Fractional order differentiation by integration and error analysis in noisy environment: Part 2 discrete case

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Abstract

In the first part of this work [1], the differentiation by integration method has been generalized from the integer order to the fractional order so as to estimate the fractional order derivatives of noisy signals. The estimation errors for the proposed fractional order Jacobi differentiators have been studied in continuous case. In this paper, the focus is on the study of these differentiators in discrete case. Firstly, the noise error contribution due to a large class of stochastic processes is studied in discrete case. In particular, it is shown that the differentiator based on the Caputo fractional order derivative can cope with a class of noises, the mean value and variance functions of which are time-variant. Secondly, by using the obtained noise error bound and the error bound for the bias term error obtained in the first part, we analyze the design parameters' influence on the obtained fractional order differentiators. Thirdly, according to the knowledge of the design parameters' influence, the fractional order Jacobi differentiators are significantly improved by admitting a time-delay. In order to reduce the calculation time for on-line applications, a recursive algorithm is proposed. Finally, numerical simulations show their accuracy and robustness with respect to corrupting noises.

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Index Terms

Digital fractional order differentiator, Error analysis, Time-delay.

EDICS: SSP-FILT; ASP-ANAL.

I. Introduction

Fractional derivatives are gaining importance in research community because of their capacity to accurately describe real world processes (see, e.g., [2], [3]). The fractional order differentiator is concerned with estimating the fractional order derivatives of an unknown signal from its discrete noisy observation. It can be used to improve the performance and robustness properties in control theory (see, e.g., [4], [5]) and in signal processing applications which can be on-line or off-line (see, e.g., [6], [7]). When designing a fractional order differentiator, the accuracy and the robustness to noise effect must be considered. Moreover, unlike the classical integer order derivative which can be estimated using a sliding window, the fractional order derivative is an hereditary operator and needs a total memory of past states [2]. Hence, the computation time must also be considered in a fractional order differentiator. In the first part of this work [1], we have proposed two kinds of fractional order Jacobi differentiators by generalizing the differentiation by integration method from the integer order to the fractional order. Accurate analysis for the corresponding estimation errors has been given in the continuous case where the noise was assumed to be integrable and locally essentially bounded. The aim of this paper is to study the fractional order Jacobi differentiators in discrete case in order to show their robustness to noises belonging to a large class of stochastic processes. We also show how they can be used for on-line applications.

This paper is organized as follows: In Section II, we provide some error bounds for the noise error contributions due to a large class of stochastic processes in discrete case. These error bounds permit to study the influence of design parameters for the proposed differentiators in Section III. In Section IV, by using the knowledge of the design parameters' influence, an improved digital fractional order differentiator is introduced. Then, a recursive algorithm is given to reduce the computation time for on-line applications. In order to show the efficiency and the stability of the proposed differentiators, numerical simulations results are given in Section V. Finally, we give some conclusions and perspectives for our future work in Section VI. To simplify the presentation, all the proofs are deferred to the appendix.

II. ERROR ANALYSIS IN DISCRETE CASE

Let $y^{\varpi} = y + \varpi$ be a noisy observation of y defined on an interval $I = [a, b] \subset \mathbb{R}$ of length $h = b - a$. If we assume $y \in \mathcal{C}^l(\mathbb{R})$ with $l \in \mathbb{N}^*$, then an estimate for the fractional order derivative $\mathrm{D}^{\alpha}_{0,h\xi} y(a + \cdot)$ ($l - 1 < \alpha < l$) of y, called fractional order Jacobi differentiator, was defined in Theorem 1 in the first part [1] as follows: $\forall\, \xi \in]0, 1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi) := \frac{1}{h^{\alpha}} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau, \xi)\, y^{\varpi}(a + h\tau)\, d\tau, \tag{1}$$

where $h \in \mathbb{R}^*_+$, $N \in \mathbb{N}$, $\mu, \kappa \in] - 1, +\infty[$,

$$Q_{\mu,\kappa,\alpha,N}(\tau, \xi) = w_{\mu,\kappa}(\tau) \sum_{i=0}^{N} \frac{P_i^{(\mu,\kappa)}(\tau)}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\, q_{\mu,\kappa,i}(\xi), \tag{2}$$

$P_i^{(\mu,\kappa)}$ is the i^{th} order shifted Jacobi orthogonal polynomial defined on $[0, 1]$, $\|\cdot\|_{\mu,\kappa}$ is the associated norm with respect to the weighted function $w_{\mu,\kappa}(\tau) = (1 - \tau)^{\mu} \tau^{\kappa}$, and $\mathrm{D}^{\alpha}_{0,\xi} P_i^{(\mu,\kappa)}(\cdot)$ is denoted by $q_{\mu,\kappa,i}(\xi)$. This differentiator refers to ${}_R D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$ (resp. ${}_C D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi)$) which estimates ${}_R\mathrm{D}^{\alpha}_{0,h\xi} y(a + \cdot)$ (resp. ${}_C\mathrm{D}^{\alpha}_{0,h\xi} y(a + \cdot)$). Moreover, we have: $\forall\, \xi \in]0, 1]$,

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi) = \frac{1}{h^{\alpha}} \sum_{i=0}^{N} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y^{\varpi}(a + h\cdot) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\, q_{\mu,\kappa,i}(\xi). \tag{3}$$

From now on, we assume $y^{\varpi}(t_i) = y(t_i) + \varpi(t_i)$ be a noisy discrete observation of y with an equidistant sampling period T_s, where $T_s = \frac{h}{M}$, $M \in \mathbb{N}^*$, and $t_i = a + h\xi_i$ with $\xi_i = \frac{i}{M}$, for $i = 0, \cdots, M$. We also consider a family of noises which are stochastic processes satisfying the following conditions:

(C_1) : for any $t, s \in I$, $t \neq s$, $\varpi(t)$ and $\varpi(s)$ are independent;

(C_2) : the mean value function of $\varpi(\cdot)$ denoted by $\mathbb{E}[\cdot]$ belongs to $\mathcal{L}^1(I)$;

(C_3) : the variance function of $\varpi(\cdot)$ denoted by $\mathrm{Var}[\cdot]$ is bounded on I.

Note that some important classes of noises such as the white Gaussian noise and the Poisson noise satisfy these conditions. We are going to study the fractional order Jacobi differentiator in this discrete case.

A. Fractional order Jacobi differentiator in discrete case

Since y^{ϖ} is a discrete observation, we need to use a numerical integration method to approximate the integral in the fractional order Jacobi differentiator given in (1).

Firstly, we consider the case where $\kappa \geq 0$ and $\mu \geq 0$. It is clear that the smaller the sampling period is, the more accurate the numerical integration method is. However, when the value of T_s decreases, the number of the observed data M increases, hence the computation time becomes larger. In order to solve this problem, we introduce a non-zero integer value θ, and we only take one sample value per θ. Hence, we fix $\tau_j = \frac{j}{\hat{M}}$ with $\hat{M} = \frac{M}{\theta}$, $(\hat{M}, \theta) \in (\mathbb{N}^*)^2$, and we denote by $w_j \geq 0$, for $j = 0, \ldots, \hat{M}$, the weights of a given numerical integration method. Then, the θ-adaptive fractional order Jacobi differentiator $D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i)$ is expressed in the discrete case as follows: $\forall\, \xi_i \in]0, 1]$,

$$D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i) := \frac{1}{h^\alpha} \sum_{j=0}^{\hat{M}} \frac{w_j}{\hat{M}} \, Q_{\mu,\kappa,\alpha,N}(\tau_j, \xi_i) \, y^{\varpi}(a + h\tau_j), \tag{4}$$

with $\kappa \geq 0,\ \mu \geq 0$. Secondly, one can remark that when κ is negative in (1) then the corresponding integral is an improper integral. Hence, if $w_0 \neq 0$ in (4), then there will be a singular value at $\tau_0 = 0$. In order to avoid this problem, we apply the following change of variable $\tau \to \tau^{\frac{1}{1+\kappa}}$ in (1) (see [8], p. 145). Thus, we get:

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi) = \frac{1}{h^\alpha} \int_0^1 \hat{Q}_{\mu,\kappa,\alpha,N}(\tau^{\frac{1}{1+\kappa}}, \xi) \, y^{\varpi}(a + h\, \tau^{\frac{1}{1+\kappa}}) \, d\tau, \tag{5}$$

where $-1 < \kappa < 0$, $\mu \geq 0$, and

$$\hat{Q}_{\mu,\kappa,\alpha,N}(\tau^{\frac{1}{1+\kappa}}, \xi) = \frac{1}{1+\kappa} (1 - \tau^{\frac{1}{1+\kappa}})^\mu \sum_{i=0}^{N} \frac{q_{\mu,\kappa,i}(\xi)}{\| P_i^{(\mu,\kappa)} \|^2_{\mu,\kappa}} \, P_i^{(\mu,\kappa)}(\tau^{\frac{1}{1+\kappa}}).$$

Since y^{ϖ} is equidistantly given, let $\hat{\tau}_j = \left(\frac{j}{\hat{M}}\right)^{1+\kappa}$, *i.e.* $\hat{\tau}_j^{\frac{1}{1+\kappa}} = \frac{j}{\hat{M}} = \tau_j$, be the new abscissas. Hence, the numerical integration steps are equal to $\frac{1}{\hat{M}^{1+\kappa}}[j^{1+\kappa} - (j-1)^{1+\kappa}]$, for $j = 1, \ldots, \hat{M}$. We denote by $\hat{w}_j \geq 0$ their corresponding weights. Then, we have: $\forall\, \xi_i \in]0, 1]$,

$$D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i) = \frac{1}{h^\alpha} \sum_{j=0}^{\hat{M}} \frac{\hat{w}_j}{\hat{M}^{1+\kappa}} \, \hat{Q}_{\mu,\kappa,\alpha,N}(\tau_j, \xi_i) \, y^{\varpi}(a + h\tau_j), \tag{6}$$

with $-1 < \kappa < 0,\ \mu \geq 0$. Finally, we can use a similar procedure in the case where μ is negative (see [8], p. 145 for more details). Then, we can apply a numerical integration method without any singular value.

The estimation error for the fractional order Jacobi differentiator in the discrete noisy case can be decomposed into three sources: $\forall\, \xi_i \in]0, 1]$,

$$\begin{aligned} D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i) - \mathrm{D}^{\alpha}_{0,h\xi_i} y(a + \cdot) = {} & \left(D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i) - D^{(\alpha),\theta}_{h,\mu,\kappa,N} y(a + h\xi_i) \right) \\ & + \left(D^{(\alpha),\theta}_{h,\mu,\kappa,N} y(a + h\xi_i) - D^{(\alpha)}_{h,\mu,\kappa,N} y(a + h\xi_i) \right) \\ & + \left(D^{(\alpha)}_{h,\mu,\kappa,N} y(a + h\xi_i) - \mathrm{D}^{\alpha}_{0,h\xi_i} y(a + \cdot) \right), \end{aligned} \tag{7}$$

where we have:

1) the discrete noise error contribution:

$$e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) = D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a + h\xi_i) - D^{(\alpha),\theta}_{h,\mu,\kappa,N} y(a + h\xi_i), \tag{8}$$

which is due to the noise in the noisy signal y^{ϖ};

2) the numerical error:

$$e^{\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) = D^{(\alpha),\theta}_{h,\mu,\kappa,N} y(a+h\xi_i) - D^{(\alpha)}_{h,\mu,\kappa,N} y(a+h\xi_i), \tag{9}$$

which is due to the used numerical integration method;

3) the N^{th} order truncated term error $e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi_i)$, which is studied in Proposition 3 in the first part [1].

By using (4), the corresponding discrete noise error contribution $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ can be written as follows: $\forall\, \xi_i \in]0,1]$,

$$e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) = \frac{1}{h^{\alpha}} \sum_{j=0}^{\hat{M}} \frac{w_j}{\hat{M}}\, Q_{\mu,\kappa,\alpha,N}(\tau_j,\xi_i)\, \varpi(a+h\tau_j), \tag{10}$$

where $\mu \geq 0,\ \kappa \geq 0$. According to the previous study, $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ can also be given for the other values of κ and μ.

In the next subsection, we are going to study the influence of the sampling period T_s to this discrete noise error contribution.

B. Influence of the sampling period on the discrete noise error contribution

The convergence in mean square with respect to the sampling period of the discrete noise error contributions was studied for the integer order Jacobi differentiator and for some parameter estimators in [9] and [10] respectively. A study was also done in [11] by using the non-standard framework. We can then extend these results in the following proposition.

Proposition 1 *Let $\varpi(\cdot)$ be a stochastic process satisfying conditions $(C_1), (C_2), (C_3)$, and $\{\varpi(t_i)\}_{i=0,\cdots,M}$ be a sequence where $t_i = iT_s$. If the values of M and θ are fixed, then we have the following convergence in mean square of the discrete noise error contribution in the fractional order Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi_i)$:*

$$e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \xrightarrow[T_s \to 0]{\mathcal{L}^2(I)} \frac{1}{h^{\alpha}} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi_i)\, \mathbb{E}\left[\varpi(a+h\tau)\right] d\tau, \tag{11}$$

where $\mathcal{L}^2(I)$ refers to the set of the square-integrable functions defined on I. Moreover, we have:

- *if $\forall\, t \in I$, $\mathbb{E}\left[\varpi(t)\right] = 0$, then*

$$_{R}e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \xrightarrow[T_s \to 0]{\mathcal{L}^2(I)} 0, \tag{12}$$

- *if* $\forall\, t \in I$, $\mathbb{E}\,[\varpi(t)] = \sum_{j=0}^{l-1} \nu_j\, t^j$ *with* $\nu_j \in \mathbb{R}$, $l \in \mathbb{N}^*$ *and* $l-1 < \alpha < l$, *then*

$$_{C}e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \xrightarrow[T_s \to 0]{\mathcal{L}^2(I)} 0, \tag{13}$$

where $_{R}e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ *and* $_{C}e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ *refer to the discrete noisy error contributions in the fractional order Jacobi differentiators* $_{R}D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h\xi_i)$ *and* $_{C}D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h\xi_i)$ *given in (1).*

According to the previous proposition, the discrete noise error contribution can be increasing with respect to the sampling period. As we can see, unlike the Riemann-Liouville fractional order differentiator, the Caputo fractional order differentiator $_{C}D^{(\alpha)}_{h,\mu,\kappa,N}y^{\varpi}(a+h\cdot)$ can cope with a large class of noises the mean value and variance functions of which are time-variant. When the sampling period is set, the discrete noise error contribution does not converge to zero. In the next subsection, we are going to study the discrete noise error contribution with a fixed sampling period.

C. Error bounds on the discrete noise error contribution

An error bound for the noise error contribution is given in Proposition 2 in the first part [1] in the continuous case where the noise is assumed to be integrable and locally essentially bounded. In this subsection, we are going to study the discrete noise error contribution $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ by using the stochastic properties of the noise. Similar to the integer order Jacobi differentiator case [9], [12], sharper noise error bounds can be given, which permit to study the influence of the parameters on $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$. To simplify our notations, we denote $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ by $e^{\varpi,\theta}(\xi_i)$.

Since the noise is a stochastic process, it is generally not bounded. However, if the noise $\varpi(\cdot)$ satisfies the conditions $(C_1), (C_2), (C_3)$, then by applying the properties of the mean value and variance functions, $\mathbb{E}[e^{\varpi,\theta}(\xi_i)]$ and $\mathrm{Var}[e^{\varpi,\theta}(\xi_i)]$ can be obtained for any $\xi_i \in]0,1]$. Consequently, by using the Bienaymé-Chebyshev inequality, we obtain: for any real number $\gamma > 0$, $\forall\, \xi_i \in]0,1]$,

$$\Pr\left(\left|e^{\varpi,\theta}(\xi_i) - \mathbb{E}[e^{\varpi,\theta}(\xi_i)]\right| < \gamma\sqrt{\mathrm{Var}[e^{\varpi,\theta}(\xi_i)]}\right) > 1 - \frac{1}{\gamma^2},$$

$i.e.$ the probability for $e^{\varpi,\theta}(\xi_i)$ to be within the interval $\left]M^{\gamma}_{l}(\xi_i)\,,M^{\gamma}_{h}(\xi_i)\right[$ is higher than $1-\frac{1}{\gamma^2}$, where

$$M^{\gamma}_{l}(\xi_i) = \mathbb{E}[e^{\varpi,\theta}(\xi_i)] - \gamma\sqrt{\mathrm{Var}[e^{\varpi,\theta}(\xi_i)]},$$
$$M^{\gamma}_{h}(\xi_i) = \mathbb{E}[e^{\varpi,\theta}(\xi_i)] + \gamma\sqrt{\mathrm{Var}[e^{\varpi,\theta}(\xi_i)]}.$$

Thus, we deduce two error bounds as follows:

$$\forall\, \xi_i \in]0,1], \quad M_l^{\gamma}(\xi_i) \overset{p_\gamma}{<} e^{\varpi,\theta}(\xi_i) \overset{p_\gamma}{<} M_h^{\gamma}(\xi_i), \tag{14}$$

where $a \overset{p_\gamma}{<} b$ means that the probability for a real number b to be larger than an other real number a is equal to p_γ with $p_\gamma > 1 - \frac{1}{\gamma^2}$. In particular, if ϖ is a white Gaussian noise, then according to the three-sigma rule, we have:

$$\forall\, \xi_i \in]0,1], \quad M_l^{\gamma}(\xi_i) \overset{p_\gamma}{\leq} e^{\varpi,\theta}(\xi_i) \overset{p_\gamma}{\leq} M_h^{\gamma}(\xi_i), \tag{15}$$

where $p_1 = 68.26\%$, $p_2 = 95.44\%$ and $p_3 = 99.73\%$, for $\gamma = 1, 2, 3$, respectively.

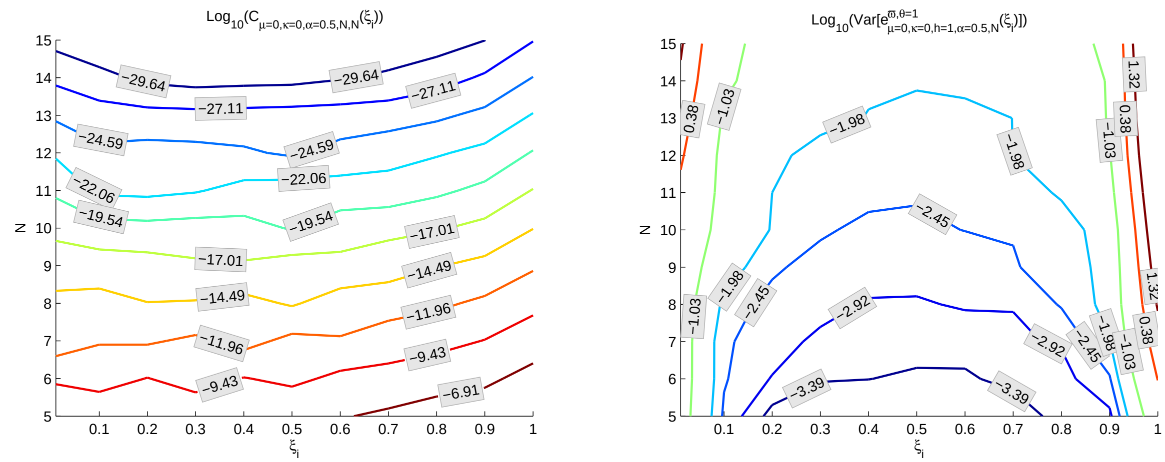
III. Analysis of the parameters' influence

As previously shown, the estimation error for the fractional order Jacobi differentiator $D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi_i)$ in the discrete noisy case can be decomposed into three sources: the numerical error $e^{\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$, the truncated term error $e^{\infty}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ and the discrete noise error contribution $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$. We assume that the numerical integration steps used in $D^{(\alpha),\theta}_{h,\mu,\kappa,N} y^{\varpi}(a+h\xi_i)$ are small enough such that the numerical error can be negligible with respect to the truncated term error and the discrete noise error contribution. We are going to study the influence of parameters on the truncated term error and the discrete noise error contribution.

An error bound for the truncated term error is given in Proposition 3 in the first part [1] and denoted by $h^{n+1-\alpha} M_{n+1} C_{\mu,\kappa,\alpha,n,N}(\xi)$ for any $\xi \in]0,1]$. Let us recall that the value of h depends on the interval where we want to estimate the fractional order derivative, and M_{n+1} depends on the original signal which is unknown. Hence, we are going to study the influence of parameters on the term $C_{\mu,\kappa,\alpha,n,N}(\xi)$ with $\xi \in]0,1]$. This can help us to generally characterize these parameters' influence on the truncated error, independently of the signal to be differentiated. For doing so, we only consider the case of the Riemann-Liouville fractional derivative. The results for the Caputo fractional derivative case can be similarly obtained.

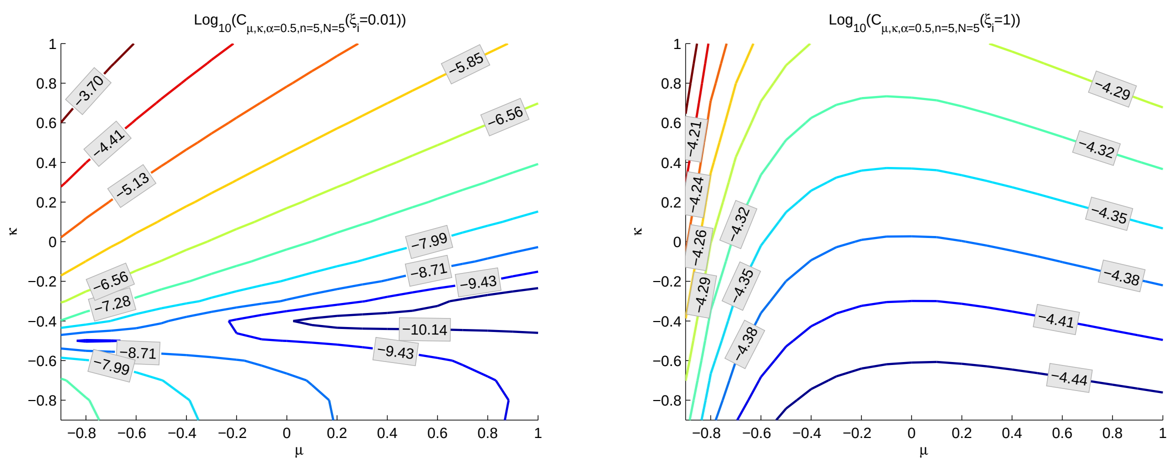
Firstly, we show in Figure 1(a) the variation of $\log_{10} C_{\mu,\kappa,\alpha,n,N}(\xi_i)$ for $N = 5, 6, \ldots, 15$ and $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$ in the case where $n = N$, $\mu = \kappa = 0$ and $\alpha = 0.5$. Hence, we can deduce that $C_{\mu,\kappa,\alpha,n,N}(\xi_i)$ is decreasing with respect to N, and the values of $C_{\mu,\kappa,\alpha,n,N}(\cdot)$ obtained near the extremities are larger than the other values.

Secondly, we show in Figure 2 the variations of $\log_{10} C_{\mu,\kappa,\alpha,n,N}(\xi_i)$ for $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$ in the case where $\alpha = 0.5$, $n = N = 5$, $\xi_i = 0.01$ and 1, respectively. Hence, we



(a) Variation of $\log_{10} C_{\mu=0,\kappa=0,\alpha=0.5,n=N,N}(\xi_i)$. (b) Variation of $\log_{10} \mathrm{Var}\left(e^{\varpi,\theta=1}_{\mu=0,\kappa=0,h=1,\alpha=0.5,N}(\xi_i)\right)$.

Fig. 1. $N = 5, 6, \ldots, 15$ and $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$, $n = N$, $\mu = \kappa = 0$ and $\alpha = 0.5$.



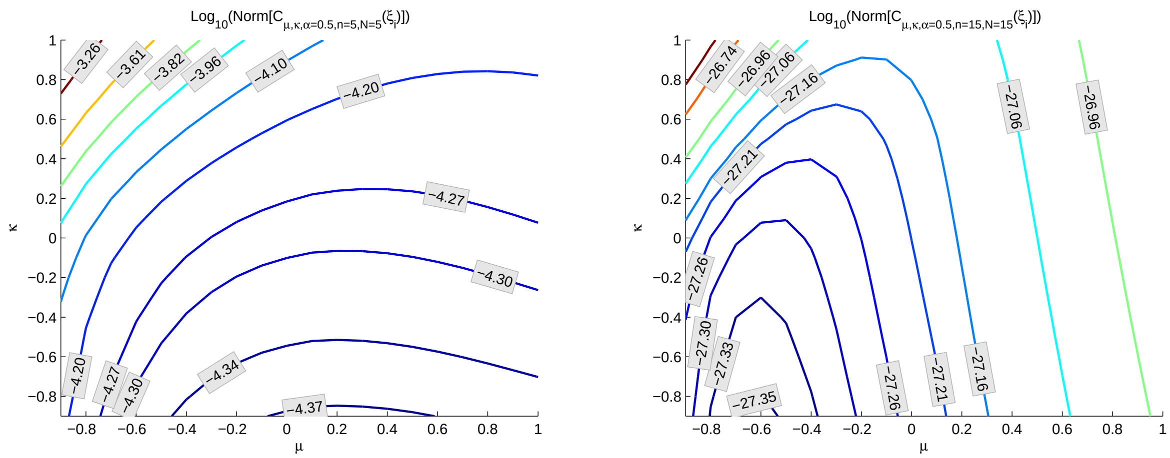
(a) $\xi_i = 0.01$. (b) $\xi_i = 1$.

Fig. 2. Variations of $\log_{10} C_{\mu,\kappa,\alpha=0.5,n=5,N=5}(\xi_i)$ for $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$.

can see that the contour level set of $\log_{10} C_{\mu,\kappa,\alpha,n,N}(\xi_i)$ varies differently with respect to κ and μ for different values of ξ_i. For this, we study the norm $\|C_{\mu,\kappa,\alpha,n,N}(\xi_i)\|_2$.

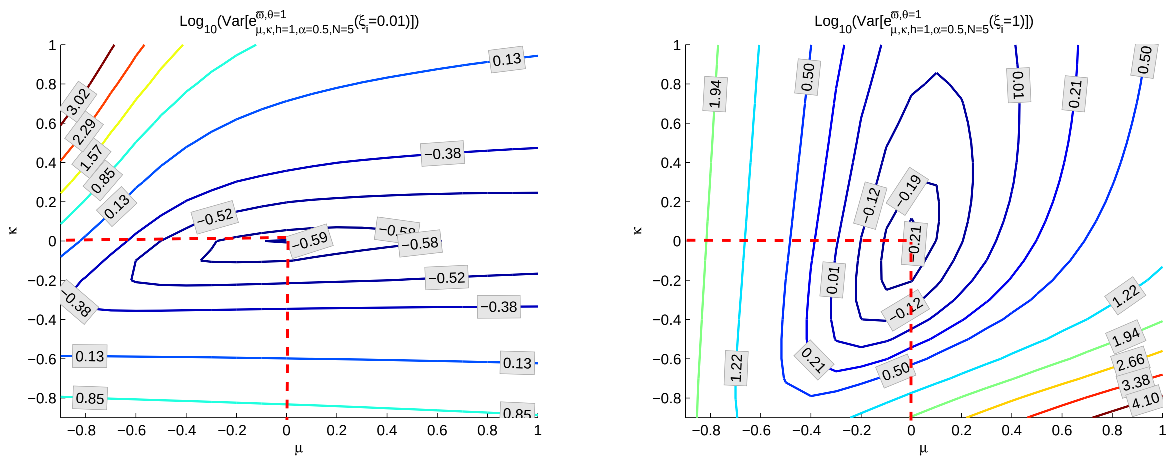
Finally, we show in Figure 3(a) (resp. Figure 3(b)) the variation of $\log_{10} \|C_{\mu,\kappa,\alpha,n,N}(\xi_i)\|_2$ for $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$, $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$ in the case where $\alpha = 0.5$, and $n = N = 5$ (reps. $n = N = 15$). Hence, $\|C_{\mu,\kappa,\alpha,n,N}(\xi_i)\|_2$ can be increasing with respect to κ. Moreover, $\|C_{\mu,\kappa,\alpha,n,N}(\xi_i)\|_2$ can have a local minimum.

Now, we study the parameters' influence on the discrete noise error contribution $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$ by using



(a) $N = 5$. (b) $N = 15$.

Fig. 3. Variations of $\log_{10} \|C_{\mu,\kappa,\alpha=0.5,n=N,N}(\xi_i)\|_2$ for $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$, $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$.



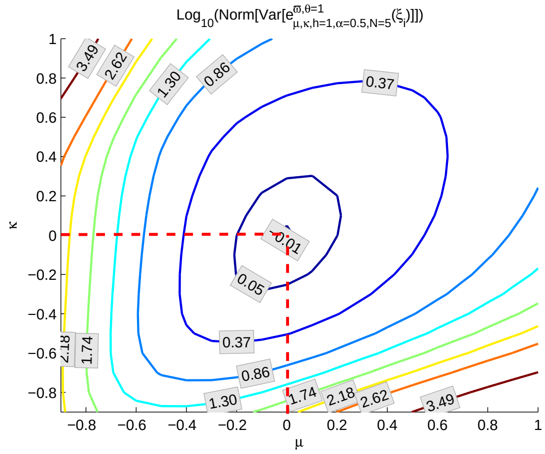
(a) $\xi_i = 0.01$. (b) $\xi_i = 1$.

Fig. 4. Variations of $\log_{10} \mathrm{Var}\left(e^{\varpi,\theta=1}_{\mu,\kappa,h=1,\alpha=0.5,N=5}(\xi_i)\right)$ for $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$.

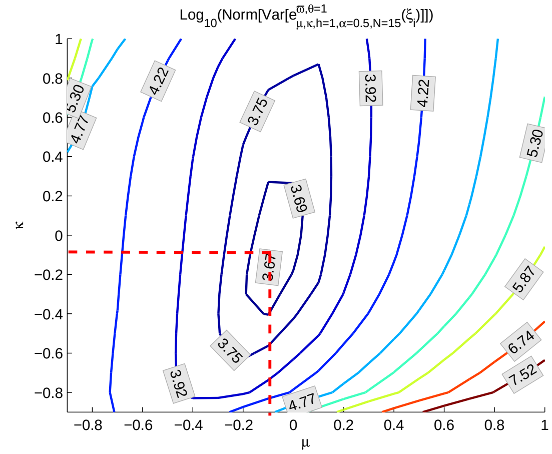
the error bounds obtained in (14). This study is also independent of the original signal to be differentiated, and it only depends on the probability properties of the noise. To simplify this study, we assume that the noises satisfy the conditions $(C_1), (C_2), (C_3)$ and the following condition:

(C_4) : $\forall\, t \in I$, $\mathbb{E}[\varpi(t)] = 0$ and $\mathrm{Var}[\varpi(t)] = \sigma^2$,

with $\sigma \in \mathbb{R}_+$. Hence, we only need to study the variance of $e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i)$, which can be obtained in the



(a) $N = 5$.



(b) $N = 15$.

Fig. 5. Variations of $\log_{10} \left\| \mathrm{Var}\left(e^{\varpi,\theta=1}_{\mu,\kappa,h=1,\alpha=0.5,N}(\xi_i) \right) \right\|_2$ for $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$, $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$.

case of $\mu \geq 0$ and $\kappa \geq 0$ by using (10): $\forall\, \xi_i \in]0, 1]$,

$$\mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right] = \frac{1}{h^{2\alpha}} \frac{\sigma^2}{\hat{M}^2} \sum_{j=0}^{\hat{M}} w_j^2 \, Q^2_{\mu,\kappa,\alpha,N}(\tau_j, \xi_i). \tag{16}$$

We use the trapezoidal rule in (16). Consequently, we study the variation of $\mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right]$ with respect to the parameters μ, κ and N by taking $\theta = h = 1$, $\hat{M} = 10^3$ and $\alpha = 0.5$.

Firstly, we show in Figure 1(b) the variation of $\log_{10} \mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right]$ for $N = 5, \ldots, 15$ and $\xi_i = 0.01, 0.1, 0 \ldots, 1$ in the case where $\mu = \kappa = 0$. Hence, $\mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right]$ can be increasing with respect to N, and $\mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\cdot) \right]$ is decreasing (resp. increasing) with respect to ξ_i when $0.01 \leq \xi_i \leq 0.5$ (resp. $0.5 \leq \xi_i \leq 1$).

Secondly, we show in Figure 4 the variations of $\log_{10} \mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right]$ for $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$ in the case where $N = 5$, $\xi_i = 0.01$ and 1, respectively. Hence, we can see that the variation of $\mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right]$ has a local minimum at $\kappa = \mu = 0$ in each case.

Finally, we show in Figure 5(a) (resp. Figure 5(b)) the variation of $\log_{10} \left\| \mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right] \right\|_2$ for $\xi_i = 0.01, 0.1, 0.2, \ldots, 1$, $\mu = -0.9, -0.8, \ldots, 1$ and $\kappa = -0.9, -0.8, \ldots, 1$ in the case where $N = 5$ (reps. $N = 15$). Hence, $\left\| \mathrm{Var}\left[e^{\varpi,\theta}_{\mu,\kappa,h,\alpha,N}(\xi_i) \right] \right\|_2$ has a local minimum at $\kappa = \mu = 0$ (resp. $\kappa = \mu = -0.1$) when $N = 5$ (resp. $N = 15$).

We conclude the parameters' influence on different error bounds in Table I. The first row means that in order to reduce the truncated term error, according to the analysis on the term $\|C_{\mu,\kappa,\alpha,n,N}(\xi_i)\|_2$ in its associated error bound, we can reduce the value of h and κ. We can also increase the value of N. Moreover, there exists an optimal value of μ.

TABLE I

INFLUENCE OF PARAMETERS ON THE ERROR BOUNDS

Parameters	h	N	κ	μ
Truncated term error $\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\nearrow\searrow$
Noise error contribution $\downarrow$	$\uparrow$	$\downarrow$	$\kappa \approx 0$	$\mu \approx 0$

According to the previous analysis, we should take a trade-off between the parameters so as to do not produce neither a large truncated term error nor a large noise error contribution. Generally, if the value of $h = b - a$ is small, then we can take a small value of N. On the one hand, this value can give a relative small truncated term error. On the other hand, a small value of N can avoid a large noise error contribution. But, when the value of h increases with time, we increase the value of N so as to reduce the truncated term error. Nevertheless this strategy consisting of increasing the value of N can increase the noise error contribution. This effect can be thwarted since a larger value of h reduces the noise error contribution. After setting the value of N, we can take $\kappa = \mu = 0$ to reduce the discrete noisy error contribution. This explains why we have increased the value of N when h was increased in the numerical examples of the frit part [1]. Consequently, the problem is reduced to how to choose the value of N. This choice needs the information on the noise level of the noisy signal and on the derivatives of the original signal, which is usually unknown in practical work. Hence, a criterion should be considered in order to choose an appropriate value of N. This is beyond the scope of this paper.

IV. FRACTIONAL ORDER LEGENDRE DIFFERENTIATOR IN DISCRETE NOISY CASE

In this section, we are going to show how to apply the fractional order Jacobi differentiator for on-line applications in discrete noisy case. According to the analysis done in the previous section, we set $\kappa = \mu = 0$ in the differentiator so as to reduce the discrete noisy error contribution. This parameters' choice implies the use of the Legendre orthogonal polynomials.

The n^{th} order shifted Legendre orthogonal polynomial defined on $[0,1]$ is given as follows (see [16]):

$$\forall\, \tau \in [0,1],\;\; P_n(\tau) := \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(k!)^2} \frac{(n+k)!}{(n-k)!}\, \tau^k. \tag{17}$$

In order to simplify our notations, we denote $D^{(\alpha)}_{h,0,0,N} y^{\varpi}(a+h\xi)$ and $Q_{0,0,\alpha,N}(\tau,\xi)$ by $D^{(\alpha)}_{h,N} y^{\varpi}(a+h\xi)$ and $Q_{\alpha,N}(\tau,\xi)$ respectively. Moreover, we denote $P^{(\alpha)}_i(\xi) = \mathrm{D}^{\alpha}_{0,\xi} P_i(\cdot)$. Then, by taking $\kappa = \mu = 0$ in

(1), we obtain: $\forall\, \xi \in]0,1]$,

$$D_{h,N}^{(\alpha)} y^{\varpi}(a+h\xi) = \frac{1}{h^{\alpha}} \int_0^1 Q_{\alpha,N}(\tau,\xi)\, y^{\varpi}(a+h\tau)\, d\tau, \tag{18}$$

where $Q_{\alpha,N}(\tau,\xi) = \sum_{i=0}^{N} (2i+1) P_i(\tau)\, P_i^{(\alpha)}(\xi)$.

Thus, we call the differentiator $D_{h,N}^{(\alpha)} y^{\varpi}(a+h\xi)$ *fractional order Legendre differentiator*. In next subsection, we are going to consider this differentiator in discrete noisy case so as to introduce a FIR-type digital fractional order Legendre differentiator.

A. A FIR-type digital fractional order Legendre differentiator

We assume that the signal y is observed in a discrete noisy case:

$$y^{\varpi}(t_i) = y(t_i) + \varpi(t_i), \text{ for } i = 0, 1, \ldots, \tag{19}$$

where $t_i = iT_s$, T_s is an equidistant sampling period, and ϖ is a noise.

Let us consider the interval $I_M = [0, t_M]$ with $M \in \mathbb{N}^*$. By taking $a = 0$, $h = t_M$ and $\xi = 1$ in (18), we can use the fractional order Legendre differentiator to estimate the value of the α^{th} order derivative of y at t_M. Then, we apply the right rectangle rule to approximate the integral in (18):

$$\mathrm{D}_{0,t_M}^{\alpha} y(\cdot) \approx D_{h,N}^{(\alpha)} y^{\varpi}(t_M) \approx \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{j=1}^{M} Q_{\alpha,N}(1,\tau_j)\, y^{\varpi}(t_j), \tag{20}$$

where $\tau_j = \frac{j}{M}$ are the associated abscissas. Then, by applying a change of indices $j \to M-j$ in (20), we get:

$$\mathrm{D}_{0,t_M}^{\alpha} y(\cdot) \approx \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{j=1}^{M} Q_{\alpha,N}(1,\tau_{M-j})\, y^{\varpi}(t_{M-j}). \tag{21}$$

Consequently, by denoting $y^{\varpi}(t_{M-j})$ by y_{M-j}^{ϖ} and using the expression of $Q_{\alpha,N}(\cdot,\cdot)$ given in (18), we obtain the following FIR digital fractional order differentiator for $\mathrm{D}_{0,t_M}^{\alpha} y(\cdot)$:

$$\tilde{y}_M^{(\alpha)}(N) := \sum_{j=1}^{M} H_j^M\, y_{M-j}^{\varpi}, \text{ for } M = 1, 2, \ldots, \tag{22}$$

where $H_j^M = \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{i=0}^{N} (2i+1) P_i^{(\alpha)}(1)\, P_i(\frac{M-j}{M})$.

B. A time-delayed digital fractional order Legendre differentiator

The digital fractional order Legendre differentiator $\tilde{y}_M^{(\alpha)}(N)$ is obtained by taking $\xi = 1$ in the fractional order Legendre differentiator given in (18) in the discrete noisy case. However, it was shown in Figure 1(a) and Figure 1(b) in Section VI that if we take $\xi = 1$ or $\xi = 0.01$ in (18), then the obtained estimation errors, both in noise-free and in noisy cases, can be much larger than the ones obtained by taking the other values of ξ.

Let us recall that an important contribution of the integer order Jacobi differentiator was to introduce a time-delay in the differentiator, which improves significatively both the truncated term error and the noise error contribution (see [13], [14], [15] for more details). Bearing this idea in mind, we consider the following digital fractional order differentiator for $\mathrm{D}^{\alpha}_{0,t_M-\vartheta}y(\cdot)$ with $\vartheta \in \mathbb{R}_+^*$:

$$\tilde{y}_{M,\vartheta}^{(\alpha)}(N) := \sum_{j=1}^{M} H_j^{M,\vartheta}\, y_{M-j}^{\varpi}, \text{ for } M = 1, 2, \ldots, \tag{23}$$

where $H_j^{M,\vartheta} = \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{i=0}^{N}(2i+1)P_i^{(\alpha)}(1-\dfrac{\vartheta}{t_M})\, P_i(\dfrac{M-j}{M})$ with $t_M = MT_s$. If we use $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$ to estimate the value $\mathrm{D}^{\alpha}_{0,t_M}y(\cdot)$, then we introduce a time-delay of value ϑ.

As we can see, if we take $\vartheta = 0$ in (23), then we can obtain (22). Hence, (23) gives a general expression of the digital fractional order Legendre differentiator $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$ with $\vartheta \in \mathbb{R}_+$.

C. A recursive algorithm

Let us remark that the coefficients $H_j^{M,\vartheta}$ given in (23) depend on the value of M. Then, we need to calculate them for each values of M. Since these coefficients do not depend on the samples of y, we can calculate them in an off-line manner. Hence, according to (23) we need $M-1$ additions to calculate the digital fractional order differentiator $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$ ($\vartheta \in \mathbb{R}_+$) in an on-line application. However, when the value of M increases, the computation time becomes larger and larger. In order to solve this problem, we give a recursive algorithm in the following proposition.

Proposition 2 *The digital fractional order Legendre differentiator $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$ given in (23) with $\vartheta \in \mathbb{R}_+$ can be calculated by:*

$$\tilde{y}_{M,\vartheta}^{(\alpha)}(N) = \sum_{k=0}^{N} \lambda_k^{\alpha,\vartheta}(N,M), \textit{ for } M = 1, 2, \ldots, \tag{24}$$

where the coefficients are given as follows:

$$\lambda_k^{\alpha,\vartheta}(N,M) = \frac{1}{T_s^{\alpha}}\, l_k^{\alpha,\vartheta}(N)\, \varphi_k^{\alpha}(M), \tag{25}$$

with

$$\varphi_k^\alpha(M) = \frac{1}{M^{\alpha+k+1}} \sum_{j=1}^{M} j^k \, y_j^\varpi, \tag{26}$$

$$l_k^{\alpha,\vartheta}(N) = \sum_{i=k}^{N} \frac{(-1)^{i+k}}{(k!)^2} \frac{(i+k)!}{(i-k)!} (2i+1) P_i^{(\alpha)}(1 - \frac{\vartheta}{t_M}). \tag{27}$$

Moreover, $\varphi_k^\alpha(M)$ *can be given by the following recursive formula:*

$$\begin{cases} \varphi_k^\alpha(1) & = y_1^\varpi, & \textit{for } M = 1, \\ \varphi_k^\alpha(M) & = \frac{(M-1)^{\alpha+k+1}}{M^{\alpha+k+1}} \varphi_k^\alpha(M-1) + \frac{1}{M^{\alpha+1}} y_M^\varpi, & \textit{for } M = 2, 3, \ldots. \end{cases} \tag{28}$$

In particular, if $\vartheta = 0$ *for any* $M \in \mathbb{N}^*$*, then the coefficients can be given in the following recursive formula:*

$$\begin{cases} \lambda_k^{\alpha,0}(N,1) & = \frac{l_k^{\alpha,0}(N)}{T_s^\alpha} \, y_1^\varpi, & \textit{for } M = 1, \\ \lambda_k^{\alpha,0}(N,M) & = \frac{(M-1)^{\alpha+k+1}}{M^{\alpha+k+1}} \lambda_k^{\alpha,0}(N, M-1) + \frac{l_k^{\alpha,0}(N)}{T_s^\alpha M^{\alpha+1}} y_M^\varpi, & \textit{for } M = 2, 3, \ldots. \end{cases} \tag{29}$$

Consequently, $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$ can be considered as a sum of $N+1$ coefficients. When the value of M increases, it is sufficient to recalculate these coefficients using the recursive algorithm. When $\vartheta = 0$, after setting the value of N, we can calculate the coefficients $\frac{l_k^{\alpha,0}(N)}{T_s^\alpha}$, $\frac{l_k^{\alpha,0}(N)}{T_s^\alpha M^{\alpha+1}}$, and $\frac{(M-1)^{\alpha+k+1}}{M^{\alpha+k+1}}$, for $k = 0, 2, \ldots, N$, in an off-line work. Then, when the signal passes form t_{M-1} to t_M, by using the old coefficients $\lambda_k^{\alpha,0}(N, M-1)$, and the new sample y_M^ϖ in (29), we only need $2(N+1)$ multiplications and $N+1$ additions to construct all the new coefficients $\lambda_k^{\alpha,0}(N,M)$. Consequently, we only need $4N+3$ operations to calculate $\tilde{y}_M^{(\alpha)}(N)$ in an on-line application for each value of $M \in \mathbb{N}^*$. Thus, comparing to (22) the calculation time is significantly improved. If $\vartheta \neq 0$, according to (28) we then need $2(N+1)$ multiplications and $N+1$ additions to construct all new coefficients $\varphi_k^\alpha(M)$. Then, by using (25) we need $N+1$ multiplications to calculate $\lambda_k^{\alpha,\vartheta}(N,M)$. Consequently, we need $5N+4$ operations to calculate the time-delayed fractional order differentiator $\tilde{y}_{M,\vartheta}^{(\alpha)}(N)$. Although the accuracy of the digital fractional order Legendre differentiator can be improved by admitting a time-delay, the price to pay is that the computation time is increased.

D. An adaptive algorithm

According to Table I in Section VI, when the length of the interval, where we estimate the fractional order derivative of a noisy signal, increases, we need to increase the value of N so as to reduce the estimation error. Hence, when the value of M increases in the estimation procedure, we should increase the value of N in the digital fractional order Legendre differentiator.

Let us substitute N in (24) by $N+1$, then we obtain:

$$\tilde{y}^{(\alpha)}_{M,\vartheta}(N+1)=\sum_{k=0}^{N+1}\lambda^{\alpha,\vartheta}_{k}(N+1,M). \tag{30}$$

Hence, we need to calculate the coefficients $\lambda^{\alpha,\vartheta}_{k}(N+1,M)$ for $k=0,1,\ldots,N+1$. By using (27), we get: for $k=0,1,\ldots,N+1$,

$$l^{\alpha,\vartheta}_{k}(N+1)=l^{\alpha,\vartheta}_{k}(N)+d^{\alpha,\vartheta}_{N+1,k}, \tag{31}$$

where $d^{\alpha,\vartheta}_{N+1,k}=\frac{(-1)^{N+1+k}}{(k!)^2}\frac{(N+1+k)!}{(N+1-k)!}(2N+3)P^{(\alpha)}_{N+1}(1-\frac{\vartheta}{t_M})$. Then, we deduce from (25) and (31) a recursive relation for the coefficients $\lambda^{\alpha,\vartheta}_{k}(\cdot,M)$:

$$\lambda^{\alpha,\vartheta}_{k}(N+1,M)=\lambda^{\alpha,\vartheta}_{k}(N,M)+\frac{\varphi^{\alpha}_{k}(M)}{T^{\alpha}_{s}}\,d^{\alpha,\vartheta}_{N+1,k}, \tag{32}$$

for $k=0,1,\ldots,N$. Moreover, by using (25) and (27) we get:

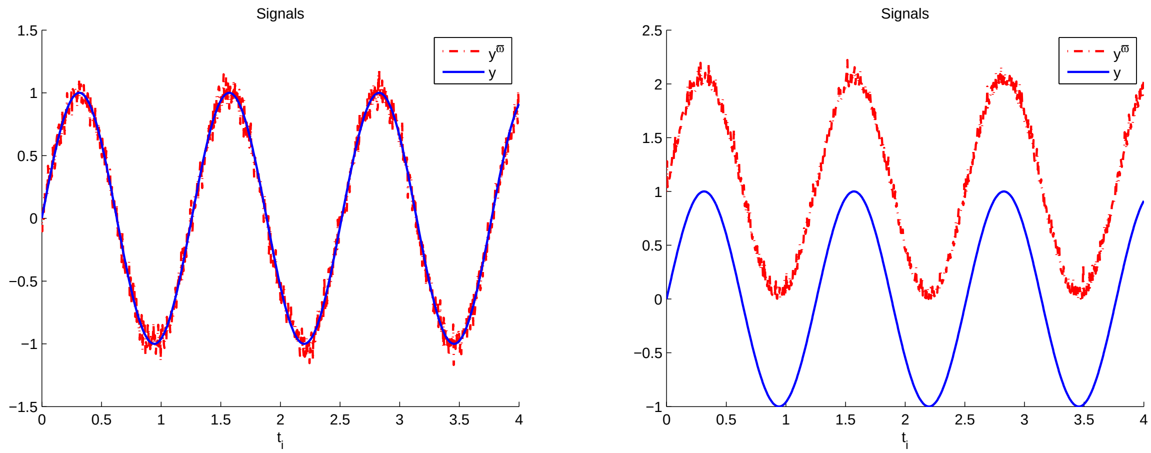
$$\lambda^{\alpha,\vartheta}_{N+1}(N+1,M)=\frac{1}{T^{\alpha}_{s}}\,l^{\alpha,\vartheta}_{N+1}(N+1)\,\varphi^{\alpha}_{N+1}(M)=\frac{1}{T^{\alpha}_{s}}\,d^{\alpha,\vartheta}_{N+1,N+1}\,\varphi^{\alpha}_{N+1}(M). \tag{33}$$

Consequently, according to (32) and (33), when the value of N passes from N to $N+1$, we only need to calculate the following terms: $\varphi^{\alpha}_{N+1}(M)$, and $d^{\alpha,\vartheta}_{N+1,k}$ for $k=0,1,\ldots,N+1$, where $\varphi^{\alpha}_{N+1}(M)$ can be given by using (26).

V. SIMULATION RESULTS

In the first part of this work [1], numerical simulations have been given to show the accuracy and the robustness with respect to corrupting noises of the fractional order Jacobi differentiators for off-line applications. In this section, we are going to show the properties of these differentiators for on-line applications.

As done in Example 2 in the first part [1], we take $y^{\varpi}(t_i)=\sin(5t_i)+\sigma\varpi(t_i)$, where $t_i=iT_s\in I=[0,h]=[0,4]$, $T_s=\frac{I}{M}$, for $i=0,\cdots,M\in\mathbb{N}^*$, and $\sigma\in\mathbb{R}^*_+$ is adjusted in such a way that the signal-to-noise ratio is equal to $SNR=20$dB. In the following examples, we fix the value of T_s to $T_s=4\times10^{-3}$. Then, we use the digital fractional order Legendre differentiator $\tilde{y}^{(\alpha)}_{i}(N)$ (DFOLD) given in (22) and the time-delayed digital fractional order Legendre differentiator $\tilde{y}^{(\alpha)}_{i,\vartheta}(N)$ (DFOLD-TD) given in (24) to estimate the values of $\mathrm{D}^{0.5}_{0,t_i}y(\cdot)$ on I. Moreover, we take different values of N for different interval's lengths. Hence, we apply the adaptive algorithm given in Subsection IV-D to these differentiators. For this purpose, we take the following values of N in different intervals: $N=5$ for $t_i\in[50T_s,1[$; $N=8$ for $t_i\in[1,2[$; $N=11$ for $t_i\in[2,3[$; $N=15$ for $t_i\in[3,4]$.



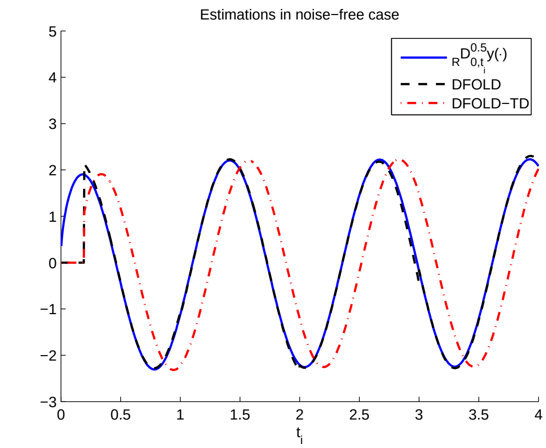
(a) $y^{\varpi}(t_i) = \sin(5t_i) + 0.074\varpi(t_i)$.

(b) $y^{\varpi}(t_i) = \sin(5t_i) + 0.052\varpi(t_i) + 1$.

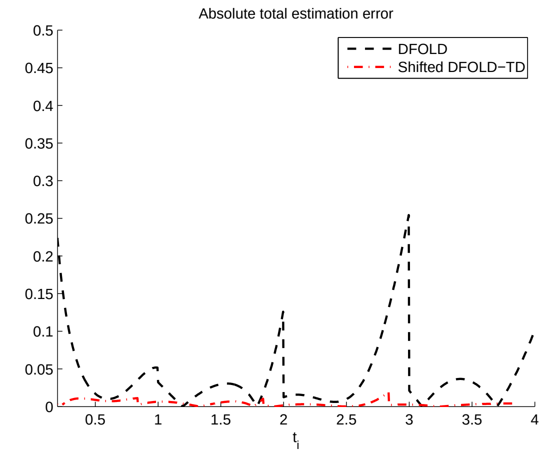
Fig. 6. Signal y and two different noisy signals y^{ϖ} ((a) $\mathbb{E}[\varpi] = 0$ and (b) $\mathbb{E}[\varpi] = 1$).

Example 1. In this example, the noise $\varpi(t_i)$ is assumed to be a zero-mean white Gaussian noise with $\sigma = 0.074$. The discrete noisy signal is given in Figure 6(a). Then, we use $\tilde{y}_i^{(\alpha)}(N)$ and $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$ with $\vartheta = 0.16$ to estimate ${}_R\mathrm{D}_{0,t_i}^{0.5} y(\cdot)$ in the noise-free and noisy cases respectively. We can see the obtained estimations in Figure 7(a) and Figure 7(c), where the dot lines (black) present the estimations obtained by $\tilde{y}_i^{(\alpha)}(N)$, and the dot-dash lines (red) present the ones obtained by $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$. Moreover, since the time-delay for $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$ is equal to ϑ, by shifting the obtained estimations, we can calculate the estimation errors for $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$. The obtained estimation errors are shown in Figure 7(b) and Figure 7(d). Consequently, the digital fractional order Legendre differentiators $\tilde{y}_i^{(\alpha)}(N)$ is significantly improved by $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$ by admitting a time-delay.

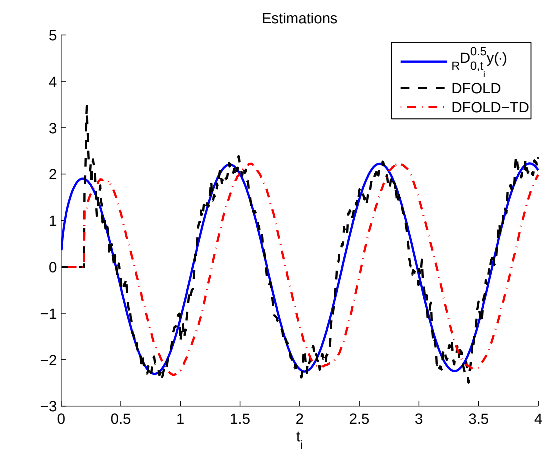
Example 2. In this example, the noise $\sigma\varpi(t_i) + 1$ is assumed to be a biased Poisson noise with $\sigma = 0.052$. The discrete noisy signal is given in Figure 6(b). According to Proposition 1, we can use the fractional order differentiator $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$ ($\vartheta = 0.16$) corresponding to the Caputo fractional derivative to cope with this non-centered noise. We obtain the estimation and the corresponding estimation error with a correction of time-delay in Figure 8. Consequently, we can see that the time-delayed digital fractional order differentiator $\tilde{y}_{i,\vartheta}^{(\alpha)}(N)$ is robust with respect to a Poisson noise.



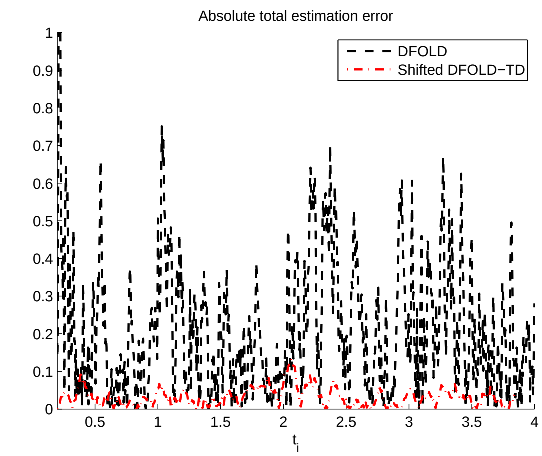
(a) DFOLD: $\tilde{y}_i^{(\alpha=0.5)}(N)$ and DFOLD-TD: $\tilde{y}_{i,\vartheta=0.16}^{(\alpha=0.5)}(N)$.



(b) $\left|\tilde{y}_i^{(\alpha=0.5)}(N) -_R \mathrm{D}_{0,t_i}^{0.5} y(\cdot)\right|$, $\left|\tilde{y}_{i,\vartheta=0.16}^{(\alpha=0.5)}(N) -_R \mathrm{D}_{0,t_i-\vartheta}^{0.5} y(\cdot)\right|$.



(c) DFOLD: $\tilde{y}_i^{(\alpha=0.5)}(N)$ and DFOLD-TD: $\tilde{y}_{i,\vartheta=0.16}^{(\alpha=0.5)}(N)$.

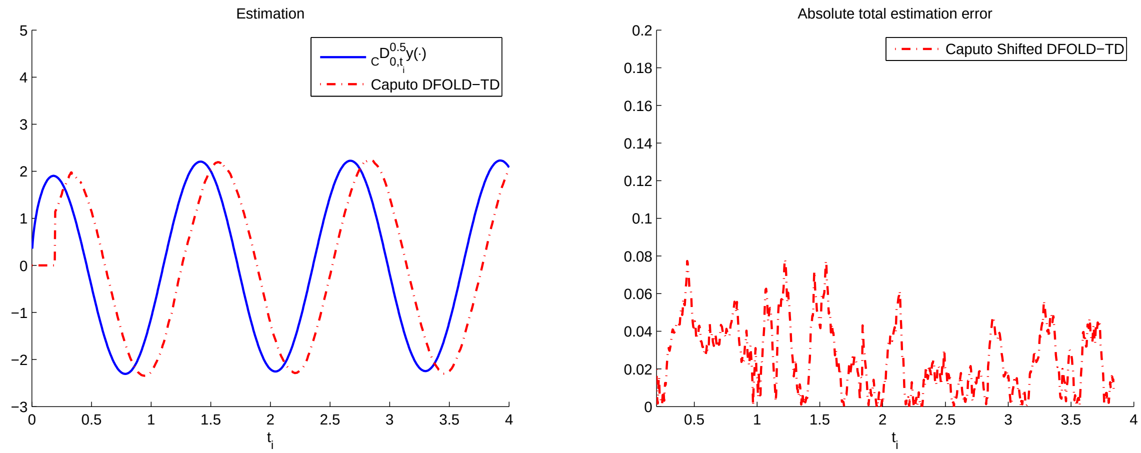


(d) $\left|\tilde{y}_i^{(\alpha=0.5)}(N) -_R \mathrm{D}_{0,t_i}^{0.5} y(\cdot)\right|$, $\left|\tilde{y}_{i,\vartheta=0.16}^{(\alpha=0.5)}(N) -_R \mathrm{D}_{0,t_i-\vartheta}^{0.5} y(\cdot)\right|$.

Fig. 7. Example 1: $\varpi(t_i)$ is a zero-mean Gaussian noise.

VI. Conclusion

In the first part of this work [1], two kinds of fractional order differentiators have been proposed and studied in continuous case. In this paper, we have studied these differentiators in discrete case. Hence, they can be used both for continuous-time and discrete-time models in noisy environment. Firstly, the noise error contribution due to a large class of stochastic processes has been studied. In particular, it has been shown that the differentiator based on the Caputo fractional order derivative can cope with a class of non-centered noises. Secondly, the analysis on the obtained error bounds has given us a guideline on how to



(a) ${}_C\mathrm{D}^{0.5}_{0,t_i}y(\cdot)$ and Caputo DFOLD-TD: $\tilde{y}^{(\alpha=0.5)}_{i,\vartheta=0.16}(N)$.

(b) $\left|\tilde{y}^{(\alpha=0.5)}_{i,\vartheta=0.16}(N) -_C \mathrm{D}^{0.5}_{0,t_i-\vartheta}y(\cdot)\right|$.

Fig. 8. Example 2: $\varpi(t_i)$ is a biased Poisson noise.

choose the design parameters on which the fractional order Jacobi differentiators depend. Thirdly, a FIR-type digital fractional order Legendre differentiator has been given. Then, according to the knowledge of the design parameters' influence, an improved digital fractional order Legendre differentiator has been proposed by introducing a time-delay. A recursive algorithm has also been obtained, which is useful for reducing the computation time for on-line applications. Finally, numerical simulations have been performed to evaluate the proposed differentiators, and the results show their efficiency. In these numerical examples, the choice of the values of N was done by experience. The objective was to show the efficiency and stability of our fractional order differentiators with these values of N, and to give a general idea on how to choose the value of N. In order to choose appropriate values of N in the case where the noisy signal is unknown, a criterion should be considered in the adapted algorithm. In a future work, the generalization for the fractional order Jacobi differentiators to multivariate case will be done for applications in image enhancement.

APPENDIX A
PROOFS

Proof of Proposition 1. If $\kappa \geq 0$ and $\mu \geq 0$, then according to (2) we have:

$$\forall\, \xi \in]0,1],\;\; Q_{\mu,\kappa,\alpha,N}(\cdot,\xi) \in \mathcal{L}^2\left([0,1]\right).$$

If $-1 < \kappa < 0$ and $\mu \geq 0$, then according to (5) we have:

$$\forall\, \xi \in]0,1],\ \hat{Q}_{\mu,\kappa,\alpha,N}(\cdot,\xi) \in \mathcal{L}^2\left([0,1]\right).$$

Consequently, by using Theorem 1 in [10], (11) can be obtained in the case where $-1 < \kappa$ and $\mu \geq 0$. It can also be similarly obtained in the case where $-1 < \mu$ and $\kappa \geq 0$. Then, (12) can be deduced. Finally, by using Lemma 2 given in the first part [1] and the definition of the Caputo fractional order derivative, (13) can be obtained. □

Proof of Proposition 2. By applying a change of indices $j \to M - j$ in (23) and using (17), the digital fractional order Legendre differentiator $\tilde{y}^{(\alpha)}_{M,\vartheta}(N)$ can also be given as follows:

$$\tilde{y}^{(\alpha)}_{M,\vartheta}(N) = \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{j=1}^{M} \left(\sum_{i=0}^{N} c_i^{\alpha,\vartheta} \sum_{k=0}^{i} d_{i,k}\, \tau_j^k \right) y_j^{\varpi}, \tag{34}$$

where $c_i^{\alpha,\vartheta} = (2i+1)P_i^{(\alpha)}(1 - \frac{\vartheta}{t_M})$, $d_{i,k} = \frac{(-1)^{i+k}}{(k!)^2}\frac{(i+k)!}{(i-k)!}$, and $\tau_j = \frac{j}{M}$. Let us regroup the terms in the following sums:

$$\sum_{i=0}^{N} c_i^{\alpha,\vartheta} \sum_{k=0}^{i} d_{i,k}\, \tau_j^k = \sum_{k=0}^{N} \left(\sum_{i=k}^{N} c_i^{\alpha,\vartheta}\, d_{i,k} \right) \tau_j^k. \tag{35}$$

Then, by denoting $l_k^{\alpha,\vartheta}(N) = \sum_{i=k}^{N} c_i^{\alpha,\vartheta}\, d_{i,k}$, (34) becomes:

$$\begin{aligned} \tilde{y}^{(\alpha)}_{M,\vartheta}(N) &= \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{j=1}^{M} \sum_{k=0}^{N} l_k^{\alpha,\vartheta}(N)\, \tau_j^k\, y_j^{\varpi} \\ &= \frac{T_s^{-\alpha}}{M^{\alpha+1}} \sum_{k=0}^{N} l_k^{\alpha,\vartheta}(N) \left(\sum_{j=1}^{M} \tau_j^k\, y_j^{\varpi} \right) \\ &= \frac{1}{T_s^{\alpha}} \sum_{k=0}^{N} l_k^{\alpha,\vartheta}(N)\, \varphi_k^{\alpha}(M), \end{aligned} \tag{36}$$

where $\varphi_k^{\alpha}(M) = \dfrac{1}{M^{\alpha+k+1}} \sum_{j=1}^{M} j^k\, y_j^{\varpi}$. Thus, (25) is obtained. Moreover, $\varphi_k^{\alpha}(\cdot)$ satisfies the recursive relation given in (28). Finally, if $\vartheta = 0$, then $l_k^{\alpha,\vartheta}(N)$ does not depend on M any more. Consequently, (29) can be obtained by using (25) and (28). □

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