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Abstract: In this paper, a mathematical analysis of a possible way to chaos for piecewise smooth systems submitted to one of it's specific bifurcations, namely the sliding , the corner and the grazing ones, is proposed. This study is based on period doubling method and Implcit function theorem.

Keywords: Dynamic bifurcations (74H60), Chaotic behavior (74H65), Bifurcation of limits cycles and periodic orbits (37G15).

1. INTRODUCTION

PieceWise Smooth systems (P.W.S) are dynamic systems given by two or more sets of differential equations such that those systems switch from one phase space (defined by smooth dynamics) to another when some switching conditions are satisfied. These phase spaces are separated by finitely smooth codimension one boundaries. PWS systems exhibit complex behavior that cannot be explained only by using the bifurcation analysis associated to smooth dynamic systems (Golubitsky & al., 1983), they can undergo all the bifurcations relied to smooth systems but they have also a specific type of bifurcations called border collision bifurcations as grazing, sliding, corner... The bifurcation phenomena relied to piecewise smooth systems are well-known and studied since at least the famous Fillipov book (Filippov, 1988), moreover, the Russian school has also proposed pioneer works in piecewise smooth bifurcations (Feigin, 1978) and many papers have concentrated on piecewise smooth maps that are continuous across the borderlines (Banerjee & *al.*, 2000; Banerjee & *al.*, 1999). Moreover, the bifurcation theory for one and two dimensional continuous piecewise maps has been developed also in (Nusse & al., 1995), Feigin obtained some results on periodic orbits for continuous maps (Feigin, 1970) and some authors extend the Feigin's approach to the case of one dimensional linear discontinuous maps (Kollar & al., 2004). On the other hand, one of the most striking feature of piecewise smooth systems is that they exhibit sudden transitions from periodic attractors to chaos in the absence of any period doubling or other bifurcation cascades usually observed in smooth systems (Nusse & al., 1996). More recently, Mario di Bernardo and coauthors give a bifurcation's classification for piecewise smooth systems (di Bernardo & *al.*, 2001-a; di Bernardo & *al.*, 2002; di Bernardo & *al.*, 2001-b) using the relied Poincaré map. On the basis of this special classification, we analyze the possibility of generating chaos via these bifurcations. In this paper, following the analysis given in (Benmerzouk & *al.*, 2006) and after treating the case of way to chaos via grazing bifurcation in (Benmerzouk & *al.*, 2007) , the sliding one in (Benmerzouk & *al.*, 2008-a) and the corner bifurcation in (Benmerzouk & *al.*, 2008-b), the paper provides a synthesis of the foundations of a rational approach for transition to chaos by period doubling based on the relied Poincaré map introduced by di Bernardo and coworkers. So, the paper is organized as follows: In section II, some recalls on the Poincaré maps associated to the sliding, corner and grazing bifurcations are recalled and the problem statement is established for each case; in section III , a route to chaos by period doubling method based on the Implicit Functions Theorem is proposed for each case. The general method is highlighted by an academical example with respect to corner bifurcations.

2. RECALLS AND PROBLEM STATEMENT:

2.1 Systems submitted to sliding bifurcations:

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x) \text{ if } H(x) \geq 0 \\ F_2(x) \text{ if } H(x) < 0 \end{cases} \tag{2.1}$$

where $x : I \longrightarrow D$, D is an open bounded domain of R^3, $I \subset R^+$, generally I is the time interval.

$$F_1, F_2 : C_{abs}(I, D) \longrightarrow C^k(I, D), \qquad \text{for } \ k \geq 4$$

where $C^k(I, D)$ is the set of C^k functions defined on I and having values in R^3, the norm on $C^k(I, D)$ is defined as follows:
$\|x\|_k = \sup_{t\in I} \|x(t)\|_e + \sup_{t\in I} \|\dot{x}(t)\|_e + ... + \sup_{t\in I} \left\|x^{(k)}(t)\right\|_e$ where $x^{(k)}(.)$ denotes the k^{th} derivative of $x(.)$ and $||.||_e$ is a norm defined on R^3. Finally $C_{abs}(I, D)$ is the set of absolutely continuous functions defined on I and having values in D provided with the norm: $||x|| = \sup_{t\in I} ||x(t)||_e$.

According to (Bresis, 1999), $(C^k(I, D), \|.\|_k)$ and $(C_{abs}(I, D), \|.\|)$ are two Banach spaces.

$H : D \longrightarrow R$ is a C^1application that defines :

$$S = \{x(t) \in D : H(x(t)) = 0\}$$

$$S^+ = \{x(t) \in D : H(x(t)) > 0\}$$

$$S^- = \{x(t) \in D : H(x(t)) < 0\}$$

Both vectors fields F_1 and F_2 are defined on both sides of S. Moreover, it is assumed that there exists a subset of switching manifold $\bar{S} \subset S$ that denotes a sliding region which is simultaneously attracting from $S^+$and S^-, so, considering any neighborhood $v_{\bar{S}}$ of $\bar{S}$, the existence of $\bar{S}$ is characterized by the following hypothesis:

H^s-1) $< \nabla H(x(t)), F_2(x(t)) - F_1(x(t)) > > 0, \forall x(t) \in v_{\bar{S}}$.

where $< ., . >$ is a usual scalar product on R^3.

Nota: In all what follows, index s (respectively g, c) are relied to sliding, grazing and corner cases.

Under H^s-1), if the system trajectory crosses $\bar{S}$, the sliding motion evolves in $\bar{S}$ until it eventually reaches its boundary. In order to present a complete sliding bifurcation analysis, it is assumed that the system (2.1) depends smoothly on a parameter ε such that at $\varepsilon = 0$, there exists a periodic orbit $x(t)$ that slides at the point x^* corresponding to t^* (where t^* is defined modulo the system periodicity), consequently, the system (2.1) is rewrite as follows:

$$\dot{x} = \begin{cases} F_1(x, \varepsilon) \text{ if } H(x) \geq 0 \\ F_2(x, \varepsilon) \text{ if } H(x) < 0 \end{cases} \tag{2.2}$$

and it is assumed that there exist a neighborhood v_ε of $\varepsilon = 0$, a neighborhood v_x of $x = x^*$ such that the following hypotheses are verified at each bifurcation point (x^*, t^*) :

H^s-2) $H(x^*) = 0$ and $\nabla H(x^*) \neq 0$.

H^s-3)$< \nabla H(x^*(t)), F_1(x^*)) > = 0$

So, without loss of generality (thanks to a coordinate change), the bifurcation point is assumed to be located at $(x^*, t^*) = (0, 0)$ and according to (di Bernardo & *al.*, 2002), four possible cases of bifurcations involving sliding are possible:

Case 1: Sliding bifurcation of type 1: here, the sliding flow is assumed to move locally towards the boundary of the sliding region when it is perturbed from bifurcation point, this yields to the following assumption:

A^s-1)

$$\frac{\partial^2 H(\phi_1(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_1(x^*(0))}{\partial x} F_1(x^*(0)) \right\rangle > 0$$

and the corresponding Poincaré normal form is given by:

$$P_1(x, \varepsilon) = \begin{cases} \varepsilon x & \text{if} < \nabla H, x > \ \leq 0 \\ \varepsilon x + \varepsilon^2 v_1(x) + o(\varepsilon^3) & \text{if} < \nabla H, x > \ > 0 \end{cases}$$

where:

$$v_1(x) = \frac{1}{2} \frac{< \nabla H, \frac{\partial F_1}{\partial x} x >^2}{< \nabla H, F_2 >< \nabla H, \frac{\partial F_1}{\partial x} F_1 >} (F_2 - F_1) \tag{2.3}$$

and $o(\varepsilon^\alpha) \longrightarrow 0$ when $\varepsilon \longrightarrow 0$, $\alpha \in R^+$.

Remark 2.1. For the sake of compactness (and also for the next cases) F_1, F_2 and ∇H stand for $F_1(0), F_2(0)$ and ∇H (0) and the projection of x on the Poincaré map is noted by x.

Case 2: Grazing sliding bifurcation: here, the sliding flow moves towards the edge of the sliding strip, this yields to the following assumption:

A^s-2)

$$\frac{\partial^2 H(\phi_2(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_2(x^*(0))}{\partial x} F_2(x^*(0)) \right\rangle > 0.$$

and the corresponding Poincaré normal form is given by:

$$P_2(x, \varepsilon) = \begin{cases} \varepsilon x & \text{if} < \nabla H, x > \geq 0 \\ \varepsilon x + \varepsilon v_2(x) + o(\varepsilon^{\frac{3}{2}}) & \text{if} < \nabla H, x > < 0 \end{cases}$$

where

$$v_2(x) = -\frac{< \nabla H,\ x >}{< \nabla H, F_2 >} (F_2 - F_1) \tag{2.4}$$

Case 3: Sliding bifurcation type 2: here, a switching-sliding bifurcation occurs and the following assumption is required:

A^s-3)

$$\frac{\partial^2 H(\phi_1(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_1(x^*(0))}{\partial x} F_1(x^*(0)) \right\rangle < 0.$$

and the corresponding Poincaré normal form is given by:

$$P_3(x, \varepsilon) = \begin{cases} \varepsilon x & \text{if} < \nabla H, x > \ \leq 0 \\ \varepsilon x + \varepsilon^3 v_3(x) + o(\varepsilon^4) & \text{if} < \nabla H, x > \ > 0 \end{cases}$$

where:

$$v_3(x) = \frac{2}{3} \frac{< \nabla H, \frac{\partial F_1}{\partial x} x >^3}{< \nabla H, F_2 >< \nabla H, \frac{\partial F_1}{\partial x} F_1 >^2} \left(\frac{\partial F_1}{\partial x} + \frac{I_d}{< \nabla H, F_2 >} (< \nabla H, \frac{\partial F_1}{\partial x} F_1 > - < \nabla H, \frac{\partial F_1}{\partial x} F_2 >) \right) (F_2 - F_1) \tag{2.5}$$

and I_d denotes the identity matrix of appropriate dimension.

Case 4: Multisliding bifurcation: here the sliding flow is tangential to the boundary of the sliding strip at the bifurcation point, so the following assumption is required:

A^s-4)

$$\frac{\partial^3 H(\phi_1(0,0)}{\partial t^3} = \left\langle \nabla H(x^*(0)), (\frac{\partial F_1(x^*(0))}{\partial x})^2 F_1(x^*(0) \right\rangle < 0.$$

and the corresponding normal form is given by:

$$P_4(x, \varepsilon) = \begin{cases} \varepsilon x & \text{if} < \nabla H, x > \geq 0 \\ \varepsilon x + \varepsilon^2 v_4(x) + o(\varepsilon^{\frac{5}{2}}) & \text{if} < \nabla H, x > < 0 \end{cases}$$

where:

$$v_4(x) = -\frac{9}{2} \frac{< \nabla H, \frac{\partial F_1}{\partial x} x >^2}{< \nabla H, F_2 >< \nabla H, (\frac{\partial F_1}{\partial x})^2 x >} (\frac{\partial F_1}{\partial x} - \frac{< \nabla H, \frac{\partial F_1}{\partial x} F_2 >}{< \nabla H, F_2 >} I_d)(F_2 - F_1) \tag{2.6}$$

Lemma 1. If ($< \nabla H, x > > 0$ for case 1 and 3) or ($< \nabla H, x > < 0$ for case 2 and 4), then the problem of finding a periodic solution of problem (2.2) is equivalent to analyze for each type of sliding bifurcations the following corresponding equations:

$$\beta_i^s(x, \varepsilon) - x = 0,$$

where:

- For case 1: $\beta_1^s(x, \varepsilon) = \varepsilon x + \varepsilon^2 v_1(x) + o(\varepsilon^3)$, $v_1(x)$ is given in (2.3), $\varepsilon \in v_\varepsilon$.
- For case 2: $\beta_2^s(x, \varepsilon) = \varepsilon x + \varepsilon v_2(x) + o(\varepsilon^{\frac{3}{2}})$, $v_2(x)$ is given in (2.4), $\varepsilon \in v_\varepsilon$.
- For case 3: $\beta_3^s(x, \varepsilon) = \varepsilon x + \varepsilon^3 v_3(x) + o(\varepsilon^4)$, $v_3(x)$ is given in (2.5), $\varepsilon \in v_\varepsilon$.
- For case 4: $\beta_4^s(x, \varepsilon) = \varepsilon x + \varepsilon^2 v_4(x) + o(\varepsilon^{\frac{5}{2}})$, $v_4(x)$ is given in (2.6), $\varepsilon \in v_\varepsilon$.

2.2 Systems submitted to Grazing bifurcations:

In this section, one considers the same piecewise smooth system (2.1) and throughout this section; for all definitions, propositions and theorems, D is considered as an open bounded and connex domain even if it is not a necessary assumption for the considered purpose but this assumption is necessary for the global result. According to (di Bernardo & *al.*, 2001-b), a grazing occurs at $x = 0$ (denoted grazing point) if the following conditions are satisfied:

A^g-1) $H(0) = 0$ and $\nabla H(0) \neq 0$.

A^g-2)$< \nabla H(0), \frac{\partial \Phi_i}{\partial t}(0,0) > = < \nabla H(0), F_i^0 > = 0, i = 1, 2.$

A^g-3) for $i = 1, 2$.

$$\frac{\partial^2 H(\Phi_i(0,0))}{\partial t^2} = (< \nabla H(0), \frac{\partial F_i^0}{\partial x} F_i^0 > + < \frac{\partial^2 H(\Phi_i(0,0))}{\partial x^2} F_i^0, F_i^0 >) > 0$$

A^g-4) $\quad (< L, F_1^0 >< L, F_2^0 >) > 0$

where Φ_i are the flows associated to F_i , $F_i^0 = F_i(\Phi_i(0,0))$, $i = 1, 2$ and L is the unit vector perpendicular to $\nabla H(0)$. The system (2.1) is assumed to depend smoothly explicitly or implicitly on a parameter ε and is defined by equation (2.2). Moreover, at $\varepsilon = 0$ and $x(0) = 0$, there is a periodic orbit $x(t)$ that grazes at the point $x(0)$. The solution is hyperbolic and hence isolated such that there is no points of grazing along the orbit other than 0. As the previous conditions are defined on an open set then there exist two sufficiently small neighborhoods of $\varepsilon = 0$ and $x(t)$ such that A^g-i)$_{(1,2,3,4)}$ are verified. Moreover, if the vector field is continuous at grazing i.e. $F_1^0 = F_2^0 := F$ but has discontinuous first derivatives, then the Poincaré map is given by:

$$P(x,\varepsilon) = \begin{cases} P_1(x,\varepsilon) & \text{if} < \nabla H, x > \, > 0 \\ P_2(x,\varepsilon) & \text{if} < \nabla H, x > \, < 0 \end{cases}$$

Where:

$$\begin{aligned} P_1(x,\varepsilon) &= Nx + M\varepsilon + o(\|x\|_{n-1}, \varepsilon) \\ P_2(x,\varepsilon) &= Nx + M\varepsilon + Nw_1(| < \nabla H, x > |)^{\frac{3}{2}} \\ &+ w_2 x(| < \nabla H, x > |)^{\frac{1}{2}} \\ &+ w_3(< \nabla H, \frac{\partial F_2}{\partial x} x >)(| < \nabla H, x > |)^{\frac{1}{2}} + o(\|x\|_{n-1}^2, \varepsilon) \end{aligned} \tag{2.7}$$

N is a nonsingular matrix 2×2, M is a nonzero 2 dimensional vector.

$$\begin{aligned} w_1 &= \frac{2}{(< \nabla H, \frac{\partial F_1}{\partial x} F >)^{\frac{3}{2}}} (\frac{2}{3}(\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x^2}) F^2 \\ &+ 2\frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial x} F - \frac{2}{3}((\frac{\partial F_1}{\partial x})^2 + 2(\frac{\partial F_2}{\partial x})^2) F \\ &- \frac{2}{\sqrt{< \nabla H, \frac{\partial F_2}{\partial x} F >}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}) \\ &F(\frac{2}{3} < \nabla H, (\frac{\partial^2 F_2}{\partial x^2} F^2 + (\frac{\partial F_2}{\partial x})^2) F > \\ &+ < \nabla H, (\frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial x} - 2(\frac{\partial F_2}{\partial x})^2) F > \\ &+ < \nabla H, (\frac{\partial^2 F_2}{\partial x^2} F^2 >))) \\ w_2 &= \frac{2}{\sqrt{< \nabla H, \frac{\partial F_1}{\partial x} F >}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}) \\ w_3 &= \frac{2}{\sqrt{< \nabla H, \frac{\partial F_1}{\partial x} F >< \nabla H, \frac{\partial F_2}{\partial x} F >}} (\frac{\partial F_2}{\partial x} F - \frac{\partial F_1}{\partial x} F) \end{aligned}$$

and $o(\alpha, \varepsilon) \longrightarrow 0$ when $(\alpha, \varepsilon) \longrightarrow (0, 0)$.

Lemma 2. Searching a periodic solution of (2.2) at grazing is equivalent to resolve the following equation:$P_2(x,\varepsilon) - x = 0$,where $P_2(x,\varepsilon)$ is given by (2.7).

2.3 Systems submitted to corner bifurcations:

For this case the following piecewise smooth system is required:

$$\dot{x} = \begin{cases} F_1(x) \text{ if } H_1(x) \geq 0 \text{ or } H_2(x) \geq 0 \\ F_2(x) \text{ if } H_1(x) < 0 \text{ and } H_2(x) < 0 \end{cases} \tag{2.8}$$

for $i = 1, 2$, $H_i : D \longrightarrow R$ are assumed to be $C^1$and define the sets: $S_1 = \{x(t) \in D : H_1(x(t)) = 0\}$ and $S_2 = \{x(t) \in D : H_2(x(t)) = 0\}$ such that S_1 and S_2 intersect along a corner C with a smooth codimension two surfaces and the following regions are not empty:

$$D_{int} = \{x(t) \in D : H_1((x(t)) < 0 \text{ and } H_2((x(t)) < 0\}$$

$$D_{out} = \{x(t) \in D : \ H_1(x(t)) \geq 0 \text{ or } H_2(x(t)) \geq 0\}$$

This generates a non-zero angle, so, let's consider that $(0, 0)$ is on the corner C i.e satisfies the following hypothesis:

H^c-1) $\nabla H_1(0,0) \times \nabla H_2(0,0) \neq 0$.

F_1 and F_2 are assumed to be defined on both sides of D_{int} and D_{out} respectively.

According to (di Bernardo & *al.*, 2001-a), there exist two types of corner -collision bifurcations: external collision and internal one and in order to avoid sliding phenomena, one requires the following assumptions for each case:

For the external corner collision:

A^c-1)$< F_i(0,0), \nabla H_1(0,0)) >< 0$ and $< F_i(0,0), \nabla H_2(0,0) >> 0$, $i = 1, 2.$

For the internal corner collision:

A^c-2)$< F_i(0,0), \nabla H_1(0,0)) > \geq 0$ or $< F_i(0,0), \nabla H_2(0,0) > \leq 0$, $i = 1, 2.$

As for the previous case of sliding bifurcations and in order to present a complete bifurcation analysis, one assumes that the system (2.8) depends smoothly on a parameter ε defined in a neighborhood V_ε of 0 such that at $\varepsilon = 0$, there exists a periodic orbit $x(t) := p(t)$ that intersects with the corner C at the point $x^* = 0 = p(0)$ corresponding to t^* (where t^* is defined modulo the periodicity of (2.8)) and that there exists a neighborhood of x noted V_x such that there is no other points of collision near x^*, so by continuity, one assumes that this will be also true locally for a sufficiently small ε, consequently, the system (2.8) is rewrite as follows:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) \text{ if } H_1(x) \geq 0 \text{ or } H_2(x) \geq 0 \\ F_2(x,\varepsilon) \text{ if } H_1(x) < 0 \text{ and } H_2(x) < 0 \end{cases} \tag{2.9}$$

Such that $p(t)$ is an T periodic orbit, $(x \ , \ \varepsilon)$ is in the neighborhood of $(0, 0)\epsilon V_x \times V_\varepsilon$.

Two different cases of bifurcations involving corner can occur:

Case 1 the considering trajectory enters in D_{int} in the neighborhood of C, the Poincaré normal form is given by:

$$P_1(x,\varepsilon) = \begin{cases} Px + Q\varepsilon + o(x,\varepsilon) \quad \text{if non-crossing} \\ Px + (F_1 - F_2) < a_2, \ x > + Q\varepsilon + o(x,\varepsilon) \\ \text{if crossing} \end{cases}$$

where:

$$a_2 = J_2 - \frac{1}{2} < J_2, F_1 > J_1, J_i = \frac{\nabla H_i}{< \nabla H_i, F_i >}, i = 1, 2 \quad (2.10)$$

and P is a 2×2 nonsingular matrix , Q is a 2 dimensional non null vector.

Case 2, external bifurcation: the corresponding Poincaré normal form is given by:

$$P_2(x, \varepsilon) = \begin{cases} Px + Q\varepsilon + o(x, \varepsilon) \text{ if non-crossing} \\ Px + (F_1 - F_2) < a_1, x > +Q\varepsilon + o(x, \varepsilon) \\ \text{ if crossing} \end{cases}$$

where

$$a_1 = J_1 - < J_1, F_2 > J_2 \quad (2.11)$$

One obtains:

Lemma 3. When a crossing occur, the problem of finding a periodic solution for problem (2.9) is equivalent to analyze for each case $i = 1, 2$, the following corresponding equations:

$$\beta_i^c(x, \varepsilon) - x = 0$$

where:

- for case 1, (internal collision-bifurcation):

$$\beta_1^c(x, \varepsilon) = Px + (F_1 - F_2) < a_2, \ x > +Q\varepsilon + o(x, \varepsilon)$$

a_2 is given by (2.10) and $(x, \varepsilon) \epsilon V_x \times V_\varepsilon$.

- for case 2, (external collision-bifurcation):

$$\beta_2^c(x, \varepsilon) = \ Px + (F_1 - F_2) < a_1, \ x > +Q\varepsilon + o(x, \varepsilon)$$

a_1 is given by (2.11) and $(x, \varepsilon) \epsilon V_x \times V_\varepsilon$.

3. ROUTE TO CHAOS:

The aim of this work is to present a mathematical analysis permitting to generate chaos for piecewise smooth systems of dimension 3 given by the form (2.2) and that are submitted to those specific bifurcations namely the sliding, the corner or the grazing ones. Note that for each of these cases of bifurcations the P.W.S system is bounded but the corresponding Poincaré map is of dimension 2, thus at this step the famous result of Li and Yorke (Glendinning, 1970): 'period three implies Chaos' can not be used. Nevertheless, the proposed analysis (based on the implicit function theorem) deals with a point x on the Poincaré map and in the neighborhood of the impact point x^*, this analysis allows to construct a local branch of continuous solutions passing through this point. At this final step, the Li and Yorke Theorem can be used since this point x^* is on the collision manifold and so is defined by only one component.

3.1 Sliding bifurcations case:

The problem becomes to determine for each map β_i^s ,$i = 1, 2$, three distinct points noted respectively x_i, y_i and z_i such that: $\beta_i^s(x_i, \varepsilon) = y_i$, $\beta_i^s(y_i, \varepsilon) = z_i$ and $\beta_i^s(z_i, \varepsilon) = x_i$, this well be done naturally in three steeps as follows:

First step: analyze of the equation:

$$\beta_i^s(x_i, \varepsilon) = y_i := x_i + \eta_i$$

where for a sake of simplicity, η_i stands for a vector defined in R^2, having only one component equal to some fixed value (noted also η_i) and the other component is zero.

This equation is equivalent to:

$$\Psi_i^s(x_i, \varepsilon, \eta_i) := \beta_i^s(x_i, \varepsilon) - x_i - \eta_i = 0 \quad (3.12)$$

Using the Implicit Functions Theorem, one obtains:

Lemma 4. Under conditions H^s-j) $j = 1, 2, 3$ and A^s-i) specific to each case $i = 1, 2, 3, 4$, there exist a neighborhood $\vartheta_{\varepsilon=0}^i \subset \upsilon_{\varepsilon=0}^i$ in R, a neighborhood υ_{η_i} in R , a neighborhood $\upsilon_{x_i=0}$ on the Poincaré section's (defined in R^{n-1}) and an unique application x_i^*: $\vartheta_{\varepsilon=0}^i \times \upsilon_{\eta_{i=0}} \longrightarrow \upsilon_{x_i=0}$ solution of $\Psi_i^s(x_i^*(\varepsilon, \eta_i), \varepsilon, \eta_i) = 0$ such that $x_i^*(0, 0) = 0$. Furthermore, x_i^* depends continuously on ε and η_i.

Second step: analyze of the equation:

$$\beta_i^s(\beta_i^s(x_i, \varepsilon), \varepsilon) = z_i := y_i + \mu_i = x_i^*(\varepsilon, \eta_i) + \eta_i + \mu_i$$

where μ_i stands for a vector defined on R^2, having only one component equal to some fixed value(noted also μ_i) and the other is zero. This equation is equivalent to:

$$\Gamma_i^s(\varepsilon, \eta_i, \mu_i) = \beta_i^s(\beta_i^s(x_i, \varepsilon), \varepsilon) - x_i^*(\varepsilon, \eta_i) - \eta_i - \mu_i = 0 \quad (3.13)$$

In order to continue the process with the same arguments, the following hypothesis according to each case is necessary:

H^s-4) $\frac{\partial \Gamma_i^s}{\partial \eta_i}(0, 0, 0) \neq 0$.

It comes:

Lemma 5. Under conditions H^s-j) $j = 1, 2, 3, 4$ and A-i), specific to each case $i = 1, 2, 3, 4$, there exist a neighborhood $\upsilon_{\varepsilon=0}^i \subset \vartheta_{\varepsilon=0}^i$, a neighborhood $\nu_{\eta_{i=0}} \subset \upsilon_{\eta_{i=0}}$, a neighborhood $\nu_{\mu_i=0}$ in R and an unique application η_i^*:$\upsilon_{\varepsilon=0}^i \times \nu_{\mu_i=0} \longrightarrow \nu_{\eta_{i=0}}$ solution of $\Gamma_i^s(\varepsilon, \eta_i^*(\varepsilon, \mu_i), \mu_i) = 0$ such that $\eta_i^*(0, 0) = 0$, furthermore, η_i^* depends continuously on ε and μ_i.

Third step: analyze of the equation:

$$\beta_i^s(\beta_i^s(\beta_i^s(x, \varepsilon), \varepsilon), \varepsilon) = x \quad (3.14)$$

The equation (3.14) is equivalent to:

$$\begin{aligned} \Pi_i^s(\varepsilon, \mu_i) := \ & \beta_i^s(x^*(\varepsilon, \eta_i^*(\varepsilon, \mu_i)) + \eta_i^*(\varepsilon, \mu_i) + \mu_i), \varepsilon) \\ & - x_i^*(\varepsilon, \eta_i^*(\varepsilon, \mu_i)) = 0 \end{aligned} \quad (3.15)$$

So the following hypothesis is required for each case $i = 1, 2, 3, 4$:

H^s-5) $\frac{\partial \Pi_i^s}{\partial \mu_i}(0, 0) \neq 0$

in order to obtain:

Lemma 6. Under conditions H^s-j), $j = 1, 2, 3, 4, 5$ and A^s-i), $i = 1, 2, 3, 4$ specific to each case , there exist a neighborhood $\omega_{\varepsilon=0}^i \subset \upsilon_{\varepsilon=0}^i$, a neighborhood $\theta_{\mu_{i=0}}$ in $\nu_{\mu_i=0}$ and an unique application μ_i^*: $\omega_{\varepsilon=0}^i \longrightarrow \theta_{\mu_{i=0}}$ solution of $\Pi_i^s(\varepsilon, \mu_i^*(\varepsilon)) = 0$ such that $\mu_i^*(0) = 0$, furthermore, μ_i^* depends continuously on ε.

The next corollary sums up the previous results:

Corollary 3.1. Under conditions H^s-j), $j = 1, 2, 3, 4, 5$, and A^s-i), $i = 1, 2, 3, 4$ specific to each case, the system (2.2) (of dimension 3) admits a chaotic behavior at sliding collision.

3.2 Grazing bifurcations case:

In order to avoid the $\frac{3}{2}$ type singularity in the Poincaré map, the topological degree theory is considered (Katok &

al., 1997) and for continuity reason, $P_2(.,\varepsilon)$ is considered to be equal to 0 for all x such that $< \nabla H, x > \geq 0$. So, in order to approach P_2 by a quadratic function having the same solutions number as P_2, the following "alternative" function β^g is considered:

$$\begin{aligned}\beta^g(x,\varepsilon) &= Nx + M\varepsilon + (\delta(x))^{\frac{1}{2}}(N(w_1(\delta(x))^{\frac{3}{2}} \\ &+ w_2 x(\delta(x))^{\frac{1}{2}} + w_3(< \nabla H, \frac{\partial F_2}{\partial x}x >)(\delta(x))^{\frac{1}{2}})) \\ &+ o(\|x\|_{n-1}^2, \varepsilon)\end{aligned}$$

Where $\delta(x) = | < \nabla H, x > |$; Thus if the line joining $\beta^g(x,\varepsilon)$ to $P_2(x,\varepsilon)$ does not contain 0 i.e. if the following assumption is satisfied:

A^g-5)
$$(Nx + M\varepsilon - x)^T (N(w_1(\delta(x))^2 + w_2 x(\delta(x)) + w_3(< \nabla H, \frac{\partial F_2}{\partial x}x >)(\delta(x))) + o(\|x\|^2, \varepsilon)) > 0$$

for all $x \in \partial D$ and any real ε in the neighborhood of zero, then:

$$deg(\beta^g, 0, D) = deg(P_2, 0, D)$$

where $deg(\beta^g, 0, D)$ and $deg(P_2, 0, D)$ are respectively the degree of the application β^g and P_2 defined on D at point 0.

Thus the problem to analyze becomes:

$$\beta^g(x,\varepsilon) - x = 0 \tag{3.16}$$

and generating chaos for the system (2.2) (and so for the corresponding equation (3.16)) is equivalent to determine three distinct points x, y and z such that: $\beta^g(x,\varepsilon) = y$, $\beta^g(y,\varepsilon) = z$ and $\beta^g(z,\varepsilon) = x$. This well be done by considering the same analysis (in three steeps) as the previous case of sliding bifurcations using the following assumptions:

A^g-6) $\frac{\partial \Gamma^g}{\partial \eta_i}(0,0,0) \neq 0$.
A^g-7) $\frac{\partial \Pi^g}{\partial \mu_i}(0,0) \neq 0$.

(where Γ_i^g and Π_i^g applications having the same form given by (3.13) and (3.15) but with β^g in stead of β^s)
are necessary to obtain:

Corollary 3.2. Under assumptions A^g-i) for i= 1, 2, 3, 4, 5, 6,7, the system (2.2) admits a chaotic behavior at grazing collision.

3.3 Corner bifurcations case:

The same analysis proposed for sliding case is available, the difference is only in the definitions of the relied Poincaré maps and so in the associated conditions. Thus, under the following assumptions for each case $i \in \{1, 2\}$:

• A^c-3)$(P - I_d)$ is a nonsingular matrix, where I_d is the 2×2 identity matrix.
• A^c-4,i) $B_i = (P - I_d + (F_1 - F_2)a_i^T)$ is a nonsingular matrix.
• A^c-5,i)$C_i = (P - 2I_d + (F_1 - F_2)a_i^T)$ is a nonsingular matrix.
• A^c-6,i)$D_i = (B_i\widetilde{C}_j + I_d^0) - \widetilde{C}_j - I_d^0$) is a vector such that one of it's two components (noted d) is non null.

where $\widetilde{C}_j$ is the j column of the inverse matrix of C_i and I_d^0 is the vector that only the j component is equal to 1 and the other component is zero.
• A^c-7,i) $E_i = (B_i(\widetilde{C}_j\alpha + \alpha I_d^0 + I_d^0) - \widetilde{C}_j\alpha)$ is a non null scalar, where $\alpha = -\frac{1}{d}, i = 1, 2$.

One obtains:

Corollary 3.3. Under conditions H^c-1), A^c-i), A^c-3), A^c-4,i), A^c-5,i), A^c-6,i) and A^c-7,i) specific to each case $i = 1, 2$, the system (2.9) admits a chaotic behavior at corner collision.

3.4 An illustrative example:

Let us consider the following piecewise smooth system:

$$\begin{cases}\dot{x}_1 = f_{11}(x_1,x_2,x_3) \\ \dot{x}_2 = f_{12}(x_1,x_2,x_3)) \\ \dot{x}_3 = f_{13}(x_1,x_2,x_3)\end{cases} \quad if\ x \epsilon D_{int} \tag{3.17}$$

and

$$\begin{cases}\dot{x}_1 = f_{21}(x_1,x_2,x_3) \\ \dot{x}_2 = f_{22}(x_1,x_2,x_3) \\ \dot{x}_3 = f_{23}(x_1,x_2,x_3)\end{cases} \quad if\ x \epsilon D_{out} \tag{3.18}$$

where:

$$\begin{aligned}f_{11} &= x_2 - (x_1^2 + x_2^2 - (1 + 10\varepsilon(\sin(777x_3) + 1)^2))x_1 \\ f_{12} &= -x_1 - (x_1^2 + x_2^2 - (1 + 10\varepsilon(\sin(777x_3) + 1)^2))x_2 \\ f_{13} &= \varepsilon(x_3 - \cos(330x_2) - (x_3 - \sin x_2)^3)\end{aligned}$$

$$\begin{aligned}f_{21} &= \frac{x_2}{\pi} - 100x_3^2((x_1+2)^2 + x_2^2 - (3 + 10x_3)^2)x_1 \\ f_{22} &= -(\frac{x_1+2}{\pi}) - 100x_3^2((x_1+2)^2 + x_2^2 - (3 + 10x_3)^2)x_2 \\ f_{23} &= 10\pi\varepsilon(x_3 + \sin(500x_1x_3)) - \varepsilon x_3^3\end{aligned}$$

$S_1 = \left\{(x_1,x_2,x_3)\epsilon\mathbb{R}^3 : H_1(x_1,x_2,x_3) = -x_1 + 1\right\}$
$S_2 = \left\{(x_1,x_2,x_3)\epsilon\mathbb{R}^3 : H_2(x_1,x_2,x_3) = -x_2\right\}$

The conditions for internal collision are satisfied and computations give us the corresponding Poincaré map:

$$P(x,\varepsilon) = \begin{cases}\begin{pmatrix} e^{2\pi} & 0 \\ 0 & -e^{-2\pi}\end{pmatrix} x + o(x,\varepsilon) \quad \text{if non-crossing} \\ \left(\begin{pmatrix} e^{2\pi} & 0 \\ 0 & -e^{-2\pi}\end{pmatrix} x + (-0.231, -0.113)^T \right. \\ \left. < (2.411, -1.013)^T,\ x > + o(x,\varepsilon)\right) \\ \text{if crossing}\end{cases}$$

where $x = (x_1, x_2)^T$ and $o(x,\varepsilon) \longrightarrow 0$ when $(x,\varepsilon) \longrightarrow (0,0)$.

For all simulations, the system is initialized in $x_1(0) = 1.1$, $x_2(0) = 0$ and $x_3(0) = 0$ (because the initial state is on the Poincaré map).

Moreover, in order to highlight the proposed way to chaos different values of ε in zero's vicinity are considered:

- For $\epsilon = 0$, the system (3.17 and 3.18) generates an attractive limit cycle see figure 1.
- For $\epsilon = 0.001$, a period doubling appears see figure 2.
- For $\epsilon = 0.1$, a chaotic behavior occurs see figure 3.

4. CONCLUSION

In this paper, we have proposed an analysis for generating chaos with respect to piecewise smooth systems submitted

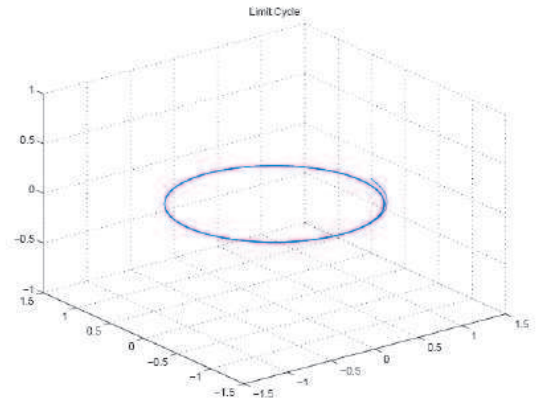

Fig. 1. Limit cycle $\epsilon = 0$

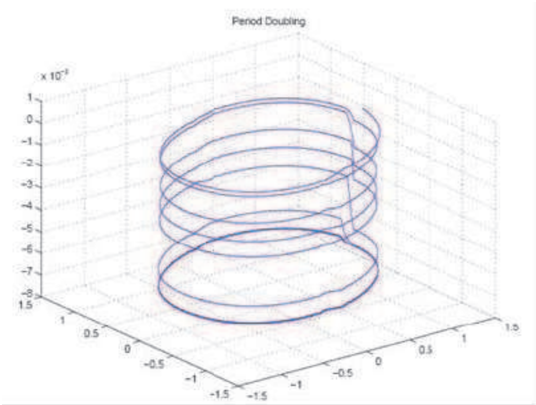

Fig. 2. Multi cycle behavior $\epsilon = 0.001$

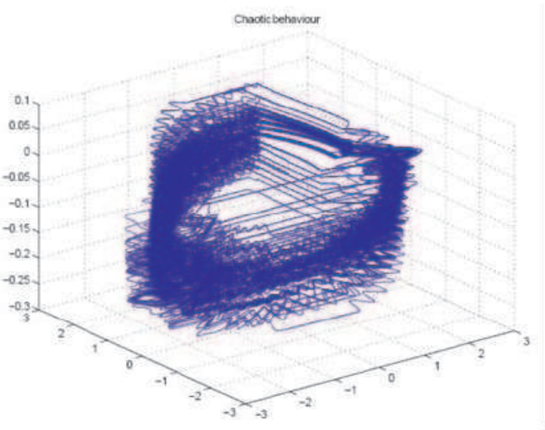

Fig. 3. Chaotic behavior $\epsilon = 0.1$

to sliding, corner or grazing bifurcations. This approach based on implicit functions theorem shows that the proposed route to chaos defined on the collision manifold is available not only for some fixed value of the bifurcation parameter but also for any value of this parameter in some small neighborhood of this value. Note that there are many possible extensions of this work, for example generating chaos for piecewise smooth systems with at least three different subsystems, more than one surfaces of collision, piecewise smooth systems submitted simultaneously to different types of border bifurcations (for example grazing, sliding, corner), piecewise smooth systems having more than one bifurcation parameter,... Moreover, this classification and chaotification of piecewise smooth system must have many applications as for example in the synchronization of non smooth chaotic systems for private communication or in the observer base control design of a multi-cell chopper.

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