# Variant Quantifier Elimination

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**Abstract**

We describe an algorithm (VQE) for a *variant* of the real quantifier elimination problem (QE). The variant problem requires the input to satisfy a certain *extra condition*, and allows the output to be *almost* equivalent to the input. The motivation/rationale for studying such a variant QE problem is that many quantified formulas arising in applications do satisfy the extra conditions. Furthermore, in most applications, it is sufficient that the output formula is almost equivalent to the input formula. The main idea underlying the algorithm is to substitute the repeated projection step of CAD by a single projection without carrying out a parametric existential decision over the reals. We find that the algorithm can tackle important and challenging problems, such as numerical stability analysis of the widely-used MacCormack's scheme. The problem has been practically out of reach for standard QE algorithms in spite of many attempts to tackle it. However the current implementation of VQE can solve it in about 12 hours. This paper extends the results reported at the conference ISSAC 2009.

*Key words:* Quantifier Elimination, Computational Real Algebraic Geometry, Stability Analysis

⋆ Hoon Hong's research was partially supported by USA NSF 0532140. Mohab Safey El Din's research was partially supported by the INRIA Program "Explorateurs" and the EXACTA grant from the French National Research Agency (ANR-09-BLAN-0371-01) and the National Science Foundation of China (NSFC) (NSFC 60911130369).

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## 1. Introduction

Real quantifier elimination (QE) is a fundamental problem in mathematical logic and computational real algebraic geometry. Furthermore, it naturally arises in many challenging problems in diverse application areas. Thus, there have been extensive research on developing mathematical theories, efficient algorithms, software systems, and applications: to cite only a few: (Tarski, 1951; Collins, 1975; McCallum, 1984; Grigoriev, 1988; Hong, 1990; Collins and Hong, 1991; Hong, 1992; Liska and Steinberg, 1993; Renegar, 1992; Basu et al., 1996; Sturm and Weispfenning, 1996; McCallum, 1999; Anai and Weispfenning, 2001; Brown, 2001; Strzebonski, 2006).

In this paper, we study a *variant* of the QE problem, obtained by strengthening the pre-condition and weakening the post-condition of the standard QE problem. Roughly speaking, we strengthen the pre-condition by requiring that the polynomials in the input formula satisfy certain natural geometric conditions (such as radicality, equidimensionality, smoothness, properness, etc). We weaken the post-condition by allowing that the input and the output are "almost" equivalent, unlike the standard QE where the input and the output are required to be exactly equivalent.

The motivation for studying a variant QE problem is that currently many important and challenging application problems are still practically out of reach for standard QE algorithms, in spite of tremendous progress made in their efficiency during last 30 years. We choose to strengthen the pre-condition because many important quantified formulas arising in real-life applications (for example, numerical stability analysis, control system design, etc) naturally satisfy the extra conditions. Furthermore, in most real-life applications, it is sufficient that the output formula is almost equivalent to the input formula.

We present an algorithm (VQE), that exploits the strengthened pre-condition and the weakened post-condition. The main idea underlying the algorithm is to substitute the repeated projection step of CAD by a single projection without carrying out a parametric existential decision over the reals.

We find that the algorithm VQE can tackle challenging problems such as stability analysis of the renowned MacCormack's scheme. The problem has been practically out of reach for standard QE algorithms implemented in Mathematica, SyNRAC or QEPCAD. However the current implementation of the algorithm VQE solves it in about 12 hours.

This paper extends the results reported at ISSAC 2009 (Hong and Safey El Din, 2009) in three aspects: (1) The paper provides a more general algorithm, widening the scope of applicability. The algorithm now allows free variables in polynomial equations and it also allows more than one polynomial inequality. This generalization required some modification of the algorithm and significant changes of the correctness proof. (2) The paper provides a bound on the degrees of crucial polynomials computed by the algorithm. (3) The paper also reports a few more challenging problems that have been successfully solved using the algorithm.

*Structure of the paper:* Section 2 provides a precise statement of the variant QE problem. Section 3 presents an algorithm VQE for the problem. Section 4 gives a proof for the algorithm's termination and correctness. Section 5 provides a bound on the degrees of the polynomials computed by the algorithm. Section 6 describes case studies where the algorithm is successfully applied to challenging problems arising from stability analysis.

## 2. Problem

In this section, we state the variant quantifier elimination problem precisely and illustrate it by a simple (toy) example.

**Notation 1.** Throughout the paper, we will use the following notations:

$X = (x_1, \ldots, x_m)$.

$Y = (y_1, \ldots, y_n)$.

$F = (f_1, \ldots, f_k) \subset \mathbb{Q}[X, Y]$.

$G = (g_1, \ldots, g_s) \subset \mathbb{Q}[X, Y]$.

$F = 0$ stands for $f_1 = 0 \wedge \cdots \wedge f_k = 0$.

$G > 0$ stands for $g_1 > 0 \wedge \cdots \wedge g_s > 0$.

$\mathsf{proj}$ stands for the canonical projection on the $X$-space: $\mathsf{proj}(x, y) = x$.

$\mathsf{solution}(\Omega) = \{p \in \mathbb{R}^m : \Omega(p) \text{ is true } \}$, where $\Omega$ is a (possibly quantified) boolean formula of polynomial equations/inequalities with $m$ free variables.

Now we are ready to state the variant quantifier elimination problem. As mentioned in the introduction, we strengthen the pre-condition ($\mathbf{S_1}$ and $\mathbf{S_2}$) and weaken the post-condition ($\mathbf{W_1}$ and $\mathbf{W_2}$).

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**Problem**: Variant Quantifier Elimination (VQE)

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*Input*: $\Psi$, a quantified formula of the form

$$\exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$$

such that

$\mathbf{S_1}$ : The ideal generated by $F$ is radical.
The complex variety of $F$ is equidimensional (co-dim $= p$) and smooth.

$\mathbf{S_2}$ : The restriction of $\mathsf{proj}$ to the real variety of $F$ is proper. [1]

*Output*: $\Phi$, a quantifier-free formula which is "almost" equivalent to $\Psi$, that is,

$\mathbf{W_1}$ : $\mathsf{solution}(\Psi) \supseteq \mathsf{solution}(\Phi)$

$\mathbf{W_2}$ : $\mathsf{solution}(\Psi) \setminus \mathsf{solution}(\Phi)$ is measure zero.

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**Remark 1.** We made several extensions to the result in (Hong and Safey El Din, 2009). One extension is that we now allow the free variables $X$ in the equations $F = 0$ and we also allow more than one polynomial inequality in $G > 0$. A careful reader would notice that we replaced the compactness condition with the properness condition. They essentially play the same role. Furthermore, we simplified the presentation by using existential quantification (instead universal one). We also simplified the condition on the output.

[1] We recall that the restriction of $\mathsf{proj}$ to $S \subset \mathbb{R}^{m+n}$ is called *proper* at a point $x \in \mathbb{R}^m$ if and only if there exists a closed ball $B$ centered at $x$ such that $\mathsf{proj}^{-1}(B) \cap S$ is compact. We say that the restriction of $\mathsf{proj}$ to $S$ is proper if and only if it is proper at any point $x \in \mathbb{R}^m$.

**Example 1.** We will illustrate the problem by a simple (toy) example. Non-trivial examples will be given later in the application section. We claim that the input and the output in the following example satisfy the conditions in the above problem statement.

*Input*: $\Psi$, the quantified formula

$$\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$$

where

$$\begin{aligned} X &= \{x\} \\ Y &= \{y_1, y_2\} \\ F &= \{y_1^2 + y_2^2 - 1\} \\ G &= \{y_1^2 x - (y_2 - 1)^2\} \end{aligned}$$

*Output*: $\Phi$, the quantifier-free formula

$$x > 0$$

To check the claim, let us take a look at the surfaces defined by the vanishing of the polynomials in $F$ and $G$ as shown in Figure 1. The cylinder is the vanishing set of $F$, the

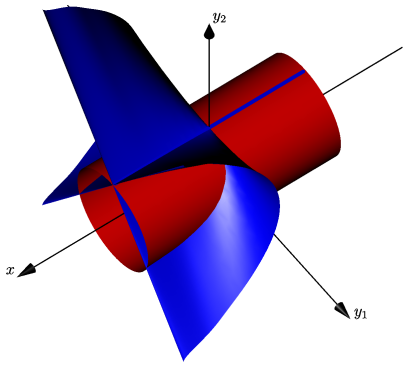


Fig. 1. Simple example

Whitney umbrella is that of $G$.

It is immediate that $\langle F \rangle$ is radical and that the complex variety defined by $F$ is equidimensional (co-dimension 1) and smooth. It is also immediate that the restriction of proj to the real variety defined by $F$ is proper. Thus $F$ satisfies the conditions ($\mathbf{S_1}$ and $\mathbf{S_2}$) in the problem statement.

It is also immediate from the drawings that the solution set of $\Psi$ is given by $x > 0$. Hence the output trivially satisfies the condition in the problem statement. □

## 3. Algorithm

We present an algorithm for the variant quantifier elimination problem. For the sake of simple presentation, we will freely use the notations introduced in the problem statement (Section 2). We will also consider (imagine) an object **A** which is initialized as an ordered list of all $s$-tuples of positive integers such that $(a_1, \ldots, a_s)$ appears before $(b_1, \ldots, b_s)$ if $\max_i a_i < \max_i b_i$.

**Algorithm**: $\Phi \leftarrow \mathrm{VQE}(\Psi)$

a. Remove the first element $(a_1, \ldots, a_s)$ from $\mathbf{A}$.

b. For each $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ with $0 \leq \ell \leq \min(s, n-p+1)$ do

   1. $P_J \leftarrow F \cup \{g_{j_1}, \ldots, g_{j_\ell}\}$
   2. $\Delta_J \leftarrow$ the set of all $(k+\ell)$-minors of the jacobian of $P_J$ w.r.t. $X \cup Y$
   3. If $F \cup \{g_{j_1} - a_{j_1}, \ldots, g_{j_\ell} - a_{j_\ell}\} \cup \Delta_J$ has a solution over $\mathbb{C}$ then go to Step (a).
   4. $\Delta_J^* \leftarrow$ the set of all $(k+\ell)$-minors of the jacobian of $P_J$ w.r.t. $Y$
   5. $P'_J \leftarrow F \cup \{a_{j_1} g_{j_2} - a_{j_2} g_{j_1}, \ldots, a_{j_1} g_{j_\ell} - a_{j_\ell} g_{j_1}\}$
   6. $S_J \leftarrow$ a set of generators of $\langle P'_J \cup \Delta_J^* \rangle : \langle \Delta_J \rangle^\infty$
   7. $e_J \leftarrow$ a non-zero element of $\langle S_J \cup \{g_{j_1}\} \rangle \cap \mathbb{Q}[X]$

c. $B \leftarrow$ the set of all $e_J$.

d. $\Phi \leftarrow \mathsf{Lift}(B, \Psi)$.

**Remark 2.** We will later prove that the boundary of the solution set of $\Psi$ is "captured" by the polynomials in the set $B$ of Step (c), that is,

$$\mathsf{boundary}(\mathsf{solution}(\Psi)) \quad \subseteq \quad W$$

where

$$W = \{x \in \mathbb{R}^m \mid b(x) = 0 \text{ for some } b \in B\}.$$

Therefore the polynomials in $B$ can be viewed as "projection" polynomials of $F$ and $G$. In that sense Step (b) plays a similar role as the projection step of the CAD algorithm or in algorithms based on the critical point method. However, unlike CAD, it carries out a *single* projection in Step b.7. Furthermore, it does not involve any computation with infinitesimals (even though the proof would utilize infinitesimals for the sake of simple presentation).

**Remark 3.** The subalgorithm Lift produces a quantifier free formula $\Phi$ which is almost equivalent to $\Psi$, by utilizing the projection polynomials in $B$. Typically, it begins by decomposing the set $\mathbb{R}^m \setminus W$, that is, by computing a set of quantifier-free formulas $\Phi_i$ such that the closure of each connected component of $\mathbb{R}^m \setminus W$ is equal to the closure of a union of sets defined by some of $\Phi_i$'s. It also samples a point $s_i$ from the set defined by each $\Phi_i$. Then, it sets $\Phi \leftarrow \bigvee_{\Psi(s_i) \text{ is true}} \Phi_i$. This paper does *not* make any contribution to this step. Hence, we encapsulate it into a subalgorithm, in order to hide irrelevant details.

**Remark 4.** We suggest a few implementational details.

Step b.3: This can be done by Gröbner bases algorithms (see e.g. (Faugère, 1999, 2002) and references therein), characteristic sets (see e.g. (Hubert, 2003) and references therein) or geometric resolutions (see e.g. (Giusti et al., 2001; Lecerf, 2003) and references therein).

Step b.6,7: The ideal theoretic operations (saturation and elimination) can be done by Gröbner bases (see e.g. (Greuel and Pfister, 2007)). Characteristic sets and geometric resolutions provide lazy algebraic representations, in the sense that they represent a Zariski-dense subset of the algebraic variety under study. In this framework, some specific techniques (see (Chen et al., 2009)) can be used to handle saturation and elimination.

Step d: This can be carried out by using open-CAD algorithm (Strzebonski, 2006). One can also use critical point methods (Safey El Din, 2007b; Faugère et al., 2008) and roadmap algorithms (Canny, 1993; Basu et al., 1999; Safey El Din and Schost, 2011) and their parameterized versions to compute semi-algebraic descriptions of the set defined by $B \neq 0$ and sample points in each of its connected components. For the moment, open-CAD seems to be the best practical choice to describe the connected components of $\mathbb{R}^m \setminus W$.

**Example 2.** We illustrate the algorithm on the toy example from Section 2. Recall that

$$\begin{aligned} X &= \{x\} \\ Y &= \{y_1, y_2\} \\ F &= \{y_1^2 + y_2^2 - 1\} \\ G &= \{y_1^2 x - (y_2 - 1)^2\} \end{aligned}$$

Note that (1) is the first element in $\mathbf{A}$. When we enter in the loop with $J = \emptyset$, we get $e_J = \{1\}$ since, when $J = \emptyset$, $\Delta_J = \Delta_J^\star$ on this example. When we enter in the loop with $J = \{1\}$, the following computations are performed:

1. $P_J \leftarrow F \cup \{g_1\}$
2. We compute the set of all $1+1$-minors of the jacobian of $F \cup \{g_1\}$ with respect to $X \cup Y$, obtaining
$$\Delta_J = \left\{-4\, y_1 \left(y_2 - 1 + y_2 x\right), \quad 2\, {y_1}^3, \quad 2\, y_2 {y_1}^2\right\}$$
3. We check easily that $\langle F \cup \{g_1 - 1\} \cup \Delta_J \rangle = \langle 1 \rangle$
4. We compute the set of all $1+1$-minors of the jacobian of $F \cup \{g_1\}$ with respect to $Y$, obtaining
$$\Delta_J^\star = \{-4\, y_1 \left(y_2 - 1 + y_2 x\right)\}$$
5. The set $P'_J$ is $F$
6. We compute a set of generators of $\langle F \cup \Delta_J^\star \rangle : \langle \Delta_J \rangle^\infty$, obtaining
$$S_J = \left\{{y_1}^2 + {y_2}^2 - 1, \quad y_2 - 1 + y_2 x\right\}$$
7. We compute a set of generators of $\langle S_J \cup \{g_1\} \rangle \cap \mathbb{Q}[X]$, obtaining
$$e_J = \left\{x^2\right\}$$

Finally, a call to Lift returns

$$\Phi \equiv x > 0$$

**Comparison to CAD:** It is instructive to observe how CAD would handle the problem. Note that $F = 0$ can be viewed as "equational constraint". Hence we use the improved version of CAD that utilizes equational constraints (Collins, 1998; McCallum, 1999). The first projection with respect to $y_2$ produces the polynomials:

$$\{\, y_1 \left(-4\, x + {y_1}^2 (x+1)^2\right),\quad 4\,(y_1 - 1)\,(y_1 + 1)\,\}$$

which are the square-free part of the resultant of $f_1$ and $g_1$ and the square-free part of the resultant of $f_1$ and $\frac{\partial f_1}{\partial y_2}$. The second projection w.r.t. $y_1$ produces the polynomials

$$\{\, x,\quad x - 1,\quad x + 1\,\}$$

The lifting phase will eventually produce, using the projection polynomials and sample point checks, a quantifier-free formula $x > 0$.

It is *crucial* to note that the projection polynomials

$$x - 1,\quad x + 1$$

are *irrelevant* to the quantifier elimination problem. They induce *useless* cells, causing inefficiency. In comparison, the VQE algorithm does not produce the irrelevant polynomials.

We explain why this happens geometrically (see Figure 2). The CAD algorithm, among

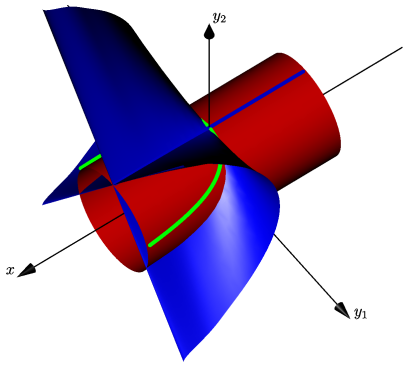


Fig. 2. Simple example continued

others, projects the intersection of the red cylinder ($F$) and the blue Whitney umbrella ($G$), which is complicated. On the other hand, the VQE algorithm projects the intersection of the green curve ($S_J$ in Step 6) and the blue Whitney umbrella ($G$), which is much simpler. This kind of advantage becomes much more pronounced for larger problems, yielding significant improvement in computing time.

## 4. Proof for Termination and Correctness

In this section, we prove the termination and the correctness of the VQE algorithm. The proof is long and hence we divide it into several lemmas (which could be interesting on their own) and two theorems (one for termination and the other for correctness). We present the lemmas and the theorems in the bottom-up order. If the readers prefer to get the overall structure of the proof first, then we suggest that the reader reads this section in the backward order, starting from Theorems 12 and 13.

We begin by fixing terminology and notations for basic concepts on infinitesimals and critical points.

*Preliminaries on infinitesimals:* Let $\mathbb{J}$ be a field containing $\mathbb{Q}$ (e.g. $\mathbb{R}$ or $\mathbb{C}$). Let $\varepsilon$ be an infinitesimal and let $\mathbb{J}\langle\varepsilon\rangle$ stand for the Puiseux series field. We say that $z = \sum_{i\geq i_0} a_i\varepsilon^{i/q} \in \mathbb{J}\langle\varepsilon\rangle$ is *bounded over* $\mathbb{J}$ if and only if $i_0 \geq 0$. We say that $z = (z_1, \ldots, z_n) \in \mathbb{J}\langle\varepsilon\rangle^n$ is *bounded over* $\mathbb{J}$ if each $z_i$ is bounded over $\mathbb{J}$. Given a bounded element $z \in \mathbb{J}\langle\varepsilon\rangle$, we denote by $\lim_{\varepsilon\to 0} z$ the number $a_0 \in \mathbb{J}$. Given a bounded element $z \in \mathbb{J}\langle\varepsilon\rangle^n$, we denote by $\lim_{\varepsilon\to 0} z$ the point $(\lim_{\varepsilon\to 0}(z_1), \ldots, \lim_{\varepsilon\to 0}(z_n)) \in \mathbb{J}^n$. Given a subset $A \subset \mathbb{J}\langle\varepsilon\rangle^n$, we denote by $\lim_{\varepsilon\to 0}(A)$ the set $\{\lim_{\varepsilon\to 0}(z) \mid z \in A \text{ and } z \text{ is bounded}\}$. Given a semi-algebraic (resp. constructible) set $A \subset \mathbb{R}^n$ (resp. $A \subset \mathbb{C}^n$) defined by a quantifier-free formula $\Phi$ with polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, we denote by $\mathrm{Ext}(A, \mathbb{R}\langle\varepsilon\rangle)$ (resp. $\mathrm{Ext}(A, \mathbb{C}\langle\varepsilon\rangle)$) the set of solutions of $\Phi$ in $\mathbb{R}\langle\varepsilon\rangle^n$ (resp. $\mathbb{C}\langle\varepsilon\rangle$).

*Preliminaries on critical points:* Let $\bar{\mathbb{J}}$ stand for an algebraic closure of $\mathbb{J}$. Let $V \subset \bar{\mathbb{J}}^n$ be an equidimensional algebraic variety. The set of regular points of $V$ is denoted by $\mathsf{reg}(V)$ and the set of singular points is denoted by $\mathsf{sing}(V)$. Given $x \in V$, the tangent space to $V$ at $x$ is denoted by $T_xV$. Let $\varphi$ be a polynomial mapping $V \to \bar{\mathbb{J}}^m$. The differential of $\varphi$ at $x \in \mathsf{reg}(V)$ is denoted by $d_x\varphi$. A point $x \in \mathsf{reg}(V)$ is a *critical point* of $\varphi$ if and only if $d_x\varphi(T_xV) \neq \bar{\mathbb{J}}^m$; we denote by $\mathsf{crit}(\varphi, V)$ the union of $\mathsf{sing}(V)$ and the set of all critical points of $\varphi$. A *critical value* of $\varphi$ is the image by $\varphi$ of a critical point. We denote by $\mathcal{D}(\varphi, V)$ the set of critical values of $\varphi$. A *regular value* is a point of $\bar{\mathbb{J}}^m$ which is a not a critical value.

*Notations:* Let $\mathbb{J}$ be a real field and $S$ be a semi-algebraic set in $\mathbb{J}^n$. We denote by $\mathsf{boundary}(S)$ the boundary of $S$ and by $\mathsf{int}(S)$ its interior (for the euclidean topology). Given a point $x \in \mathbb{J}^n$ and $r \in \mathbb{J}$ positive, $\mathsf{ball}(x, r)$ denotes the ball centered at $x$ of radius $r$. Given $(f_1, \ldots, f_k) \subset \mathbb{Q}[Z]$, $\mathsf{jacobian}_Z(f_1, \ldots, f_k)$ denotes the jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial Z_1} & \cdots & \frac{\partial f_1}{\partial Z_r} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial Z_1} & \cdots & \frac{\partial f_k}{\partial Z_r} \end{bmatrix}.$$

**Lemma 1.** Let $S$ be a connected component of the set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 > 0, \ldots, g_s > 0.$$

Let $x \in \mathsf{boundary}(\mathsf{proj}(S))$. For all $r > 0$, there exists $e_0 > 0$ such that for all $e \in ]0, e_0[$ there exists a connected component $S_e$ of the set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 \geq e, \ldots, g_s \geq e$$

such that $\mathsf{ball}(x, r) \cap \mathsf{boundary}(\mathsf{proj}(S_e))$ is non-empty.

*Proof.* Let $r > 0$. Since $x \in \mathsf{boundary}(\mathsf{proj}(S))$, we see that $\mathsf{ball}(x,r) \cap \mathsf{proj}(S)$ is non-empty. Let $(x',y') \in S$ such that $x' \in \mathsf{ball}(x,r) \cap \mathsf{proj}(S)$ and $e_0 > 0$ be less than $\min(g_1(x',y'),\ldots,g_s(x',y'))$. Then, for all $e \in ]0,e_0[$, we see that $(x',y')$ is in the set $T_e$ defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 \geq e, \ldots, g_s \geq e.$$

Hence there exists a connected component $S_e$ of $T_e$ such that $\mathsf{ball}(x,r) \cap \mathsf{proj}(S_e)$ is non-empty.

Consider $e \in ]0,e_0[$. We prove that $\mathsf{ball}(x,r) \cap \mathsf{boundary}(\mathsf{proj}(S_e))$ is not empty by distinguishing the cases where $x \notin \mathsf{proj}(S_e)$ and $x \in \mathsf{proj}(S_e)$.

Suppose that $x \notin \mathsf{proj}(S_e)$. Let $C_e$ be a connected component of $\mathsf{int}(\mathsf{ball}(x,r)) \cap \mathsf{proj}(S_e)$ and $x' \in C_e$. Then, any semi-algebraic path $\gamma \subset \mathsf{int}(\mathsf{ball}(x,r))$ linking $x'$ to $x$ meets $\mathsf{int}(\mathsf{ball}(x,r)) \cap \mathsf{boundary}(C_e)$. Since $C_e$ is a connected component of $\mathsf{int}(\mathsf{ball}(x,r)) \cap \mathsf{proj}(S_e)$, its boundary is contained in $\mathsf{boundary}(\mathsf{ball}(x,r)) \cup \mathsf{boundary}(\mathsf{proj}(S_e))$. Then $\gamma$ has a non-empty intersection with $\mathsf{boundary}(\mathsf{proj}(S_e))$ since it meets $\mathsf{boundary}(C_e)$ at a point in $\mathsf{int}(\mathsf{ball}(x,r))$. We conclude that $\mathsf{ball}(x,r) \cap \mathsf{boundary}(\mathsf{proj}(S_e))$ is non-empty.

Suppose that $x \in \mathsf{proj}(S_e)$. Since $S_e \subset S$, $\mathsf{proj}(S_e) \subset \mathsf{proj}(S)$ and $\mathsf{int}(\mathsf{proj}(S_e)) \subset \mathsf{int}(\mathsf{proj}(S))$. By assumption, $x \in \mathsf{boundary}(\mathsf{proj}(S))$, hence $x \notin \mathsf{int}(\mathsf{proj}(S))$ which implies that $x \notin \mathsf{int}(\mathsf{proj}(S_e))$. We deduce that $x \in \mathsf{proj}(S_e) - \mathsf{int}(\mathsf{proj}(S_e))$ which is contained in $\mathsf{boundary}(\mathsf{proj}(S_e))$. We conclude that $\mathsf{ball}(x,r) \cap \mathsf{boundary}(\mathsf{proj}(S_e))$ is non-empty. □

**Lemma 2.** Let $S_e$ be a connected component of the set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 \geq e, \ldots, g_s \geq e.$$

Let $x \in \mathsf{boundary}(\mathsf{proj}(S_e))$. There exists $\{j_1,\ldots,j_\ell\} \subset \{1,\ldots,s\}$ and a connected component $C_e$ of the real algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_{j_1} = \cdots = g_{j_\ell} = e$$

such that $x \in \mathsf{boundary}(\mathsf{proj}(C_e))$.

*Proof.* Denote by $\{j_1 \ldots, j_\ell\}$ a subset of $\{1,\ldots,s\}$ such that there exists a connected component $C_e$ of the real algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_{j_1} = \cdots = g_{j_\ell} = e$$

meeting the following condition: for all $r > 0$, $\mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap C_e$ is non-empty and contained in $\mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap S_e$. One concludes immediately that $x$ belongs to the closure (for the euclidean topology) of $\mathsf{proj}(C_e)$. In order to prove that $x \in \mathsf{boundary}(\mathsf{proj}(C_e))$, we prove below that there exists $r > 0$ such that $x$ does not belong to $\mathsf{ball}(x,r) \cap \mathsf{int}(\mathsf{proj}(C_e))$.

By construction, there exists $r > 0$ such that $\mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap C_e$ is not empty and contained in $\mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap S_e$. Let $x' \in \mathsf{ball}(x,r) \cap \mathsf{proj}(C_e)$. Then, there exists $y'$ such that $(x',y') \in \mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap C_e \subset \mathsf{proj}^{-1}(\mathsf{ball}(x,r)) \cap S_e$. Consequently, $x' \in \mathsf{ball}(x,r) \cap \mathsf{proj}(S_e)$. We deduce that $\mathsf{ball}(x,r) \cap \mathsf{proj}(C_e) \subset \mathsf{ball}(x,r) \cap \mathsf{proj}(S_e)$.

Moreover, by assumption, $x \in \mathsf{boundary}(\mathsf{proj}(S_e))$ which implies $x \notin \mathsf{int}(\mathsf{proj}(S_e))$, $x \notin \mathsf{ball}(x,r) \cap \mathsf{int}(\mathsf{proj}(S_e))$ and $x \notin \mathsf{int}(\mathsf{proj}(C_e)) \cap \mathsf{ball}(x,r)$ since $\mathsf{ball}(x,r) \cap \mathsf{proj}(C_e) \subset \mathsf{ball}(x,r) \cap \mathsf{proj}(S_e)$ implies $\mathsf{ball}(x,r) \cap \mathsf{int}(\mathsf{proj}(C_e)) \subset \mathsf{ball}(x,r) \cap \mathsf{int}(\mathsf{proj}(S_e))$. □

**Lemma 3.** Let $S$ be a connected component of the set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 > 0, \ldots, g_s > 0.$$

Let $x \in \mathsf{boundary}(\mathsf{proj}(S))$. For all $r > 0$ there exists $e_0 > 0$ and $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ such that for all $e \in ]0, e_0[$ there exists a connected component $C_e$ of the real algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_{j_1} = \cdots = g_{j_\ell} = e$$

such that $\mathsf{ball}(x, r) \cap \mathsf{boundary}(\mathsf{proj}(C_e))$ is non-empty.

*Proof.* Immediate from Lemma 1 and 2. □

**Lemma 4.** Let $\mathsf{R}$ be a real closed field and $\mathsf{C}$ be its algebraic closure. Consider a smooth algebraic variety $V \subset \mathsf{C}^{m+n}$ and a semi-algebraically connected component $C$ be of $V \cap \mathsf{R}^{m+n}$. Suppose that the restriction of $\mathsf{proj}$ to $V$ is proper. Then, $\mathsf{boundary}(\mathsf{proj}(C))$ is contained in $\mathsf{proj}(\mathsf{crit}(\mathsf{proj}, V))$.

*Proof.* This lemma is similar to (Safey El Din and Schost, 2003, Proposition 4). Let $x \in \mathsf{boundary}(\mathsf{proj}(C))$. Suppose that $x \notin \mathsf{proj}(\mathsf{crit}(\mathsf{proj}, V))$. Since the restriction of $\mathsf{proj}$ to $V \cap \mathsf{R}^{m+n}$ is proper, by the semi-algebraic Ehresmann's theorem (Coste and Shiota, 1992, Theorem 3.4), there exists a neighborhood $U$ of $x \in \mathsf{R}^m$ such that the restriction of $\mathsf{proj}$ to $V \cap \mathsf{R}^{m+n}$ realizes a locally trivial fibration over $\mathsf{proj}^{-1}(U) \cap V \cap \mathsf{R}^{m+n}$. In particular for all $(x', x'') \in U \times U$, $\mathsf{proj}^{-1}(x') \cap V \cap \mathsf{R}^{m+n}$ is diffeomorphic to $\mathsf{proj}^{-1}(x'') \cap V \cap \mathsf{R}^{m+n}$. In particular, for all $x' \in U$, $\mathsf{proj}^{-1}(x') \cap V \cap \mathsf{R}^{m+n}$ is empty if and only if $\mathsf{proj}^{-1}(x) \cap V \cap \mathsf{R}^{m+n}$ is empty. This contradicts that $x \in \mathsf{boundary}(\mathsf{proj}(C))$. □

From now on, let $S \subset \mathbb{R}^{m+n}$ be a connected component of the semi-algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 > 0, \ldots, g_s > 0$$

with $(f_1, \ldots, f_k, g_1, \ldots, g_s) \subset \mathbb{Q}[X, Y]$ satisfying the assumptions $\mathbf{S_1}$ and $\mathbf{S_2}$.

Consider $\mathbf{a} = (a_1, \ldots, a_s) \in {\mathbb{Z}_+^\star}^s$. Remark that $S$ is still a connected component of the set defined by

$$f_1 = \cdots = f_k = 0, \quad \frac{g_1}{a_1} > 0, \ldots, \frac{g_s}{a_s} > 0.$$

Let $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$. We denote by $V_{J,\varepsilon}^{\mathbf{a}} \subset \mathbb{C}\langle\varepsilon\rangle^{m+n}$ the algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0.$$

**Lemma 5.** Let $x \in \mathsf{boundary}(\mathsf{proj}(S))$ and $\mathbf{a} = (a_1, \ldots, a_s) \in {\mathbb{Z}_+^\star}^s$. Then, there exists $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ such that $x \in \mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V_{J,\varepsilon}^{\mathbf{a}})))$.

*Proof.* By the transfer principle (see (Basu et al., 2006, Chapter 2)), the statement of Lemma 3 applied to the system

$$f_1 = \cdots = f_k = 0, \quad g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0$$

can be rephrased as follows: there exists $J \subset \{1, \ldots, s\}$ such that

$$x \in \lim_{\varepsilon \to 0} \mathsf{boundary}(\mathsf{proj}(C_\varepsilon))$$

where $C_\varepsilon$ is a semi-algebraically connected component of $V_{J,\varepsilon}^{\mathbf{a}} \cap \mathbb{R}\langle\varepsilon\rangle^{m+n}$.

Denote by $V \subset \mathbb{C}^{m+n}$ the algebraic variety defined by $f_1 = \cdots = f_k = 0$ and by $B_\varepsilon$ the set $\mathrm{Ext}(\mathsf{proj}^{-1}(\mathsf{ball}(x, r)), \mathbb{R}\langle\varepsilon\rangle)$. Since the restriction of $\mathsf{proj}$ to $V$ is proper (assumption $\mathbf{S_2}$) and $B_\varepsilon \cap C_\varepsilon \subset B_\varepsilon \cap \mathrm{Ext}(V, \mathbb{R}\langle\varepsilon\rangle)$, Lemma 4 implies that $\mathsf{boundary}(\mathsf{proj}(C_\varepsilon)) \subset \mathsf{proj}(\mathsf{crit}(\mathsf{proj}, V_{J,\varepsilon}^{\mathbf{a}}))$. Thus, $x \in \lim_{\varepsilon \to 0}(\mathsf{proj}(\mathsf{crit}(\mathsf{proj}, V_{J,\varepsilon}^{\mathbf{a}})))$.

Still using assumption $\mathbf{S_2}$, there exists $r > 0$ such that $B_\varepsilon \cap \mathsf{crit}(\mathsf{proj}, V^a_{J,\varepsilon})$ is bounded over $\mathbb{R}$ and let $A$ be a semi-algebraically connected component of $B_\varepsilon \cap \mathsf{crit}(\mathsf{proj}, V^a_{J,\varepsilon})$ such that $x$ belongs to $\lim_{\varepsilon\to 0} \mathsf{proj}(A)$. Since $A$ is semi-algebraically connected and bounded, (Basu et al., 2006, Proposition 12.43) implies that $\lim_{\varepsilon\to 0}(A)$ exists and is semi-algebraically connected, closed and bounded. Thus $\mathsf{proj}(\lim_{\varepsilon\to 0}(A))$ is closed (see (Basu et al., 2006, Theorem 3.20)) and contains $x$ since $x \in \lim_{\varepsilon\to 0}(\mathsf{proj}(A))$ (see (Basu et al., 2006, Lemma 3.21)). Now, remark that $A \subset \mathsf{crit}(\mathsf{proj}, V^a_{J,\varepsilon})$ implies $\lim_{\varepsilon\to 0}(A) \subset \lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj}, V^a_{J,\varepsilon}))$. This implies that

$$x \in \lim_{\varepsilon\to 0}(\mathsf{proj}(A)) = \mathsf{proj}(\lim_{\varepsilon\to 0}(A)) \subset \mathsf{proj}(\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj}, V^a_{J,\varepsilon}))).$$

□

**Remark 5.** Note that in the above lemma, the set $J$ may depend on $\mathbf{a}$.

**Lemma 6.** Let $\mathbb{J}$ be a field containing $\mathbb{Q}$. Let $H = \{h_1, \ldots, h_r\} \subset \mathbb{J}[X]$, and let $V$ be the algebraic variety defined by $H = 0$. Let $\varphi : x \in V \to (\varphi_1(x), \ldots, \varphi_s(x))$ be a polynomial mapping. Suppose that $\langle H\rangle$ is radical and that $V$ is smooth and equidimensional with co-dimension $k$ and that $s \leq n - k$. Let $\Delta$ be the set of $(k + s)$-minors of $\mathsf{jacobian}_X(H, \varphi_1, \ldots, \varphi_s)$. Then $\mathsf{crit}(\varphi, V)$ is the algebraic variety associated to $\langle H\rangle + \langle\Delta\rangle$.

*Proof.* Well known. □

**Lemma 7.** Let $\{h_1, \ldots, h_r\} \subset \mathbb{J}[X]$ such that the algebraic variety $V \subset \bar{\mathbb{J}}^n$ defined by $h_1 = \cdots = h_r = 0$ is equidimensional of co-dimension $p$ and for all $x \in V(h_1, \ldots, h_r)$, the rank of $\mathsf{jacobian}_X(h_1, \ldots, h_r)(x)$ is $p$. Then, the ideal $\langle h_1, \ldots, h_r\rangle$ is radical.

*Proof.* Denote by $I$ the ideal $\langle h_1, \ldots, h_r\rangle$. Consider an irredundant primary decomposition $Q_1, \ldots, Q_\ell$ of $I$ so that the prime ideals associated to the $Q_i$'s are pairwise distinct. We prove below that each isolated component $Q_i$ is prime, which will imply that $I$ is radical.

For $1 \leq i \leq \ell$, consider an isolated primary component $Q_i$ of $I$. Since $V(Q_i)$ is an irreducible component of $V$ which is equidimensional of co-dimension $p$ we deduce the $Q_i$ has co-dimension $p$.

Since $Q_i$ is isolated, one can choose $x \in V(Q_i)$ such that $x \notin V(Q_j)$ for $j \neq i$. Let $\mathfrak{m}$ be the maximal ideal at $x$. Supposing that $I_\mathfrak{m} = Q_{i_\mathfrak{m}}$ and $I_\mathfrak{m}$ is prime, $Q_{i_\mathfrak{m}}$ is prime which implies that $Q_i$ itself is prime by (Atiyah and MacDonald, 1969, Proposition 3.11 (iv)).

We prove now that $I_\mathfrak{m} = Q_{i_\mathfrak{m}}$ and $I_\mathfrak{m}$ is prime. By (Atiyah and MacDonald, 1969, Proposition 4.9), $I_\mathfrak{m} = Q_{1_\mathfrak{m}} \cap \cdots \cap Q_{s_\mathfrak{m}}$. Since $Q_i$ is the unique primary ideal of the considered minimal primary decomposition of $I$ such that $x \in V(Q_i)$, $Q_i$ is the unique isolated ideal of that decomposition which is contained in $\mathfrak{m}$. Thus, $I_\mathfrak{m} = Q_{i_\mathfrak{m}}$.

Since $\mathsf{jacobian}_X(h_1, \ldots, h_r)(x)$ has rank $p$, Part $b$ of (Eisenbud, 1995, Theorem 16.19) shows that the local ring $\bar{\mathbb{J}}[X]_\mathfrak{m}/I_\mathfrak{m}$ is regular and hence an integral ring, so that $I_\mathfrak{m}$ is prime. □

**Lemma 8.** Let $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ and $\Delta_J$ be the set of $p + \ell$ minors of $\mathsf{jacobian}_{(X,Y)}(f_1, \ldots, f_k, g_{j_1}, \ldots, g_{j_\ell})$. There exists a non-empty Zariski-open subset $O_J \subset \mathbb{C}^\ell$ such that for all $\mathbf{a} = (a_{j_1}, \ldots, a_{j_\ell}) \in O_J$ the ideal

$$\langle f_1, \ldots, f_k, g_{j_1} - a_{j_1}, \cdots, g_{j_\ell} - a_{j_\ell}\rangle \subset \mathbb{Q}[X, Y]$$

is equidimensional, radical and defines either an empty set or a smooth algebraic variety of co-dimension $p+\ell$ which has an empty intersection with the variety defined by $\Delta_J$.

*Proof.* Denote by $\mathcal{I}$ the set of subsets of $\{1,\ldots,k\}$ having cardinality $p$. For $I = \{i_1,\ldots,i_p\} \in \mathcal{I}$, consider the constructible set $V_I$ defined as

$$\{(x,y) \mid f_1(x,y) = \cdots = f_k(x,y) = 0 \text{ and } \mathsf{rank}(\mathsf{jacobian}_{(X,Y)}(f_{i_1},\ldots,f_{i_p})(x,y)) = p\}.$$

Remark that $V(f_1,\ldots,f_k) = \cup_{I\in\mathcal{I}} V_I$ and that each $V_I$ is a smooth constructible set of co-dimension $p$. Let $\varphi_{IJ}$ be the polynomial mapping

$$(x,y) \in V_I \to (g_{j_1}(x,y),\ldots,g_{j_\ell}(x,y)) \in \mathbb{C}^\ell.$$

If the constructible set $\varphi_{IJ}$ is not dense in $\mathbb{C}^\ell$, denote by $O_{IJ}$ the complementary of its Zariski-closure. Then for $\mathbf{a}$ outside its Zariski-closure, $\varphi_{IJ}^{-1}(\mathbf{a})$ is empty.

If the constructible set $\varphi_{IJ}(V_I)$ is dense in $\mathbb{C}^\ell$. Then, by Sard's theorem (see (Shafarevich, 1977, Theorem 2, and Lemmas 1 and 2, Chapter 6)), there exists a non-empty Zariski-open subset $O_{IJ}$ such that for all $\mathbf{a} \in O_{IJ}$ and for all $(x,y) \in \varphi_{IJ}^{-1}(\mathbf{a})$, $\varphi_{IJ}^{-1}(\mathbf{a})$ is smooth and $d_{(x,y)}\varphi_{IJ}$ is surjective. In particular, this implies that for all $\mathbf{a} = (a_{j_1},\ldots,a_{j_\ell}) \in O_{IJ}$ and $(x,y) \in \varphi_{IJ}^{-1}(\mathbf{a})$,

$$\mathsf{rank}(\mathsf{jacobian}_{(X,Y)}(f_{i_1},\ldots,f_{i_p},g_{j_1},\ldots,g_{j_\ell})(x,y)) = p+\ell.$$

This implies that for all $(x,y)$ in $\varphi_{IJ}^{-1}(\mathbf{a})$, the rank of

$$\mathsf{jacobian}_{(X,Y)}(f_{i_1},\ldots,f_{i_p},g_{j_1},\ldots,g_{j_\ell})(x,y)$$

is $p+\ell$. Thus, $\varphi_{IJ}^{-1}(\mathbf{a})$ has co-dimension at least $p+\ell$ at all $(x,y) \in \varphi_{IJ}^{-1}(\mathbf{a})$. Since $V_I$ has dimension $p$ and its Zariski-closure is equidimensional, $\varphi_{IJ}^{-1}(\mathbf{a})$ has co-dimension less than or equalled to $p+\ell$. We conclude that $\varphi_{IJ}^{-1}(\mathbf{a})$ has co-dimension $p+\ell$ and its Zariski-closure is equidimensional and for all $(x,y) \in \varphi_{IJ}^{-1}(\mathbf{a})$

$$\mathsf{jacobian}_{(X,Y)}(f_{i_1},\ldots,f_{i_p},g_{j_1},\ldots,g_{j_\ell})(x,y)$$

has rank $p+\ell$.

By choosing $O_J = \cap_{I\in\mathcal{I}} O_{IJ}$ and since $V = \cup_{I\in\mathcal{I}} V_I$, $(a_{j_1},\ldots,a_{j_\ell}) \notin O_J$ implies that the algebraic variety defined by $f_1 = \cdots = f_k = g_{j_1} - a_{j_1} = \cdots = g_{j_\ell} - a_{j_\ell} = 0$ is either empty, or smooth equidimensional of co-dimension $p+\ell$ and it has an empty intersection with the variety defined by $\Delta_J$.

Moreover, the ideal $\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1},\ldots,g_{j_\ell}-a_{j_\ell}\rangle$ is radical by Lemma 7. □

**Lemma 9.** Let $J = \{j_1,\ldots,j_\ell\} \subset \{1,\ldots,s\}$ and $\mathbf{a} = (a_{j_1},\ldots,a_{j_\ell}) \in O_J$. Then, the ideal

$$\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon\rangle \subset \mathbb{Q}(\varepsilon)[X,Y]$$

is radical and defines either an empty set or a smooth equidimensional algebraic variety $V_{J,\varepsilon}^{\mathbf{a}}$ of co-dimension $p+\ell$ in $\mathbb{C}\langle\varepsilon\rangle^{m+n}$.

*Proof.* Let $\mathbf{a} = (a_{j_1},\ldots,a_{j_\ell}) \in O_J$ and consider the line $L_{\mathbf{a}}$ containing the origin and $\mathbf{a}$. Since $O_J$ is a non-empty Zariski open set, the intersection of $L_{\mathbf{a}} \subset \mathbb{C}^\ell$ with the complementary of $O_J$ is a finite set of points in $\mathbb{C}^\ell$.

Hence, the point $\mathbf{a}\varepsilon = (a_{j_1}\varepsilon,\ldots,a_{j_\ell}\varepsilon)$ belongs to $\mathrm{Ext}(O_J,\mathbb{C}\langle\varepsilon\rangle)$. This implies that for all $I = \{i_1,\ldots,i_p\} \subset \{1,\ldots,k\}$ and all $(x,y)$ in the variety $V_{J,\varepsilon}^{\mathbf{a}}$ defined by

$$f_1 = \cdots = f_k = g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0$$

the rank of $\mathsf{jacobian}_{(X,Y)}(f_{i_1},\ldots,f_{i_p},g_{j_1},\ldots,g_{j_\ell})(x,y)$ is $p+\ell$. Thus, following the same argumentation used in the proof of Lemma 8, we conclude that $V^{\mathbf{a}}_{J,\varepsilon}$ has co-dimension $p+\ell$.

Moreover, the ideal $\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon\rangle$ is radical by Lemma 7. $\square$

**Lemma 10.** We use the notations of Algorithm VQE. Let $J=\{j_1,\ldots,j_\ell\}\subset\{1,\ldots,s\}$ and $\mathbf{a}=(a_{j_1},\ldots,a_{j_\ell})\in O_J$. Then, the algebraic variety defined by $\langle S_J\cup\{g_{j_1}\}\rangle$ equals $\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$;

*Proof.* We suppose in the sequel that $(a_{j_1},\ldots,a_{j_\ell})$ belongs the non-empty Zariski-open set $O_J$ defined in Lemma 8. This implies that $V^{\mathbf{a}}_{J,\varepsilon}$ is smooth, equidimensional of co-dimension $p+\ell$ and the ideal $\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon\rangle\subset\mathbb{C}(\varepsilon)[X,Y]$ is radical.

We start by proving that the algebraic variety associated to $\langle S_J\rangle+\langle g_{j_1}\rangle$ is contained in $\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$. Consider an element $\mathbf{z}$ of the algebraic variety associated to $\langle S_J\rangle+\langle g_{j_1}\rangle$ and denote by $Z$ an irreducible component of this variety containing $\mathbf{z}$. Given $r>0$, denote by $\mathsf{ball}(\mathbf{z},r)\subset\mathbb{C}^{n+k}$ the ball centered at $\mathbf{z}$ of radius $r$. We prove that for all $r>0$, $\mathrm{Ext}(Z\cap B(\mathbf{z},r),\mathbb{C}\langle\varepsilon\rangle^{n+k})$ has a non-empty intersection with $\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon})$ which implies that $\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$ contains $\mathbf{z}$.

Since $\langle S_J\rangle=(\langle f_1,\ldots,f_k,a_{j_1}g_{j_2}-a_{j_1}g_{j_1},\ldots,a_{j_1}g_{j_\ell}-a_{j_\ell}g_{j_1}\rangle+\langle\Delta_J^*\rangle):\langle\Delta_J\rangle^\infty$, $Z$ contains points such that $\mathsf{jacobian}_{X,Y}(f_1,\ldots,f_k,g_{j_1},\ldots,g_{j_\ell})$ has rank $p+\ell$. Denote by $\mathfrak{S}$ the algebraic variety associated to $\langle\Delta_J\rangle$. Then, $Z\setminus\mathfrak{S}$ is not empty. Moreover, $\{t\in\mathbb{C}\mid\exists\mathbf{z}'\in Z,\ \frac{g_{j_1}}{a_{j_1}}(\mathbf{z}')=t\}$ has dimension 1 which implies that $Z\setminus\mathfrak{S}$ can not have dimension 0. Thus, for all $r>0$, $\mathsf{ball}(\mathbf{z},r)\cap Z\setminus\mathfrak{S}$ is positive dimensional and it is connected for $r$ small enough. Remark now that $g_{j_1}(\mathbf{z})=0$. Thus, from the intermediate value theorem, there exists $\mathbf{z}'\in\mathrm{Ext}((Z\setminus\mathfrak{S})\cap B(\mathbf{z},r),\mathbb{C}\langle\varepsilon\rangle^{m+n})$ such that $\frac{g_{j_1}}{a_{j_1}}(\mathbf{z}')=\varepsilon$. To summarize, we have $\frac{g_{j_1}}{a_{j_1}}(\mathbf{z}')=\varepsilon$ and $\mathbf{z}'\in V(\Delta_J^*)$ and

$$f_1(\mathbf{z}')=\cdots=f_k(\mathbf{z}')=0\text{ and }a_{j_1}g_{j_2}(\mathbf{z}')-a_{j_2}g_{j_1}(\mathbf{z}')=\cdots=a_{j_1}g_{j_\ell}(\mathbf{z}')-a_{j_\ell}g_{j_1}(\mathbf{z}')=0.$$

From Lemma 6, we conclude that $\mathbf{z}'\in\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon})$.

Now, we prove that $\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$ is contained in the intersection of the algebraic variety $V(S_J)$ associated to $\langle S_J\rangle$ and the hypersurface defined by $g_{j_1}=0$. Remember that $V(S_J)$ is the Zariski-closure of $V(f_1,\ldots,f_k,a_{j_1}g_{j_2}-a_{j_1}g_{j_1},\ldots,a_{j_1}g_{j_\ell}-a_{j_\ell}g_{j_1},\Delta_J^*)-V(\Delta)$.

Let $\mathbf{z}\in\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$. By continuity of $g_{j_1}$, this implies that $g_{j_1}(\mathbf{z})=0$. Thus, it remains to prove that $\mathbf{z}$ belongs to $V(S_J)$. Since $\mathbf{z}\in\lim_{\varepsilon\to 0}(\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\varepsilon}))$, from the Transfer Principle, for all $r>0$, there exists an open set $U\in\mathbb{C}\setminus\{0\}$ whose closure contains 0 such that for all $\mathbf{e}\in U$, $\mathsf{ball}(\mathbf{z},r)$ has a non-empty intersection with $\mathsf{crit}(\mathsf{proj},V^{\mathbf{a}}_{J,\mathbf{e}})$, where $V^{\mathbf{a}}_{J,e}$ denotes the algebraic variety defined by the system

$$f_1=\cdots=f_k=0,\quad g_{j_1}-a_{j_1}\mathbf{e}=\cdots=g_{j_\ell}-a_{j_\ell}\mathbf{e}=0.$$

Note also that since the ideal $\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon\rangle\subset\mathbb{C}(\varepsilon)[X,Y]$ is radical, one can suppose that for all $\mathbf{e}\in U$, the ideal $\langle f_1,\ldots,f_k,g_{j_1}-a_{j_1}\mathbf{e},\ldots,g_{j_\ell}-a_{j_\ell}\mathbf{e}\rangle\subset\mathbb{C}[X,Y]$ is radical.

Thus, by Lemma 6, we conclude that at all $\mathbf{z}_\mathbf{e}\in V^{\mathbf{a}}_{J,\mathbf{e}}$, all polynomials in $\Delta_J^*$ vanish and that one of the minors in $\Delta_J$ does not vanish. This implies that $\mathbf{z}_\mathbf{e}$ does not belong to the algebraic variety associated to $\langle\Delta_J\rangle$. Thus $\mathbf{z}_\mathbf{e}$ belongs to $V(f_1,\ldots,f_k,a_{j_1}g_{j_2}-$

$a_{j_1}g_{j_1}, \ldots, a_{j_1}g_{j_\ell} - a_{j_\ell}g_{j_1}, \Delta_J^*) - V(\Delta_J)$, and for all $r > 0$, there exists an open set $U \in \mathbb{C} \setminus \{0\}$ whose closure contains 0 such that for all $\mathbf{e} \in U$, $V_{J,\mathbf{e}}^{\mathbf{a}}$ is contained in $V(S_J)$ and it has a non-empty intersection with $\mathsf{ball}(\mathbf{z}, r)$. Since $V(S_J)$, as an algebraic set, is closed, this implies that $\mathbf{z}$ belongs to $V(S_J)$. □

**Lemma 11.** Let $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ and $\mathbf{a} = (a_{j_1}, \ldots, a_{j_\ell}) \in O_J$. Then, the Zariski-closure of $\mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V_{J,\varepsilon}^{\mathbf{a}})))$ has co-dimension greater than 0.

*Proof.* We prove that the algebraic variety associated to $(\langle S_J \rangle + \langle g_{j_1} \rangle) \cap \mathbb{Q}[X]$ has dimension less than $m$. As above, we suppose that $(a_{j_1}, \ldots, a_{j_\ell})$ does not belong to the Zariski-open set $O_J$ defined in Lemma 8. In the sequel, $\mathsf{crit}(\mathsf{proj}, V_{J,\varepsilon}^{\mathbf{a}})$ is denoted by $C_{J,\varepsilon}^{\mathbf{a}}$.

Since $(a_{j_1}, \ldots, a_{j_\ell}) \notin O_J$, we conclude by Lemma 6 that $C_{J,\varepsilon}^{\mathbf{a}}$ is defined by $f_1 = \cdots = f_k = 0, g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0$ and the vanishing of all polynomials in $\Delta_J^*$. Then, by Sard's theorem, $\langle f_1, \ldots, f_k, g_{j_1} - a_{j_1}\varepsilon, \ldots, g_{j_1} - a_{j_1}\varepsilon, \Delta_J^* \rangle \cap \mathbb{Q}(\varepsilon)[X]$ is non-empty. Note that this implies that $\mathsf{proj}(C_{J,\varepsilon}^{\mathbf{a}})$ has dimension less than $m$ and there exists $h \in (\langle S_J \rangle + \langle g_{j_1} - a_{j_1}\varepsilon \rangle) \cap \mathbb{Q}(\varepsilon)[X]$.

Given $f \in \mathbb{Q}(\varepsilon)[X, Y]$ and $\mathbf{e} \in \mathbb{C}\langle\varepsilon\rangle$, denote by $\varphi_{\mathbf{e}}(f)$ the polynomial obtained by substituting $\varepsilon$ by $\mathbf{e}$ in $f$.

Consider $\bar{h}$ the primitive part of the polynomial obtained by multiplying $h$ by the ppcm of its coefficients. Remark that $\bar{h} \in (\langle S_J \rangle + \langle g_{j_1} - a_{j_1}\varepsilon \rangle)\mathbb{Q}[\varepsilon][X]$ and that the set of solutions of $\bar{h}$ in $\mathbb{C}\langle\varepsilon\rangle^k$ contains $\mathsf{proj}(C_{J,\varepsilon}^{\mathbf{a}})$. Denote by $h_0 \in \mathbb{Q}[X]$ the polynomial $\varphi_0(\bar{h})$ and note that $h_0 \neq 0$ (since, by construction, $\bar{h} \in \mathbb{Q}[\varepsilon][X]$ has no content). The set of solutions of $h_0$ has dimension less than $m$ since $h_0 \neq 0$ and it contains obviously $\lim_{\varepsilon \to 0}(\{\mathbf{z} \in \mathbb{C}\langle\varepsilon\rangle^k \mid h_0(\mathbf{z}) = 0\}$. To summarize, we have proved that $\lim_{\varepsilon \to 0}(\mathsf{proj}(C_{J,\varepsilon}^{\mathbf{a}}))$ has dimension less than $m$. Since it obviously contains $\mathsf{proj}(\lim_{\varepsilon \to 0}(C_{J,\varepsilon}^{\mathbf{a}}))$, we are done. □

**Theorem 12** (Termination)**.** The algorithm VQE terminates.

*Proof.* Suppose now that there exists $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ such that $(a_{j_1}, \ldots, a_{j_\ell})$ does not belong to the non-empty Zariski-open set defined in Lemma 8. Then, the polynomial family $f_1, \ldots, f_k, g_{j_1} - a_{j_1}, \ldots, g_{j_\ell} - a_{j_\ell}, \Delta_I$ has a common complex solution. This degenerate situation is detected at Step (b).3 and a new point $(a_1, \ldots, a_s)$ is chosen. The algorithm terminates since all unlucky choices of $(a_1, \ldots, a_s)$ are enclosed in a Zariski-closed subset of $\mathbb{C}^s$. □

**Remark 6.** From the proof of the above theorem, one can suppose, without loss of generality, that the first choice of $(a_1, \ldots, a_s)$ (Step a.) is such that for all $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ (with $0 \leq \ell \leq \min(s, n-p+1)$), $(a_{j_1}, \ldots, a_{j_\ell})$ belongs to the non-empty Zariski open set $O_J$ defined in Lemma 8.

**Theorem 13** (Correctness)**.** The algorithm VQE is correct.

*Proof.* Given $(a_1, \ldots, a_s) \in {\mathbb{Z}_+^\star}^s$, remark that the semi-algebraic set defined by

$$f_1 = \cdots = f_k = 0, \quad g_1 > 0, \ldots, g_s > 0$$

is the same as the one defined by

$$f_1 = \cdots = f_k = 0, \quad \frac{g_1}{a_1} > 0, \ldots, \frac{g_s}{a_s} > 0.$$

Let $S$ be a connected component of this semi-algebraic set. By Remark 6, one can suppose that for all $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ (with $0 \leq \ell \leq \min(s, n-p+1)$), $(a_{j_1}, \ldots, a_{j_\ell})$ belongs to the non-empty Zariski-open set $O_J$ defined in Lemma 8.

Let $x \in \mathsf{boundary}(\mathsf{proj}(S))$. In order to prove the correctness of VQE, it is sufficient to prove that there exists a polynomial $h \in \mathbb{Q}[X]$ contained in the set $B$ (see Step c) such that $h(x) = 0$.

By Lemma 5, there exists $J \subset \{1, \ldots, s\}$ such that $x \in \mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})))$. By Lemmas 10 and 11, if the cardinality of $J$ is not greater than $\min(s, n-p+1)$, a polynomial $h$ whose solution set contains $\mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})))$ is computed at Steps (b.6-b.7). Hence, we have proved that $x$ belongs to the solution set of $h$.

Suppose now that the cardinality of $J$ is greater than $\min(s, n-p+1)$. Then, there exists $J'$ of cardinality $\min(s, n-p+1)$ contained in $J$. Remark that $V^{\mathbf{a}}_{J,\varepsilon} \subset V^{\mathbf{a}}_{J',\varepsilon}$ and that $\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J',\varepsilon}) = V^{\mathbf{a}}_{J',\varepsilon}$ because the assumption on $\mathbf{a}$ implies that $V^{\mathbf{a}}_{J',\varepsilon}$ has dimension less than or equalled to $m-1$. Hence, $x \in \mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})))$ implies that $x \in \mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J',\varepsilon})))$.

Thus, by Lemma 5 and Lemma 10, the set $B$ contains polynomials such that the union $\mathcal{B}$ of their solution set contains $\mathsf{boundary}(\mathsf{proj}(S))$. Consider now a connected component $C$ of $\mathbb{R}^m \setminus \mathcal{B}$ having a non-empty intersection with the interior of $S$. This implies that there exist some connected components $C_1, \ldots, C_q$ of $\mathbb{R}^m \setminus \mathcal{B}$ such that

- $(C_1 \cup \cdots \cup C_q) \subset S$ and
- $\mathsf{proj}(S) \setminus (C_1 \cup \cdots \cup C_q)$ is contained in $\mathcal{B}$ and hence has measure 0.

From the specification of $\mathsf{Lift}$, we conclude that algorithm VQE is correct. □

## 5. Degree bounds

Algorithm VQE computes a set $B$ containing polynomials such that the union of their solution contains the boundary of the solution set of the input quantified formula

$$\exists Y \in \mathbb{R}^n \quad f_1 = 0 \wedge \cdots \wedge f_k = 0 \wedge g_1 > 0 \wedge \cdots \wedge g_s > 0$$

where the set $\{f_1, \ldots, f_k, g_1, \ldots, g_s\} \subset \mathbb{Q}[X, Y]$ satisfies the assumptions $\mathbf{S_1}$ and $\mathbf{S_2}$. Given $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ and $\mathbf{a} = (a_1, \ldots, a_s) \subset {\mathbb{Z}_+^\star}^s$, recall that $V^{\mathbf{a}}_{J,\varepsilon}$ denotes the algebraic variety defined by

$$f_1 = \cdots = f_k = 0,\ g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon.$$

Each polynomial in $B$ is obtained by computing the projection on the $X$-space of $\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon}))$ for all $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$ with $0 \leq \ell \leq \min(s, n+1-p)$. In this section, we give degree bounds for $\mathsf{proj}(\lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})))$. Before stating the main result of this section, we recall some basic definitions:

- Let $Z \subset \mathbb{C}^n$ be an irreducible algebraic set of dimension $d$, then by definition $\deg(Z)$ is the maximal cardinality of a finite set obtained by intersecting $Z$ with an $(n-d)$-dimensional affine linear subspace (this maximal cardinality is reached for a generic choice of the $(n-d)$-dimensional affine linear subspace);
- Let $Z$ be an algebraic set and $Z_1, \ldots, Z_r$ be its irreducible components; following (Heintz, 1979), we extend this definition by $\deg(Z) = \sum_{i=1}^r \deg(Z_i)$.

We use the following notations:

- $D_{\mathbf{f}} = \max(\deg(f_1), \ldots, \deg(f_k))$,
- $D_{\mathbf{g}} = \max(\deg(g_1), \ldots, \deg(g_s))$ and
- $D = \max(D_{\mathbf{f}}, D_{\mathbf{g}})$,

and we also use the notations of Algorithm VQE.

Recall that $V^{\mathbf{a}}_{J,\varepsilon}$ is defined by

$$f_1 = \cdots = f_k = 0,\ g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0$$

and that in Lemma 8, we have defined a non-empty Zariski-open set such that if $\mathbf{a} \in O_J$, then $V^{\mathbf{a}}_{J,\varepsilon}$ is either empty or smooth equidimensional of co-dimension $p+\ell$. Now we are ready to state the main result of this section.

**Theorem 14.** Consider $J = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, s\}$, $\mathbf{a} = (a_1, \ldots, a_s) \in {\mathbb{Z}_+^\star}^s$ such that $(a_{j_1}, \ldots, a_{j_\ell}) \in O_J$ and $Z = \lim_{\varepsilon \to 0}(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})) \subset \mathbb{C}^{n+m}$. The degree of the Zariski-closure of $\mathsf{proj}(Z)$ is bounded by

$$D_{\mathbf{f}}^p D_{\mathbf{g}}^\ell \left((p+\ell)D\right)^{n+m-(p+\ell)}.$$

*Proof.* It is sufficient to prove that $\deg(Z) \leq D_{\mathbf{f}}^p D_{\mathbf{g}}^\ell \left((p+\ell)D\right)^{n+m-(p+\ell)}$. We start by proving that the following inequality holds

$$\deg(Z) \leq \deg(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})).$$

Let $K \subset \mathbb{Q}(\varepsilon)[X,Y]$ be the ideal generated by the above system of equations defining $V^{\mathbf{a}}_{J,\varepsilon}$, $\bar{K} = K \cap \mathbb{Q}[\varepsilon][X,Y]$ and $K_0 = \left(\bar{K} + \langle\varepsilon\rangle\right) \cap \mathbb{Q}[X,Y]$. By definition, it is clear that $\deg(V(K)) \geq \deg(V(K_0))$.

We prove now that $V(K_0) = Z$. Let $z \in Z$, hence for all $r > 0$, $\mathsf{ball}(z,r)$ has a non-empty intersection with $\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})$ and consequently with $V(K)$. This implies that for all $r > 0$ $\mathsf{ball}(z,r) \cap V(\bar{K}) \neq \emptyset$. Moreover, by definition of $Z$, $(z,0)$ belongs $V(\bar{K}) \cap V(\varepsilon)$ and then $z \in V(K_0)$. Suppose now that $z \in V(K_0)$. Then, for all $r > 0$, $\mathsf{ball}(z,r)$ has a non-empty intersection with $V(\bar{K})$ and hence with $V(K)$. We conclude that $z \in Z$.

We prove now that the following inequality holds

$$\deg(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})) \leq D_{\mathbf{f}}^p D_{\mathbf{g}}^\ell \left((p+\ell)D\right)^{n+m-(p+\ell)}$$

which is sufficient to end the proof.

Since $\mathbf{a} \in O_J$, Lemma 6 implies that $\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})$ is defined by the vanishing of the polynomials in $\Delta^\star_J$ (which have degree bounded by $(p+\ell)D$) and the polynomial system

$$f_1 = \cdots = f_k = 0,\ g_{j_1} - a_{j_1}\varepsilon = \cdots = g_{j_\ell} - a_{j_\ell}\varepsilon = 0$$

(which defines $V^{\mathbf{a}}_{J,\varepsilon}$).

Hence, following (Heintz and Schnorr, 1980, Proposition 2.3),

$$\deg(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})) \leq \deg(V^{\mathbf{a}}_{J,\varepsilon}) \left((p+\ell)D\right)^{\dim(V^{\mathbf{a}}_{J,\varepsilon})}.$$

Since $(a_{j_1}, \ldots, a_{j_\ell}) \in O_J$, Lemma 9 implies that $V^{\mathbf{a}}_{J,\varepsilon}$ is either empty or smooth equidimensional of co-dimension $p+\ell$.

If $V^{\mathbf{a}}_{J,\varepsilon}$ is empty, the claimed inequality holds trivially. Else, it has co-dimension $p+\ell$, so that $\dim(V^{\mathbf{a}}_{J,\varepsilon}) = n+m-(p+\ell)$. We prove now that $\deg(V^{\mathbf{a}}_{J,\varepsilon}) \leq D_{\mathbf{f}}^p D_{\mathbf{g}}^\ell$ which is sufficient to conclude the proof.

By Bézout's inequality (see e.g. (Heintz and Schnorr, 1980, pp. 265)),

$$\deg(V^{\mathbf{a}}_{J,\varepsilon}) \leq \deg(V(f_1,\ldots,f_k))\deg(V(g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon)).$$

Still using Bézout's inequality, we obtain $\deg(V(g_{j_1}-a_{j_1}\varepsilon,\ldots,g_{j_\ell}-a_{j_\ell}\varepsilon)) \leq D^{\ell}_{\mathbf{g}}$. It remains to prove that $\deg(V(f_1,\ldots,f_k)) \leq D^{p}_{\mathbf{f}}$. By assumption $\mathbf{S}_1$, $V(f_1,\ldots,f_k)$ is equidimensional of co-dimension $p$. Hence, its degree is the maximal cardinality of a finite set obtained by intersecting it with a $p$-dimensional affine linear subspace $L$. Therefore, $L$ is defined by $n+m-p$ linear equations. By Gaussian elimination, one can eliminate $n+m-p$ variables in $f_1,\ldots,f_k$ and we get $\tilde{f}_1,\ldots,\tilde{f}_k$. Without loss of generality, one can suppose that $\tilde{f}_i \in \mathbb{Q}[X_1,\ldots,X_p]$. Note that the degree of $V(\tilde{f}_1,\ldots,\tilde{f}_k)$ is the one of $V(f_1,\ldots,f_k)$. Remark also that $V(\tilde{f}_1,\ldots,\tilde{f}_k)$ has dimension 0. By Bézout's theorem, its degree is bounded by $D^{p}_{\mathbf{f}}$. □

**Remark 7.** Suppose that $\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})$ is equidimensional of dimension $m-1$. This situation occurs frequently since it is the generic case (see (Bank et al., 2004, 2010)). When this generic situation holds, one can prove that

$$\deg(Z) \leq D^{p}_{\mathbf{f}} D^{\ell}_{\mathbf{g}}((p+\ell)D)^{n+1-p-\ell}$$

by remarking that $\deg(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon})) = \deg(\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon}) \cap L_{n+1}$ where $L_{n+1}$ is an affine linear subspace of dimension $n+1$. Hence, denoting by $C$ the finite set of points $\mathsf{crit}(\mathsf{proj}, V^{\mathbf{a}}_{J,\varepsilon}) \cap L_{n+1}$, $\deg(C) \leq \deg(V^{\mathbf{a}}_{J,\varepsilon} \cap L_{n+1})((p+\ell)D)^{\dim(V^{\mathbf{a}}_{J,\varepsilon} \cap L_{n+1})}$. Since for a generic choice of $L_{n+1}$, $\dim(V^{\mathbf{a}}_{J,\varepsilon} \cap L_{n+1}) = n+1-p-\ell$. Proving that $\deg(V^{\mathbf{a}}_{J,\varepsilon} \cap L_{n+1})$ is dominated by $D^{p}_{\mathbf{f}} D^{\ell}_{\mathbf{g}}$ is done as above.

**Corollary 15.** Let $\Psi$ be a quantified formula satisfying the pre-condition of Algorithm VQE and $\mathcal{B}$ the boundary of its solution set. Then the Zariski-closure of $\mathcal{B}$ has a degree bounded by

$$D^{p}_{\mathbf{f}} D^{m} \sum_{i=0}^{\min(s,n-p+1)} D^{\ell}_{\mathbf{g}} \left((p+\ell)D\right)^{n-p-\ell}$$

*Proof.* Immediate from Theorems 13 and 14. □

**Remark 8.** Note that the above bound is singly exponential in the number of variables.

## 6. Application

In this section, we report on our experience in using the proposed VQE algorithm to tackle several challenging application problems. In particular, we tackle stability analysis problems (listed below) for solving differential equations. We chose those problems because of the following reasons.

- They are fundamental in the application field.
- They can be naturally reduced to quantifier elimination problems (Liska and Steinberg, 1993; Hong et al., 1997).
- The polynomials mostly satisfy the pre-conditions ($\mathbf{S_1}$ and $\mathbf{S_2}$). Some do not fully satisfy the assumption $\mathbf{S}_1$: the ideal generated by the input equations may not be equidimensional. In this case, we substitute the equations by equations defining

the equidimensional components, solve the VQE problem for each formula, and return a disjunction of the outputs we obtained. For the examples we considered, the equidimensional decomposition took less than 5 sec.

- Some of them have been out of reach for all the previous quantifier elimination algorithms.

Now we list the test problems. We will use the notations in Section 2.

**IBVP:** Example 6.1.2 in (Hong et al., 1997), Example 11.4.1 in (Strikwerda, 1976), Example 8.4.1 in (Kreiss and Lorenz, 1989)

*Input*: $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$
\begin{aligned}
X &= \{\, a_1, a_2 \,\} \\
Y &= \{\, \lambda_1, \lambda_2, \eta_{1,1}, \eta_{1,2}, \xi_1, \xi_2 \,\} \\
F &= \{\, f_1, f_2, f_3, f_4, f_5 \,\} \\
G &= \{\, \lambda_1, -\eta_1 \,\}
\end{aligned}
$$

$$
\begin{aligned}
f_1 &= {\lambda_1}^2 - {\lambda_2}^2 - {\eta_{1,1}}^2 + {\eta_{1,2}}^2 + {\xi_1}^2 - 2\,\xi_1\xi_2 + {\xi_2}^2 \\
f_2 &= 2\,\lambda_1\lambda_2 - 2\,\eta_{1,1}\eta_{1,2} \\
f_3 &= -\lambda_1 + \eta_{1,1} + \xi_1 a_2 - \xi_2 a_2 \\
f_4 &= -\lambda_2 + \eta_{1,2} - \xi_1 a_1 + \xi_2 a_1 \\
f_5 &= {\lambda_1}^2 + {\lambda_2}^2 + {\xi_1}^2 + {\xi_2}^2 + {\eta_{1,1}}^2 + {\eta_{1,2}}^2 - 1
\end{aligned}
$$

*Output*: $a_1^2 + a_2^2 > 1 \wedge a_1 \neq 0 \wedge a_2 \neq 0$

**Lax-Wendroff:** (Lax and Wendroff, 1960).

*Input*: $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$
\begin{aligned}
X &= \{\, a, b \,\} \\
Y &= \{\, c_1, s_1, c_2, s_2 \,\} \\
F &= \{\, c_1^2 + s_1^2 - 1,\, c_2^2 + s_2^2 - 1 \,\} \\
G &= \{g\}
\end{aligned}
$$

$$
\begin{aligned}
g = {} & -2\,c_2 a^2 b^2 - 2\,c_1 a^2 b^2 + 2\,a b^3 s_1 s_2 + 2\,a^2 b^2 c_1 c_2 + 2\,a^3 b s_1 s_2 + a^2 b^2 {c_1}^2 {c_2}^2 + \\
& 3\,a^2 b^2 + 2\,c_1 a^2 - 2\,c_1 a^4 + 2\,c_2 b^2 - 2\,c_2 b^4 - a^2 - b^2 + b^4 + a^4 - a^2 b^2 {c_2}^2 - \\
& a^2 b^2 {c_1}^2 - 2\,a b^3 s_1 s_2 c_2 - 2\,a^3 b s_2 s_1 c_1 - a^2 {c_1}^2 + a^4 {c_1}^2 - b^2 {c_2}^2 + b^4 {c_2}^2
\end{aligned}
$$

*Output*: $a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$

**LeVeque:** (LeVeque, 1996).

*Input*: $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$X = \{\, a, b \,\}$$
$$Y = \{\, c_1, s_1, c_2, s_2 \,\}$$
$$F = \{\, c_1^2 + s_1^2 - 1,\ c_2^2 + s_2^2 - 1 \,\}$$
$$G = \{g\}$$

$$g = 2\,ba^2c_2 + 2\,ab^3c_1 + 4\,a^3bc_1 + 2\,b^2ac_2{}^2 - 2\,ab^3c_2{}^2 - 2\,abc_1 + 2\,ba^2c_1{}^2 + 2\,b^2ac_1 - 2\,a^3bc_1{}^2 + 2\,a^3bc_2 - 6\,a^2b^2c_1 + 4\,c_2ab^3 - 2\,c_2ab - 6\,c_2a^2b^2 - 2\,b^2a - 2\,b^2as_1s_2 - 2\,ba^2s_1s_2 + 2\,abc_1c_2 - 2\,b^2ac_1c_2{}^2 - 2\,ba^2c_2c_1{}^2 - 4\,a^3bc_1c_2 - 4\,ab^3c_1c_2 + 6\,a^2b^2c_1c_2 + 2\,a^3bc_2c_1{}^2 + 2\,ab^3c_1c_2{}^2 + 2\,ab^3s_1s_2 + 2\,a^3bs_2s_1 - b^2 + a^4 + b^4 - a^2 - 2\,ba^2 + 2\,ab - 2\,a^3b - 2\,ab^3 + 6\,a^2b^2 + 2\,b^2as_1s_2c_2 + 2\,ba^2s_1s_2c_1 - 2\,ab^3s_1s_2c_2 - 2\,a^3bs_2s_1c_1 + 2\,c_1a^2 - a^2c_1{}^2 + a^4c_1{}^2 - b^2c_2{}^2 + b^4c_2{}^2 + 2\,c_2b^2 - 2\,c_2b^4 - 2\,c_1a^4$$

*Output*: Too long to be printed here.

**MacCormack:** (MacCormack, 1969), (Hong, 1996).

*Input*: $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$X = \{\, a, b \,\}$$
$$Y = \{\, c_1, s_1, c_2, s_2 \,\}$$
$$F = \{\, c_1^2 + s_1^2 - 1,\ c_2^2 + s_2^2 - 1 \,\}$$
$$G = \{g\}$$

$$g = 4\,a^6b^2c_1{}^4c_2{}^2 - 8\,a^5b^3s_1s_2c_1{}^3c_2 - 8\,a^5b^3s_1s_2c_1{}^2c_2{}^2 + 4\,a^4b^4c_1{}^4c_2{}^2 + 16\,a^4b^4c_1{}^3c_2{}^3 + 4\,a^4b^4c_1{}^2c_2{}^4 - 8\,a^3b^5s_1s_2c_1{}^2c_2{}^2 - 8\,a^3b^5s_1s_2c_1c_2{}^3 + 4\,a^2b^6c_1{}^2c_2{}^4 - 4\,a^7bs_1s_2c_1{}^3 + 4\,a^6b^2c_1{}^4c_2 - 4\,a^6b^2c_1{}^3c_2{}^2 + 8\,a^5b^3s_1s_2c_1{}^3 + 12\,a^5b^3s_1s_2c_1{}^2c_2 + 16\,a^5b^3s_1s_2c_1c_2{}^2 - 8\,a^4b^4c_1{}^4c_2 - 24\,a^4b^4c_1{}^3c_2{}^2 - 24\,a^4b^4c_1{}^2c_2{}^3 - 8\,a^4b^4c_1c_2{}^4 + 16\,a^3b^5s_1s_2c_1{}^2c_2 + 12\,a^3b^5s_1s_2c_1c_2{}^2 + 8\,a^3b^5s_1s_2c_2{}^3 - 4\,a^2b^6c_1{}^2c_2{}^3 + 4\,a^2b^6c_1c_2{}^4 - 4\,ab^7s_1s_2c_2{}^3 + a^8c_1{}^4 + 12\,a^7bs_1s_2c_1{}^2 - 8\,a^6b^2c_1{}^4 - 12\,a^6b^2c_1{}^3c_2 - 12\,a^6b^2c_1{}^2c_2{}^2 - 4\,a^5b^3s_1s_2c_1{}^2 - 8\,a^5b^3s_1s_2c_2{}^2 + 4\,a^4b^4c_1{}^4 + 22\,a^4b^4c_1{}^2c_2{}^2 + 4\,a^4b^4c_2{}^4 - 4\,a^4b^2c_1{}^4c_2{}^2 - 8\,a^3b^5s_1s_2c_1{}^2 - 4\,a^3b^5s_1s_2c_2{}^2 + 8\,a^3b^3s_1s_2c_1{}^2c_2{}^2 - 12\,a^2b^6c_1{}^2c_2{}^2 - 12\,a^2b^6c_1c_2{}^3 - 8\,a^2b^6c_2{}^4 - 4\,a^2b^4c_1{}^2c_2{}^4 + 12\,ab^7s_1s_2c_2{}^2 + b^8c_2{}^4 - 4\,a^8c_1{}^3 - 12\,a^7bs_1s_2c_1 + 16\,a^6b^2c_1{}^3 + 12\,a^6b^2c_1{}^2c_2 + 20\,a^6b^2c_1c_2{}^2 - 16\,a^5b^3s_1s_2c_1 - 4\,a^5b^3s_1s_2c_2 + 4\,a^5bs_1s_2c_1{}^3 + 8\,a^4b^4c_1{}^3 + 12\,a^4b^4c_1{}^2c_2 + 12\,a^4b^4c_1c_2{}^2 + 8\,a^4b^4c_2{}^3 + 4\,a^4b^2c_1{}^4c_2 + 4\,a^4b^2c_1{}^3c_2{}^2 - 4\,a^3b^5s_1s_2c_1 - 16\,a^3b^5s_1s_2c_2 - 12\,a^3b^3s_1s_2c_1{}^2c_2 - 12\,a^3b^3s_1s_2c_1c_2{}^2 + 20\,a^2b^6c_1{}^2c_2 + 12\,a^2b^6c_1c_2{}^2 + 16\,a^2b^6c_2{}^3 + 4\,a^2b^4c_1{}^2c_2{}^3 + 4\,a^2b^4c_1c_2{}^4 - 12\,ab^7s_1s_2c_2 + 4\,ab^5s_1s_2c_2{}^3 - 4\,b^8c_2{}^3 + 6\,a^8c_1{}^2 + 4\,a^7bs_1s_2 - 4\,a^6b^2c_1c_2 - 8\,a^6b^2c_2{}^2 - 2\,a^6c_1{}^4 + 12\,a^5bs_1s_2c_1{}^2 - 14\,a^4b^4c_1{}^2 + 8\,a^4b^4c_1c_2 - 14\,a^4b^4c_2{}^2 - 4\,a^4b^2c_1{}^3c_2 + 10\,a^4b^2c_1{}^2c_2{}^2 + 12\,a^3b^5s_1s_2 + 4\,a^3b^3s_1s_2c_1{}^2 + 16\,a^3b^3s_1s_2c_1c_2 + 4\,a^3b^3s_1s_2c_2{}^2 - 8\,a^2b^6c_1{}^2 - 4\,a^2b^6c_1c_2 + 10\,a^2b^4c_1{}^2c_2{}^2 - 4\,a^2b^4c_1c_2{}^3 + 4\,ab^7s_1s_2 - 12\,ab^5s_1s_2c_2{}^2 + 6\,b^8c_2{}^2 - 2\,b^6c_2{}^4 - 4\,a^8c_1 - 16\,a^6b^2c_1 + 8\,a^6c_1{}^3 + 12\,a^5bs_1s_2c_1 - 12\,a^4b^4c_1 - 12\,a^4b^4c_2 - 8\,a^4b^2c_1{}^2c_2 - 16\,a^4b^2c_1c_2{}^2 - 4\,a^3b^3s_1s_2c_1 - 4\,a^3b^3s_1s_2c_2 - 16\,a^2b^6c_2 - 16\,a^2b^4c_1{}^2c_2 - 8\,a^2b^4c_1c_2{}^2 + 12\,ab^5s_1s_2c_2 - 4\,b^8c_2 + 8\,b^6c_2{}^3 + a^8 + 8\,a^6b^2 - 12\,a^6c_1{}^2 - 4\,a^5bs_1s_2 + 14\,a^4b^4 - 2\,a^4b^2c_1{}^2 + 12\,a^4b^2c_1c_2 + 6\,a^4b^2c_2{}^2 + a^4c_1{}^4 + 8\,a^2b^6 + 6\,a^2b^4c_1{}^2 + 12\,a^2b^4c_1c_2 - 2\,a^2b^4c_2{}^2 + 2\,a^2b^2c_1{}^2c_2{}^2 - 4\,ab^5s_1s_2 + b^8 - 12\,b^6c_2{}^2 + b^4c_2{}^4 + 8\,a^6c_1 + 4\,a^4b^2c_1 - 4\,a^4b^2c_2 - 4\,a^4c_1{}^3 - 4\,a^3bs_1s_2c_1 - 4\,a^2b^4c_1 + 4\,a^2b^4c_2 - 4\,ab^3s_1s_2c_2 + 8\,b^6c_2 - 4\,b^4c_2{}^3 - 2\,a^6 - 2\,a^4b^2 + 8\,a^4c_1{}^2 + 4\,a^3bs_1s_2 - 2\,a^2b^4 - 2\,a^2b^2c_1{}^2 + 4\,a^2b^2c_1c_2 - 2\,a^2b^2c_2{}^2 + 4\,ab^3s_1s_2 - 2\,b^6 + 8\,b^4c_2{}^2 - 8\,a^4c_1 - 4\,a^2b^2c_1 - 4\,a^2b^2c_2 - 8\,b^4c_2 + 3\,a^4 + 6\,a^2b^2 - 2\,a^2c_1{}^2 + 3\,b^4 - 2\,b^2c_2{}^2 + 4\,a^2c_1 + 4\,b^2c_2 - 2\,a^2 - 2\,b^2$$

*Output*: $a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$

**Stab1:** Based on the stabilizability problem (Jirstrand, 1997).

*Input:* $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$
\begin{aligned}
X &= \{\, x_1, x_2 \,\} \\
Y &= \{\, y_1, y_2, y_3, y_4 \,\} \\
F &= \{f_1, f_2, f_3\} \\
G &= \{2\,{x_2}^2 - 2\,x_2 y_1 + 2\,x_2 y_4, x_1 - y_2 - y_4\}
\end{aligned}
$$

$$
\begin{aligned}
f_1 &= -1 - y_2 + {x_2}^2 - y_1 y_2 - y_3 y_4 \\
f_2 &= -x_1 x_2 + x_1 y_1 - x_1 y_4 + x_2 y_2 + x_2 y_4 + y_2 y_4 \\
f_3 &= {y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2 - 1
\end{aligned}
$$

*Output*: Too long to be printed here.

**Stab2:** Based on the stabilizability problem (Jirstrand, 1997).

*Input:* $\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$

$$
\begin{aligned}
X &= \{\, x_1, x_2 \,\} \\
Y &= \{\, y_1, y_2, y_3, y_4 \,\} \\
F &= \{f_1, f_2, f_3\} \\
G &= \{g_1, g_2\}
\end{aligned}
$$

$$
\begin{aligned}
f_1 &= -1 + y_4 + x_2 y_1 + x_2 y_3 - x_2 y_4 - y_1 y_4 + {y_2}^2 - y_2 y_3 + {y_3}^2 - y_3 y_4 \\
f_2 &= y_1 - {x_1}^2 - x_1 y_1 - x_1 y_2 + x_1 y_3 + x_2 y_1 + x_2 y_2 - x_2 y_3 + {y_2}^2 - y_2 y_3 \\
f_3 &= {y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2 - 1 \\
g_1 &= 2\,x_1 y_1 + {y_1}^2 + y_1 y_2 + 2\,x_1 y_3 + y_2 y_3 - {y_3}^2 - 2\,x_1 y_4 - \\
&\qquad y_1 y_4 - y_2 y_4 + y_3 y_4 \\
g_2 &= -y_1 - y_2 + y_3
\end{aligned}
$$

*Output*: Too long to be printed here.

See Table 1. In order to evaluate the practical performance of the VQE algorithm, we have compared its computing times against several state-of-the-art general purpose QE software packages: QEPCAD (Brown, 2003; Collins and Hong, 1991), Mathematica (Strzebonski, 2006) and SyNRAC (Yanami and Anai, 2007).

The line QEPCAD-Opt (Brown, 2009) reports timings for *simplified* input formula obtained by making linear substitutions and/or the half-tangent parameterization whenever possible (e.g. IBVP, Lax-Wendroff, LeVeque and MacCormack). Such simplification yields formula with *less* quantified variables. QEPCAD-Opt also uses "`measure-zero-error`" option to allow measure-zero error in the output formula (in a similar way to VQE).

**Table 1.** Computing Times

| | IBVP | Lax-Wendroff | LeVeque | MacCormack | Stab1 | Stab2 |
|---|---|---|---|---|---|---|
| Mathematica | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| SyNRAC | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| QEPCAD | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| QEPCAD-Opt | 2 s | 45 s | 56 s | $\infty$ | $\infty$ | $\infty$ |
| VQE | 15 s | 20 s | 63 s | 12.2 h | 1.6 h | 2.3 h |
| VQE Step (b).3 | $< 1$ s | $< 1$ s | $< 1$ s | 5 s | $< 1$ s | $< 1$ s |
| VQE Step (b).6 | 4 s | 2 s | 2 s | 10 m | 5 s | 30 s |
| VQE Step (b).7 | 2 s | 1 s | 5 s | 3 h | 30 s | 12 m |
| VQE Step (d) | 8 s | 16 s | 55 s | 9 h | 1.5 h | 2 h |

Timings for VQE are for the *original* (un-simplified) inputs. In fact, half-tangent parameterization could not be used for VQE since it would remove the equations, which VQE requires.

The symbol $\infty$ means that the computation was stopped after 2 days of computations. When stopped, they were still carrying out the projection phase of CAD. We also provide detail timings for the non-trivial steps of the VQE algorithm. All other steps are trivial and thus their computing times are negligible.

The computations have been performed on a PC Intel(R) Xeon(R) 2.50GHz with 6144 KB of Cache and 20 GB of RAM. The implementation was done on top of the following packages:

**FGb** ( Faugère ) in C, by J.C. Faugère, for Gröbner bases computations for Step (b).

**RS** ( Rouillier ) in C, by F. Rouillier, for isolating the real solutions of zero-dimensional ideals for Step (d)

**OpenCAD** (Moroz and Rouillier, 2007) in Maple, by G. Moroz and F. Rouillier, for Step (d).

**RAGlib** (Safey El Din, 2007a) in Maple, by M. Safey El Din, for Step (d).

We remind the reader that the comparison is between the *special* QE package (VQE) and the *general* QE packages (Mathematica, QEPCAD, SyNRAC). Thus, it was expected that the computing time of VQE would be generally smaller. However it is interesting to see that the reduction is quite significant for some problems (such as MacCormack, Stab1 and Stab2).

**Acknowledgements**

This work was partially carried out while both authors were visiting Korea Institute for Advanced Studies (KIAS), Seoul, South-Korea. We thank the support of KIAS. We also thank A. Strzebonski, H. Yanami, H. Anai, and C. Brown for their help in the use of their general purpose QE software.

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