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Abstract

The integer order differentiation by integration method based on the Jacobi orthogonal polynomials for noisy signals was originally introduced by Mboup, Join and Fliess [1], [2]. We are going to generalize this method from the integer order to the fractional order so as to estimate the fractional order derivatives of noisy signals. For sake of clarity, this work has been divided into two parts. The first part presented in this paper focuses on the continuous case while the second part that has been presented in another paper [3] deals with the discrete case with on-line applications. In this paper, two kinds of fractional order differentiators are deduced from the Jacobi orthogonal polynomial filter, using the Riemann-Liouville and the Caputo fractional order derivative definitions respectively. Exact and simple formulae for these differentiators are given by integral expressions. Hence, they can be used both for continuous-time and discrete-time models in on-line or off line applications. Then, some error bounds are provided for the corresponding estimation errors in continuous case. These bounds will be used to study the design parameters' influence on the obtained fractional order differentiators in the second part. Finally, numerical simulations are given to show the accuracy and the robustness with respect to corrupting noises

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of the proposed differentiators in off-line applications. The properties of our differentiators in on-line applications will be shown in the second part.

Index Terms

Fractional order differentiator, Riemann-Liouville fractional derivative, Caputo fractional derivative, Jacobi orthogonal polynomial filter, Error analysis.

EDICS: SSP-FILT; ASP-ANAL.

I. Introduction

Fractional order derivatives have a long history and are now very useful in many scientific and engineering fields, including flow, control (papers [4], [5], [6], [7] showed that using fractional order derivatives in the control design can improve the performances and robustness properties), signal processing, electrical networks, and etc. (see, e.g., [8], [9]). The fractional order differentiator is concerned with estimating the fractional order derivatives of an unknown signal from its noisy observed data. This motivated the interest of fractional order differentiators in signal processing applications, such as edge detection [10], electrocardiogram signal detection [11], biological signal processing [12], and image signal processing [13]. Various methods have been developed during the last years. They are divided into two classes: those dealing with continuous-time models (see, e.g., [14], [15], [16]) and those dealing with discrete-time models (see, e.g., [17], [18], [19], [20], [21]). Nevertheless, the real case of signals with noises was somewhat overlooked. In order to solve this problem, an optimization formulation using genetic algorithms was proposed in [22] to reduce the noise effect in the estimation of the fractional order derivatives. But, the complex mathematical deduction restricts its application. A novel digital fractional order Savitzky-Golay differentiator was introduced in [23], which was deduced from the Riemann-Liouville fractional order derivative definition and the Savitzky-Golay filter [24], [25]. The use of the Savitzky-Golay filter guarantees its accuracy and simplicity for estimating the fractional order derivatives for noisy signals. However, let us recall that the Legendre polynomial filter is more recommended than the Savitzky-Golay filter for reasons of simplicity and convergence speed [26]. In particular, it has the advantage of being suitable for irregularly spaced or missing data.

The *integer order differentiation by integration* method is concerned with estimating the integer order derivatives of an unknown signal by using an integral formula of its noisy observation. This method was firstly studied for noise-free signals by Cioranescu [28] in 1938 and became well known for the

Lanczos generalized derivative [29] (p. 324) in 1956 (see [27] for more details). The idea of the Lanczos generalized derivative is to use an integral of an unknown signal and the Legendre orthogonal polynomial to estimate the first order derivative of the signal. Recently, Mboup, Join and Fliess [1], [2] applied an algebraic setting to estimate high order derivatives by integration in noisy case, where the Jacobi orthogonal polynomials were introduced in the integral. Hence, we call the obtained differentiator *integer order Jacobi differentiator*. This differentiator greatly improves the convergence rate of the Lanczos generalized derivative [30]. Moreover, it was shown in [31] that it can also be obtained by taking the integer order derivative of the Jacobi orthogonal polynomial filter considered as the extension of the Legendre polynomial filter [26], [32]. Bearing this idea in mind, a fractional order Jacobi differentiator was introduced in [33] by taking the Riemann-Liouville fractional order derivative of the Jacobi orthogonal polynomial filter. Similarly, the Jacobi orthogonal polynomial filter was used to estimate the Caputo fractional order derivatives so as to solve fractional order differential equations in [34]. In [33], we showed that the fractional order Jacobi differentiator is more accurate than the one based on the classical Savitzky-Golay filter. Nevertheless, no error analysis was provided for this differentiator.

Let us mention that the algebraic differentiation method used in [1], [2] was introduced in [35], where the idea was inspired from the algebraic parametric estimation methods [36], [37], [38], [39]. The reader may find additional theoretical foundations in [40], [41]. In particular, by using non standard analysis techniques, Fliess [40], [42] showed that these methods exhibit good robustness properties with respect to corrupting noises without the need of knowing their statistical properties. Moreover, some accurate error analysis were given in [43], [44] to show the robustness properties of the integer order Jacobi differentiator with respect to noises issued from stochastic process models. Consequently, the fractional order Jacobi differentiator proposed in [33] can have the same robustness properties. In [46], by applying these methods to a truncated fractional Taylor series expansion, we estimated the Jumarie's modified Riemann-Liouville derivative defined in [45] without any error analysis.

This work has been divided into two parts presented in two different papers for sake of clarity. This first part presented in this paper focuses on the continuous case where we first extend the differentiation by integration method from the integer order to the fractional order so as to estimate both the Riemann-Liouville and the Caputo fractional order derivatives for noisy signals. Then, we provide some analysis for the corresponding estimation errors in continuous case. In Section II, we recall the definitions and some useful properties of the Riemann-Liouville and the Caputo fractional order derivatives. Then, two fractional order differentiators are given by exact and simple integral formulae in Section III. They are obtained by calculating the Riemann-Liouville and the Caputo fractional order derivatives of the

Jacobi orthogonal polynomial filter respectively. In Section IV, by providing some error bounds, we study different sources of errors for these differentiators in continuous case. These bounds will be used to study the design parameters' influence on the obtained fractional order differentiators in the second part [3]. In order to show the efficiency and the stability of these differentiators in off-line applications, numerical simulations results are given in Section V. Finally, we give some conclusions for this part and the perspectives for the second part in Section VI. To simplify the presentation, all the proofs are deferred to the appendix.

II. Preliminary

In this section, we are going to recall the definitions and some useful properties of the Riemann-Liouville fractional derivative, the Caputo fractional derivative and the Jacobi orthogonal polynomials, respectively.

A. Riemann-Liouville and Caputo fractional order derivatives

Let $l \in \mathbb{N}^*$, $a \in \mathbb{R}$, and $f \in \mathcal{C}^l(\mathbb{R})$ where $\mathcal{C}^l(\mathbb{R})$ refers to the set of functions being l-times continuously differentiable on $\mathbb{R}$. Then, the Riemann-Liouville fractional order derivative (see [8] p. 62) of f is defined as follows: $\forall\, t > a$,

$$ {}_R\mathrm{D}^{\alpha}_{a,t} f(\cdot) := \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_a^t (t-\tau)^{l-\alpha-1} f(\tau)\, d\tau, \tag{1} $$

where $l-1 \leq \alpha < l$, and $\Gamma(\cdot)$ is the Gamma function (see [47] p. 255). Moreover, the Caputo fractional order derivative (see [8] p. 79) of f is defined as follows: $\forall\, t > a$,

$$ {}_C\mathrm{D}^{\alpha}_{a,t} f(\cdot) := \frac{1}{\Gamma(l-\alpha)} \int_a^t (t-\tau)^{l-\alpha-1} f^{(l)}(\tau)\, d\tau, \tag{2} $$

where $l-1 < \alpha < l$. If $f \in \mathcal{C}^{l+1}(\mathbb{R})$, then the Caputo fractional order derivative is also valid for $\alpha = l$.

In this paper, we only consider the case where $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. By applying integration by parts to (1), we get (see [8] pp. 75-76): $\forall\, t > a$,

$$ {}_R\mathrm{D}^{\alpha}_{a,t} f(\cdot) = \sum_{i=0}^{l-1} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} f^{(i)}(a) + {}_C\mathrm{D}^{\alpha}_{a,t} f(\cdot), \tag{3} $$

where $l-1 < \alpha < l$ with $l \in \mathbb{N}^*$. As an example, if we take $f(t) = (t-a)^n$ with $n \in \mathbb{N}$ and $a < t \in \mathbb{R}$, then by using (1) and (2), we obtain (see [8] p. 72):

$$ {}_R\mathrm{D}^{\alpha}_{a,t} f(\cdot) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, (t-a)^{n-\alpha}, \tag{4} $$

$$ {}_C\mathrm{D}^{\alpha}_{a,t} f(\cdot) = \begin{cases} 0, & \text{if } n < \alpha, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, (t-a)^{n-\alpha}, & \text{if } n > \alpha, \end{cases} \tag{5} $$

where $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. As we can see, unlike the Riemann-Liouville fractional order derivative the Caputo fractional order derivative of the polynomial f can be defined at $t = a$, *i.e.* $\left[{}_C\mathrm{D}^{\alpha}_{a,t} f(\cdot)\right]_{t=a} = 0$.

From now on, we denote the α^{th} ($\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$) order derivative of f by $\mathrm{D}^{\alpha}_{a,t} f(\cdot)$. This notation can refer both to the Riemann-Liouville and the Caputo fractional order derivatives when the context is clear. Otherwise, we use the specific notations introduced in (1) and (2). Some useful properties of these fractional order derivatives are recalled as follows:

- Linearity (see [8] p. 91): $\forall\, t > a$,

$$\mathrm{D}^{\alpha}_{a,t} \left\{\lambda_1 f_1(\cdot) + \lambda_2 f_2(\cdot)\right\} = \lambda_1 \mathrm{D}^{\alpha}_{a,t} f_1(\cdot) + \lambda_2 \mathrm{D}^{\alpha}_{a,t} f_2(\cdot), \tag{6}$$

- Scale change (see [9] p. 76):

$$\forall\, t > a,\ \mathrm{D}^{\alpha}_{a,\lambda t} f(\cdot) = \frac{1}{\lambda^{\alpha}} \mathrm{D}^{\alpha}_{\frac{a}{\lambda},t} f(\lambda\, \cdot), \tag{7}$$

- Translation (see [9] p. 89):

$$\forall\, t \in \mathbb{R}^*_+,\ \mathrm{D}^{\alpha}_{a,a+t} f(\cdot) = \mathrm{D}^{\alpha}_{0,t} f(\cdot + a), \tag{8}$$

where $l - 1 < \alpha < l$ with $l \in \mathbb{N}^*$, $\lambda \in \mathbb{R}^*_+$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f, f_1, f_2 \in \mathcal{C}^l(\mathbb{R})$.

B. *Jacobi orthogonal polynomials*

In this article, we use the n^{th} ($n \in \mathbb{N}$) order shifted Jacobi orthogonal polynomial defined on $[0, 1]$ as follows (see [47] p. 775):

$$P_n^{(\mu,\kappa)}(\tau) := \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (\tau - 1)^{n-j}\, \tau^j, \tag{9}$$

where $\mu, \kappa \in]-1, +\infty[$, and

$$\binom{n+\mu}{j} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu-j+1)\Gamma(j+1)},$$
$$\binom{n+\kappa}{n-j} = \frac{\Gamma(n+\kappa+1)}{\Gamma(\kappa+j+1)\Gamma(n-j+1)}.$$

Let g_1 and g_2 be two continuous functions defined on $[0, 1]$, then we define the scalar product $\langle \cdot, \cdot \rangle_{\mu,\kappa}$ of these functions by (see [47] p. 773):

$$\langle g_1(\cdot), g_2(\cdot) \rangle_{\mu,\kappa} = \int_0^1 w_{\mu,\kappa}(\tau)\, g_1(\tau)\, g_2(\tau)\, d\tau, \tag{10}$$

where $w_{\mu,\kappa}(\tau) = (1-\tau)^{\mu} \tau^{\kappa}$ is the associated weighted function. Thus, by denoting its associated norm by $\| \cdot \|_{\mu,\kappa}$, we obtain:

$$\|P_n^{(\mu,\kappa)}\|^2_{\mu,\kappa} = \frac{\Gamma(\mu+n+1)\,\Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\,\Gamma(n+1)\,(2n+\mu+\kappa+1)}. \tag{11}$$

By applying the linearity of the fractional order derivative given in (6) to (9) and using (4) and (5), we obtain the fractional order derivatives of the Jacobi polynomial in the following lemma.

Lemma 1 *The α^{th} ($\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$) order derivatives of the n^{th} order Jacobi orthogonal polynomial $P_n^{(\mu,\kappa)}$ defined in (9) are given as follows: $\forall\, \tau \in]0,1]$,*

$$ {}_R\mathrm{D}_{0,\tau}^{\alpha} P_n^{(\mu,\kappa)}(\cdot) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} c_{\mu,\kappa,n,j,i} \, \frac{\Gamma(n-i+1)}{\Gamma(n-i+1-\alpha)} \tau^{n-i-\alpha}, \tag{12} $$

$$ {}_C\mathrm{D}_{0,\tau}^{\alpha} P_n^{(\mu,\kappa)}(\cdot) = \begin{cases} 0, & \text{if } n < \alpha, \\ \displaystyle\sum_{j=0}^{n} \sum_{i=0}^{\min(n-j,n-l)} \frac{c_{\mu,\kappa,n,j,i}\, \Gamma(n-i+1)}{\Gamma(n-i+1-\alpha)} \, \tau^{n-i-\alpha}, & \text{if } n \geq \alpha, \end{cases} \tag{13} $$

where $c_{\mu,\kappa,n,j,i} = (-1)^i \binom{n+\mu}{j} \binom{n+\kappa}{n-j} \binom{n-j}{i}$, and $l \in \mathbb{N}^*$ with $l-1 < \alpha < l$.

III. Fractional order differentiation by integration

In this section, we are going to extend the method of differentiation by integration involving some Jacobi polynomials, introduced in [1], [2], from the integer order to the fractional order. Let y be a continuous function defined on $\mathbb{R}$, and y^{ϖ} be a noisy observation of y on an interval $I = [a,b] \subset \mathbb{R}$ of length $h = b - a$:

$$ \forall\, t \in I,\; y^{\varpi}(t) = y(t) + \varpi(t), \tag{14} $$

where the noise ϖ is integrable and locally essentially bounded, $i.e.$ locally bounded except on a set of measure zero. We are going to estimate the α^{th} ($\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$) order derivative of y using the observation y^{ϖ}.

In this part, we ignore the noise. The Jacobi orthogonal series expansion of $y(a+h\cdot)$ ([48] p. 6) is given by:

$$ \forall\, \xi \in [0,1],\; y(a+h\xi) = \sum_{i=0}^{+\infty} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(a+h\cdot) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2} \; P_i^{(\mu,\kappa)}(\xi). \tag{15} $$

By taking the truncated Jacobi orthogonal series expansion of $y(a+h\cdot)$, we define an N^{th} ($N \in \mathbb{N}$) order polynomial which approximates y on I: $\forall\, \xi \in [0,1]$,

$$ D_{h,\mu,\kappa,N}^{(0)} y(a+h\xi) := \sum_{i=0}^{N} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(a+h\cdot) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2} \; P_i^{(\mu,\kappa)}(\xi). \tag{16} $$

If we take $\kappa = \mu = 0$, then the Jacobi orthogonal polynomials become the Legendre orthogonal polynomials. Hence, this Jacobi polynomial filter can be considered as the extension of the Legendre polynomial filter (see [26], [32] p. 29).

Let us recall that the integer order derivative of the previous approximation polynomial was used for the integer order differentiation by integration in [1], [2], [31], [30]. Hence, by taking the fractional order derivative of this polynomial, we extend this method of differentiation by integration from the integer order to the fractional order in the following theorem.

Theorem 1 *Let y^{ϖ} be the noisy observation defined in (14). If we assume $y \in \mathcal{C}^{l}(\mathbb{R})$ with $l \in \mathbb{N}^*$, then an estimate for the fractional order derivative $\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot)$ $(l-1<\alpha<l)$ of y is defined as follows: $\forall\, \xi \in]0,1]$,*

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) := \frac{1}{h^{\alpha}} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, y^{\varpi}(a+h\tau)\, d\tau, \tag{17}$$

where $h \in \mathbb{R}^*_+$, $N \in \mathbb{N}$, $\mu,\kappa \in]-1,+\infty[$, and

$$Q_{\mu,\kappa,\alpha,N}(\tau,\xi) = w_{\mu,\kappa}(\tau) \sum_{i=0}^{N} \frac{P_i^{(\mu,\kappa)}(\tau)}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\, q_{\mu,\kappa,i}(\xi), \tag{18}$$

with $q_{\mu,\kappa,i}(\xi) = \mathrm{D}^{\alpha}_{0,\xi} P_i^{(\mu,\kappa)}(\cdot)$. This differentiator refers to ${}_{R}D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ (resp. ${}_{C}D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$) which estimates ${}_{R}\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot)$ (resp. ${}_{C}\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot)$). Moreover, we have: $\forall\, \xi \in]0,1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) = \frac{1}{h^{\alpha}} \sum_{i=0}^{N} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y^{\varpi}(a+h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\, q_{\mu,\kappa,i}(\xi). \tag{19}$$

On the one hand, according to (17), this proposed differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ uses an integral involving the Jacobi polynomials to estimate the fractional order derivative of a noisy signal. We will call this differentiator *fractional order Jacobi differentiator*, and we will call this method *fractional order differentiation by integration*. It can be used both for continuous-time and discrete-time models. In continuous case, we need to calculate the integral of the noisy signal y^{ϖ} and a sum of the Jacobi polynomials. This integral corresponds to a convolution in discrete case.

On the other hand, it can be used both for off-line and on-line applications. For the off-line applications, we fix the length of the interval $I = [a,b]$. Hence, by setting the parameters κ, μ and N, we can calculate the scalar product and norm values in each term of the sum in (19). Then, the values of $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ can be easily obtained for any $\xi \in]0,1]$. For the on-line applications, the length of the interval I varies with time ($b=t$). If we fix the value of ξ to 1, then in this case the fractional Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ only estimates the value of $\mathrm{D}^{\alpha}_{0,t} y(a+\cdot)$ at point $t=b=a+h$.

In order to give a guideline on how to choose the parameters κ, μ and N, we are going to study the estimation error for the fractional order Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ in the next section.

IV. Error analysis in continuous case

The estimation error for the integer order Jacobi differentiator has been studied in [30], [31], [1] in the continuous case. We will use a similar procedure to determine some error analysis for the fractional order Jacobi differentiator in this section.

By writing the following equality: $\forall\, \xi \in]0,1]$,

$$\begin{aligned} D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = {} & \left(D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) - D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) \right) \\ & + \left(D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) \right), \end{aligned} \tag{20}$$

the estimation error for the fractional order Jacobi differentiator in the noisy case can be divided into two sources:

- the noise error contribution:

  $$e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi) = D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) - D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi),$$

  which is due to the noise of signal y^{ϖ};
- the N^{th} order truncated term error:

  $$e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi) = D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot),$$

  which comes from the truncated part in the Jacobi series expansion of y in (15).

Since the fractional order Jacobi differentiator is a linear operator, then the noise error contribution can be studied in the following proposition.

Proposition 1 *Let y^{ϖ} be the noisy observation defined in (14), where $y \in \mathcal{C}^{l}(\mathbb{R})$ with $l \in \mathbb{N}^*$ and $l-1 < \alpha < l$. Then, the noise error contribution in the fractional order Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\cdot)$ can be given as follows: $\forall\, \xi \in]0,1]$,*

$$e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi) = \frac{1}{h^{\alpha}} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, \varpi(a+h\tau)\, d\tau, \tag{21}$$

where $h \in \mathbb{R}^*_+$, $Q_{\mu,\kappa,\alpha,N}(\cdot,\cdot)$ is given by (18) with $N \in \mathbb{N}$ and $\mu, \kappa \in]-1,+\infty[$. Moreover, if the essential supremum[1] of $|\varpi|$ exists on I, i.e. $\delta = \operatorname{ess\,sup} |\varpi| < \infty$, then we have: $\forall\, \xi \in]0,1]$,

$$\left| e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi) \right| \leq \frac{\delta}{h^{\alpha}} E_{\mu,\kappa,\alpha,N}(\xi), \tag{22}$$

where $E_{\mu,\kappa,\alpha,N}(\xi) = \int_0^1 |Q_{\mu,\kappa,\alpha,N}(\tau,\xi)|\, d\tau$.

[1]The essential supremum of $|\varpi|$, denoted by $\operatorname{ess\,sup}|\varpi|$, is defined by: $\operatorname{ess\,sup}|\varpi| := \inf\{c \in \mathbb{R} : \eta(\{t \in I : |\varpi(t)| > c\}) = 0\}$, where I is a set, and η is a measure on I.

Consequently, it is shown that the noise error contribution is decreasing with respect to h. By studying the influence of the parameters κ, μ and N on the value $E_{\mu,\kappa,\alpha,N}(\xi)$, we can also deduce their influence on the noise error contribution $e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi)$. In the following proposition, we study the truncated term error.

Proposition 2 *Let y^{ϖ} be the noisy observation defined by (14), where we assume that $y \in \mathcal{C}^{\hat{N}+1}(\mathbb{R})$ with $\hat{N} \in \mathbb{N}^*$. If the Taylor series expansion of y at the point a converges on I, then the N^{th} ($N \in \mathbb{N}^*$) order truncated term error in the fractional order Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\cdot)$, where $\alpha < n = \min(N, \hat{N})$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, can be given as follows: $\forall\, \xi \in]0,1]$,*

$$e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi) = h^{n+1-\alpha} \left(\int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, I^y_{n,h}(\tau)\, d\tau + I^y_{n,\alpha,h}(\xi) \right), \tag{23}$$

where $h \in \mathbb{R}^*_+$, and $Q_{\mu,\kappa,\alpha,N}(\cdot,\cdot)$ is given by (18) with $\mu, \kappa \in]-1,+\infty[$, and

$$I^y_{n,h}(\tau) = \frac{\tau^{n+1}}{n!} \int_0^1 (1-u)^n\, y^{(n+1)}\,(a+uh\tau)\; du,$$

$$I^y_{n,\alpha,h}(\xi) = \frac{\xi^{n+1-\alpha}}{\Gamma(n+1-\alpha)} \int_0^1 (1-\tau)^{n-\alpha}\, y^{(n+1)}(a+\xi h\tau)\, d\tau.$$

Moreover, if $M_{n+1} = \left\| y^{(n+1)} \right\|_{\infty}$ exists on I, then $e^{\infty}_{\mu,\kappa,h,\alpha,N}(\cdot)$ can be bounded as follows: $\forall\, \xi \in]0,1]$,

$$\left| e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi) \right| \leq h^{n+1-\alpha} M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi), \tag{24}$$

where $C_{\mu,\kappa,\alpha,n,N}(\xi) = \frac{I_{\mu,\kappa,\alpha,n,N}(\xi)}{(n+1)!} + \frac{\xi^{n+1-\alpha}}{\Gamma(n+2-\alpha)}$ with $I_{\mu,\kappa,\alpha,n,N}(\xi) = \int_0^1 \left| Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, \tau^{n+1} \right| d\tau$.

Consequently, the fractional order Jacobi differentiator can give the exact fractional order derivatives of an n^{th} order polynomial in continuous noise-free case. Moreover, the convergence rate for the truncated term error can be improved by increasing $n = \min\left(N, \hat{N}\right)$. Let us recall that N refers to the order of the approximation polynomial defined in (16) which can be an arbitrary non-zero integer, and $\hat{N}$ depends on the smoothness of the function y. We can take $n = N = \hat{N}$. Hence, the convergence rate is $h^{\hat{N}+1-\alpha}$. Consequently, by providing some additional smoothness hypotheses on y the convergence rate can be improved. In the case where y is analytic, it is sufficient to take $n = N$.

Unlike the noise error contribution, the truncated term error can be increasing with respect to h. An appropriate value of h is then proposed in the following corollary.

Corollary 1 *Let y^{ϖ} be the noisy observation defined by (14). We also assume that ϖ and y satisfy the hypotheses given in Proposition 1 and Proposition 2 respectively. Then, the total estimation error in the fractional order Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\cdot)$ can be bounded as follows:*

$$\left| D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) \right| \leq h^{n+1-\alpha} M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi) + \frac{\delta}{h^{\alpha}} E_{\mu,\kappa,\alpha,N}(\xi). \tag{25}$$

Moreover, if we choose

$$h^* = \left(\frac{\alpha\, E_{\mu,\kappa,\alpha,N}(\xi)\,\delta}{(n+1-\alpha)M_{n+1}C_{\mu,\kappa,\alpha,n,N}(\xi)}\right)^{\frac{1}{n+1}},$$

then we have:

$$\left|D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h^*\xi) - \mathrm{D}^{\alpha}_{0,h^*\xi}y(a+\cdot)\right| = \mathcal{O}(\delta^{\frac{n+1-\alpha}{n+1}}).$$

In the second part of this work [3], we will give some error bounds for the noise error contributions due to a large class of stochastic processes in discrete case. Then, these noise error bounds and the error bound for the truncated term error obtained in this section will be used to study the design parameters' influence on the obtained fractional order differentiators.

V. Simulation results

In order to show the accuracy and robustness with respect to corrupting noises of the fractional order Jacobi differentiator, we present some numerical results in this section. The reader may find some comparisons between the fractional order Jacobi differentiator and some exiting fractional order differentiators in [33], [23].

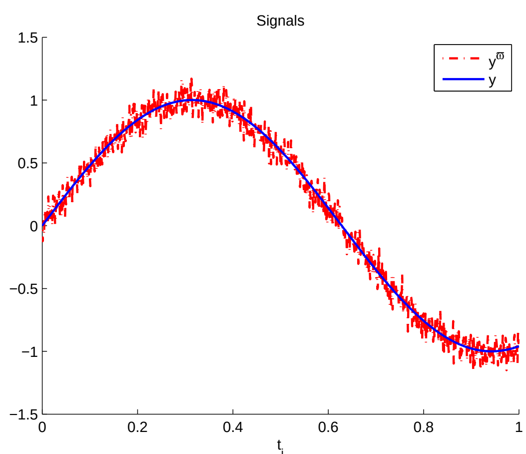
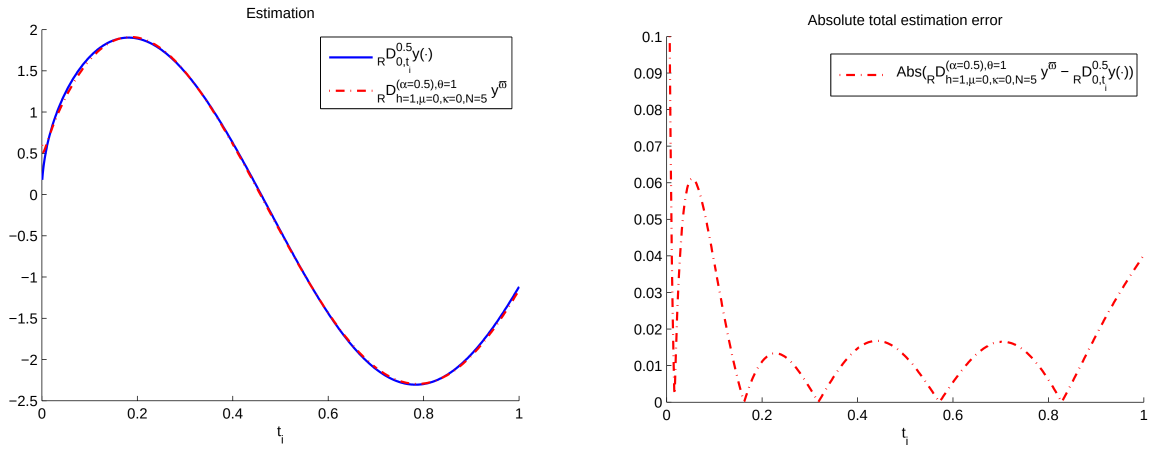


Fig. 1. Signal y and the noisy signal $y^{\varpi}(t_i) = \sin(5t_i) + 0.077\varpi(t_i)$.

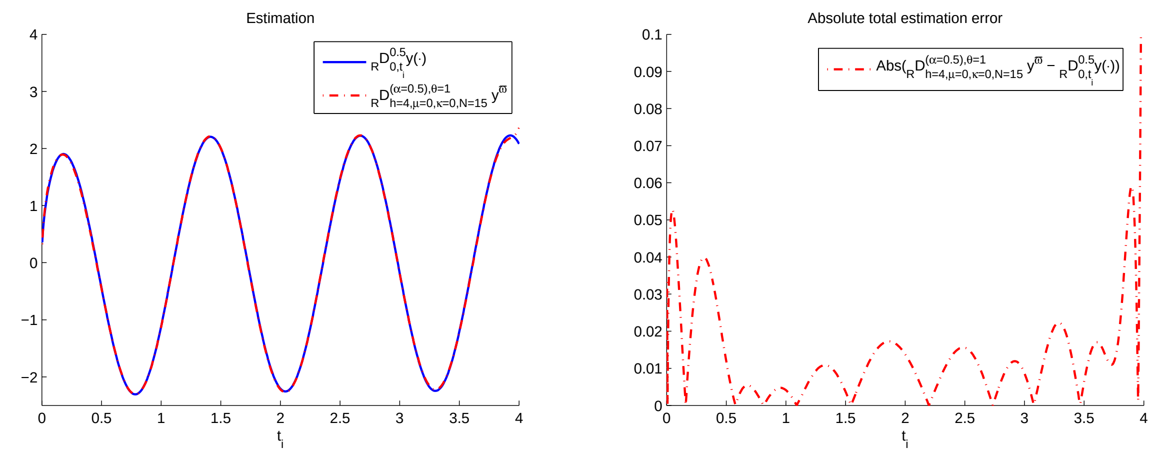
Let $y^{\varpi}(t_i) = \sin(5t_i) + \sigma\varpi(t_i)$ be a noisy signal with σ adjusted such that the signal-to-noise ratio $SNR = 10\log_{10}\left(\frac{\sum|y^{\varpi}(t_i)|^2}{\sum|\sigma\varpi(t_i)|^2}\right)$ is equal to $SNR = 20$dB (see [49] for this well known concept in signal processing). We also set $t_i = iT_s \in I = [0,h]$ with $T_s = \frac{I}{M}$, for $i = 0,\cdots,M \in \mathbb{N}^*$. Let us recall the



(a) ${}_R\mathrm{D}^{0.5}_{0,t_i}y(\cdot)$ and ${}_R D^{(\alpha=0.5),\theta=1}_{h=1,\mu=0,\kappa=0,N=5}y^{\varpi}(t_i)$.

(b) $\left|{}_R D^{(\alpha=0.5),\theta=1}_{h=1,\mu=0,\kappa=0,N=5}y^{\varpi}(t_i) - {}_R\,\mathrm{D}^{0.5}_{0,t_i}y(\cdot)\right|$.

Fig. 2. Example 1: $I = [0,1]$ and $\varpi(t_i)$ is a zero-mean Gaussian noise.



(a) ${}_R\mathrm{D}^{0.5}_{0,t_i}y(\cdot)$ and ${}_R D^{(\alpha=0.5),\theta=1}_{h=4,\mu=0,\kappa=0,N=15}y^{\varpi}(t_i)$.

(b) $\left|{}_R D^{(\alpha=0.5),\theta=1}_{h=4,\mu=0,\kappa=0,N=15}y^{\varpi}(t_i) - {}_R\,\mathrm{D}^{0.5}_{0,t_i}y(\cdot)\right|$.

Fig. 3. Example 2: $I = [0,4]$ and $\varpi(t_i)$ is a zero-mean Gaussian noise.

classical Riemann-Liouville fractional order derivative of $\sin(5\cdot)$ (see [50] p. 83):

$$ {}_R\mathrm{D}^{\alpha}_{0,t}\sin(5\cdot) = \frac{5t^{1-\alpha}}{\Gamma(2-\alpha)}\,{}_1\mathrm{F}_2\left(1; \frac{1}{2}(2-\alpha), \frac{1}{2}(3-\alpha); -\frac{1}{4}5^2t^2\right), \tag{26} $$

where ${}_1\mathrm{F}_2$ is the generalized hypergeometric function (see [50] p. 303). Then, the Caputo fractional order derivative of $\sin(5\cdot)$ can be deduced by using (3).

In the following examples, we fix the value of M to $M = 10^3$. Then, we use the fractional order Jacobi differentiator ${}_R D^{(\alpha),\theta}_{h,\mu,\kappa,N}y^{\varpi}(t_i)$ given in Theorem 1 to estimate the half order ($\alpha = 0.5$) derivative of y. In the estimation procedure, we use the trapezoidal rule to numerically calculate the scalar products in (19). Then, the values of ${}_R\mathrm{D}_{0,t_i}^{0.5}y(\cdot)$ can be estimated by taking different values of ξ_i in $q_{\mu,\kappa,i}(\xi_i)$, where $\xi_i = \frac{t_i}{T_s}$, for $i = 1, 2, \cdots, M$.

Example 1. In this example, we take $h = 1$, $\sigma = 0.077$. The noise $\varpi(t_i)$ is simulated from a zero-mean white Gaussian iid sequence by the Matlab function 'randn' with STATE reset to 0. We can see this discrete noisy signal in Figure 1. By taking $\mu = \kappa = 0$ and $N = 5$ in ${}_RD_{h,\mu,\kappa,N}^{(\alpha),\theta}y^{\varpi}(t_i)$, the obtained estimation and the corresponding estimation error are shown in Figure 2. Hence, we can see that the proposed fractional order Jacobi differentiator is accurate and robust with respect to a zero-mean Gaussian noise. However, the estimation errors near the extremities of the interval are relatively larger. This phenomena will be explained in the second part [3].

Example 2. In this example, we increase the length of the interval I by taking $h = 4$. The noise $\sigma\varpi(t_i)$ is still a zero-mean white Gaussian noise with $\sigma = 0.074$. As done in the previous example, we take $\mu = \kappa = 0$ in ${}_RD_{h,\mu,\kappa,N}^{(\alpha),\theta}y^{\varpi}(t_i)$. In this case, we take $N = 15$. Then, we obtain the estimation and the corresponding estimation error in Figure 3. Hence, we can see that this value of N guarantees the validity of the fractional order Jacobi differentiator.

In these two examples, on the one hand, we assume that the noisy signals are given in the whole interval I, then the values of the fractional order derivatives are estimated on each point of I by taking different value of ξ. This corresponds to an off-line application of the fractional order Jacobi differentiator. In the second part of this work [3], we will show how to use the fractional order Jacobi differentiator for on-line applications. On the other hand, we take $\mu = \kappa = 0$ in ${}_RD_{h,\mu,\kappa,N}^{(\alpha),\theta}y^{\varpi}(t_i)$. The value of N is equal to 5 when $h = 1$, and it is increased to 15 when $h = 4$. The analysis on these choices of μ, κ and N will also be given in the second part [3].

VI. Conclusion

In this paper, firstly we have extended the differentiation by integration method from the integer order to the fractional order by proposing two kinds of fractional order Jacobi differentiators. These differentiators can accurately estimate the Riemann-Liouville and the Caputo fractional order derivatives by using exact integral formulae, without complex mathematical deduction. Secondly, accurate error analysis has been proposed by proving some error bounds for the corresponding estimation errors in continuous case. Finally, numerical simulations have been given to show the accuracy and the robustness with respect to a corrupting Gaussian noise of the fractional order Jacobi differentiator in off-line applications. The discrete case is considered in a second paper [3] where we give a guideline on how to choose the design parameters on which the fractional order Jacobi differentiators depend. Then, according to the knowledge of the design parameters' influence, an improved digital fractional order Jacobi differentiator is given by a recursive algorithm for on-line applications.

APPENDIX A
PROOFS

Proof of Theorem 1. We denote the α^{th} order derivative of the polynomial $D^{(0)}_{h,\mu,\kappa,N}y(a+\cdot)$ defined in (16) as follows: $\forall\,\xi\in]0,1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N}y(a+h\xi) := \mathrm{D}^{\alpha}_{0,h\xi}\left\{D^{(0)}_{h,\mu,\kappa,N}y(a+\cdot)\right\}. \tag{27}$$

By using the scale change propriety of the fractional order derivative given in (7), we obtain: $\forall\,\xi\in]0,1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N}y(a+h\xi) = \frac{1}{h^{\alpha}}\mathrm{D}^{\alpha}_{0,\xi}\left\{D^{(0)}_{h,\mu,\kappa,N}y(a+h\cdot)\right\}. \tag{28}$$

Then, we apply the linearity of the fractional order derivative to (16). It yields: $\forall\,\xi\in]0,1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N}y(a+h\xi) = \frac{1}{h^{\alpha}}\sum_{i=0}^{N}\frac{\left\langle P_i^{(\mu,\kappa)}(\cdot),y(a+h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\mathrm{D}^{\alpha}_{0,\xi}P_i^{(\mu,\kappa)}(\cdot). \tag{29}$$

By using Lemma 1 in (29), we obtain: $\forall\,\xi\in]0,1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N}y(a+h\xi) = \frac{1}{h^{\alpha}}\int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\,y(a+h\tau)\,d\tau. \tag{30}$$

Finally, this proof can be completed by substituting y by y^{ϖ} in (30). □

Proof of Proposition 1. By using $y^{\varpi}=y+\varpi$ in (17), the noise error contribution for $D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h\cdot)$ can be obtained: $\forall\,\xi\in]0,1]$,

$$\begin{aligned} e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi) &= D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h\xi) - D^{(\alpha)}_{h,\mu,\kappa,N}y(a+h\xi) \\ &= \frac{1}{h^{\alpha}}\int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\,\varpi(a+h\tau)\,d\tau. \end{aligned}$$

Then, this proof can be completed by taking the absolute value of $e^{\varpi}_{\mu,\kappa,h,\alpha,N}(\xi)$. □

Before giving the proof of Proposition 2, we need the two following lemmas.

Lemma 2 *We assume that y_n is an n^{th} order polynomial defined on I. If $n\le N$ with $N\in\mathbb{N}$, then we have:*

$$\forall\,\xi\in]0,1],\;\; D^{(\alpha)}_{h,\mu,\kappa,N}y_n(a+h\xi) = \mathrm{D}^{\alpha}_{0,h\xi}y_n(a+\cdot), \tag{31}$$

where $D^{(\alpha)}_{h,\mu,\kappa,N} y_n(a+h\cdot)$ *is given by substituting* y *in (30) by* y_n.

Proof. Since y_n is an n^{th} order polynomial defined on I with $n \leq N$, its Jacobi orthogonal series expansion can be given by (16), $i.e.$ we have $y_n(a+h\cdot) \equiv D^{(0)}_{h,\mu,\kappa,N} y_n(a+h\cdot)$. Then, this proof can be completed by taking the α^{th} order derivative of $y_n(a+\cdot)$ and using (27). $\square$

Lemma 3 *We assume that* $y \in \mathcal{C}^{n+1}(\mathbb{R})$ *with* $n \in \mathbb{N}^*$, *and* $\alpha < n$ *with* $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. *Then we have:* $\forall\, \xi \in]0,1]$,

$$ {}_R\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = \sum_{i=0}^{n} \frac{(h\xi)^{i-\alpha}}{\Gamma(i+1-\alpha)} y^{(i)}(a) + \frac{h^{n+1-\alpha}}{\Gamma(n+1-\alpha)} \int_0^{\xi} (\xi-\tau)^{n-\alpha}\, y^{(n+1)}(a+h\tau)\, d\tau, \quad (32)$$

$$ {}_C\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = \sum_{i=l}^{n} \frac{(h\xi)^{i-\alpha}}{\Gamma(i+1-\alpha)} y^{(i)}(a) + \frac{h^{n+1-\alpha}}{\Gamma(n+1-\alpha)} \int_0^{\xi} (\xi-\tau)^{n-\alpha}\, y^{(n+1)}(a+h\tau)\, d\tau, \quad (33)$$

where $l-1 < \alpha < l \leq n$ *with* $l \in \mathbb{N}^*$.

Proof. By substituting $f(\cdot)$ in (3) by $y(a+\cdot)$, we obtain:

$$ {}_R\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = \sum_{i=0}^{l-1} \frac{(h\xi)^{i-\alpha}}{\Gamma(i+1-\alpha)}\, y^{(i)}(a) + \ {}_C\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot). \quad (34)$$

By using (2), the following change of variables $\tau \to h\tau$ gives us:

$$ {}_C\mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = \frac{h^{l-\alpha}}{\Gamma(l-\alpha)} \int_0^{\xi} (\xi-\tau)^{l-1-\alpha}\, y^{(l)}(a+h\tau)\, d\tau. \quad (35)$$

Finally, this proof can be completed by applying $n-l+1$ times integration by parts in (35). $\square$

Consequently, by using (4) and (5), the formulae given in (32) and (33) can be considered as the Riemann-Liouville and the Caputo fractional order derivative of the Taylor series expansion of $y(a+h\cdot)$ with an integral remainder, respectively.

Proof of Proposition 2. If the Taylor series expansion of y at point a converges on I, then the Taylor's formula gives us (see [47] p. 14): $\forall\, \xi \in [0,1]$,

$$ y(a+h\xi) = \sum_{i=0}^{n} \frac{(h\xi)^{i}}{i!} y^{(i)}(a) + \frac{(h\xi)^{n+1}}{n!} \int_0^{1} (1-u)^{n}\, y^{(n+1)}(a+uh\xi)\, du. \quad (36)$$

Then, we define the following truncated Taylor series expansion: $\forall\, \xi \in [0,1]$,

$$ y_n(a+h\xi) = \sum_{i=0}^{n} \frac{(h\xi)^{i}}{i!} y^{(i)}(a). \quad (37)$$

Hence, the α^{th} order derivative of $y_n(a+\cdot)$ can be defined and calculated as follows: $\forall\, \xi \in]0,1]$,

$$ {}_R\mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) = \sum_{i=0}^{n} \frac{(h\xi)^{i-\alpha}}{\Gamma(i+1-\alpha)} y^{(i)}(a), \tag{38} $$

$$ {}_C\mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) = \sum_{i=l}^{n} \frac{(h\xi)^{i-\alpha}}{\Gamma(i+1-\alpha)} y^{(i)}(a). \tag{39} $$

Let us consider the following equality: $\forall\, \xi \in]0,1]$,

$$ \begin{aligned} D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) = &\left(D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) \right) \\ &+ \left(\mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) \right). \end{aligned} \tag{40} $$

Then, by using Lemma 2 and (30) we get:

$$ D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) = \frac{1}{h^{\alpha}} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, \left(y(a+h\tau) - y_n(a+h\tau) \right) d\tau. \tag{41} $$

Hence, the utilization of (36) in (41) gives us:

$$ D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y_n(a+\cdot) = h^{n+1-\alpha} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, I^{y}_{n,h}(\tau)\, d\tau, \tag{42} $$

where $I^{y}_{n,h}(\tau) = \frac{\tau^{n+1}}{n!} \int_0^1 (1-u)^n\, y^{(n+1)}(a+uh\tau)\, du$.

Consequently, by using (40), (38) (resp. (39)) and applying the following change of variables $\tau \to \xi\tau$ in (32) (resp. (33)) we get: $\forall\, \xi \in]0,1]$,

$$ \begin{aligned} e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi) =& D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi) - \mathrm{D}^{\alpha}_{0,h\xi} y(a+\cdot) \\ =& h^{n+1-\alpha} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, I^{y}_{n,h}(\tau)\, d\tau + h^{n+1-\alpha} I^{y}_{n,\alpha,h}(\xi), \end{aligned} \tag{43} $$

where $I^{y}_{n,\alpha,h}(\xi) = \frac{\xi^{n+1-\alpha}}{\Gamma(n+1-\alpha)} \int_0^1 (1-\tau)^{n-\alpha}\, y^{(n+1)}(a+\xi h\tau)\, d\tau$.

Finally, this proof can be completed by taking the absolute value of $e^{\infty}_{\mu,\kappa,h,\alpha,N}(\cdot)$ and the following inequalities:

$$ \begin{aligned} \left| I^{y}_{n,h}(\tau) \right| &\leq M_{n+1} \frac{\tau^{n+1}}{n!} \int_0^1 \left| (1-u)^n \right| du = M_{n+1} \frac{\tau^{n+1}}{(n+1)!}, \\ \left| I^{y}_{n,\alpha,h}(\xi) \right| &\leq M_{n+1} \frac{\xi^{n+1-\alpha}}{\Gamma(n+1-\alpha)} \int_0^1 \left| (1-\tau)^{n-\alpha} \right| d\tau = M_{n+1} \frac{\xi^{n+1-\alpha}}{\Gamma(n+2-\alpha)}. \end{aligned} $$

□

Proof of Corollary 1. Let us denote the error bound by:

$$ \psi(h) = h^{n+1-\alpha} M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi) + \frac{\delta}{h^{\alpha}} E_{\mu,\kappa,\alpha,N}(\xi). \tag{44} $$

Consequently, we can calculate its minimum value. It is obtained for $h^* = \left(\frac{\alpha\, E_{\mu,\kappa,\alpha,N}(\xi)\,\delta}{(n+1-\alpha) M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi)}\right)^{\frac{1}{n+1}}$ and

$$\psi(h^*) = \frac{n+1}{n+1-\alpha}\left(\frac{n+1-\alpha}{\alpha}\right)^{\frac{\alpha}{n+1}} \left(M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi)\right)^{\frac{\alpha}{n+1}} E_{\mu,\kappa,\alpha,N}(\xi)^{\frac{n+1-\alpha}{n+1}}\, \delta^{\frac{n+1-\alpha}{n+1}}. \tag{45}$$

The proof is thus complete. □

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