A Hajós-like theorem for weighted coloring

Julio Araujo, Claudia Linhares Sales

Noname manuscript No.
(will be inserted by the editor)

A Hajós-like Theorem for Weighted Coloring

J. Araujo · C. Linhares Sales

December 15, 2012

Abstract The Hajós' Theorem [8] shows a necessary and sufficient condition for the chromatic number of a given graph G be at least k: G must contain a k-constructible subgraph. A graph is k-constructible if it can be obtained from a complete graph of order k by successively applying a set of well-defined operations.

Given a vertex-weighted graph G and a (proper) r-coloring $c = \{C_1, \ldots, C_r\}$ of G, the weight of a color class C_i is the maximum weight of a vertex colored i and the weight of c is the sum of the weights of its color classes. The objective of the Weighted Coloring Problem [7] is, given a vertex-weighted graph G, to determine the minimum weight of a proper coloring of G, that is its weighted chromatic number.

In this article, we prove that the Weighted Coloring Problem admits a version of the Hajós' Theorem and so we show a necessary and sufficient condition for the weighted chromatic number of a vertex-weighted graph G be at least k, for any positive real k.

Partially supported by CNPq/Brazil (Conselho Nacional de Desenvolvimento Científico e Tecnológico), FUNCAP/Brazil (Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico) and INRIA/France (Institut Nationale de Recherche en Informatique et Automatique), project numbers INR-0046-00014.01.00/11, PRN-0040-00040.01.00/10 and CNPq 202049/2012-4.

J. Araujo
Projet Mascotte I3S (CNRS & UNS) and INRIA
INRIA Sophia-Antipolis
2004 route des Lucioles
BP 93 - F06902 Sophia-Antipolis Cedex France
E-mail: julio.araujo@inria.fr

C. Linhares Sales
Universidade Federal do Ceará
UFC - Campus do Pici, Bloco 910
60455-760 - Fortaleza, CE - Brazil
E-mail: linhares@lia.ufc.br

1 Introduction

For the terms not defined here the reader can be referred to [2].

Given a graph $G = (V, E)$, a *k-coloring* c of G is an assignment of colors to the vertices of G, $c : V(G) \to \{1, \ldots, k\}$, such that if $(u, v) \in E$, then $c(u) \neq c(v)$. The chromatic number of a graph G, $\chi(G)$, is the smallest integer k such that G admits a k-coloring.

The Vertex Coloring Problem consists in, for a given a graph G, determining $\chi(G)$. It is a well known fact that it is a NP-hard problem [11].

The characterization of the *k-chromatic graphs*, i.e., graphs G such that $\chi(G) = k$, has been a challenging problem for many years. In 1961, Hajós [8] gave a characterization of graphs with chromatic number at least k by proving that they must contain a k-constructible subgraph. In order to present this characterization, we need to recall some definitions. The *identification* of two vertices a and b of a graph G means the removal of a and b followed by the inclusion of a new vertex $a \circ b$ adjacent to each vertex in $N_G(a) \cup N_G(b)$, where $N_G(x)$ is the set of neighbors of a vertex x in a graph G. A graph $G = (V, E)$ is *complete* if it is simple and for every pair of vertices u, v of G, $uv \in E(G)$. We denote by K_n the complete graph with n vertices.

Definition 1 The set of *k-constructible graphs* is defined recursively as follows:

1. The complete graph on k vertices is k-constructible.
2. **Hajós' Sum:** If G_1 and G_2 are disjoint graphs which are k-constructible, (a_1, b_1) is an edge of G_1 and (a_2, b_2) is an edge of G_2, then the graph G obtained from $G_1 \cup G_2$ by removing (a_1, b_1) and (a_2, b_2), identifying a_1 with a_2, and adding an edge

between (b_1, b_2), is a k-constructible graph (see Figure 2).

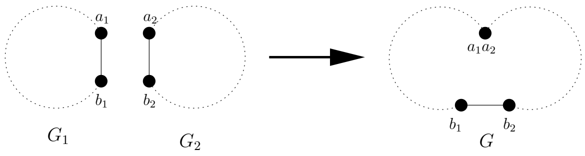

Fig. 1 Hajós' Sum.

3. **Identification:** If G is k-constructible and a and b are two non-adjacent vertices of G, then the graph obtained by the identification of a and b is a k-constructible graph.

Theorem 11 (Hajós [8]) *$\chi(G) \geq k$ if and only if G is a supergraph of a k-constructible graph.*

The Hajós' Theorem determines a set of operations that we can use to obtain, from a complete graph with k vertices, all k-chromatic graphs, including the k-critical ones. A graph H is *k-critical* if $\chi(H) \geq k$ and for every proper subgraph I of H, $\chi(I) < k$. Clearly, for a given graph G, the difficulty of determining whether $\chi(G) \geq k$ is equivalent to the difficulty of determining if G contains a k-critical subgraph H. Thanks to the Hajós' Theorem, we know that all the k-critical graphs can be built from complete graphs on k vertices by successive applications of Hajós Sum and Identification of vertices. Because of this, it has been subject of interest to obtain Hajós-like Theorem for several variations of the classical coloring problem. Gravier [6] proved an extension of Hajós' Theorem for List Coloring. Král [12] gave a simplified proof of Gravier's result. Zhu [17] found an extension of this theorem for chromatic circular number. Mohar [14] demonstrated two new versions of the referred theorem for chromatic circular number and an extension of Hajós' Theorem for the channel assignment problem, i.e., a coloring of edge-weighted graphs.

In this paper, we are interested in a variation of the classical Vertex Coloring Problem, which is called the Weighted Coloring Problem. In order to present it, we need some definitions.

Given a simple graph $G = (V, E)$, an assignment w of non-negative real weights to its vertices and a proper k-coloring $c = \{C_1, \ldots, C_k\}$ of G, the *weight of a color class C_i* in the coloring c is the maximum weight of a vertex in this color class. Moreover, the *weight of a k-coloring c*, denoted by $w(c)$, is the sum of the weights of its color classes.

The Weighted Coloring Problem [7] consists in, for a given graph G with non-negatives real weights associated to its vertices, determining the minimum weight of a proper coloring of G, that is, the *weighted chromatic number* of G, denoted by $\chi_w(G)$:

$$\chi_w(G) = \min_{c \ coloring \ of \ G} w(c)$$

When all the vertices of the input graph G have equal weights, the Weighted Coloring Problem is equivalent to the Vertex Coloring Problem. Then, the former is particular case of the second one. Consequently, Weighted Coloring is also NP-Hard.

The complexity of this problem was already studied in some previous works. It was shown that the weighted chromatic number of a graph G can be calculated in polynomial-time whenever G is a bipartite graph with only two different weights associated to its vertices [4], or G is a P_5-free bipartite graph [3], or G belongs to a subclass of P_4-sparse graphs that properly contains the cographs [1].

On the other hand, it was proved that the decision version of Weighted Coloring is NP-complete for bipartite graphs [4], split graphs [4], planar graphs [3], interval graphs [5] and others graph classes.

In the next section, we show an extension of Hajós' Theorem for Weighted Coloring.

2 Hajós' Theorem for Weighted Coloring

In this section, we deal with simple (vertex-)weighted graphs. We denote by $G = (V, E, w)$ a vertex-weighted graph $G = (V, E)$ together with its function weight $w : V \rightarrow \mathbb{R}^+$.

Notation 21 *The infinity family of complete weighted graphs $G = (V, E, w)$ with order $n = |V(G)|$ and such that $\sum_{v \in V} w(v) = k$ is denoted by $\mathcal{K}_n^k$.*

Notation 22 *Given a weighted graph $G = (V, E, w)$ and a proper coloring c of G, we choose as the* ***representative*** *of the color i in c, $rep_c(i)$, a unique vertex $v \in V$ satisfying the inequality $w(v) \geq w(x)$, for all $x \in V$ such that $c(x) = c(v) = i$.*

Definition 2 Given weighted graphs $G = (V, E, w)$ and $H = (V', E', w')$, we say that $H \subseteq G$ (H is a *weighted subgraph* of G) if $V' \subseteq V$, $E' \subseteq E$, and, for all $v \in V'$, we have $w'(v) \leq w(v)$.

Now, we redefine the Hajós' construction to the weighted case:

Definition 3 The *weighted* k-constructible graphs are defined recursively as follows:

1. The graphs in $\bigcup_{i \in \mathbb{N}} \mathcal{K}_i^k$ are weighted k-constructible.

2. **Weighted Hajós' Sum:** If G_1 and G_2 are disjoint weighted k-constructible graphs, (a_1, b_1) is an edge of G_1 and (a_2, b_2) is an edge of G_2, then the graph G obtained from $G_1 \cup G_2$ by removing (a_1, b_1) and (a_2, b_2), identifying a_1 with a_2 into a vertex $a_1 \circ a_2$, doing $w(a_1 \circ a_2) = \max\{w(a_1), w(a_2)\}$, and adding the edge (b_1, b_2) is a weighted k-constructible graph.
3. **Weighted Identification:** If a and b are two non-adjacent vertices of a weighted k-constructible G, then the graph that we obtain by the identification of two vertices a and b into a vertex $a \circ b$, doing $w(a \circ b) = \max\{w(a), w(b)\}$, is a weighted k-constructible graph.

The main result of this paper is:

Theorem 21 *Let $G = (V, E, w)$ be a weighted graph and k be a positive real. Therefore, $\chi_w(G) \geq k$ if and only if G has a weighted k-constructible weighted subgraph H.*

Proof In order to simplify the notation, whenever we refer to *subgraph* in this proof, we mean *weighted subgraph*, as in Definition 2.

We prove first that if $\chi_w(G) \geq k$, then G has a weighted k-constructible subgraph H. Suppose, by contradiction, that this statement is false and consider as counter-example a graph $G = (V, E, w)$ which is maximal with respect to its number of edges. It means that $\chi_w(G) \geq k$ and G does not contain any weighted k-constructible subgraph and for any pair of non-adjacent vertices of G, let us say u, v, $G' = G + uv$ contains a weighted k-constructible subgraph.

We claim that G is not isomorphic to a complete multipartite graph. Suppose the contrary and let p be the number of stable sets in the partition $\mathcal{P}$ of $V(G)$. For each color class C_i of an optimal weighted coloring c of G, there is no pair of vertices of C_i in distinct sets of $\mathcal{P}$, since there is an edge between any two vertices of distinct parts.

Moreover, we cannot have more than one color class in one stable set. As a matter of fact, suppose by contradiction that there are two color classes, say C_i and C_j, whose vertices belong to the same stable set in the partition $\mathcal{P}$ of $V(G)$. Without loss of generality, suppose that $w(rep_c(i)) \geq w(rep_c(j))$. Then a coloring c', obtained from c by the union of the color classes C_i and C_j, has cost exactly $\chi_w(G) - w(rep_c(j))$, and this contradicts the optimality of c.

Consequently, the vertices with the greatest weight in every part are exactly the representatives of each color class. Observe that the subgraph induced by the representatives is an element of the set $\mathcal{K}_p^k$, because $\chi_w(G) \geq k$ and $p \in \mathbb{N}$. This contradicts the hypothesis that G has no weighted k-constructible subgraph.

Therefore, the counter-example G is not a complete multipartite graph. Thus, there are at least three vertices in G, let us say a, b and c, such that $ab, bc \notin E(G)$ and $ac \in E(G)$. Consider now the graphs $G_1 = G + ab$ and $G_2 = G + bc$. Because of the maximality of G, G_1 and G_2 have, respectively, weighted k-constructible subgraphs H_1 and H_2. Obviously, the edges ab and bc belong, respectively, to H_1 and H_2. Consider then the application of Hajós Sum in two disjoint graphs isomorphic to H_1 and H_2, respectively. Let us choose edge ab of H_1 and bc of H_2 to remove and let us identify the vertices labeled b. Finally, identify all the vertices in H_1 with their correspondent vertices in H_2, if they exist. Observe that a graph isomorphic to a subgraph of G is obtained at the end of this sequence of operations. Then, G has a weighted k-constructible subgraph, a contradiction.

Conversely, we prove now that, if G has a weighted k-constructible subgraph H, then $\chi_w(G) \geq k$. First, observe that $\chi_w(G) \geq \chi_w(H)$. Then, we just have to show that $\chi_w(H) \geq k$. The proof is by induction on the number of Hajós' operations applied to obtain H.

If H is isomorphic to a graph $K_i^k \in \mathcal{K}_i^k$, for some $i \in \mathbb{N}$, then its weighted chromatic number is trivially k, since it is a complete graph whose sum of weights is equal to k.

Suppose then that H was obtained by the Weighted Identification of two non-adjacent vertices a and b of a weighted k-constructible graph H' into a vertex $a \circ b$. By induction hypothesis, H' has a weighted chromatic number at least k. Suppose, by contradiction, that $\chi_w(H) < k$ and let c be an optimal weighted coloring of H. Then, a coloring c' of H' can be obtained from c, by assigning to a and b the color assigned to $a \circ b$ in c, and letting all the other vertices of H' be assigned to the same color they have been assigned in c. Observe that, except the color i of $a \circ b$, for all the other colors j, $rep_{c'}(j) = rep_c(j)$. For the color i, the vertex $rep_c(i)$ has weight greater than or equal to the weight of $a \circ b$, that is greater than or equal to the weight of a and b. Therefore, the coloring c' has weight equal to the coloring c, that is less than k. This contradicts the hypothesis of $\chi_w(H') \geq k$.

Finally, suppose that H was obtained from weighted k-constructible graphs H_1 and H_2 using the Weighted Hajós' Sum on the edges (a_1, b_1) and (a_2, b_2) from H_1 and H_2, respectively. Let $a_1 \circ a_2$ be the vertex of H obtained by the identification of a_1 e a_2. Suppose, by contradiction, that $\chi(H) < k$, while $\chi(H_1) \geq k$ and $\chi(H_2) \geq k$. Consider an optimal weighted coloring c of H. Observe that either $c(a_1 \circ a_2) \neq c(b_1)$ or $c(a_1 \circ a_2) \neq c(b_2)$ (because b_1 and b_2 are adjacent). Without loss of generality, suppose that $c(a_1 \circ a_2) \neq c(b_1)$. Then, consider now the restriction c' of c to H_1, assigning to a_1 the color of $a_1 \circ a_2$. We have that, for all color class C_j of c', the weight of $rep_{c'}(j) \leq rep_c(j)$ (including the color class of a_1, because the weight of a_1 is less than or equal to the weight of $a_1 \circ a_2$). Consequently, $w(c') \leq w(c) < k$, contradicting the hypothesis that $\chi_w(H_1) \geq k$.

3 Conclusions

Ore [15] has proved that the Hajós construction can be simplified. He has shown that by using only one operation that collapses the two Hajós operations, one may construct any k-colorable graph from complete graphs of order k.

It is not difficult to see that the same simplification can be done for the weighted case. It is just necessary use the same adaptation we did whenever two vertices u and v are identified, i.e., the weight of the new vertex must be the maximum value between the weight of u and the weight of v.

Moreover, by extending the ideas of Urquhart [16] for the non-weighted case, it is also not hard to establish the equivalence between the class of the weighted k-constructible graphs and the class obtained by using the adapted Ore's operation described above.

Finally, there is a well known problem that is to study the complexity of the construction of k-chromatic graphs of a given size by Hajós operations [9, 10, 13]. Since the Vertex Coloring Problem is a particular case of Weighted Coloring, it is obvious that the complexity of this problem in the weighted case is as hard as in the non-weighted one and can also be let as open question.

References

1. Araujo, J., Linhares Sales, C., and Sau, I. Weighted coloring on p_4-sparse graphs. In *11es Journées Doctorales en Informatique et Réseaux* (Sophia Antipolis, France, Mar. 2010).
2. Bondy, J. A., and Murty, U. S. R. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2008.
3. de Werra, D., Demange, M., Escoffier, B., Monnot, J., and Paschos, V. T. Weighted coloring on planar, bipartite and split graphs: Complexity and improved approximation. *Lecture Notes in Computer Science 3341* (2005), 896–907.
4. Demange, M., de Werra, D., Monnot, J., and Paschos, V. T. Weighted node coloring: When stable sets are expensive. *Lecture Notes in Computer Science 2573* (2002), 114 – 125.
5. Escoffier, B., Monnot, J., and Paschos, V. T. Weighted coloring: futher complexity and approximability results. *Information Processing Letters 97* (2006), 98–103.
6. Gravier, S. A hajós-like theorem for list coloring. *Discrete Math. 152* (1996), 299–302.
7. Guan, D., and Zhu, X. A coloring problem for weighted graphs. *Inform. Proc. Letters 61* (1997), 77–81.
8. Hajós, G. Über eine konstruktion nicht n-färbbarer graphen. *Wiss. Z. Martin Luther Univ. Math.-Natur.Reihe 10* (1961), 116–117.
9. Hanson, D., Robinson, G. C., and Toft, B. Remarks on the graph colour theorem of hajós. *Cong. Numer. 55* (1986), 69 – 76.
10. Jensen, T. R., and Toft, B. *Graph Coloring Problems.* Wiley-Interscience, New York, 1995.
11. Karp, R. M. Reducibility among combinatorial problems. *Complexity of Computer Computations Plenum* (1972), 85–103.
12. Kral, D. Hajós' theorem for list coloring. *Discrete Math. 287* (2004), 161–163.
13. Mansfield, A., and Welsh, D. Some colouring problems and their complexity. In *Graph Theory, Proceedings of the Conference on Graph Theory*, B. Bollobás, Ed., vol. 62 of *North-Holland Mathematics Studies*. North-Holland, 1982, pp. 159 – 170.
14. Mohar, B. Hajós' theorem for colorings of edge-weighted graphs. *Combinatorica 25* (2005), 65–76.
15. Ore, O. *The four color problem.* Academic Press, New York, 1967.
16. Urquhart, A. The graph constructions of hajós and ore. *J. Graph Theory 26* (December 1997), 211–215.
17. Zhu, X. An analogue of hajós' theorem for the circular chromatic number (ii). *Graphs and Combinatorics 19* (2003), 419–432.