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**Abstract**

This paper deals with the design of quadratic and higher order normal forms for the left invertibility problem. The linearly observable case and one-dimensional linearly unobservable case are investigated. The interest of such a study in the design of a delayed discrete-time observer is examined. The example of the Burgers map with unknown input is treated and a delayed discrete-time observer is designed. Finally, some simulated results are commented.

**Keywords:** Discrete-time normal forms, Left invertibility problem, Output injection, Homological equations.

## 1 Introduction

Since the last decade the concept of normal form in control theory was introduced by W. Kang and A. Krener in [21] and [22], (see also [11] for an algebraic point of view). On this basis the appearance of bifurcations under lost of controllability was studied [2, 15, 17, 20, 23, 39]. Following this way of thinking and the original concept of normal form introduced by H. Poincaré in [33], the observability normal form for continuous-time system was introduced in [8].
In the well known paper [31] the authors demonstrated that unidirectional synchronization of chaotic systems is equivalent to an observer design problem. Starting from this result and considering chaotic systems with unknown input, the problem of synchronization with recovering of the input can be seen as a left invertibility problem. Left invertibility was studied in several papers (see [12, 35, 36, 37, 38]...).

This work deals with left invertibility for discrete-time systems with observability singularity or/and left invertibility singularity. The results are based on the concept of discrete-time observability normal form [3]. The group of transformations associated with the left invertibility normal form is similar to the one associated with the observability normal form; the only difference is that input injection is not allowed in the first case. This slight difference generates extra resonant terms in the normal form. Clearly, when considering systems without input, both forms are identical. The normal form approach is facilitated in discrete-time by the possible use of geometric differential tools as proposed by S. Monaco and D. Normand-Cyrot (see [25, 27, 28]). Even though exact properties can be satisfied such as linearization or matching conditions, it is often enough in practice to consider approximate solutions so enlarging the domain of applicability. This is one of the main motivations of our approach (see the first paper of A. Krener regarding approximate design [22]). In the sequel, left invertibility normal forms are introduced and, depending on the resonant terms, their use is proposed in the design a delayed discrete-time observer. An application to an academic private communication is given as an illustrative example.

This paper follows the lines of our previous work [2, 5, 6, 8] and it is organized as follows. In section 2 the definitions and a settlement of the problem are stated. Quadratic observability and the left invertibility problem are presented in section 3. Section 4 is dedicated to the linearly unobservable case followed by the left invertibility problem in the one-dimensional unobservable case. An illustrative example ends the paper by showing the efficiency of the proposed approach.

# 2 Problem Statement and Quadratic Equivalence

We are interested in solving the **L**eft **In**vertibility **P**roblem (**LInP**) for a nonlinear **SISO** (Single Input Single Output) discrete-time system of the form:

$$z^+ = \bar{\Gamma}(z, u) \qquad y = h(z) = Cz \tag{1}$$

where the state vector $z(k)$ is denoted $z$ and $z^+$ denotes $z(k+1)$, $k \in N$. The unknown input is $u(k) \in D \subset \Re$ and $y(k) \in \Re$ is the output. The vector fields $\bar{\Gamma} : U \times D \longrightarrow M \subset \Re^n$ and the function $h : U \subset \Re^n \longrightarrow W \subset \Re$ are assumed to be real analytic. Without loss of generality, we assume that $\bar{\Gamma}(0, 0) = 0$.

The left invertibilty notion used in this paper is the following one. Given system (1), recover the state $z(k)$ and the input $u(k)$ from the outputs $y(k), ..., y(k+n), y(k+n+1)$..... Such a problem (LInP), is motivated by the fact that usually, in a control scheme, $u$ is on the left side and $y$ is on the right side of the block diagram. If (1) is invertible with respect to the unknown input $u$, the construction of a delayed observer like in [5] allows us to completely recover the information $u$. Such a delayed observer was implemented as a deciphering process for a secure communication application.

In [32], the so-called *observability matching condition* is given. This condition ensures the existence of a unique solution for the LInP in a neighborhood of an equilibrium point. In this paper, we deal with discrete-time systems and study how to design an equivalent class to (1) modulo an extended output injection, under the so-called Discrete-Time Observability Matching Condition (**DTOMC**). Each class is characterized by a discrete-time normal form for LInP. This normal form is reduced to the main terms of the original system while preserving its structural properties. These so-called resonant terms are shown to be the key for recovering the unknown input in the observer.

Firstly, the case where system (1) is linearly observable is analyzed. This means that the corresponding observer may be designed on the basis of the linear residue. Secondly, systems which do not satisfy linear observability in one direction are studied. As the linear residue does not give enough information about the state vector, we have to look for more pertinent information, in higher order terms. In both cases, the linear residue represents the linear part of the equivalent normal form. Moreover, the linear resonant terms are the only ones which can not be cancelled by linear transformation and output injection. These terms characterize the observability and detectability properties.

The system is rewritten as:

$$\left\{\begin{array}{lcl} z^{+} & = & Az+Bu+F^{[2]}(z) \\ & & +g^{[1]}(z)u+\gamma^{[0]}u^{2}+O^{3}\left(z,u\right) \\ y & = & Cz:=h(z) \end{array}\right. \tag{2}$$

with $A=\frac{\partial\bar{\Gamma}}{\partial z}(0,0)$, $B=\frac{\partial\bar{\Gamma}}{\partial u}(0,0)$ where:

$$F^{[2]}\left(z\right)=\left[\begin{array}{c} F_{1}^{[2]}\left(z\right) \\ \vdots \\ F_{n}^{[2]}\left(z\right) \end{array}\right] \text{ and } g^{[1]}\left(z\right)=\left[\begin{array}{c} g_{1}^{[1]}\left(z\right) \\ \vdots \\ g_{n}^{[1]}\left(z\right) \end{array}\right]$$

for $1\leq i\leq n$ , $F_{i}^{[2]}\left(z\right)$ and $g_{i}^{[1]}\left(z\right)$ are respectively homogeneous polynomials of degree 2 and 1 in $z$. Roughly speaking the resonant term is that one which can not be cancelled by coordinates change and output injection.

The following definition also sets a criterion which ca be used to check the "Quadratic Discrete-Time Observability Matching Condition" **QDTOMC**. This criterion means that[1], $y^{+},...,y^{(n-1)+}$ should not depend on the unknown input, contrary to $y^{(n)+}$. We will see later how this condition allows to recover $u$ in the normal form for the LInP.

Definition 2.1 *The **QDTOMC** holds a neighborhood of the equilibrium point $(z_{e},u_{e})$ of system (2) if*

$$J^{[1]}.G^{[2]}=(0,...,0,\star)^{T}+O^{3}(z,u) \tag{3}$$

*where*

$$J^{[1]}=\left[(dh)^{T^{[1]}},...,\{d((\sigma^{[2]})_{\circ}^{n-1}\circ h)\}^{T^{[1]}}\right]^{T}$$

[1] $\forall\, j\in\mathbb{N}$, $y^{(j)+}$ denotes $y(k+j)$ and $y^{(j)-}$ denotes $y(k-j)$

$$G^{[2]}(z,u) = Bu + g^{[1]}(z)u + \gamma^{[0]}u^2$$

$$\sigma^{[2]} = Az + F^{[2]}(z)$$

*"o" denotes the usual composition operator and* $(\sigma^{[2]})^j_o$ *denotes the expansion of* $\sigma$ *up to order 2 in* $z$ *and composed* $j$ *times (i.e.* $(\sigma^{[2]})^j_o = \sigma^{[2]} \circ (\sigma^{[2]})^{j-1}_o$ *for* $2 \leq j \leq n$ *with* $(\sigma^{[2]})^1_o = \sigma^{[2]}$*).*
*The symbol* $\star$ *in (3) represents a first or second order non-trivial function of* $x$ *and* $u$*. This function is non-null almost everywhere around the equilibrium point, containing -if it exists- observability singularity.*

In order to analyze the local invertibility of system (2), an equivalence modulo an extended output injection must be established at each order $i$ ($2 \leq i \leq m$). In the sequel, for simplicity of presentation, only quadratic equivalence is investigated.

## 2.1 Quadratic Equivalence

Let us first define the notion of quadratic equivalence under coordinates change and output injection. Note that, in this paper output injection denotes injection of all available variables, output and known input, and strict output injection denotes usual injection from the output. This last injection will be used to solve the left invertibility problem. Let us first define the so-called quadratic equivalence.

Definition 2.2 *The system (2) is said to be quadratically equivalent to the system:*

$$\left\{\begin{array}{rcl} x^+ & = & Ax + Bu + \bar{F}^{[2]}(x) + \bar{g}^{[1]}(x)u + \bar{\gamma}^{[0]}u^2 \\ & & +\beta^{[2]}(y) + \alpha^{[1]}(y)u + \tau^{[0]}u^2 + O^3(x,u) \\ y & = & Cx \end{array}\right. \tag{4}$$

*Modulo an Output Injection (**MOI**) if there exists a coordinates change of the form:*

$$x = z - \Phi^{[2]}(z) \tag{5}$$

*and an output injection:*

$$\beta^{[2]}(y) + \alpha^{[1]}(y)u + \tau^{[0]}u^2 \tag{6}$$

*which transforms the quadratic part of (2) into the quadratic part of (4) modulo the injection (6) and* $\Phi^{[2]}(z) = \left[\Phi_1^{[2]}(z), ......, \Phi_n^{[2]}(z)\right]^T$ *with* $\Phi_i^{[2]}(z)$ *a quadratic homogeneous term in* $z$*.*

In the next proposition, we give necessary and sufficient conditions for quadratic equivalence MOI:

Proposition 2.1 *System (2) is quadratically equivalent modulo an output injection to system (4), if and only if there exist* ($\Phi^{[2]}$, $\beta^{[2]}$ , $\alpha^{[1]}$, $\gamma^{[0]}$) *which satisfy the three following homological equations:*

*i)* $F^{[2]}(x) \ - \bar{F}^{[2]}(x) = \Phi^{[2]}(Ax) - A\Phi^{[2]}(x) + \beta^{[2]}(x_1)$

*ii)* $g^{[1]}(x) - \bar{g}^{[1]}(x) = \widehat{\Phi}^{[2]} \ (Ax, B) + \alpha^{[1]}(x_1)$

*iii)* $\gamma^{[0]} - \bar{\gamma}^{[0]} = \Phi^{[2]}(B) + \tau^{[0]}$

*where* $y = x_1$ *and* $\widehat{\Phi}^{[2]} \ (Ax, B) = (Ax)^T\phi \ Bu + (Bu)^T\phi \ Ax$, *and* $\phi$ *is a square symmetric matrix such that:*

$$\phi = \frac{1}{2}\frac{\partial^2\Phi^{[2]}(x)}{\partial x\partial x^T}$$

**Proof.** By applying the coordinates change (5) to system (2) we obtain the following equality:

$$\begin{array}{rcl} x^+ & = & Az + Bu + F^{[2]}(z) + g^{[1]}(z)u + \gamma^{[0]}u^2 \\ & - & \Phi^{[2]}(Az + Bu) + O^3\,(z, u) \end{array}$$

Since $\Phi^{[2]}$ satisfies $z = x + \Phi^{[2]}\,(x) + O^3(x)$ , from the computations one obtais:

$$\begin{array}{rcl} x^+ & = & Ax + Bu + F^{[2]}(x) + A\Phi^{[2]}(x) - \Phi^{[2]}(Ax) \\ & & +g^{[1]}(x)u - ((Ax)^T\phi\ Bu + (Bu)^T\phi\ Ax) \\ & & +\gamma^{[0]}u^2 - \Phi^{[2]}(B)u^2 + O^3\,(x, u) \end{array} \tag{7}$$

System (7) deduced from (2) (via the coordinates change (5)) is quadratically equivalent to (4) if and only if their quadratic parts coincide, that is:

$$\begin{array}{rl} & F^{[2]}(x) + A\Phi^{[2]}(x) - \Phi^{[2]}(Ax) + g^{[1]}(x)u - \widehat{\Phi}^{[2]}(Ax, B)u + \gamma^{[0]}u^2 - \Phi^{[2]}(B)u \\ = & \bar{F}^{[2]}(x) + \beta^{[2]}(x_1) + \bar{g}^{[1]}(x)u + \alpha^{[1]}(x_1)u + \bar{\gamma}^{[0]}u^2 + \tau^{[0]}u^2 \end{array}$$

and by identification, we obtain:

$$\left\{ \begin{array}{rcl} A\Phi^{[2]}(x) - \Phi^{[2]}(Ax) + F^{[2]}(x) & = & \bar{F}^{[2]}(x) + \beta^{[2]}(x_1) \\ -\widehat{\Phi}^{[2]}(Ax, B) + g^{[1]}(x) & = & \bar{g}^{[1]}(x) + \alpha^{[1]}(x_1) \\ -\Phi^{[2]}(B) + \gamma^{[0]} & = & \bar{\gamma}^{[0]} + \tau^{[0]} \end{array} \right.$$

This ends the proof. ■

In what follows we will determine the normal forms associated to system (2) modulo the output injection (6), at first for the linearly observable case.

Remark 2.1 *Throughout the paper and without loss of generality we deal with systems with linearly observable part in the Brunovsky form. Moreover, as the output is structurally fixed, we have set* $\Phi_1^{[2]}\,(z) = 0$ *in the coordinates change (5).*

# 3 Linearly observable case

## 3.1 Observability normal form

Under the assumption that $(A, C)$ is observable, we can assume the linear part of system (2) in the observable Brunovsky form, knowing that there exist a linear change of coordinates $z = T\xi$ and a Taylor expansion which transforms the system (1) into the following form [10] :

$$\left\{ \begin{array}{lcl} z^+ & = & A_{obs}\, z + B_{obs}\, u + F^{[2]}(z) + g^{[1]}(z)u + \gamma^{[0]}u^2 + O^3\,(z,u) \\ y & = & C_{obs}\, z = z_1 \end{array} \right. \tag{8}$$

where:

$$A_{obs} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \cdots & 0 & 1 \\ a_n & 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ and } B_{obs} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

On this basis we can establish the following theorem:

THEOREM 3.1 *The quadratic normal form associated to system (8), modulo an output injection is:*

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-1}^+ \end{bmatrix} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} u + \begin{bmatrix} \sum_{i=2}^n k_{i1}x_i \\ \sum_{i=2}^n k_{i2}x_i \\ \vdots \\ \sum_{i=2}^n k_{i(n-1)}x_i \end{bmatrix} u$$

$$x_n^+ = a_n x_1 + b_n u + \sum_{j>i=1}^{n} h_{ij}x_i x_j + \left( \sum_{i=2}^{n} k_{in}x_i \right) u$$

REMARK 3.1 *The normal form defined in the previous theorem is slightly different from the normal form for continuous-time systems: here we are able to cancel any term in $x_i^2 (n \geq i \geq 1)$ in the last row of the dynamics (instead of any term $x_1 x_j$ , $n \geq j \geq 2$).*

To prove *Theorem* 3.1 we use the following lemma:

LEMMA 3.1 *Consider system (8), then:*

**1** *In the last component of $F^{[2]}$, the terms $x_i x_j$ $(n \geq j > i \geq 1)$ are resonant.*

**2** *In $g^{[1]}$, terms $x_i$ ($n \geq i \geq 2$) are resonant.*

**3** *There are no resonant terms in the vector field $\gamma^{[0]}$.*

REMARK 3.2 *We recall that resonant terms -according to Poincaré's works- are those which are invariant under quadratic transformations and output injection (additional transformation due to the study of observability). Obviously for high-order approximation, some high order resonant terms must be considered.*

**Proof.** See the appendix A for the proof of Theorem 3.1. ∎

## 3.2 Left invertibility normal form

Since in the left invertibility problem the input is unknown, we used the strict output injection (6)

$$\beta^{[2]}(y) \tag{9}$$

From Theorem 3.1 and its proof, we deduce the following corollary:

COROLLARY 3.1 *The quadratic normal form associated to system (8), modulo a strict output injection (9) is:*

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-1}^+ \end{bmatrix} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} u$$
$$+ \begin{bmatrix} \sum_{i=1}^n k_{i1} x_i \\ \sum_{i=1}^n k_{i2} x_i \\ \vdots \\ \sum_{i=1}^n k_{i(n-1)} x_i \end{bmatrix} u + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} u^2$$

$$x_n^+ = a_n x_1 + b_n u + \sum_{j>i=1}^{n} h_{ij} x_i x_j + \left( \sum_{i=1}^{n} k_{in} x_i \right) u + \gamma_n u^2$$

Now, in order to recover the state and the unknown input, let us consider the following lemma and proposition.

LEMMA 3.2 *The **QDTOMC** is invariant by a quadratic coordinates change and strict output injection (9).*

**Proof.** Using the following conditions

$$\begin{aligned} \left.\frac{\partial y}{\partial u}\right|_{x_1} &= O^3(x,u) \\ &\vdots \\ \left.\frac{\partial y^{(n-1)+}}{\partial u}\right|_{x} &= O^3(x,u) \end{aligned}$$

the particular form of the coordinates change $z = I_d + \Phi^{[2]}(x)$, and the fact that the output injection (9) do not change the **QDTOMC**, we obtain directly

$$\begin{array}{rcl} \dfrac{\partial y}{\partial u} & = & O^3(z,u) \\ & \vdots & \\ \dfrac{\partial y^{(n-1)+}}{\partial u} & = & O^3(z,u) \end{array}$$

■

From theorem 3.1 and lemma 3.2, we obtain:

PROPOSITION 3.1 *The quadratic normal form, modulo a strict output injection, for system (8) verifying the **QDTOMC**, is:*

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-1}^+ \end{bmatrix} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_n^+ = a_n x_1 + b_n u + \sum_{j>i=1}^{n} h_{ij} x_i x_j + \left( \sum_{i=1}^{n} k_{in} x_i \right) u + \gamma_n u^2$$

From the previous form it will be easier to analyze if the system is Left invertible and if so to design an observer. In the next section we will study the case of linearly unobservable systems in one dimension.

# 4 One-dimensional linearly unobservable case

Let us consider system (1) and assume that the pair $(A, C)$ has one unobservable mode. Then there is a linear change of coordinates ($z = T\xi$) and a Taylor expansion which transform the system (1) into:

$$\left\{ \begin{array}{rcl} \tilde{z}^+ & = & A_{obs}\tilde{z} + B_{obs}u + \tilde{F}^{[2]}(z) + \tilde{g}^{[1]}(z)u + \tilde{\gamma}^{[0]}u^2 + O^3(z,u) \\ z_n^+ & = & \eta z_n + \sum_{i=1}^{n-1} \lambda_i z_i + b_n u + F_n^{[2]}(z) + g_n^{[1]}(z)u + \gamma_n^{[0]} u^2 + O^3(z,u) \\ y & = & C_{obs}\, z \end{array} \right. \quad (10)$$

where: $\tilde{z} = [z_1, z_2, ..., z_{n-1}]^T$ and $z = [\tilde{z}^T, z_n]^T$.

REMARK 4.1

*i)- System (10) is the general **linear** canonical form of the unobservable system in one direction.*

*ii)- If $\eta$ is a unique eigenvalue for the linear approximation of (1), then there exists a linear transformation ($z = T\xi$) which transforms (1) in (10), with $\lambda_i = 0$ for any $i \in \{1, \ldots, n-1\}$.*

*iii)- The normal form which follows is structurally different from the controllability discrete-time normal form, given in [15, 14], in the last state dynamics $x_n^+$. For the observability analysis the main structural information is not in the $x_n^+$ dynamics but in the previous state evolution $\left(x_i^+ \text{ for } n-1 \geq i \geq 1\right)$. The terms $\lambda_i x_i$, $b_n u$, $F_n^{[2]}(x)$, $g_n^{[1]}(x)u$ are only important in the case of detectability analysis when $\eta = \pm 1$.*

## 4.1 Observability normal form

Hereafter, we particularize the definition of quadratic equivalence to those systems with one unobservable mode.

Now, following definition 2.2, system (10) is said to be quadratically equivalent to:

$$\left\{ \begin{array}{lcl} \tilde{x}^+ & = & A_{obs}\tilde{x} + B_{obs}u + \bar{\tilde{F}}^{[2]}(x) + \bar{\tilde{g}}^{[1]}(x)u + \bar{\tilde{\gamma}}^{[0]}u^2 \\ & & +\tilde{\beta}^{[2]}(y) + \tilde{\alpha}^{[1]}(y)u + \tilde{\tau}^{[0]}u^2 + O^3(x,u) \\ x_n^+ & = & \eta x_n + \sum_{i=1}^{n-1} \lambda_i x_i + b_n u + \bar{F}_n^{[2]}(x) + \bar{g}_n^{[1]}(x)u \\ & & +\bar{\gamma}_n^{[0]}u^2 + \beta_n^{[2]}(y) + \alpha_n^{[1]}(y)u + \tau^{[0]}u^2 + O^3(x,u) \\ y & = & C_{obs}\, x \end{array} \right. \tag{11}$$

modulo an output injection:

$$\left\{ \begin{array}{l} \tilde{\beta}^{[2]}(y) + \tilde{\alpha}^{[1]}(y)u + \tilde{\tau}^{[0]}u^2 \\ \beta_n^{[2]}(y) + \alpha_n^{[1]}(y)u + \tau_n^{[0]}u^2 \end{array} \right.$$

if there exists a coordinates change of the form:

$$\left\{ \begin{array}{lcl} \tilde{x} & = & \tilde{z} - \tilde{\Phi}^{[2]}(z) \\ x_n & = & z_n - \Phi_n^{[2]}(z) \end{array} \right.$$

which transforms the quadratic part of (10) into the quadratic part of (11), with:

$$\tilde{\Phi}^{[2]}(z) = [\Phi_1^{[2]}(z), ..., \Phi_{n-1}^{[2]}(z)]^T.$$

Now we determine the set of homological equations which will allow us to construct the quadratic normal form associated to system (10).

Proposition 4.1 *System (10) is quadratically equivalent to (11), modulo an output injection if and only if there exists ($\tilde{\Phi}^{[2]}$, $\tilde{\beta}^{[2]}$, $\tilde{\alpha}^{[1]}(y)$, $\tilde{\tau}^{[0]}$ and $\Phi_n^{[2]}$, $\beta_n^{[2]}$, $\alpha_n^{[1]}$, $\tau_n^{[0]}$) which satisfy the following sets of homological equations:*

**i)**

$$\left\{ \begin{array}{lcl} \tilde{F}^{[2]}(x) - \bar{\tilde{F}}^{[2]}(x) & = & \tilde{\Phi}^{[2]}(\bar{A}x) - A_{obs}\tilde{\Phi}^{[2]}(x) + \tilde{\beta}^{[2]}(x_1) \\ \tilde{g}^{[1]}(x) - \bar{\tilde{g}}^{[1]}(x) & = & \hat{\tilde{\Phi}}^{[2]}(\bar{A}x, \bar{B}) + \tilde{\alpha}^{[1]}(x_1) \\ \tilde{\gamma}^{[0]} - \bar{\tilde{\gamma}}^{[0]} & = & \tilde{\Phi}^{[2]}(\bar{B}) + \tilde{\tau}^{[0]} \end{array} \right. \tag{12}$$

**ii)**

$$\left\{\begin{array}{lcl} F_n^{[2]}(x)-\bar{F}_n^{[2]}(x) & = & \Phi_n^{[2]}(\bar{A}x)-\eta\Phi_n^{[2]}(x)-\sum_{i=1}^{n-1}\lambda_i\Phi_i^{[2]}(x)+\beta_n^{[2]}(x_1) \\ g_n^{[1]}(x)-\bar{g}_n^{[1]}(x) & = & \widehat{\Phi}_n^{[2]}(\bar{A}x,\bar{B})+\alpha_n^{[1]}(x_1) \\ \gamma_n^{[0]}-\bar{\gamma}_n^{[0]} & = & \Phi_n^{[2]}(\bar{B})+\tau_n^{[0]} \end{array}\right. \tag{13}$$

*where,*

$$\bar{A}=\left[\begin{array}{cc} A_{obs} & 0_{n-1} \\ 0_{n-1}^T & \eta \end{array}\right],\ \bar{B}=\left[\begin{array}{c} B_{obs} \\ b_n \end{array}\right];$$

*and*

$$\left\{\begin{array}{lcl} \widehat{\tilde{\Phi}}^{[2]}(\bar{A}x,\bar{B}) & = & (\bar{A}x)^T\phi\ Bu+(\bar{B}u)^T\phi\ Ax \\ \widehat{\Phi}_n^{[2]}(\bar{A}x,\bar{B}) & = & (\bar{A}x)^T\phi_n\ Bu+(\bar{B}u)^T\phi_n\ Ax \end{array}\right.$$

*with* $\phi=\frac{1}{2}\frac{\partial^2\Phi^{[2]}(x)}{\partial x\partial x^T};\phi_n=\frac{1}{2}\frac{\partial^2\Phi_n^{[2]}(x)}{\partial x\partial x^T}$ .

**Proof.** The proof is obvious; by considering the vector fields $\tilde{\Phi}^{[2]}$, $\tilde{\beta}^{[2]}$, $\tilde{\alpha}^{[1]}(y)$, $\tilde{\tau}^{[0]}$, the homological equations (12) imply, by applying then only to the observable part of system (11), the same assumption of *proposition* (2.1). The set of homological equations (13) is deduced from the unobservability line of the system, by considering the output injection: $\beta_n^{[2]}(y)+\alpha_n^{[1]}(y)u+\tau_n^{[0]}u^2$ and the change of coordinates: $x_n=z_n-\Phi_n^{[2]}(z)$. ■

Thanks to the notion of quadratic observability equivalence presented above, we are able to define an equivalent class for every system of the type (2), reduced to a unique system under the quadratic observability normal form, possessing the same structural properties as those of the corresponding equivalent system. This unique form will be described in the following theorem.

THEOREM 4.1 *The normal form with respect to the quadratic equivalence modulo an output injection of system (10) is:*

$$\left[\begin{array}{c} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-2}^+ \end{array}\right]=\left(\begin{array}{ccccc} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-2} & 0 & \cdots & 0 & 1 \end{array}\right)\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{array}\right]+\left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_{n-2} \end{array}\right]u+\left[\begin{array}{c} \sum_{i=2}^n k_{i1}x_i \\ \sum_{i=2}^n k_{i2}x_i \\ \vdots \\ \sum_{i=2}^n k_{i(n-2)}x_i \end{array}\right]u$$

$$x_{n-1}^+=a_{n-1}x_1+b_{n-1}u+\sum_{j>i=1}^n h_{ij}x_ix_j+h_{nn}x_n^2+\left(\sum_{i=2}^n k_{i(n-1)}x_i\right)u$$

*Moreover, by setting:* $R=A^T\phi_n\bar{A}-\eta\ \phi_n$; *and by considering condition:*

$$\exists\ \ (i,j)\in I\subseteq\ \{1,\ldots,n\}\ \text{such that}\ R_{i,j}=0^2 \tag{14}$$

*Given* $\eta$, $\mathbf{a}_i$ *and* $\lambda_i$ ($\forall\ n-1\geq i\geq 1$)*;*
*if* $\nexists\phi_n\neq 0$ *such that, (14) is verified,* $\exists\ \phi_n$ *such that all* $l_{ij}$ *can be cancelled, and consequently the dynamic is:*

$$x_n^+=\eta x_n+\sum_{i=1}^{n-1}\lambda_ix_i+b_nu+\left(\sum_{i=2}^n k_{ni}x_i\right)u$$

**Proof.** See Appendix B for the proof of Theorem 4.1. ■

Remark 4.2

- *i)- If the term $k_{n(n-1)} \neq 0$, then the quadratic one $k_{n(n-1)}x_n u$ in the normal form described above restores the observability for a well chosen input $u$.*
- *ii)- In the normal form, let us focus more closely on the state quadratic part: $\sum_{j>i=1}^{n} h_{ij}x_ix_j$ which appears in the $(n-1)^{th}$ line. By isolating the terms in the unobservable direction $x_n$, as follows: $\sum_{j>i=1}^{n-1} h_{ij}x_ix_j + \left(\sum_{i=1}^{n} h_{in}x_i\right)x_n$ we can deduce the manifold of local unobservability: $S_n = \left\{x, \text{such that } \sum\limits_{i=1}^{n} h_{in}x_i = 0\right\}$. Then, outside $S_n$ we recover the observability.*
- *iii)- If $\eta \in ]-1,1[$ then locally, the system (10) is detectable.*

Now, as in section 3 we will study the left invertibility problem.

## 4.2 Left invertibility normal form

From Theorem 4.1 and its proof, one deduces the following corollary:

Corollary 4.1 *The quadratic normal form associated to (8), modulo a strict output injection (9) is:*

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-2}^+ \end{bmatrix} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-2} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-2} \end{bmatrix} u$$

$$+ \begin{bmatrix} \sum_{i=1}^{n} k_{i1}x_i \\ \sum_{i=1}^{n} k_{i2}x_i \\ \vdots \\ \sum_{i=1}^{n} k_{i(n-2)}x_i \end{bmatrix} u + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{n-2} \end{bmatrix} u^2$$

$$x_{n-1}^+ = a_{n-1}x_1 + b_{n-1}u + \sum_{j>i=1}^{n} h_{ij}x_ix_j + h_{nn}x_n^2 + \left(\sum_{i=2}^{n} k_{i(n-1)}x_i\right) u + \gamma_{n-1}u^2$$

Similarly to section 3, we will consider, for the left invertibility problem, a strict output injection of the form (9).

Remark 4.3 *In this paper, only the case of left invertibility, without "approximated zero dynamics"* [2] *is investigated [16].*

[2] $\frac{\partial y^{j+}}{\partial u} = O^{j+1}(x,u)$ for $j < n$ and $\frac{\partial y^{n+}}{\partial u} \neq 0$, where $j$ is the considered approximation order.

Since lemma 3.1 is independent of the considered case, we can set the following proposition

PROPOSITION 4.2 *For system (8) verifying the **DTOMC**, the quadratic normal form modulo strict output injection (9) is:*

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_{n-2}^+ \end{bmatrix} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-2} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$x_{n-1}^+ = a_{n-1}x_1 + b_{n-1}u + \sum_{j>i=1}^{n} h_{ij}x_ix_j + h_{nn}x_n^2 + \left(\sum_{i=2}^{n} k_{i(n-1)}x_i\right) u + \gamma_{n-1}u^2$$

To conclude the paper, an algorithm is established in order to compute the $m^{\text{th}}$ ($m \longrightarrow \infty$) order **DTNF**.

**The algorithm**

**Initialization**: In this phase, we give the order of **DTOMC**:

$$Order(DTOMC) = k$$

and set $p$, the desired approximation order, then

$$m = \min\{k,\ p\}$$

**Beginning of the algorithm**

**Question one**: is the system linearly observable?
If the answer is positive then go to **Case one** otherwise go to **question two**.

**Question two**: is the system linearly unobservable in one dimension?
If the answer is positive then go to **Case two** otherwise go to the **End of the algorithm**.
Case one: Compute the output injection $\beta^{[i]}(y)$ and the coordinates change $I_d + \Phi^{[i]}$, in order to obtain the Left Invertible normal form of $i^{\text{th}}$ order.
If $i = m$ then go to the **End of the algorithm**.
Else, $i := i + 1$. Go to **Case one**.

Case two: Compute the output injection $\beta^{[i]}(y)$ and the coordinates change $I_d + \Phi^{[i]}$, in order to obtain the Left Invertible normal form of $i^{\text{th}}$ order.
If $i = m$ then, go to the **End of the algorithm**.
Else, $i := i + 1$. Go to **Case Two**.

**End of the algorithm**.

In figure 1, we sum up the previous algorithm.

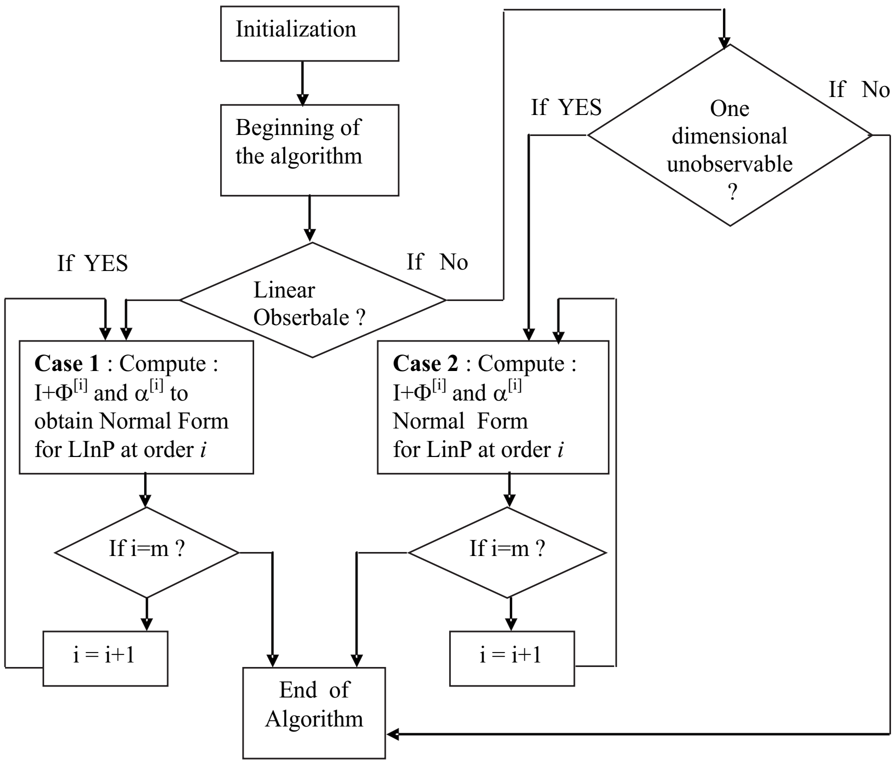


Fig. 1 : Flow chart of the algorithm

# 5 An illustrative example: The Burgers map

As known from the work of H.Nijmeijer and I. Mareels [31] synchronization of a chaotic system can be reformulated as an observer design problem. In this section by means of an example we will show how the quadratic normal form can be used to solve the problem of synchronization with unknown input.

Let us study the following two-dimensional system (modulo $O^3(z,u)$):

$$\left\{ \begin{array}{l} z^+ = Az + Bu + f^{[2]}(z) \\ \qquad +g^{[1]}(z)u + \gamma^{[0]}u^2 \\ y = C\ z = z_1, \end{array} \right. \tag{15}$$

where $A = diag\{1+a,\ 1-b\}$, $B = [1,0]^T$.

$$\begin{array}{l} f^{[2]}(z) = \left[z_1 z_2,\ (a^2+2a+b-1)z_1^2\right]^T \\ g^{[1]}(z) = \left[0,\ 2(a+1)z_1\right]^T,\ \gamma^{[0]} = [0,1]^T \end{array}$$

where $a$ and $b$ are real parameters and $u$ is the unknown input. Let us assume that state $z_1$ is measured so $y = z_1$ is the output. Firstly, the observability of (15) will be studied.

System (15) is linearly unobservable in the neighborhood of the equilibrium point $(0,0)$. Moreover, system (15) satisfies the **DTOMC** everywhere except in $z_1 = 0$. This allows to recover $u$ outside the set $S=\{z/z_1=0\}$ of left invertibility an observability singularity.

By applying the change of coordinates
$x=z-\Phi^{[2]}(z)$ to system (15); $\Phi^{[2]}$ is deduced as follows:

$$\Phi_1^{[2]}(z)=0,\ \Phi_2^{[2]}(z)=z_1^2 \quad \text{and} \quad \delta^{[2]}=\bar{0}_{\Re^2}.$$

The associated Quadratic discrete-time Normal Form (**QDTNF**), is then obtained:

$$\begin{aligned} x_1^+ &= (1+a)\,x_1+x_1x_2 \\ x_2^+ &= (1-b)x_2-x_1^2+u \end{aligned} \tag{16}$$

with $y=x_1$. This system, representing a quadratic chaotic system, arises in hydrodynamics patterns [19]: the Burgers map (figure.2, with $a=0.548$, $b=2.28$, and $x_1^0=1.05$, $x_2^0=-0.66$.).

**Delayed Observer Design:** A model of an observer is proposed for system (16) allowing to recover $x_2$ from $y$. Thus, using delayed corrections on the first observer state, it is possible to ensure "the synchronization" of both systems. The observer is represented by the following equations:

$$\begin{aligned} \hat{x}_1^+ &= (1+a)\,y+y\hat{x}_2. \\ \hat{x}_2^+ &= (1-b)\hat{x}_2-y^2 \end{aligned}$$

*Computation of* $\tilde{x}_2$: From the previous equation, the state $x_2$ is approximated by $\tilde{x}_2$. $\tilde{x}_2$ can be easily recovered, from the following observation error and thanks to the resonant term $yx_2$:

$$e_2^-=\frac{e_1}{y^-},\quad \forall\, y\neq 0$$

The singularity $y=0$ could be by-pass as follows:

$$\tilde{x}_2^-=\hat{x}_2^-+\frac{e_1\,y^-}{(y^-)^2+\varepsilon},\ \forall\, y$$

where $\varepsilon$ is a constant in the neighborhood of zero. The recovered state

$$\tilde{x}_2=(1-b)\tilde{x}_2^- -(y^-)^2,$$

is implemented in the observer in order to synchronize it with (16), so $\hat{x}_2^+$ the prediction of $\tilde{x}_2$, is given by the following equation:

$$\hat{x}_2^+=(1-b)\tilde{x}_2-y^2$$

*Recovering* $u$: The unknown input $u$ is recovered from the observation error $e_2$ with **two delays** (for the sake of causality) as follows:

$$u^{--}=e_2^- -(1-b)e_2^{--}$$

Remark 5.1 *The observer synchronizes with Burgers map in three steps as shown in figure 3 (for* $u = 0.02$*) and with* $\tilde{x}_2^0 = \hat{x}_2^0 = \hat{x}_1^0 = 1$*). Since, the initial conditions are such that* $y(0) \neq 0$*, after one step* $\tilde{x}_2^+ = x_2^+$ *and then,* $\hat{x}_2^{++} = x_2^{++}$*. Consequently from the first equation of the observer, after three steps we obtain* $\hat{x}_1^{+++} = x_1^{+++}$*.*

In [18] the authors have proved that the design of observers for systems with unknown input can be profitably used in the context of secure communications. So, the observer described above can be used in a secure communication context, with $u$ denoting the confidential data one wants to transmit.
The previous example underlines the efficiency of the normal form for LInP, especially the resonant terms in the design and validation of observers.

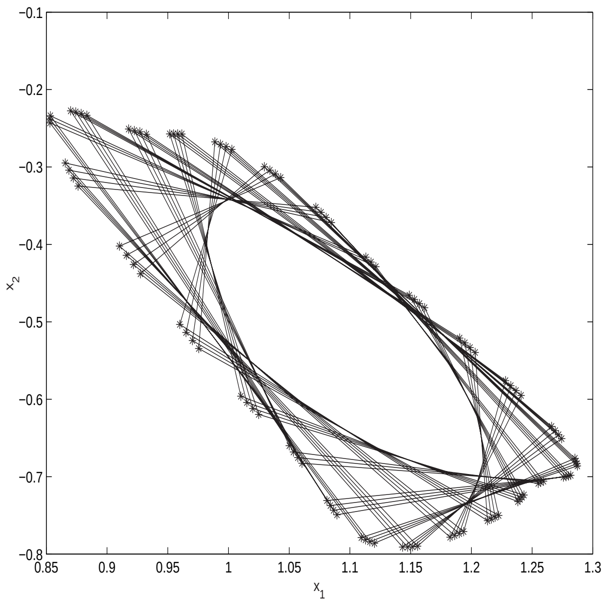


Fig. 2 : Burgers map phase portrait

# 6 Conclusion

In this paper we have investigated the left invertibility problem for discrete-time systems making use of the normal-form methodology. The results obtained highlight the efficiency of the Poincaré normal form method in solving such a problem. Several questions remain open; between the others we cite the MIMO case (see [12] for the linear case and [29, 1] for nonlinear case), and the case of systems with zero dynamics (an important subject for system under sampling; see [26, 4, 30]). In the opinion of the authors several concrete problems should benefit from the application of the proposed technique: private communication, fault detection, parameters identification are some of them.

**Acknowledgements:** the authors are grateful to S. Monaco, D. Normand-Cyrot and W. Kang for their helpful discussions and their perceptive suggestions.

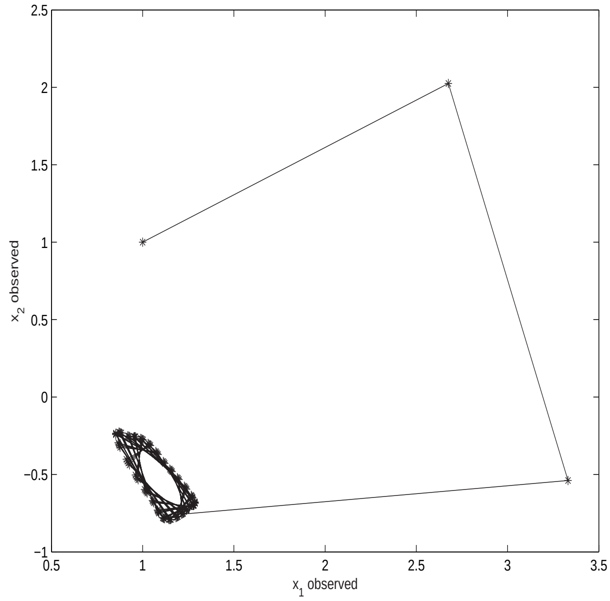


Fig. 3 : Observer synchronization

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## Appendix A: Proof of theorem 3.1

**Proof.** The homological equations associated to system (8), are:

**$1^{st}$ homological equation**:

$$F^{[2]}(x) = \Phi^{[2]}(A_{obs}\, x) - A_{obs}\Phi^{[2]}(x) + \beta^{[2]}(x_1)$$

with $\bar{F}^{[2]}$ wished equal to zero. By considering the structure of matrix $A_{obs}$, and using the fact that $\beta^{[2]}$ is a homogeneous vector field (i.e. $\beta_i^{[2]}(x_1) = \beta_i\, x_1^2$ , $\forall\ n \geq i \geq 1$ ), we can write the equation in a more explicit form:

$$\left\{\begin{array}{lcl} F_1^{[2]}(x) & = & -\Phi_2^{[2]}(x) + \beta_1\, x_1^2 \\ F_2^{[2]}(x) & = & \Phi_2^{[2]}(A_{obs}x) - \Phi_3^{[2]}(x) + \beta_2 x_1^2 \\ \vdots & = & \vdots \\ F_{n-1}^{[2]}(x) & = & \Phi_{n-1}^{[2]}(A_{obs}x) - \Phi_n^{[2]}(x) + \beta_{n-1}\, x_1^2 \\ F_n^{[2]}(x) & = & \Phi_n^{[2]}(A_{obs}x) + \beta_n\, x_1^2 \end{array}\right. \tag{17}$$

Now, in any row $i$ ($\forall\ n \geq i \geq 2$) of system (17) let us substitute $\Phi_i^{[2]}(A_{obs}x)$ by its expression deduced from the $i-1^{th}$ row; it follows:

The $1^{st}$ equation do not change:

$$F_1^{[2]}(x) = -\Phi_2^{[2]}(x) + \beta_1 x_1^2$$

Then, by induction the $i^{th}$ equation ($\forall\ n-1 \geq i \geq 2$) may be written as:

$$\begin{array}{lcl} F_i^{[2]}(x) & = & -\Phi_{i+1}^{[2]}(x) - \sum_{k=1}^{i-1} F_k^{[2]}(A_{obs}^{i-k}\, x) + \sum_{k=2}^{i} d_{1k}^i\, x_1 x_k \\ & & + \sum_{k=1}^{i-1} d_{kk}^i\, x_k + \delta_i^{(3)}(x_1, \ldots, x_i) + \sum_{k=1}^{i} \beta_k x_{i-k+1}^2 \end{array}$$

where $d_{1k}^i$ ($\forall\ i \geq k \geq 2$), $d_{kk}^i$ ($\forall\ i-1 \geq k \geq 1$) may be written as a linear combination of: $\beta_j\, a_l a_m$ and/or $\beta_j\, a_l$ ($\forall\ \ i \geq j \geq 1,\ i-1 \geq,\ m \geq l \geq 1$); and $\delta_i^{(3)}$ is a polynomial function of $(x_1, \ldots, x_i)$ of degree $\geq 3$.

So, we remark that in each row $i$ ($\forall\ n-1 \geq i \geq 1$) of system (17):

► We can determine by identification, the coordinates change component $\Phi^{[2]}_{i+1}$ so as to cancel the quadratic terms $F^{[2]}_i$.

► Consequently, we can isolate $i$ degrees of freedom: $\beta_1, \ldots,\ \beta_i$.

Finally the last equation is:

$$\begin{aligned} F^{[2]}_n(x) \; &= \; -\textstyle\sum_{k=1}^{n-1} F^{[2]}_k(A^{n-k}_{obs}\, x) \; + \sum_{k=2}^{n} d^n_{1k}\, x_1 x_k + \sum_{k=1}^{n-1} d^n_{kk} \\ &\quad + \delta^{(3)}_n(x_1, \ldots, x_n) + \textstyle\sum_{k=1}^{n} \beta_k x^2_{n-k+1} \end{aligned}$$

where $d^n_{1k}, d^n_{kk}$ are defined by a linear combination of: $\beta_j\ a_l a_m$ and/or $\beta_j\ a_l$ ($\forall\ \ i \geq j \geq 1,\ i-1 \geq,\ m \geq l \geq 1$); and $\delta^{(3)}_n$ is a polynomial function $(x_1, \ldots, x_n)$ of degree $\geq 3$ . We can see that in the row $n$ of system (17), there are $n$ degrees of freedom: $\beta_1, \ldots,\ \beta_n$, for $\frac{n(n+1)}{2}$ degrees of freedom in $F^{[2]}_n$. So the coefficients $\beta_i$ ($\forall\ n \geq i \geq 1$) can cancel only $n$ quadratic terms in $F^{[2]}_n$, such that each $\beta_i$ cancel $x^2_{n-i+1}$($\forall\ n \geq i \geq 1$). We chose to cancel these terms because they are independent from the entries: $a_i$ of the matrix $A_{obs}$.

We conclude the following result about the $\mathbf{1}^{st}$ **homological equation**:

- ► The coordinates change $\Phi^{[2]}$ is completely determined in the $(n-1)$ first rows, which cancel all quadratic terms in $F^{[2]}_1 \ldots\ F^{[2]}_{n-1}$; hence there is no resonant terms in $F^{[2]}_i$ ($\forall\ n-1 \geq i \geq 1$).

  ► In the last row, the free vector field $\beta^{[2]}$ allows to cancel only the quadratic terms in $x^2_i$ ($\forall\ n \geq i \geq 1$); consequently the terms $x_i x_j$ ( $\forall\ n \geq j > i \geq 1$) are resonant in the component $F^{[2]}_n$.

- $\mathbf{2}^{nd}$ **homological equation**: (with $\bar{g}^{[1]}$ desired equal to zero)

  $$g^{[1]}(x) = \widehat{\Phi}^{[2]}(A_{obs}x, B_{obs}) + \alpha^{[1]}(x_1)$$

  We deduce from this equation, that only the quadratic terms in $x_1 u$ may be cancelled by the free vector field $\alpha^{[1]}u$. Thus, all terms in $x_i u$ ($\forall\ n \geq i \geq 2$) are resonant in the dynamics.

- $\mathbf{3}^{rd}$ **homological equation**: ($\bar{\gamma}^{[0]}$ wished equal to zero)

  $$\gamma^{[0]} \;= \Phi^{[2]}(B_{obs}) + \tau^{[0]}$$

  This equation is trivial, since the free vector field $\tau^{[0]}u^2$ may cancel all the quadratic terms in $u^2$. So, we conclude that there is no resonant terms in $u^2$.

Finally, thanks to the coordinates change $\Phi^{[2]}$ we construct the equivalent system of (8) modulo the output injection (6), restricted to the key dynamics in order to study and analyze this last; these dynamics are no others than the resonant terms established in *Lemma* 3.1. The equivalent system in question is under the quadratic normal form described in *Theorem* 3.1. ■

## Appendix B: Proof of theorem 4.1

**Proof.**

- The quadratic normal form associated to the linearly observable part of the system (10) is deduced from the theorem 3.1's proof, by considering the first system of homological equations (12) in Proposition (4.1).

- For the linearly unobservable part; let us consider the two last homological equations of (13) in Proposition (4.1), it is easy to deduce that: $x_i u$ ($\forall\ n \geq i \geq 2$) are the resonant terms issued from $g_n^{[1]}u$, and finally there is no resonance in $u^2$. So, let us analyze the $\mathbf{1}^{st}$ **homological equation** in (13) of the Proposition (4.1):

  Given $\eta$, $\mathbf{a}_i$ and $\lambda_i$ ($\forall\ n-1 \geq i \geq 1$);

  **Case 1** *If* $\exists\ \phi_n$, *verifying (14) so,* $F_n^{[2]}(x) - \bar{F}_n^{[2]}(x) = -\sum_{i=1}^{n-1} \lambda_i \Phi_i^{[2]}(x) + \beta_n {x_1}^2$. *We remark that* $\beta_n^{[2]}$ *allows us to cancel only the quadratic terms in* ${x_1}^2$. *Consequently, we have* $x_i x_j$ *and* $x_1 x_j$ *(*$\forall\ n \geq j \geq i \geq 2$*) as resonant terms due to* $F_n^{[2]}$.

  **Case 2** *If* $\nexists\ \phi_n \neq 0$, *such that (14) is verified, we obtain* $\frac{n(n+1)}{2}$ *degrees of freedom in both* $\Phi_n^{[2]}$ *and* $F_n^{[2]}$, *which allows us cancelling all the quadratic terms in* $F_n^{[2]}$. *Consequently, in this case there is no resonant terms in* $x_i x_j$ *(* $\forall\ n \geq j > i \geq 1$*).*

■