that the relation $n \leq \sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} f_{k-1}(t,\mathcal{E})$, must hold for every deterministic, oblivious algorithm solving k-Selection in time t, and maximum energy consumption $\mathcal{E}_{max}$. We can partition vectors into groups of the same Hamming weight, i.e., $w(v) \in W_i$ if $h(w(v)) = i$, where $h(w)$ is Hamming weight of the vector w. Set $W = \{w(v) : v \in V\}$ is $(k-1)$-superimposed code, because algorithm solves k-Selection. Thus each set W_i is also $(k-1)$-superimposed. From the definition of the function f_k, $|W_i| \le f_{k-1}(t,i)$. On the other hand $n = \sum_{i=1}^{\mathcal{E}_{max}} |W_i|$. Thus $n \le \sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} f_{k-1}(t,\mathcal{E})$. From Fact 1 we obtain $f_{k-1}(t,\mathcal{E}) \le \binom{t}{\lceil \frac{\mathcal{E}}{k-1} \rceil} / \binom{\mathcal{E}-1}{\lceil \frac{\mathcal{E}}{k-1} \rceil - 1}$. Using an identity for the binomial coefficient and applying Fact 2 we have:

$$\begin{aligned}
\binom{\mathcal{E}-1}{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1} &= \frac{\mathcal{E} - \left\lceil \frac{\mathcal{E}}{k-1} \right\rceil + 1}{\mathcal{E}} \binom{\mathcal{E}}{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1} \\
&\ge \left(1 - \frac{1}{k-1}\right) \left(\frac{\mathcal{E}}{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1} \right)^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1} \\
&\ge \left(1 - \frac{1}{k-1}\right) (k-1)^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1}.
\end{aligned}$$

Again we use Fact 2 directly to the sum $\sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} f_{k-1}(t,\mathcal{E})$ and we apply inequality obtained above. It is easy to see that following observations are satisfied:

$$\begin{aligned}
n &\le \sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} \frac{\binom{t}{\lceil \frac{\mathcal{E}}{k-1} \rceil}}{\left(1 - \frac{1}{k-1}\right) (k-1)^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1}} \\
&\le \left(1 + \frac{1}{k-2}\right) \sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} \frac{t^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil}}{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil ! \, (k-1)^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil - 1}} \\
&= \left(1 + \frac{1}{k-2}\right) (k-1) \sum_{\mathcal{E}=1}^{\mathcal{E}_{max}} \frac{\left(\frac{t}{k-1}\right)^{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil}}{\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil !}.
\end{aligned}$$

Since $\mathcal{E}$ occurs in the above sum only in term $\left\lceil \frac{\mathcal{E}}{k-1} \right\rceil$, we have the same $(k-1)$ summands. Thus,

$$\begin{aligned}
n &\le \left(1 + \frac{1}{k-2}\right) (k-1)^2 \sum_{s=1}^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil} \frac{\left(\frac{t}{k-1}\right)^s}{s!} \\
&\le \left(1 + \frac{1}{k-2}\right) (k-1)^2 \frac{\Gamma\left(\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil + 1, \frac{t}{k-1}\right)}{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil ! e^{-\frac{t}{k-1}}},
\end{aligned}$$

where $\Gamma(s,x)$ is the incomplete gamma function defined as follows $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} \, \mathrm{d}t$. If s is a positive integer, function $\Gamma(s,x)$ has following expansion $\Gamma(s,x) = (s-1)! \, e^{-x} \sum_{k=0}^{s-1} \frac{x^k}{k!}$. From [20] we know, that $\frac{\Gamma(s,x)}{x^{s-1} e^{-x}} \to 1$, as $x \to \infty$. But it is proved in [19], that $t \in \Omega(k \log n)$, even without energy restriction. Thus $\frac{t}{k} = \Omega(\log n)$. From asymptotic behavior of $\Gamma(s,x)$, we know, that:

$$\Gamma\left(\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil + 1, \frac{t}{k-1}\right) \in O\left(\left(\frac{t}{k-1}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil} e^{-\frac{t}{k-1}}\right).$$

From the fact that $n! \ge (\frac{n}{e})^n$ after some simplifications we get:

$$\begin{aligned}
\frac{\left(\frac{t}{k-1}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil} e^{-\frac{t}{k-1}}}{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil ! e^{-\frac{t}{k-1}}} &\le \frac{\left(\frac{t}{k-1}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}}{\left(\frac{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}{e}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}} \\
&\le e^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil} \left(\frac{t}{\mathcal{E}_{max}}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}.
\end{aligned}$$

From calculations above we obtain following facts

$$n \in O\left((k-1)^2 \left(\frac{et}{\mathcal{E}_{max}}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}\right),$$

$$(k-1)^2 \left(\frac{et}{\mathcal{E}_{max}}\right)^{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil} \in \Omega(n).$$

and finally $t \in \Omega\left(\mathcal{E}_{max} \left(\frac{n}{(k-1)^2}\right)^{\frac{1}{\left\lceil \frac{\mathcal{E}_{max}}{k-1} \right\rceil}}\right)$. ∎

The above theorem yields a spectrum of time-energy complexity trade-offs for oblivious, deterministic k-Selection algorithms. For example, it implies following corollary.

Corollary 5.1: Let $k \in O\left(n^{1/4}\right)$ and $\mathcal{E}_{max} \in O\left(k \frac{\log n}{\alpha \log \log n}\right)$, then for any $\alpha > 0$

$$t \in \Omega\left(k \frac{\log^{1+\frac{\alpha}{2}} n}{\alpha \log \log n}\right).$$

VI. Computer Simulations

In addition to the analysis of the protocols presented for random model, we show empirical results obtained by means of computer simulations. We have evaluated the performance of Protocol 1 for networks consisting of $k = 10$ and $k = 10^4$ activated stations. The results allow us to speculate on tightness of the analysis, as well as to see how the protocol behaves in a case of a small number of activated stations. We have also run simulations of Martel algorithm to compare the difference between maximum energy usage and the energetic effort of the last station.

A. Energy efficient protocol

Table I shows results of simulations of Protocol 1 solving $10-$Selection problem for different values of the ϵ parameter. The $time = max_{iter} \cdot \lceil 2k^{1+\epsilon} \rceil$ is a total number of rounds needed by the protocol to complete. The number of stations left activated after consecutive iterations, $iter_i$, were obtained by averaging outcomes of 10^6 simulation runs. The last row shows how many (out of 10^6) runs ended with failure, which is a case when after max_{iter} iterations there are some stations, that were unable to broadcast their messages.

Table II were obtained in a similar manner as Table I, but for $k = 10^4$ stations and 10^5 simulation runs. It can be seen that the Protocol 1 behaves much better for larger number

Table I
SIMULATION RESULTS OF PROTOCOL 1 FOR $k = 10$ STATIONS.

ϵ	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{128}$
max_{iter}	4	4	5	10
$time$	800	256	180	210
$iter_1$	0.44242	1.32249	2.24018	3.55332
$iter_2$	0.00258	0.038901	0.153025	0.548462
$iter_3$	0	0.000628	0.005062	0.038523
$iter_4$	0	0.00002	0.000134	0.002036
$iter_5$			0.000002	0.000096
$iter_6$				0.000008
$iter_7$				0
$iter_8$				0
$iter_9$				0
$iter_{10}$				0
$failed$	0	10	1	0

Table II
SIMULATION RESULTS OF PROTOCOL 1 FOR $k = 10^4$ STATIONS.

ϵ	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{128}$
max_{iter}	4	4	5	10
$time$	800000000	8000000	1000000	214930
$iter_1$	0.4965	49.8927	487.765	3720.12
$iter_2$	0	0.0013	1.1899	591.196
$iter_3$	0	0	0	16.0588
$iter_4$	0	0	0	0.01214
$iter_5$			0	0
$iter_6$				0
$iter_7$				0
$iter_8$				0
$iter_9$				0
$iter_{10}$				0
$failed$	0	0	0	0

of stations, as one could expect based on the results of the analysis.

B. Martel algorithm

In Section IV we proved $\Omega(\log k)$ lower bound on energy usage of the last station in the Martel algorithm. While this result obviously translates to the lower bound of the energetic complexity of the algorithm, one could ask how big is the difference between maximum energy usage and the energetic effort of the last station. Figure 1 shows results of 10^5 simulations for different number of stations (logarithmic scale).

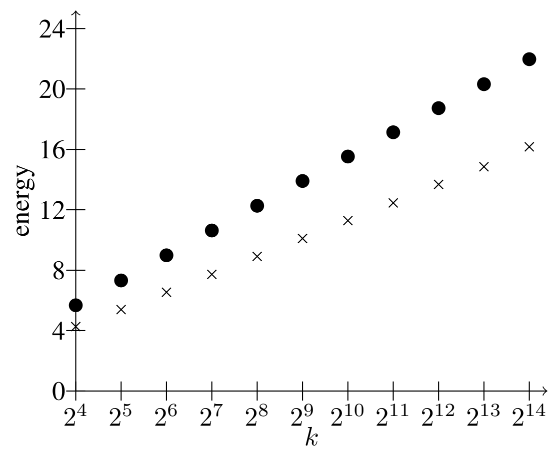

Figure 1. Energy usage in Martel algorithm. $\bullet$ is an average maximum energy usage and $\times$ is an average energy usage of the last station.

VII. CONCLUSIONS AND FURTHER RESEARCH

In our paper we presented several results about energetic aspects of k-Selection protocols in a single–hop radio network. We believe that presented approach can be applied to more realistic scenarios. In particular, it is clear that some results can be easily applied for dynamic counterparts of k-Selection problem (described e.g., in [6]) at least for some models.

We believe that most interesting and most challenging task is to find general relation between energy consumption and time necessary for completion of k-Selection in randomized model. We tried to obtain such result, without effects, using information theory approach techniques. We strongly believe that the presented Protocol 1 is efficient in sense of time complexity. That is, we think that in randomized model, there is no k-Selection algorithm with constant energy usage working in $O(k \operatorname{polylog}(k))$ time.

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