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**Abstract**

Let $f_1, \ldots, f_p$ be in $\mathbb{Q}[\mathbf{X}]$, where $\mathbf{X} = (X_1, \ldots, X_n)^t$, that generate a radical ideal and let $V$ be their complex zero-set. Assume that $V$ is smooth and equidimensional. Given $f \in \mathbb{Q}[\mathbf{X}]$ bounded below, consider the optimization problem of computing $f^\star = \inf_{x \in V \cap \mathbb{R}^n} f(x)$. For $\mathbf{A} \in GL_n(\mathbb{C})$, we denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{A}\mathbf{X})$ and by $V^{\mathbf{A}}$ the complex zero-set of $f_1^{\mathbf{A}}, \ldots, f_p^{\mathbf{A}}$.

We construct families of polynomials $\mathsf{M}_0^{\mathbf{A}}, \ldots, \mathsf{M}_d^{\mathbf{A}}$ in $\mathbb{Q}[\mathbf{X}]$: each $\mathsf{M}_i^{\mathbf{A}}$ is related to the section of a linear subspace with the critical locus of a linear projection. We prove that there exists a non-empty Zariski-open set $\mathscr{O} \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in \mathscr{O} \cap GL_n(\mathbb{Q})$, $f(x)$ is positive for all $x \in V \cap \mathbb{R}^n$ if, and only if, $f^{\mathbf{A}}$ can be expressed as a sum of squares of polynomials on the truncated variety generated by the ideal $\langle \mathsf{M}_i^{\mathbf{A}} \rangle$, for $0 \leq i \leq d$.

Hence, we can obtain algebraic certificates for lower bounds on $f^\star$ using semidefinite programs. Some numerical experiments are given. We also discuss how to decrease the number of polynomials in $\mathsf{M}_i^{\mathbf{A}}$.

*Key words:* Global constrained optimization, polynomials, sum of squares, polar varieties

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## 1. Introduction

*Motivation and Problem statement.* Consider the *global constrained optimization problem*

$$f^\star := \inf_{x \in V \cap \mathbb{R}^n} f(x)$$

where $f \in \mathbb{Q}[X_1, \ldots, X_n]$ is bounded below and $V \subset \mathbb{C}^n$ is an algebraic variety given by a set of defining equations $f_1 = \cdots = f_p = 0$ in $\mathbb{Q}[X_1, \ldots, X_n]$.

Given $a \in \mathbb{R}$, providing algebraic certificates of positivity for $f - a$ over $V \cap \mathbb{R}^n$ allowing certification of lower bounds on $f^\star$ (i.e. $a \leq f^\star$) is a question of first importance since it arises in several applications of engineering sciences (e.g. control theory Henrion and Garulli (2005); Henrion et al. (2003) or static analysis of programs Cousot (2005); Monniaux (2010)).

This problem can be solved in theory through the Positivstellensatz (Bochnak et al., 1998, Chapter 4). The issue is that computing such an algebraic certificate of positivity is empirically known to be computationally expensive. Our approach fits in the framework of sums of squares decompositions of multivariate polynomials through a relaxation to semi-definite programming (see Shor (1987); Parrilo (2000); Lasserre (2001); Parrilo and Sturmfels (2003) for the semi-definite relaxations methods). The goal is to obtain algebraic certificates of positivity by means of sums-of-squares decompositions which could be easier to compute.

In this context, the issue is to provide results ensuring the existence of algebraic certificates of positivity by means of sums of squares decompositions. For instance, it is well-known that not all positive polynomials are sums-of-squares of polynomials. Nevertheless, in the univariate case, positive polynomials are sums-of-squares (see Hilbert (1888)). This gives the intuition that over regions of "small dimension" positive polynomials can be written as sums-of-squares of polynomials.

Thus, the idea is to consider additional constraints to define subsets of $V \cap \mathbb{R}^n$ of smaller dimension so that one can ensure two properties:

- if $f - a$ is positive over these subsets then $a \leq f^\star$;
- There exist sum-of-squares certificates for the positivity of $f - a$ on these subsets.

Under these conditions, one can certify that $a$ is a lower bound for $f^\star$.

*Prior works.* This approach has been previously developed in the case where $f^\star$ is reached. We denote by $\langle \nabla f \rangle$ the ideal $\left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \right\rangle$. Nie et al. (2006) prove that either $f$ is positive over $V\left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\right)$, or $f$ is non-negative over $V\left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\right)$ and $\langle \nabla f \rangle$ is radical, then $f$ is a sum of squares of polynomials modulo $\langle \nabla f \rangle$. Note that if the infimum is reached, it is reached over $V\left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\right) \cap \mathbb{R}^n$. Then over the gradient variety, $f - f^\star$ can be written as a sum of squares and outside the gradient variety, it is necessarily greater than 0. Here the local certificate is actually a global certificate of non-negativity. These results have been recently generalized for the constrained case that we are considering in this paper in Nie (2010) but still with the assumption that the global infimum $f^\star$ is reached.

When one does not know *a priori* if $f$ attains a minimum, one has to take into account asymptotic phenomena. To do that, Schweighofer (2006) replaces the gradient variety with its gradient tentacle. Over the gradient tentacle, a positive polynomial for which its values "at infinity" is a finite subset of $\mathbb{R}_{>0}$, (see point (3) in our Proposition 1.3 for a formal definition) belongs to the preordering generated by the polynomials defining the gradient tentacle.

Hà and Phạm (2009) follow the approach initiated by Schweighofer with their truncated tangency variety, which are subsets of the region defined by the constraints of smaller dimension and on which the target function $f$ has a finite number of values "at infinity". These truncated tangency varieties are related to critical loci of the square of distance functions to a given point, say $(a_1, \ldots, a_n)$. They are defined by considering $(n-d+2, n-d+2)$ minors of the Jacobian matrix associated to $f_1, \ldots, f_p, f, \sum_{i=1}^{n}(X_i - a_i)^2$.

Considering simpler critical loci of linear projections leads to consider only $(n - d + 1, n - d + 1)$-minors of the Jacobian matrix associated to $f_1, \ldots, f_p, f$. This may lead to simpler algebraic certificates and a better numerical behavior of programs computing numerical approximations of sums-of-squares decompositions via semi-definite programming.

In Guo et al. (2010), we successfully reached this goal *in the unconstrained case*. In this paper, we go further and investigate the constrained case which is conceptually harder.

The subsets of $V$ that we consider are related to critical loci of linear projections. This is related to the notion of polar variety already investigated for the real root finding problem in the solution of polynomial systems using Computer Algebra techniques (see e.g. Bank et al. (1997); Safey El Din and Schost (2003); Bank et al. (2005, 2010)). We provide several numerical experiments showing the relevance of our approach. Before describing in detail our contributions we need to introduce some definitions.

*Basic definitions, assumptions and notations.* We need a few definitions and refer to Zariski and Samuel (1958); Mumford (1976); Shafarevich (1977); Eisenbud (1995) for standard notions which are not recalled here. An algebraic variety $V \subset \mathbb{C}^n$ is the set of common zeros of some polynomial equations $f_1, \ldots, f_p$ in variables $X_1, \ldots, X_n$; we write $V = V(f_1, \ldots, f_p)$ and $d$ its dimension. Moreover, we assume in the sequel that the ideal $\langle f_1, \ldots, f_p \rangle$ is radical.

The Zariski-tangent space to $V$ at $x \in V$ is the vector space $T_xV$ defined by the equations $\frac{\partial f}{\partial X_1}(x)v_1 + \cdots + \frac{\partial f}{\partial X_n}(x)v_n = 0$, for all polynomials $f$ that vanish on $V$.

We will only consider equidimensional algebraic varieties. In this context, the *regular points* on $V$ are those points $x$ where $\dim(T_xV) = \dim(V)$; the *singular points* are all other points. The set of singular points is defined as the set of points on $V$ where all $(n-d, n-d)$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{1 \leq i \leq p, 1 \leq j \leq n}$ vanish. An equidimensional variety $V$ such that its set of singular points is empty will be said to be smooth.

For $\mathbf{A} \in GL_n(\mathbb{Q})$ and $g \in \mathbb{Q}[X_1, \ldots, X_n]$, we denote by $g^{\mathbf{A}}$ the polynomial $g(\mathbf{AX})$ where $\mathbf{X} = (X_1, \ldots, X_n)^t$. In the sequel, the algebraic variety $V(f_1^{\mathbf{A}}, \ldots, f_p^{\mathbf{A}})$ is denoted by $V^{\mathbf{A}}$. Note that $f^\star = \inf_{x \in V^{\mathbf{A}} \cap \mathbb{R}^n} f^{\mathbf{A}}(x)$.

Given a polynomial family $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ and a non-negative integer $k \leq n$, $\mathsf{jac}(\mathbf{F}, [X_k, \ldots, X_n])$ denotes the truncated Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{1 \leq i \leq p, k \leq j \leq n}$.

Given a matrix $\mathbf{M}$ and an integer $r$, we denote by $\mathsf{Minors}(\mathbf{M}, r)$ the set of $(r, r)$-minors of $\mathbf{M}$.

In the sequel, we suppose that the set of polynomials $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfies the following regularity assumptions $\mathbf{R}$:

$\mathbf{R}_1$: the ideal $\langle f_1, \ldots, f_p \rangle$ is radical and equidimensional; we denote its dimension by $d$;

$\mathbf{R}_2$: the algebraic variety $V = V(f_1, \ldots, f_p) \subset \mathbb{C}^n$ is smooth.
Now, consider an additional polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$.

**Notations 1.1.** For $i = d$, let $\mathsf{M}_d^{\mathbf{A}} = \left\{f_1^{\mathbf{A}}, \ldots, f_p^{\mathbf{A}}, X_1, \ldots, X_{d-1}\right\}$. Then for $0 \leq i \leq d-1$, we denote by $\mathsf{M}_i^{\mathbf{A}}$ the set of polynomials which is the union of

- the polynomials $f_1^{\mathbf{A}}, \ldots, f_p^{\mathbf{A}}$;
- the set $\mathsf{Minors}(\mathsf{jac}([\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}], [X_{i+1}, \ldots, X_n]), n - d + 1)$;
- the sequence of variables $X_1, \ldots, X_{i-1}$.

In the sequel, $W^{\mathbf{A}}$ denotes the algebraic set $\bigcup_{i=0}^{d} V(\mathsf{M}_i^{\mathbf{A}})$.

*Statement of the main results.* Given two real numbers $B \in \mathbb{R}$ and $a \in \mathbb{R}$, we will say that property $\mathsf{SOS}(f^{\mathbf{A}} - a, \mathsf{M}_i^{\mathbf{A}}, B)$ holds if and only if there exist sums of squares of polynomials $S_i^{\mathbf{A}}$ and $T_i^{\mathbf{A}}$ in $\mathbb{R}[X_1, \ldots, X_n]$ satisfying

$$f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}(B - f^{\mathbf{A}}) \mod \langle \mathsf{M}_i^{\mathbf{A}} \rangle.$$

We will say that property $\mathsf{SOS}(f^{\mathbf{A}} - a, \mathsf{M}^{\mathbf{A}}, B)$ holds if for all $0 \leq i \leq d$, properties $\mathsf{SOS}(f^{\mathbf{A}} - a, \mathsf{M}_i^{\mathbf{A}}, B)$ hold.

We are now ready to state the main results of this paper using Notations 1.1.

**Theorem 1.2.** Let $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfying assumption $\mathbf{R}$, $V = V(\mathbf{F})$, $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and $f^\star = \inf_{x \in V \cap \mathbb{R}^n} f(x)$. Let $B \in f(V \cap \mathbb{R}^n)$. There exists a non-empty Zariski open set $\mathscr{O} \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}$:

(a) If property $\mathsf{SOS}(f^{\mathbf{A}} - a, \mathsf{M}^{\mathbf{A}}, B)$ holds then $a \leq f^\star$.
(b) If $a < f^\star$ then property $\mathsf{SOS}(f^{\mathbf{A}} - a, \mathsf{M}^{\mathbf{A}}, B)$ holds.

Define $f_i^{\mathbf{sos}}$ as the real number

$$\sup\left\{a \in \mathbb{R} \mid f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}\left(B - f^{\mathbf{A}}\right) \mod \left\langle \mathsf{M}_i^{\mathbf{A}} \right\rangle\right\},$$

where $S_i^{\mathbf{A}}$ and $T_i^{\mathbf{A}}$ are sums of squares of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$.

Then Theorem 1.2 implies that $f^\star = \min_{0 \leq i \leq d} f_i^{\mathbf{sos}}$. Hence, the initial constrained optimization problem is reduced to the problem of computing the numbers $f_i^{\mathbf{sos}}$. Computational aspects of Theorem 1.2 are discussed hereafter. Its proof is a straightforward consequence of (Schweighofer, 2006, Theorem 9) and the result below.

**Proposition 1.3.** Let $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfying assumption $\mathbf{R}$, $V = V(\mathbf{F})$ and $f \in \mathbb{Q}[X_1, \ldots, X_n]$. There exists a non-empty Zariski open set $\mathscr{O} \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}$, the following holds:

(1) there exists a non-empty Zariski-open set $\mathscr{T}_{\mathbf{A}}$ such that for all $t \in \mathbb{R} \cap \mathscr{T}_{\mathbf{A}}$, $V(f^{\mathbf{A}} - t) \cap V(M_i^{\mathbf{A}})$ has dimension at most 0 for $1 \leq i \leq d$ and $V^{\mathbf{A}} \cap V(f^{\mathbf{A}} - t) \cap \mathbb{R}^n$ is empty if and only if $V(f^{\mathbf{A}} - t) \cap V(M_i^{\mathbf{A}}) \cap \mathbb{R}^n$ is empty for $1 \leq i \leq d$;
(2) denoting by $W^{\mathbf{A}}$ the algebraic set $\cup_{i=0}^{d} V(\mathsf{M}_i^{\mathbf{A}})$, $f^\star$ equals $\inf_{x \in W^{\mathbf{A}} \cap \mathbb{R}^n} f(x)$;
(3) the set of values $t \in \mathbb{C}$ such that there exists $(x_k)_{k \in \mathbb{N}} \subset V(M_i^{\mathbf{A}})$ satisfying $\lim_k ||x_k|| = \infty$ and $\lim_k f^{\mathbf{A}}(x_k) = t$ is finite.

It is implied by (Schweighofer, 2006, Theorem 9) and Proposition (1.3) that

$$f_i^{\mathbf{sos}} = \inf\{f^{\mathbf{A}}(x) \mid x \in V(M_i^{\mathbf{A}}) \cap \mathbb{R}^n\},\ 1 \leq i \leq d.$$

*Proof of Theorem 1.2.* Let $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$ such that assertions 1, 2 and 3 of Proposition 1.3 apply. Consider the semi-algebraic sets

$$E_B^{\mathbf{A}} = V^{\mathbf{A}} \cap \{x \in \mathbb{R}^n \mid f^{\mathbf{A}}(x) \leq B\} \text{ and } E_{B,i}^{\mathbf{A}} = E_B^{\mathbf{A}} \cap V(\mathsf{M}_i^{\mathbf{A}}) \quad (0 \leq i \leq d).$$

Note that by definition of $E_B^{\mathbf{A}}$ and since $B \in f(V \cap \mathbb{R}^n)$, $f^\star = \inf_{x \in E_B^{\mathbf{A}}} f^{\mathbf{A}}(x)$. Moreover, the definition of $E_{B,i}^{\mathbf{A}}$ and Proposition 1.3 (assertion 1) imply that $\cup_{i=0}^d E_{B,i}^{\mathbf{A}} \neq \emptyset$ and $\inf_{x \in W^{\mathbf{A}}} f(x) = \inf_{x \in \cup_{i=0}^d E_{B,i}^{\mathbf{A}}} f^{\mathbf{A}}(x)$. Consequently, by Proposition 1.3 (assertion 2), $f^\star = \inf_{x \in \cup_{i=0}^d E_{B,i}^{\mathbf{A}}} f^{\mathbf{A}}(x)$.

If there exist sums of squares of polynomials $S_i^{\mathbf{A}}$ and $T_i^{\mathbf{A}}$ in $\mathbb{R}[X_1, \ldots, X_n]$ such that

$$f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}(B - f^{\mathbf{A}}) \mod \langle \mathsf{M}_i^{\mathbf{A}} \rangle \text{ for } 0 \leq i \leq d$$

then $f^{\mathbf{A}}(x) - a \geq 0$ for all $x \in E_{B,i}^{\mathbf{A}}$. Since $f^\star = \inf_{x \in \cup_{i=0}^d E_{B,i}^{\mathbf{A}}} f^{\mathbf{A}}(x)$, this implies that $a \leq f^\star$ and proves assertion (a).

Suppose now that $a < f^\star$. We prove in the sequel that this implies that there exist sums of squares of polynomials $S_i^{\mathbf{A}}$ and $T_i^{\mathbf{A}}$ in $\mathbb{R}[X_1, \ldots, X_n]$ such that

$$f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}(B - f^{\mathbf{A}}) \mod \langle \mathsf{M}_i^{\mathbf{A}} \rangle \text{ for } 0 \leq i \leq d.$$

By definition of $E_B^{\mathbf{A}}$,

*(i)* $f^{\mathbf{A}}$ is bounded on $E_B^{\mathbf{A}}$ and $E_{B,i}^{\mathbf{A}}$ for $0 \leq i \leq d$.

Since by assumption $a < f^\star$ the following property holds

*(ii)* $f^{\mathbf{A}}(x) - a > 0$ for all $x \in E_{B,i}^{\mathbf{A}}$ for $0 \leq i \leq d$.

Moreover, Proposition 1.3 (assertion 3) implies that

*(iii)* $\{t \in \mathbb{R} \mid \exists (x_k)_{k \in \mathbb{N}} \subset E_{B,i}^{\mathbf{A}} s.t. \lim_k ||x_k|| = \infty \text{ and } \lim_k f^{\mathbf{A}}(x_k) = t\}$ is finite.

Now, let $(h_{i,1}, \ldots, h_{i,m}) = \mathsf{M}_i^{\mathbf{A}}$. By (Schweighofer, 2006, Theorem 9), Properties *(i)*, *(ii)* and *(iii)* imply that

$$f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}(B - f^{\mathbf{A}}) + \sum_{j=1}^{m} \theta_j^{\mathbf{A}} h_{i,j}$$

where $S_i^{\mathbf{A}}$, $T_i^{\mathbf{A}}$ and the $\theta_j^{\mathbf{A}}$'s are polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ and $S_i^{\mathbf{A}}$, $T_i^{\mathbf{A}}$ are sums of squares in $\mathbb{R}[X_1, \ldots, X_n]$, which proves assertion (b). □

*Computational aspects of the contribution.* Note that numerical approximations of the algebraic certificates of positivity given by Theorem 1.2 can be computed through the use of semi-definite programming (see, among others, Schweighofer (2006); Hà and Phạm (2009)).

**Proposition 1.4.** For $i \in \{1, \ldots, d\}$ and $k \in \mathbb{N}$, let $g_1^i, \ldots, g_{m_i}^i$ be the polynomials in the set $\mathsf{Minors}\left(\mathsf{jac}\left(\left[\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}\right], [X_{i+1}, \ldots, X_n]\right), n - d + 1\right)$. Let $B$ be any value in $f^{\mathbf{A}}\left(V^{\mathbf{A}} \cap \mathbb{R}^n\right)$. Then define $f_{i,k}^{\mathbf{sos}}$ as the real number

$$\sup \left\{ a \in \mathbb{R} \mid f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}\left(B - f^{\mathbf{A}}\right) + \sum_{j=1}^{p} \phi_j^{\mathbf{A}} f_j^{\mathbf{A}} + \sum_{j=1}^{m_j} \varphi_j^{\mathbf{A}} g_j^i + \sum_{j=1}^{i-1} \psi_j^{\mathbf{A}} X_j \right\}, \tag{1}$$

where $S_i^{\mathbf{A}}$, $T_i^{\mathbf{A}}$, $\phi_j^{\mathbf{A}}$, $\varphi_j^{\mathbf{A}}$ and $\psi_j^{\mathbf{A}}$ are polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ such that each term on the right side of the equation in (1) has degree $\leq 2k$ and $S_i^{\mathbf{A}}$ and $T_i^{\mathbf{A}}$ are sums of squares of polynomials. Then the sequence $\left(f_{i,k}^{\mathbf{sos}}\right)_{k\in\mathbb{N}}$ converges monotonically increasing to $f_i^{\mathbf{sos}}$.

Since the sets of polynomials $\mathsf{Minors}(\mathsf{jac}([\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}], [X_{i+1}, \ldots, X_n]), n-d+1)$ may contain a large number of polynomials, we also show how to use results on determinantal ideals to reduce the number of polynomials to be considered in order to define $\mathsf{M}_i^{\mathbf{A}}$. Using Bruns and Schwänzl (1990), one can prove the following.

**Lemma 1.5.** The set $\mathsf{Minors}(\mathsf{jac}([\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}], [X_{i+1}, \ldots, X_n]), n-d+1)$ can be replaced with $(n-i)(p+1)-(n-d+1)^2+1$ equations.

Note that for big $n$, this is much smaller than the initial number of minors, that is $\binom{n-i}{n-d+1}\binom{p+1}{n-d+1}$.

**Remark 1.6.** Notice that $\mathsf{M}_0^{\mathbf{A}} \supset \mathsf{M}_1^{\mathbf{A}}$ implies $V(\mathsf{M}_0^{\mathbf{A}}) \subset V(\mathsf{M}_1^{\mathbf{A}})$, then $f_1^{\mathbf{sos}} \leq f_0^{\mathbf{sos}}$ and $f^\star = \min_{1\leq i\leq d} f_i^{\mathbf{sos}}$. One can skip the computations with $\mathsf{M}_0^{\mathbf{A}}$ which is the variety used in Nie (2010) to guarantee the exact SDP relaxations, and start with $\mathsf{M}_1^{\mathbf{A}}$. According to Lemma 1.5, $\mathsf{M}_1^{\mathbf{A}}$ contains fewer polynomials than $\mathsf{M}_0^{\mathbf{A}}$.

*Structure of the paper.* Section 2 is devoted to proving Proposition 1.3. It uses genericity properties of the varieties $V(\mathsf{M}_i^{\mathbf{A}})$ which are proved in Section 3. In Section 4, we discuss computational aspects of Theorem 1.2 by proving Proposition 1.4 and providing numerical experiments showing the effectiveness of our approach.

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## 2. Proof of Proposition 1.3

### 2.1. *Auxiliary results on polar varieties*

This paragraph aims at recalling properties about polar varieties proved in Safey El Din and Schost (2003) which play a crucial role in the proof of Proposition 1.3 and some auxiliary results that will be helpful in the sequel.

We consider the canonical projections $\Pi_i : (x_1, \ldots, x_n) \to (x_1, \ldots, x_i)$ and a polynomial family $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfying the regularity assumption $\mathbf{R}$ and we let $d$ be the dimension of $V^{\mathbf{A}}$.

In the sequel, for $0 \leq i \leq d-1$, we denote by $W_i^{\mathbf{A}}$ the algebraic variety

$$V\Big(F^{\mathbf{A}}, \mathsf{Minors}\left(\mathsf{jac}\left(\mathbf{F}^{\mathbf{A}}, [X_{i+2}, \ldots, X_n]\right), n-d\right)\Big).$$

Then for $i = d$, we denote by $W_d^{\mathbf{A}}$ the algebraic variety $V^{\mathbf{A}} = V\left(\mathbf{F}^{\mathbf{A}}\right)$.

(Safey El Din and Schost, 2003, Theorem 1): Under the above assumptions, there exists a non-empty Zariski-open set $\mathscr{O}'$ such that for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}'$, the restriction of $\Pi_i$ to $W_i^{\mathbf{A}}$ is proper for $0 \leq i \leq d$.

(Safey El Din and Schost, 2003, Theorem 2): Suppose that the polynomial family $\mathbf{F}$ satisfies the regularity assumption $\mathbf{R}$ and that the restriction of $\Pi_i$ to $W_i^{\mathbf{A}}$ is proper for $0 \leq i \leq d$. Then, for $0 \leq i \leq d$, the algebraic sets $W_i^{\mathbf{A}}$ (resp. $W_i^{\mathbf{A}} \cap V(X_1, \ldots, X_i)$) have dimension at most $i$ (resp. 0) and the union $\bigcup_{i=0}^{d} W_i^{\mathbf{A}} \cap V(X_1, \ldots, X_i)$ has a non-empty intersection with each connected component of $V^{\mathbf{A}} \cap \mathbb{R}^n$.

We will also need the following lemmas.

**Lemma 2.1.** Suppose that the polynomial family $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfies assumption $\mathbf{R}$. Let $V = V(\mathbf{F})$, $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and let $f^\star = \inf_{x \in V \cap \mathbb{R}^n} f(x)$. If there exists $x \in V \cap \mathbb{R}^n$ such that $f(x) = f^\star$ then $x \in V(\mathsf{M}_0)$.

*Proof.* Recall that $\mathsf{M}_0$ is the polynomial family containing $\mathbf{F}$ and all the $(n-d+1, n-d+1)$-minors of $\mathsf{jac}\left([\mathbf{F}, f], [X_1, \ldots, X_n]\right)$. Since by assumption $x \in V$, we need to prove that $\mathsf{jac}([\mathbf{F}, f], [X_1, \ldots, X_n])$ has rank $\leq n - d$.

Since $\mathbf{F}$ satisfies assumption $\mathbf{R}$, $\langle \mathbf{F} \rangle$ is radical equidimensional and $V$ is smooth and of dimension $d$. Since $x \in V$, the Jacobian criterion (Eisenbud, 1995, Theorem 16.19 pp. 402) implies that $\mathsf{jac}(\mathbf{F}, [X_1, \ldots, X_n])$ has rank $n - d$ at $x$. Without loss of generality, we suppose in the sequel that $\mathsf{jac}([f_1, \ldots, f_{n-d}], [X_1, \ldots, X_n])$ has rank $n-d$. We denote by $U$ the subset of points in $V$ at which $\mathsf{jac}([f_1, \ldots, f_{n-d}], [X_1, \ldots, X_n])$ has rank $n - d$. Note that $U$ is not empty since $x \in U$.

Now, suppose by contradiction that $\mathsf{jac}([f_1, \ldots, f_{n-d}, f], [X_1, \ldots, X_n])$ has rank greater than $n - d$ at $x$. Since it has $n - d + 1$ rows and $n$ columns, this implies that it has rank $n - d + 1$ at $x$. Without loss of generality, one can suppose that

$$J = \mathsf{jac}([f_1, \ldots, f_{n-d}, f], [X_1, \ldots, X_{n-d+1}])$$

is invertible at $x$. Denoting by $x_i$ the $i$-th coordinate of $x$, note that

$$\tilde{J} = \mathsf{jac}([f_1, \ldots, f_{n-d}, f, (X_k - x_k)_{n-d+2 \leq k \leq n}], [X_1, \ldots, X_n])$$

is invertible at $x$. We denote by $\tilde{U}$ the set of points in $U \cap V(X_{n-d+2} - x_{n-d+2}, \ldots, X_n - x_n)$ at which $\tilde{J}$ is invertible. Since $x \in \tilde{U}$, $\tilde{U}$ is not empty. Now, applying the inverse function theorem (Lee, 2002, Theorem 7.10 pp. 166) to the projection to $t$ on $\{(y, t) \mid y \in \tilde{U} \cap \mathbb{R}^n, t = f(y)\}$ yields the existence of an open interval $]a, b[\subset \mathbb{R}$ containing $f^\star$ such that for all $\vartheta \in ]a, b[$, $V(f - \vartheta) \cap \tilde{U} \cap \mathbb{R}^n \neq \emptyset$. Since $V(f - \vartheta) \cap \tilde{U} \cap \mathbb{R}^n \subset V(f - \vartheta) \cap V \cap \mathbb{R}^n$, this implies that there exists $x' \in V \cap \mathbb{R}^n$ such that $f(x') < f^\star$ with $f^\star = \inf_{x \in V \cap \mathbb{R}^n} f(x)$ which is a contradiction. □

### *2.2. Genericity Lemmas and proof of Proposition 1.3*

The proof of Proposition 1.3 is based on the results presented in the previous paragraph and the following lemmas. They provide genericity properties of geometric nature on the algebraic sets defined by the polynomial families $\mathsf{M}_i^{\mathbf{A}}$. The proofs of these lemmas are postponed to Section 3.

**Lemma 2.2.** Let $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfying assumption $\mathbf{R}$ and $f \in \mathbb{Q}[X_1, \ldots, X_n]$. The following property holds:

$\mathbf{P}_1$: for all $t \in \mathbb{R} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$, the ideal generated by $\mathbf{F}, f - t$ is radical equidimensional and its associated algebraic variety is either smooth of dimension $d-1$ or it is empty.

Moreover, the set $\{f(x) \mid x \in V(\mathsf{M}_0)\}$ has dimension at most 0.

**Lemma 2.3.** Let $\mathbf{F} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ satisfying assumption $\mathbf{R}$ and $f \in \mathbb{Q}[X_1, \ldots, X_n]$. There exists a non-empty Zariski-open set $\mathscr{O}_1 \subset \mathrm{GL}_n(\mathbb{C})$ such that, for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}_1$, there exists a non-empty Zariski-open set $\mathscr{U}_\mathbf{A} \subset \mathbb{C}$ such that:

$\mathbf{P}_2$: for all $t \in \mathbb{R} \cap \mathscr{U}_\mathbf{A}$, the restriction of $\Pi_{i-1}$ to $V^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap V(\mathsf{M}_i^\mathbf{A})$ is proper for $1 \leq i \leq d$.

We can now prove Proposition 1.3.

By Lemma 2.3, there exists a non-empty Zariski-open set $\mathscr{O}_1 \subset \mathrm{GL}_n(\mathbb{C})$ such that, for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}_1$, there exists a non-empty Zariski-open set $\mathscr{U}_\mathbf{A} \subset \mathbb{C}$ such that:

$\mathbf{P}_2$: for all $t \in \mathbb{R} \cap \mathscr{U}_\mathbf{A}$, the restriction of $\Pi_{i-1}$ to $V^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap V(\mathsf{M}_i^\mathbf{A})$ is proper for $1 \leq i \leq d$.

We set in the sequel $\mathscr{O} = \mathscr{O}_1$ and fix $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}$. Then, we set $\mathscr{T}_\mathbf{A} = \mathscr{U}_\mathbf{A} \setminus \{f^\mathbf{A}(x) \mid x \in V(\mathsf{M}_0^\mathbf{A})\}$. Note that by Lemma 2.2, $\{f^\mathbf{A}(x) \mid x \in V(\mathsf{M}_0^\mathbf{A})\}$ has dimension at most 0; consequently $\mathscr{T}_\mathbf{A}$ is a non-empty Zariski-open set since $\mathscr{U}_\mathbf{A}$ is also non-empty and Zariski-open.

*Proof of assertion (1).* By Lemma 2.2 applied to $F^\mathbf{A}$ and $f^\mathbf{A}$, for all $t \in \mathbb{R} \setminus \{f^\mathbf{A}(x) \mid x \in V(\mathsf{M}_0^\mathbf{A})\}$, the ideal generated by $F^\mathbf{A}, f^\mathbf{A} - t$ is radical and equidimensional and its associated algebraic variety is smooth (property $\mathbf{P}_1$) and $\{f^\mathbf{A}(x) \mid x \in V(\mathsf{M}_0^\mathbf{A})\}$ has dimension at most 0.

Moreover, for all $t \in \mathbb{R} \cap \mathscr{U}_\mathbf{A}$, the properness property $\mathbf{P}_2$ (Lemma 2.3) holds. Now let $\mathscr{T}_\mathbf{A} = \mathscr{U}_\mathbf{A} \setminus \{f^\mathbf{A}(x) \mid x \in V(\mathsf{M}_0^\mathbf{A})\}$ which is non-empty and Zariski-open. By Lemma 2.3, for all $t \in \mathbb{R} \cap \mathscr{T}_\mathbf{A}$ one can apply (Safey El Din and Schost, 2003, Theorem 2) to $\mathbf{F}^\mathbf{A}, f^\mathbf{A} - t$ which states that under $\mathbf{P}_1$ and $\mathbf{P}_2$ the algebraic sets defined by $V^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap V\left(\mathsf{M}_i^\mathbf{A}\right)$ for $1 \leq i \leq d$ have a non-empty intersection with each connected component of $V^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap \mathbb{R}^n$ and dimension at most 0.

*Proof of assertion (2).* Note first that $f^\star = \inf_{x \in V \cap \mathbb{R}^n} f(x) = \inf_{x \in V^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x)$. Recall that $W^\mathbf{A} = \cup_{i=0}^d V(\mathsf{M}_i^\mathbf{A})$. Since $W^\mathbf{A} \subset V^\mathbf{A}$, the inequality $f^\star \leq \inf_{x \in W^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x)$ holds. In the sequel, we prove that $\inf_{x \in W^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x) \leq f^\star$.

Suppose first that there exists $x \in V^\mathbf{A} \cap \mathbb{R}^n$ such that $f^\mathbf{A}(x) = f^\star$. Then, by Lemma 2.1, $x \in V(\mathsf{M}_0^\mathbf{A}) \cap \mathbb{R}^n \subset W^\mathbf{A} \cap \mathbb{R}^n$ which implies that $\inf_{x \in W^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x) \leq f^\star$.

Suppose now that for all $x \in V^\mathbf{A} \cap \mathbb{R}^n$, $f^\mathbf{A}(x) > f^\star$. Since $f^\star = \inf_{x \in V^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x)$, this implies that there exists a real number $c > f^\star$ such that for all $t \in ]f^\star, c[$, $V^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap \mathbb{R}^n$ is not empty.

Without loss of generality, one can suppose that $c$ is small enough so that $]f^\star, c[ \cap \mathscr{U}_\mathbf{A} \neq \emptyset$. Using assertion 1 of Proposition 1.3 which is proved above, this implies that $W^\mathbf{A} \cap V(f^\mathbf{A} - t) \cap \mathbb{R}^n$ is not empty for $t \in ]f^\star, c[$. Consequently, the inequality $\inf_{x \in W^\mathbf{A} \cap \mathbb{R}^n} f^\mathbf{A}(x) \leq f^\star$ holds which ends the proof of Assertion 2.

*Proof of assertion (3).* Let $Z^{\mathbf{A}}$ be an irreducible component of $V(\mathsf{M}_i^{\mathbf{A}})$ and consider the map $x \in Z^{\mathbf{A}} \to f^{\mathbf{A}}(x) \in \mathbb{C}$. In the sequel, we denote by $V_\infty(f^{\mathbf{A}}, Z^{\mathbf{A}}) \subset \mathbb{C}$ the set

$$\{t \in \mathbb{C} \mid \exists (x_k)_{k\in\mathbb{N}} \subset Z^{\mathbf{A}} \ \lim_k ||x_k|| = \infty \text{ and } \lim_k f^{\mathbf{A}}(x_k) = t\}.$$

Suppose first that $f^{\mathbf{A}}(Z^{\mathbf{A}})$ has dimension 0. Then, $R_\infty(f^{\mathbf{A}}, Z^{\mathbf{A}}) \subset f^{\mathbf{A}}(Z^{\mathbf{A}})$ which has dimension 0.

Suppose now that $f^{\mathbf{A}}(Z^{\mathbf{A}})$ has dimension 1. By the theorem on the dimension of fibers, (Shafarevich, 1977, Theorem 7, Chapter 1, pp. 76), there exists a non-empty Zariski-open set $\mathscr{W} \subset \mathbb{C}$ such that for all $t \in \mathscr{W}$, $\dim(Z^{\mathbf{A}} \cap V(f^{\mathbf{A}} - t)) = \dim(Z^{\mathbf{A}}) - 1$. By assertion 1 of Proposition 1.3 which is proved above, $Z^{\mathbf{A}} \cap V(f^{\mathbf{A}} - t)$ is either empty or 0-dimensional.

Hence, two situations may occur:

- either $Z^{\mathbf{A}} \cap V(f^{\mathbf{A}} - t)$ is empty and then $\dim(Z^{\mathbf{A}}) = 0$ which is not possible since, by assumption, $\dim(f^{\mathbf{A}}(Z^{\mathbf{A}})) = 1$;
- or $Z^{\mathbf{A}} \cap V(f^{\mathbf{A}} - t)$ has dimension 0 and then $\dim(Z^{\mathbf{A}}) = 1$ which implies that $V_\infty(f^{\mathbf{A}}, Z^{\mathbf{A}}) \subset \mathbb{C}$ is the set of non-properness of the map $x \in Z^{\mathbf{A}} \to f^{\mathbf{A}}(x)$ which has dimension at most 0 by (Jelonek, 1999, Theorem 3.8).

Since $V(\mathsf{M}_i^{\mathbf{A}})$ has finitely many irreducible components, the last assertion of Proposition 1.3 is proved.

## 3. Genericity properties

### *3.1. Proof of Lemma 2.2*

We first prove that $\{f(x) \mid x \in V(\mathsf{M}_0)\}$ is finite. The proof below is inspired by the one of (Shafarevich, 1977, Theorem 2, Chapter 6, pp. 141).

Let $X \subset V$ be the set of points $x \in V$ at which the differential of the map $x \in V \to f(x)$ is surjective. Note that $V \setminus X$ is defined by the vanishing of all $(n-d+1, n-d+1)$-minors of $\mathsf{jac}([\mathbf{F}, f], [X_1, \ldots, X_n])$, i.e. $V \setminus X = V(\mathsf{M}_0)$.

Suppose that $f(V(\mathsf{M}_0))$ is dense in $\mathbb{C}$. Then, applying (Shafarevich, 1977, Lemma 2, pp. 141), this would mean that there exists a non-empty Zariski-open set $Z \subset V(\mathsf{M}_0)$ such that at all points $x \in Z$ the differential of the map $x \in Z \to f(x)$ is surjective. This would imply the surjectivity of the differential of $x \in V \to f(x)$ at $x \in Z \subset V(\mathsf{M}_0)$, which is a contradiction.

Thus, $\{f(x) \mid x \in V(\mathsf{M}_0)\}$ is finite. Note also that for all $t \in \mathbb{C} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$ and at all points $x \in V \cap V(f - t)$, the matrix $\mathsf{jac}([\mathbf{F}, f - t], [X_1, \ldots, X_n])$ has rank $n - d + 1$.

By (Eisenbud, 1995, Theorem 16.19, Chapter 16, pp. 404), this implies that for all $t \in \mathbb{C} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$, the co-dimension of $V(\mathbf{F}) \cap V(f - t)$ is greater than or equal to $n - d + 1$. For $t \in \mathbb{C} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$, let $Z$ be an irreducible component of $V(\mathbf{F}) \cap V(f - t)$. Then, there exists an irreducible component $Z'$ of $V(\mathbf{F})$ such that $Z$ is an irreducible component of $Z' \cap V(f - t)$. By assumption, $Z'$ has co-dimension $n - d$; consequently by Krull's Principal Ideal Theorem $Z$ has co-dimension $n - d + 1$ or is empty. Since $V(\mathbf{F}) \cap V(f - t)$ has finitely many irreducible components, this proves that for all $t \in \mathbb{C} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$

- $V(\mathbf{F}) \cap V(f - t)$ is equidimensional and has dimension $d - 1$ or is empty;
- $\mathsf{jac}([\mathbf{F}, f - t], [X_1, \ldots, X_n])$ has rank $n - d + 1$ at all points $x \in V \cap V(f - t)$.

Note that the two properties above imply that $V(\mathbf{F}) \cap V(f-t)$ is smooth.

We prove below that it also implies that for $t \in \mathbb{C} \setminus \{f(x) \mid x \in V(\mathsf{M}_0)\}$, the ideal $I_t = \langle \mathbf{F}, f-t \rangle$ is radical.

Suppose that $I_t \neq \langle 1 \rangle$ (otherwise the announced claim is immediate). Let $I_t = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_s$ be a minimal primary decomposition of $I_t$. We assume that the $Q_i$'s are isolated for $0 \leq i \leq r$. It is then sufficient to prove that for $1 \leq i \leq r$, $Q_i$ is a prime ideal.

Let $i \in \{1, \ldots, r\}$. There exists $x \in V(Q_i)$ such that $x \notin V\left(\bigcap_{i \neq j} Q_j\right)$. Let $\mathfrak{m}$ be the maximal ideal at $x$. For an ideal $I$ (resp. a ring $R$), we denote by $I_\mathfrak{m}$ (resp. $R_\mathfrak{m}$) its localization at $\mathfrak{m}$.

Consider the ring $\frac{\mathbb{Q}[X_1,\ldots,X_n]_\mathfrak{m}}{(I_t)_\mathfrak{m}}$. Because $\mathsf{jac}([\mathbf{F}, f-t], [X_1, \ldots, X_n])$ has rank $n-d+1$ at all points of $V(\mathbf{F}) \cap V(f-t)$, according to (Eisenbud, 1995, Theorem 16.19, Chapter 16, pp. 404), it is regular. Hence, by (Atiyah and MacDonald, 1969, Lemma 11.23 p. 123)), it is integral, which implies that the ideal $(I_t)_\mathfrak{m}$ is prime. Note that, since $Q_i$ is the unique isolated primary component contained in $\mathfrak{m}$, the following equalities hold:

$$(I_t)_\mathfrak{m} = (Q_i)_\mathfrak{m} \cap \bigcap_{Q_j \subset \mathfrak{m}, j \geq r+1} (Q_j)_\mathfrak{m} = (Q_i)_\mathfrak{m}.$$

Thus $(Q_i)_\mathfrak{m} = (I_t)_\mathfrak{m}$ is also prime and using (Atiyah and MacDonald, 1969, Prop. 3.11 pp. 41), we conclude that so is $Q_i$. Finally, as an intersection of prime ideals, $I_t$ is a radical ideal.

### *3.2. Proof of Lemma 2.3*

The proof is strongly inspired by the one of (Safey El Din and Schost, 2003, Theorem 1) and uses intermediate results in its proof. For clarity and simplicity we refer to those results which can be used *mutatis mutandis* and focus on steps requiring a specific treatment to prove Lemma 2.3.

Let $\mathfrak{A} = (\mathfrak{A}_{\mathrm{i,j}})_{1 \leq i,j \leq n}$ be a matrix whose entries are new indeterminates and let $\mathfrak{t}$ be another indeterminate. Given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ we define $f^\mathfrak{A} \in \mathbb{Q}(\mathfrak{A}_{i,j})[X_1, \ldots, X_n]$ as $f^\mathfrak{A} = f(\mathfrak{A}X_1, \ldots, \mathfrak{A}X_n)$. For $i = d$, we denote by $\Delta_d^\mathfrak{A}(\mathfrak{t})$ the ideal $\left\langle f_1^\mathfrak{A}, \ldots, f_p^\mathfrak{A}, f^\mathfrak{A} - \mathfrak{t} \right\rangle$. Then for $i \in \{1, \ldots, d-1\}$, let $\Delta_i^\mathfrak{A}(\mathfrak{t})$ be the ideal generated by $f_1^\mathfrak{A}, \ldots, f_p^\mathfrak{A}, f^\mathfrak{A} - \mathfrak{t}$ and the set $\mathsf{Minors}\left(\mathsf{jac}\left(\left[\mathbf{F}^\mathfrak{A}, f^\mathfrak{A}\right], [X_{i+1}, \ldots, X_n]\right), n-d+1\right)$. For an ideal $I^\mathfrak{A} = \left\langle g_1^\mathfrak{A}, \ldots, g_s^\mathfrak{A} \right\rangle \subset \mathbb{Q}(\mathfrak{A}_{i,j})[X_1, \ldots, X_n]$ and a matrix $\mathbf{A} \in GL_n(\mathbb{C})$, we denote by $I^\mathbf{A} \subset \mathbb{C}[X_1, \ldots, X_n]$ the ideal $\left\langle g_1^\mathbf{A}, \ldots, g_s^\mathbf{A} \right\rangle$.

Then we can restate (Safey El Din and Schost, 2003, Section 2.3, Prop. 1), replacing $\mathbb{Q}$ with $\mathbb{Q}(\mathfrak{t})$. Indeed, the tools used in this proof, namely Nœther normalization, Krull's Principal Ideal Theorem, Quillen-Suslin's Theorem and algebraic Bertini's Theorem can be used with any field of characteristic 0.

**Lemma 3.1.** Let $i \in \{1, \ldots, d\}$, let $P_\mathfrak{t}$ be one of the prime components of the radical of the ideal $\Delta_i^\mathfrak{A}(\mathfrak{t})$ and let $r$ be its dimension. Then $r$ is at most $i-1$ and the extension $\mathbb{Q}(\mathfrak{t})(\mathfrak{A}_{i,j})[X_1, \ldots, X_r] \to \mathbb{Q}(\mathfrak{t})(\mathfrak{A}_{i,j})[X]/P_\mathfrak{t}$ is integral.

The next Proposition shows that this result remains true specializing the indeterminates $\mathfrak{A}_{i,j}$ and $\mathfrak{t}$ in a suitable non-empty Zariski-open set. This is similar to (Safey El Din and Schost, 2003, Proposition 2), the only difference is that we have to manage the parameter $\mathfrak{t}$.

**Lemma 3.2.** There exists a nonempty Zariski-open set $\mathscr{O}_1 \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \cap \mathscr{O}_1$, there exists a non-empty Zariski-open set $\mathscr{U}_\mathbf{A} \subset \mathbb{C}$ such that for all $t \in \mathscr{U}_\mathbf{A}$, the following holds:

> Let $i \in \{1, \ldots, d\}$, let $P_t^\mathbf{A}$ be one of the prime components of the radical of $\Delta_i^\mathbf{A}(t)$ and $r$ its dimension. Then $r$ is at most $i-1$ and the extension $\mathbb{C}[X_1, \ldots, X_r] \to \mathbb{C}[X_1, \ldots, X_n]/P_t^\mathbf{A}$ is integral.

*Proof.* Let $i$ be in $\{1, \ldots, d\}$. Since $i$ is fixed, we write $\Delta = \Delta_i^{\mathfrak{A}}(\mathfrak{t})$. Applying (Safey El Din and Schost, 2003, Proposition 2) with $\mathbb{C}(\mathfrak{t})$ as a ground field yields the existence of a non-empty Zariski-open set $\mathscr{O}_1$ such that for all $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}_1$ and all prime component $P$ of $\Delta^\mathbf{A}$ the following holds:

- the dimension $r$ of $P$ is at most $i-1$;
- the extension $\mathbb{C}(\mathfrak{t})[X_1, \ldots, X_r] \to \mathbb{C}(\mathfrak{t})[X_1, \ldots, X_n]/P$ is integral.

Thus it is sufficient to prove that the ideal $P_t$ obtained specializing $\mathfrak{t}$ to $t$ contains a monic polynomial in $X_r$. Since the extension $\mathbb{C}(\mathfrak{t})[X_1, \ldots, X_r] \to \mathbb{C}(\mathfrak{t})[X_1, \ldots, X_n]/P$ is integral, as an ideal in $\mathbb{Q}(\mathfrak{t})[X_1, \ldots, X_n]$, $P$ contains a non-identically zero monic polynomial in $\mathbb{Q}(\mathfrak{t})[X_1, \ldots, X_{r-1}][X_r]$ that we denote by $m_P$. Let $\alpha(\mathfrak{t}) \in \mathbb{Q}[\mathfrak{t}]$ be the least common multiple of the denominators of $m_P$ in $\mathbb{Q}[\mathfrak{t}]$.

Now, let $\mathscr{T}_{\mathbf{A},P}$ be the non-empty Zariski-open set such that for all $t \in \mathscr{T}_{\mathbf{A},P}$, $P_t$ is equidimensional of dimension the one of $P$ and contains the polynomial $m_{P,t}$ obtained when instantiating $\mathfrak{t}$ to $t$ in $m_P$: such a Zariski-open set exists since

- one can perform equidimensional decomposition without factorization;
- one can decide that a polynomial belongs to an ideal without factorization.

Thus, $\mathscr{T}_{\mathbf{A},P}$ can be obtained as the non-vanishing of all the denominators appearing in the execution of such algorithms with input polynomials defining $P$ for the first algorithm and a Gröbner basis of $P$ and $m_P$ for the second algorithm.

Consider now the non-empty Zariski open set $\mathscr{V}_{\mathbf{A},P}$ defined by the non-vanishing of $\alpha$ and let $\mathscr{U}_{\mathbf{A},P}$ be $\mathscr{T}_{\mathbf{A},P} \cap \mathscr{V}_{\mathbf{A},P}$. For $t \in \mathscr{U}_{\mathbf{A},P}$, we instantiate $\mathfrak{t}$ to $t$: since $t \in \mathscr{T}_{\mathbf{A},P}$, $P_t$ is equidimensional and contains $m_{P,t}$. Moreover, since $t \in \mathscr{V}_{\mathbf{A},P}$, $m_{P,t}$ is monic.

Consequently, for all $t \in \mathscr{U}_{\mathbf{A},P}$, the extension $\mathbb{C}[X_1, \ldots, X_r] \to \mathbb{C}[X_1, \ldots, X_n]/P_t$ is integral. We conclude by defining $\mathscr{U}_\mathbf{A} = \bigcap \mathscr{U}_{\mathbf{A},P}$, where the intersection is taken for the finitely many prime components of $\Delta^\mathbf{A}$. □

One can now conclude the proof of Lemma 2.3. According to (Safey El Din and Schost, 2003, Section 2.5, Prop. 3), Lemma 3.2 and (Jelonek, 1999, Lemma 3.10), the following holds for $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathscr{O}_1$ and $t \in \mathscr{U}_\mathbf{A}$:

- For every prime component $P_t^\mathbf{A}$ of the radical of $\Delta_i^\mathbf{A}(t)$, the following holds. Let $r$ be the dimension of $P_t^\mathbf{A}$; then $r$ is at most $i-1$ and the extension $\mathbb{C}[X_1, \ldots, X_r] \to \mathbb{C}[X_1, \ldots, X_n]/P_t^\mathbf{A}$ is integral.
- The restriction of $\Pi_{i-1}$ to $V\left(\Delta_i^\mathbf{A}(t)\right)$ is proper.

## 4. Computational aspects of Theorem 1.2

### *4.1. Proof of Proposition 1.4*

We start with the proof of Proposition 1.4 that we restate: for $i \in \{1, \ldots, d\}$ and $k \in \mathbb{N}$, let $g_1^i, \ldots, g_{m_i}^i$ be the polynomials in the set $\mathsf{Minors}(\mathsf{jac}([\mathbf{F}^\mathbf{A}, f^\mathbf{A}], [X_{i+1}, \ldots, X_n]), n-d+$

1). Let $B$ be any value in $f^{\mathbf{A}}\left(V^{\mathbf{A}} \cap \mathbb{R}^n\right)$. Then define $f_{i,k}^{\mathbf{sos}}$ as the real number

$$\sup\left\{a \in \mathbb{R} \,|\, f^{\mathbf{A}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}\left(B - f^{\mathbf{A}}\right) + \sum_{j=1}^{p} \phi_j^{\mathbf{A}} f_j^{\mathbf{A}} + \sum_{j=1}^{m} \varphi_j^{\mathbf{A}} g_j^i + \sum_{j=1}^{i-1} \psi_j^{\mathbf{A}} X_j\right\},$$

where $S_i^{\mathbf{A}}$, $T_i^{\mathbf{A}}$, $\phi_j^{\mathbf{A}}$, $\varphi_j^{\mathbf{A}}$ and $\psi_j^{\mathbf{A}}$ are polynomials in $\mathbb{R}[X_1,\ldots,X_n]$ such that each term on the right side of the equation above has degree $\leq 2k$, and $S_i$ and $T_i^{\mathbf{A}}$ are sums of squares of polynomials. Then the sequence $\left(f_{i,k}^{\mathbf{sos}}\right)_{k\in\mathbb{N}}$ converges monotonically increasing to $f_i^{\mathbf{sos}}$.

*Proof of Proposition 1.4.* Let $i$ be a fixed integer in $\{1,\ldots,d\}$. First we show that the sequence $\left(f_{i,k}^{\mathbf{sos}}\right)_{k\in\mathbb{N}}$ is monotonically increasing. For $k \in \mathbb{N}^*$, let $\mathcal{P}_{\leq 2k}$ be the set of polynomials in $\mathbb{R}\left[X_1,\ldots,X_n\right]$ of degree $\leq 2k$. Let $k_1 \leq k_2$. It is clear that $\mathcal{P}_{\leq 2k_1} \subset \mathcal{P}_{\leq 2k_2}$. Thus, $f_{i,k_1}^{\mathbf{sos}} \leq f_{i,k_2}^{\mathbf{sos}}$ and the sequence is monotonically increasing. Then the fact that $\mathbb{R}\left[X_1,\ldots,X_n\right] = \bigcup_k \mathcal{P}_{\leq 2k}$ implies that the sequence tends to $f_i^{\mathbf{sos}}$. □

Note that practically, Proposition 1.4 is used to compute the supremum

$$\sup\left\{a \in \mathbb{R} \,|\, \widetilde{f^{\mathbf{A}}} - a = S_i^{\mathbf{A}} + T_i^{\mathbf{A}}\left(B - \widetilde{f^{\mathbf{A}}}\right) + \sum_{j=1}^{p} \phi_j^{\mathbf{A}} \widetilde{f_j^{\mathbf{A}}} + \sum_{j=1}^{m} \varphi_j^{\mathbf{A}} \widetilde{g_j^i}\right\},$$

where for a polynomial $h$, $\widetilde{h}$ denotes the polynomial $h(0,\ldots,0,X_i,\ldots,X_n)$. This allows to manipulate a smaller number of variables, which gives better numerical results.

### *4.2. Proof of Lemma 1.5*

Let $N = (N_{ij})$ be an $m \times n$ matrix of indeterminates over $\mathbb{C}$, $\Delta(N)$ its set of minors. Define the determinantal variety

$$D_{t-1}^{m,n} = \left\{N \in \mathbb{C}^{m\times n} : \text{ rank } N < t.\right\}$$

For indices $a_1,\ldots,a_t,b_1,\ldots,b_t$ such that $t \leq \min(m,n)$, $1 \leq a_1 < \cdots < a_t \leq m$, $1 \leq b_1 < \cdots < b_t \leq n$, we define $[a_1,\ldots,a_t|b_1,\ldots,b_t]$ to be the $t$-minor of matrix $N$, i.e., the determinant of the submatrix $N$ whose row indices are $a_1,\ldots,a_t$ and column indices are $b_1,\ldots,b_t$. So we have

$$D_{t-1}^{m,n} = \left\{N \in \mathbb{C}^{m\times n} : [a_1,\ldots,a_t|b_1,\ldots,b_t] = 0, \ \forall [a_1,\ldots,a_t|b_1,\ldots,b_t] \in \Delta(N)\right\}$$

We define a partial ordering on $\Delta(N)$ as follows, see also (Bruns and Vetter, 1988, pp. 46):

$$\begin{aligned}&[a_1,\ldots,a_u|b_1,\ldots,b_u] \leq [c_1,\ldots,c_v|d_1,\ldots,d_v]\\ &\iff u \geq v,\ a_1 \leq c_1,\ldots,a_v \leq c_v,\ b_1 \leq d_1,\ldots,b_v \leq d_v.\end{aligned}$$

For an arbitrary minor $\xi = [a_1,\ldots,a_u|b_1,\ldots,b_u]$ in $\Delta(N)$, we define its *length* by:

$$\begin{aligned}\text{len}(\xi) = k \iff &\text{there is a chain } \xi = \xi_k > \xi_{k-1} > \ldots > \xi_1,\ \xi_i \in \Delta(N),\\ &\text{and no longer chain starting with } \xi \text{ exits.}\end{aligned}$$

We prefer the notation of the *length* instead of the *rank* defined in (Bruns and Vetter, 1988, pp. 55).

Let $\Omega(N)$ denote the set of all $k$-minors of $N$ with $k \geq t$. For every $1 \leq l \leq mn-t^2+1$, define

$$\theta_l(N) = \sum_{\xi \in \Omega(N), \mathrm{len}(\xi)=l} \xi.$$

**Lemma 4.1.** (Bruns and Vetter, 1988, Lemma 5.9) We have that

$$D_{t-1}^{m,n} = \{N \in \mathbb{C}^{m\times n} : \ \theta_l(N) = 0, \ l = 1, \dots, mn - t^2 + 1\}.$$

In (Bruns and Schwänzl, 1990, Theorem 2), they also proved that $mn - t^2 + 1$ is the smallest number of polynomials for defining the determinantal variety $D_{t-1}^{m,n}$.

To find all minors of a given length, it is convenient to generate all chains composed by minors in $\Omega(N)$. The following proposition gives the minor of the maximal length in $\Omega(N)$. Furthermore, we show in its proof how to construct all chains in $\Omega(N)$ starting with this minor.

**Proposition 4.2.** The minor of the maximal length in $\Omega(N)$ is $[m-t+1, \dots, m|n-t+1, \dots, n]$ and its length is $mn - t^2 + 1$.

Before the proof is given, we illustrate the construction of all chains for a special case where $m = 3, n = 4$ and $t = 2$. First we generate the set of chains consisting of 2-minors. Starting with the minor of the maximal length, if we decrease one of the indices of the previous minor by 1 and keep the indices of the new minor in strictly ascending order, a new minor of smaller length is generated. All chains consisting of 2-minors are shown in Figure 1, where the arrows point to minors of higher orderings. Then we collect all 3-minors and add them to the chains we have already constructed. The set of chains consisting of all minors in $\Omega(N)$ for $m = 3, n = 4, t = 2$ is shown in Figure 2.

From Figure 1 and 2, we notice the following two facts:

(a) The $k$-minors in the same column have the same summation of their indices which is one less than that of the previous column.

(b) The $(k+1)$-minors that can increase the length of chains consisting of $k$-minors are the ones with the form $[1, 2, \dots, k, a|1, 2, \dots, k, b]$, where $k+1 \leq a \leq m$ and $k+1 \leq b \leq n$.

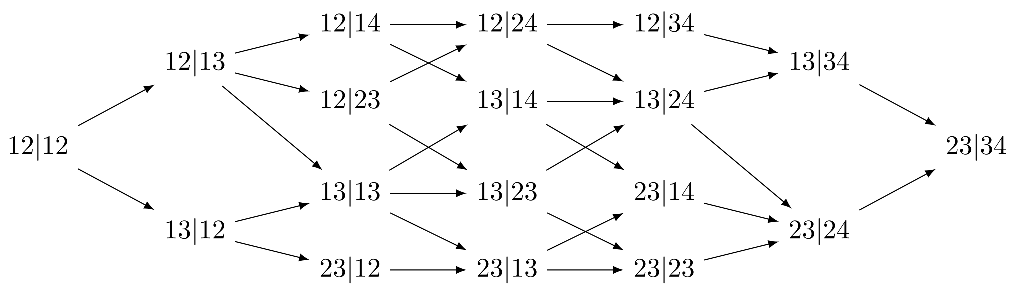


Fig. 1. All chains consisting only of the 2-minors.

*Proof of Proposition 4.2.* The first part of the statement is obvious. We prove the second part in the following. Without loss of generality, we assume that $m \leq n$.

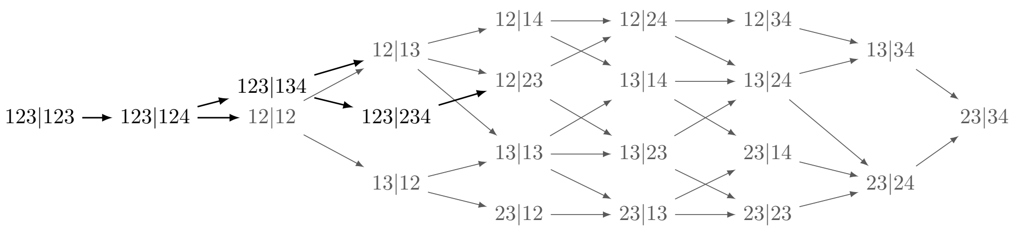


Fig. 2. All chains consisting of the 2-minors and 3-minors.

First, we show how to generate the set of chains consisting of $t$-minors, denoted by $\mathfrak{C}_t$. Starting with $\xi = [m-t+1, \ldots, m|n-t+1, \ldots, n]$, the $t$-minor with the maximal length, we construct new $t$-minors by decreasing one of the indices in $\xi$ by 1 and keeping the indices of new minors in strictly ascending order. This process continues until we reach the minor $\xi_1 = [1, 2, \ldots, t|1, 2, \ldots, t]$ with the lowest ordering. Based on the observation (a), we can show that the maximal length of the chain $\chi_t$ from $\xi$ to $\xi_1$ is

$$(2m - t + 1)t/2 + (2n - t + 1)t/2 - (1 + t)t + 1 = (m + n)t - 2t^2 + 1.$$

Secondly, we show how to add the $(t + 1)$-minors in $\Omega(N)$ to the set of chains $\mathfrak{C}_t$ constructed above. Notice that for every $(t+1)$-minor $\xi = [a_1, \ldots, a_t, a_{t+1}|b_1, \ldots, b_t, b_{t+1}]$, the $t$-minor $\eta = [a_1, \ldots, a_t|b_1, \ldots, b_t]$ has already appeared in $\mathfrak{C}_t$. Since $\xi < \eta$, we put $\xi$ in the column next (on the left) to the column consisting of $\eta$. Therefore, we generate the set of chains consisting of all $t + 1$-minors in $\Omega(N)$, denoted by $\mathfrak{C}_{t+1}$. According to (a) and (b), we obtain that the maximal length of the chain $\chi_{t+1}$ from $[1, \ldots, t, m|1, \ldots, t, n]$ to $[1, \ldots, t, t+1|1, \ldots, t, t+1]$ is $m+n-2(t+1)+1$. Since all minors in $\chi_{t+1}$ are smaller than minors in $\chi_t$, we can add the chain $\chi_{t+1}$ to the end of the chain $\chi_t$.

Going through the same process, we can generate the chains $\chi_{t+2}, \ldots, \chi_m$. It is clear that the chain $\chi_m \to \ldots \to \chi_{t+1} \to \chi_t$ consists of minors in $\Omega(N)$ from $[1, \ldots, m|1, \ldots, m]$ to $\xi$ and has the largest length

$$(m + n)t - 2t^2 + 1 + \sum_{s=t+1}^{m} (m + n - 2s + 1) = mn - t^2 + 1,$$

which is the length of $\xi$. $\square$

Now we return to the construction of $\mathsf{M}_i^{\mathbf{A}}$.

*Proof of Lemma 1.5.* The size of the Jacobian matrix $\mathsf{jac}([\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}], [X_{i+1}, \ldots, X_n])$ is $(p+1) \times (n-i)$. Applying Lemma 4.1 to it for $t = n-d+1$, we can reduce the number of equations in the set $\mathsf{Minors}(\mathsf{jac}([\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}], [X_{i+1}, \ldots, X_n]), n-d+1)$ from $\binom{n-i}{n-d+1}\binom{p+1}{n-d+1}$ to $(n-i)(p+1) - (n-d+1)^2 + 1$. $\square$

## *4.3. Numerical Results*

In this section, our method is applied to solve some constrained global optimization problems. We set $\mathbf{A}$ to be the identity matrix and call the command IsRadical in the Maple package PolynomialIdeals to test if an ideal $I$ is radical and the command HilbertDimension in the package Groebner to get the dimension of the variety $V(I)$. The Matlab software SOSTOOLS Prajna et al. (2004) is used to solve (1).

*Optimization with only equality constraints.* We consider polynomial optimization with only equality constraints for which we can apply our method directly,

$$\inf_{x\in\mathbb{R}^n} f(x) \quad s.t. \quad f_1(x) = \ldots = f_p(x) = 0. \tag{2}$$

The main contributions of our approach compared with Lasserre (2001), Demmel et al. (2007), and Nie (2010) are:

- There is no compactness requirement of the feasible set.
- We do not assume that the KKT conditions are satisfied at the minimizer or the minimum $f^\star$ is reached.
- Our regularity assumptions $\mathbf{R}$ are weaker than the assumptions in Nie (2010).

**Example 4.3.** (Nie, 2010, Example 5.2) Consider the optimization problem

$$\begin{aligned}&\inf_{x\in\mathbb{R}^3} x_1^6 + x_2^6 + x_3^6 + 3x_1^2x_2^2x_3^2 - x_1^2(x_2^4 + x_3^4) - x_2^2(x_3^4 + x_1^4) - x_3^2(x_1^4 + x_2^4)\\ &s.t.\ x_1 + x_2 + x_3 - 1 = 0.\end{aligned}$$

The feasible set is non-compact. The objective function is the Robinson polynomial which is nonnegative everywhere but not SOS. We have $f^\star = 0$. Let $g := X_1 + X_2 + X_3 - 1$, then the dimension of the ideal $\langle g\rangle$ is 2.

- To compute $f_1^{\mathbf{sos}}$, we have $\mathsf{M}_1 = \{g, h\}$ where

$$\begin{aligned}h :=&6X_2^5 + 6X_1^2X_2X_3^2 - 4X_1^2X_2^3 - 2X_2X_3^4 - 2X_2X_1^4 - 4X_3^2X_2^3\\ &- 6X_3^5 - 6X_1^2X_2^2X_3 + 4X_1^2X_3^3 + 4X_2^2X_3^3 + 2X_3X_1^4 + 2X_3X_2^4.\end{aligned}$$

  Setting $B = f(1, 0, 0) = 1$, the lower bounds we computed are: $f_{1,3}^{\mathbf{sos}} = -5.8186 \times 10^{-2}$, $f_{1,4}^{\mathbf{sos}} = -1.6531 \times 10^{-2}$, $f_{1,5}^{\mathbf{sos}} = -4.1363 \times 10^{-4}$, $f_{1,6}^{\mathbf{sos}} = 4.2929 \times 10^{-10}$. The sign of the last lower bound is not correct due to the numerical issues.
- To compute $f_2^{\mathbf{sos}}$, we have $\mathsf{M}_2 = \{g, X_1\}$. It is equivalent to solving

$$\begin{aligned}&\inf_{x_2,x_3\in\mathbb{R}} x_2^6 + x_3^6 - x_2^2x_3^4 - x_3^2x_2^4\\ &s.t.\ x_2 + x_3 - 1 = 0.\end{aligned}$$

  Setting $B = f(1, 0) = 1$, the lower bounds we obtained are: $f_{2,2}^{\mathbf{sos}} = -8.0658\times10^{-12}$, $f_{2,3}^{\mathbf{sos}} = -9.1665 \times 10^{-12}$. It is clear that $f_2^{\mathbf{sos}}$ is also equal to $f^\star$.

**Example 4.4.** Consider the optimization problem

$$\begin{aligned}&\inf_{x\in\mathbb{R}^2} (x_1 + 1)^2 + x_2^2\\ &s.t.\ -x_1^3 + x_2^2 = 0.\end{aligned}$$

Obviously, we have $x^\star = (0, 0)$ and $f^\star = 1$. It is easy to check that the feasible set is non-compact and the KKT conditions are not satisfied at the minimizer. The regularity assumption $\mathbf{R}$ is satisfied and $d = 1$. With $\mathsf{M}_1 = \{-X_1^3 + X_2^2\}$ and $B = f(0, 0) = 1$, the lower bounds we obtained are: $f_{1,2}^{\mathbf{sos}} = 0.99842$, $f_{1,3}^{\mathbf{sos}} = 0.9989$, $f_{1,4}^{\mathbf{sos}} = 0.99865$, $f_{1,5}^{\mathbf{sos}} = 0.99844$. Although there are numerical errors, we do get good approximations of the minimum $f^\star$.

**Example 4.5.** Consider the constrained optimization problem

$$\begin{aligned} \inf_{x\in\mathbb{R}^2} \ & x_1 \\ s.t. \ & x_1x_2^2 - 1 = 0. \end{aligned}$$

The KKT system $\{1-\lambda X_2^2,\ -2X_1X_2\lambda,\ X_1X_2^2-1\}$ has no solution. Applying our method, $d = 1$ and $\mathsf{M}_1 = \{X_1^2X_2^2 - 1\}$. With $B = f(1, 1) = 1$, the lower bounds we obtained are: $f_{1,3}^{\mathbf{sos}} = 2.5255 \times 10^{-3}$, $f_{1,4}^{\mathbf{sos}} = 1.902 \times 10^{-2}$, $f_{1,5}^{\mathbf{sos}} = 8.1335 \times 10^{-2}$. Obviously, there are big numerical problems: $X_2 \to \infty$, which leads to some elements of the moment matrices used to solve the associated SDP's tending toward infinity. We can employ the sparse support monomials in (1) to fight against this problem. Similar analysis can be found in Guo et al. (2010).

*Optimization with inequality constraints.* In the following we consider the general optimization problem

$$\begin{aligned} \inf_{x\in\mathbb{R}^n} \ & f(x) \\ s.t. \ & f_1(x) = \cdots = f_p(x) = 0, \\ & g_1(x) \geq 0, \ldots, g_q(x) \geq 0. \end{aligned} \tag{3}$$

Although our method applies to the global optimization of polynomials restricted to a smooth variety, it can be used to solve the problem (3) if we introduce new variables $T = [T_1, \ldots, T_q]$ and turn inequalities into equality constraints:

$$\begin{aligned} \inf_{x\in\mathbb{R}^n, t\in\mathbb{R}^q} \ & f(x) \\ s.t. \ & f_1(x) = \cdots = f_p(x) = 0, \\ & g_1(x) - t_1^2 = 0, \ldots, g_q(x) - t_q^2 = 0. \end{aligned}$$

However, we notice that related SDP problems may become very ill-conditioned because of these extra variables. Here are some techniques we used to handle numerical difficulties in order to improve the accuracy of a computed solution:

- Scaling the problem to make the magnitudes of all nonzero components of optimal solutions close to 1. Although it is impossible to make an ideal scaling before we know the optimal solutions, sometimes we can still do so by performing a linear transformation of the variables if we know finite lower and upper bounds constraints on them.
- Choosing $B$ as close to the optimum as possible.
- Normalizing the coefficients of the polynomials in (3).

For more details about these techniques, see Waki et al. (2009).

**Example 4.6.** (Demmel et al., 2007, Example 4.3) Consider the optimization problem under constraints

$$\begin{aligned} \inf_{x\in\mathbb{R}^2} \ & (-4x_1^2 + x_2^2)(3x_1 + 4x_2 - 12) \\ s.t. \ & 3x_1 - 4x_2 \leq 12,\ 2x_1 - x_2 \leq 0,\ -2x_1 - x_2 \leq 0. \end{aligned}$$

The semialgebraic set defined by the constraints is non-compact. The global minimum is $f^\star = -\frac{1024}{55} \approx -18.6182$ and the minimizer is $x^\star = (24/55,\ 128/55) \approx (-0.4364, 2.3273)$. Let $g_1 := 12 - 3X_1 + 4X_2 - T_1^2,\ g_2 := X_2 - 2X_1 - T_2^2,\ g_3 := X_2 + 2X_1 - T_3^2$, then the dimension of the ideal $\langle g_1, g_2, g_3\rangle$ is 2.

- To compute $f_1^{\mathbf{sos}}$, we have $\mathsf{M}_1 = \{g_1, g_2, g_3, h\}$, where $h := (-16X_1^2 + 6X_2X_1 + 12X_2^2 - 24X_2)T_1T_2T_3$. Setting $B = f(0,0,0) = 0$, the lower bounds we computed are: $f_{1,3}^{\mathbf{sos}} = -20.184$, $f_{1,4}^{\mathbf{sos}} = -18.618$.
- To compute $f_2^{\mathbf{sos}}$, we have $\mathsf{M}_1 = \{g_1, g_2, g_3, X_1\}$. It is equivalent to solving

$$\inf_{x\in\mathbb{R}^4, t\in\mathbb{R}^3} x_2^2(4x_2 - 12)$$
$$s.t. \; -4x_2 + t_1^2 = 12, \; -x_2 + t_2^2 = 0, \; -x_2 + t_3^2 = 0.$$

  It is easy to see that $f_2^{\mathbf{sos}} = -16$ which is not equal to $f^\star$.

**Example 4.7.** (Demmel et al., 2007, Example 4.5) Consider the following non-convex quadratic optimization

$$\begin{aligned} \inf_{x\in\mathbb{R}^2} \; & x_1^2 + x_2^2 \\ s.t. \; & x_2^2 - 1 \geq 0, \\ & x_1^2 - Nx_1x_2 - 1 \geq 0, \\ & x_1^2 + Nx_1x_2 - 1 \geq 0. \end{aligned}$$

It is shown in Demmel et al. (2007) that the global minimum is $f^\star = \frac{1}{2}(N^2 + N\sqrt{N^2+4}) + 2$. Let $g_1 := X_2^2 - 1 - T_1^2$, $g_2 := X_1^2 - NX_1X_2 - 1 - T_2^2$, $g_3 := X_1^2 + NX_1X_2 - 1 - T_3^2$, then the dimension of the ideal $\langle g_1, g_2, g_3\rangle$ is 2. It can be checked that $V(\mathsf{M}_2) = \emptyset$. Hence, in the following we only compute $f_1^{\mathbf{sos}}$ for some given constants $N$. We have $\mathsf{M}_1 = \{g_1, g_2, g_3, h\}$, where $h = X_2T_1T_2T_3$.

- $N = 2$, then we have $f^\star = 6.8284$. For $B = f(3,1) = 10$, the results are: $f_{1,2}^{\mathbf{sos}} = 4$, $f_{1,3}^{\mathbf{sos}} = 6.7692$, $f_{1,4}^{\mathbf{sos}} = 6.8284$.
- $N = 3$, then we have $f^\star = 11.9083$. For $B = f(4,1) = 17$, the results are: $f_{1,2}^{\mathbf{sos}} = 5$, $f_{1,3}^{\mathbf{sos}} = 11.316$, $f_{1,4}^{\mathbf{sos}} = 11.908$.
- $N = 4$, then we have $f^\star = 18.9443$. For $B = f(5,1) = 26$, the results are: $f_{1,2}^{\mathbf{sos}} = 6$, $f_{1,3}^{\mathbf{sos}} = 17.2$, $f_{1,4}^{\mathbf{sos}} = 22.168$. If we set $B = f(4.3,1) = 19.49$, the results are: $f_{1,2}^{\mathbf{sos}} = 15.333$, $f_{1,3}^{\mathbf{sos}} = 18.944$.

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