Normal Forms for the Algebraic Lambda-Calculus

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Abstract

We study the problem of defining normal forms of terms for the algebraic λ-calculus, an extension of the pure λ-calculus where linear combinations of terms are first-class entities: the set of terms is enriched with a structure of vector space, or module, over a fixed semiring. Towards a solution to the problem, we propose a variant of the original reduction notion of terms which avoids annoying behaviours affecting the original version, but we find it not even locally confluent. Finally, we consider reduction of linear combinations of terms over the semiring of polynomials with non-negative integer coefficients: terms coefficients are replaced by indeterminates and then, after reduction has taken placed, restored back to their original value by an evaluation function. Such a special setting permits us to talk about normal forms of terms and, via an evaluation function, to define such notion for any semiring.

1. Introduction

The principal aim of this paper is to investigate on normalization properties of the algebraic λ-calculus and, in particular, to give a first proper definition of normal form of terms of the calculus. The algebraic λ-calculus has been introduced by Vaux [8] extending the pure λ-calculus with an algebraic structure of module which permits to express linear combinations of terms. The origins of its algebraic extension have to be identified in the work done by Ehrhard and Regnier [1] on the differential λ-calculus. Unlike the latter, the algebraic λ-calculus focuses on a meticulous development of the algebraic structure of terms and the interaction between coefficients and reduction notions.

Linearity and the $\boldsymbol{\lambda}$-calculus. The computational interpretation of linearity of Girard [4] linear logic has made possible to relate resource aware reduction notions of the λ-calculus with the more usual algebraic notion. Considering the λ-calculus, a term is said to be linear if it uses its argument exactly once, an intuition formalized by making a term application linear in the function position but not in the argument position. This is related with head reduction strategy which requires subterms in function position to be evaluated exactly once: contrary to those in argument position, they are not copied nor discarded.

Algebraic linearity is generally thought of as commutation with sums. Following Girard's [5] basic idea on quantitative semantics, Ehrhard [2, 3] introduced denotational models of linear logic where formulas are interpreted as particular vector spaces, or modules, and proofs corresponding to λ-terms are interpreted as analytic functions defined by power series on these spaces. This guided Ehrhard and Regnier [1] to specify the differential λ-calculus which not only introduced a syntactic operator of differentiation into the λ-calculus but also offered serious grounding to endow the set of terms with a structure of vector space, or of R-module, with R a semiring: one can form linear combinations of terms, subject to the following identities:

$$\lambda x\left(\sum_{i=1}^{n} a_i s_i\right) = \sum_{i=1}^{n} a_i \lambda x\, s_i \tag{1}$$

and

$$\left(\sum_{i=1}^{n} a_i s_i\right) t = \sum_{i=1}^{n} a_i (s_i) t \tag{2}$$

for all linear combinations $\sum_{i=1}^{n} a_i s_i$ of terms (with $a_i \in \mathsf{R}$ to be considered as coefficients). Remarking the fact that in the above identities application is linear in the function and not in the argument, it is clear how the work on differential λ-calculus makes the two notions of linearity compatible.

Reducing Linear Combinations of $\boldsymbol{\lambda}$-terms. One important feature of the calculus formalized by Ehrhard and Regnier [1] is the way β-reduction is extended to linear combinations of terms. Among terms, those which are not subject to the identities (1) and (2) are called simple terms since they do not contain sum in function position. Of course, every term can be made simple exploiting the identities (1) and (2). Then, by naturally extending the classical term substitution to the case of a sum term, β-reduction can be widened to the present setting: $\to$ is the least contextual relation such that, if s is a simple term then

$$(\lambda x\, s)t \to s[t/x] \tag{3}$$

and, if a is a non-zero scalar, then

$$s \to s' \text{ implies } as + t \to as' + t\ . \tag{4}$$

The requirement that s is simple in (3) and (4), along with the condition $a \neq 0$ in (4), prevents $\to$ from being trivially reflexive, thus ensuring the reduction of something. This reduction notion exhibits good properties, most notably in the case of positive coefficients. For instance, it has been shown to be confluent via the usual Tait-Martin-Löf technique.

Motivating the Present Work. Once a notion of reduction is defined for the calculus, one might want to study its normalization properties. Quite obviously, since λ-calculus can be seen as a fragment of the algebraic one, the untyped version of the latter suffers of usual normalization issues of the former.

Unsurprisingly, linear combinations of terms show normal forms only in case the set of coefficients fulfills some properties, for one that of being defined on non-negative coefficients only. Such forms are, intuitively, those of the pure λ-calculus extended to sums.

In all other cases, even if $\to$ has been proved to be confluent, it does not even make sense to talk about the normal form of terms. As a matter of fact, if we also allow coefficients to be negative, every term can reduce implying that there is no normal term: for all terms s and t, $s \to^* t$. Indeed, take a fixpoint operator Y of the λ-calculus such that $(Y)s \to^* (s)(Y)s$, for all λ-terms s. Then, write $\infty_s = (Y)\lambda x\,(s + x)$ which reduces as $\infty_s \to^* s + \infty_s$, hence meaning an infinite amount of s. We easily get:

$$s = s + \infty_s - \infty_s + \infty_t - \infty_t \to^* s - s + t = t$$

Normalization issues are subtler than one might think. Indeed, assume coefficients to be in $\mathbb{Q}^+$, the set of non-negative rational numbers, and $s \to s'$. Then, there is an infinite sequence of reductions from s:

$$s = \frac{1}{2}s + \frac{1}{2}s \to \frac{1}{2}s + \frac{1}{2}s' \to \frac{1}{4}s + \frac{3}{4}s' \to \dots$$

implying that, even if coefficients are of positive sign, it may be the case that the only normalizable terms are normal ones. Obviously, such normalization issues of a reduction notion crush the related reductional equivalence relation and how it identifies terms. As pointed out by Vaux [7, 8], an equivalence relation works as we expect only in case the terms can take non-negative coefficients. In fact, if the coefficient 1 has an opposite, i.e. -1, such that $1 + (-1) = 0$, then any reductional equivalence relation defined is unsound. In other words, the interaction between β-reduction and algebraic rewriting cause the reductional equality to collapse as soon as the set of coefficients admits negative elements. Indeed, it is easy to verify that the term $\infty_s + (-1)\infty_s$ is equivalent to both s and the zero sum $\mathbf{0}$:

$$\infty_s + (-1)\infty_s = (1 + (-1))\infty_s = \mathbf{0}$$

and

$$\infty_s + (-1)\infty_s \to^* (s + \infty_s) + (-1)\infty_s = s + (\infty_s + (-1)\infty_s) = s + (1 + (-1))\infty_s = s + \mathbf{0} = s$$

which implies, for all λ-terms s, $s = \mathbf{0}$.

Summing up, the calculus coming from the integration of algebraic aspects into the λ-calculus exhibits two sources of failure to normalization: the first one is due to the possibility to describe fixpoint operators and infinite computation, the second one is due to the algebraic properties of the set of coefficients used. In the original work on the algebraic λ-calculus, it has been presented a Curry-style simple type system to address the first issue. Afterwards, necessary and sufficient conditions are discussed for strong normalization of typed terms to hold, addressing the second issue. However, the result holds imposing strong restrictions on the set of coefficients.

To this day, in the setting of the algebraic λ-calculus, there are no satisfying term reduction definitions useful to properly give the notion of normal form of a term nor that of reductional equivalence relation. The aim of the present work is to report on the state of the art concerning these issues and, following some ideas and techniques already mentioned by Vaux [8] or even by Ehrhard and Regnier [1], to provide a first solution.

Outline and Contributions. In section 2, we recall the definitions and basic results from Vaux's work [8] that are necessary in the remaining of the paper. In section 3, we briefly review the original definition of reduction, together with an alternative notion, then discuss their respective properties w.r.t. normalization and confluence. In particular, we prove the latter, defined on canonical terms, non-confluent, though strongly normalizing in the typed setting. Section 4 presents the free module of terms over the semiring of polynomials with non-negative integer coefficients. Even if it is not a new idea [1, 8], theorem 32 and the related machinery are original. We then discuss the consequences of such result with an eye to possible future developments, described in section 5.

2. Constructing the Free R-module of Terms

In this section we introduce the set of terms of the algebraic λ-calculus in several steps. Firstly, we define a language extending the pure λ-calculus with formal sums and coefficients. Then, we refine it by considering consecutive quotient sets of terms towards a language providing linear combinations of terms. In particular, in subsection 2.1 we give the grammar of *raw terms* which we quotient by α-equivalence after having extended it, along with term substitution, to the new setting. On the resulting language, in subsection 2.2 we define a notion of algebraic equality between terms by means of an equivalence relation $\triangleq$. The associated new quotient set is what we call a free R-module, moreover validating identities (1) and (2), whose elements are the so called *algebraic λ-terms*. Finally, in subsection 2.3, we give an inductive formulation of terms defining their *canonical forms*, which we emphasize as being the distinguished elements of $\triangleq$-equivalence classes.

Since we construct the set of terms of the algebraic λ-calculus from scratch, we will almost use the same notations and definitions employed by Vaux [8].

Preliminary Notions and Notations. We call *rig* any commutative unital semiring R, which is the same thing as a commutative unital ring without the condition that every element must have an additive inverse. Let $\mathsf{R} = (\mathsf{R}, +, 0, \times, 1)$ be a rig: $(\mathsf{R}, +, 0)$ is a commutative monoid, $(\mathsf{R}, \times, 1)$ is a monoid, $\times$ distributes over $+$ and 0 annihilates R. We denote by letters a, b, c the elements of R, and say that R is *positive* if, for all $a, b \in \mathsf{R}$, $a + b = 0$ implies $a = 0$ and $b = 0$. We write $\mathsf{R}^{\bullet}$ for $\mathsf{R} \setminus \{0\}$. A typical example of positive rig is the set of natural numbers $\mathbb{N}$, equipped with usual operations.

In general, a module over the unital ring A, abbreviated A-*module*, is a set of mathematical objects which are linear combinations of elements of a commutative group with coefficients in A. Moreover, if A is actually a field, we get the definition of a vector space. In the present work, we focus on modules over rigs: for all set Φ, the free R-module over Φ, denoted $\mathsf{R}\langle\Phi\rangle$, is the set of formal finite linear combinations of elements of Φ with coefficients in R.

2.1. Raw Terms

Let $\mathcal{V}$ be a denumerable set of variables which we denote with letters among x, y, z.

Definition 1. The language Λ^0_{R} of the *raw terms* of the algebraic λ-calculus over R (denoted by capital letters L, M, N) is given by the following grammar:

$$M, N, \ldots ::= x \mid \lambda x\, M \mid (M)N \mid \mathbf{0} \mid aM \mid M + N$$

We naturally extend to the current setting the usual notion of free occurrences of a variable in a raw term: lambda is the only binder. From the latter we derive the common notion of free variable, denoting with $\mathsf{FV}(L)$ the set of free variables of a raw term L. Finally, we obtain notions of α-equivalence (denoted $\sim$) and term substitution as in Krivine's [6]. From now on, we consider raw terms up-to α-equivalence. More formally:

Definition 2. The set Λ^1_{R} of raw terms of the algebraic λ-calculus over R is the quotient set $\Lambda^0_{\mathsf{R}}/\sim$.

Definition 3. A binary relation r on raw terms is said to be *contextual* if it satisfies the following conditions:

- $x \;\mathsf{r}\; x$;
- $\lambda x\, M \;\mathsf{r}\; \lambda x\, M'$ as soon as $M \;\mathsf{r}\; M'$;
- $(M)N \;\mathsf{r}\; (M')N'$ as soon as $M \;\mathsf{r}\; M'$ and $N \;\mathsf{r}\; N'$;
- $\mathbf{0} \;\mathsf{r}\; \mathbf{0}$;
- $aM \;\mathsf{r}\; aM'$ as soon as $M \;\mathsf{r}\; M'$;
- $M + N \;\mathsf{r}\; M' + N'$ as soon as $M \;\mathsf{r}\; M'$ and $N \;\mathsf{r}\; N'$;

This notion of contextual relation is the analogue of the λ-compatible relation for the pure λ-calculus. In fact, we are able to derive the following result.

Proposition 4. If r is a contextual relation, then $L[M/x] \;\mathsf{r}\; L[M'/x]$ as soon as $M \;\mathsf{r}\; M'$.

2.2. The Module of Terms

We now refine the language of raw terms towards identifying terms up to usual identities concerning linear combinations, together with (1) and (2). We carry out this idea introducing the actual algebraic content of the calculus by means of an equivalence relation we denote as $\triangleq$.

Definition 5. *Algebraic equality* $\triangleq$ is defined on raw terms as the least contextual equivalence relation such that the following identities hold:

- axioms of commutative monoid:

$$\mathbf{0} + M \triangleq M \tag{5a}$$
$$(M + N) + L \triangleq M + (N + L) \tag{5b}$$
$$M + N \triangleq N + M \tag{5c}$$

- axioms of module over rig R:

$$a(M + N) \triangleq aM + aN \tag{6a}$$
$$aM + bM \triangleq (a + b)M \tag{6b}$$
$$a(bM) \triangleq (ab)M \tag{6c}$$
$$1M \triangleq M \tag{6d}$$
$$0M \triangleq \mathbf{0} \tag{6e}$$
$$a\mathbf{0} \triangleq \mathbf{0} \tag{6f}$$

- linearity in the λ-calculus:

$$\lambda x\, \mathbf{0} \triangleq \mathbf{0} \tag{7a}$$
$$\lambda x\, (aM) \triangleq a(\lambda x\, M) \tag{7b}$$
$$\lambda x\, (M + N) \triangleq \lambda x\, M + \lambda x\, N \tag{7c}$$
$$(\mathbf{0})L \triangleq \mathbf{0} \tag{7d}$$
$$(aM)L \triangleq a((M)L) \tag{7e}$$
$$(M + N)L \triangleq (M)L + (N)L \tag{7f}$$

We call *algebraic λ-terms* the elements of $\Lambda^1_{\mathsf{R}}/\triangleq$, i.e. the $\triangleq$-classes of raw terms. If $L \in \Lambda^1_{\mathsf{R}}$, then we write $\underline{L}$ for its $\triangleq$-class.

Identities (7a) through (7c) subsume (1) and those from (7d) to (7f) subsume (2). Then the quotient set $\Lambda^1_{\mathsf{R}}/\triangleq$ is an R-module validating (1) and (2).

Definition 6. For all $L_1, \ldots, L_n \in \Lambda^1_{\mathsf{R}}$, we write $L_1 + \cdots + L_n$ or even $\sum_{i=1}^n L_i$ for the term $L_1 + (\ldots + L_n)$ (or $\mathbf{0}$ if $n = 0$).

Intuitively, each raw term can be thought as a *writing* of its $\triangleq$-class, which is an element of the free R-module $\Lambda^1_{\mathsf{R}}/\triangleq$. Among raw terms, one would like to distinguish some of them as the canonical writings. We work then, towards an inductively definition of the syntax of the algebraic λ-calculus which permits each term $L \in \Lambda^1_{\mathsf{R}}$ to be uniquely written as $L \triangleq \sum_{i=1}^n a_i s_i$, where the s_i's are pairwise distinct base elements and the a_i's are non zero.

The approach we follow subsumes a rewriting system obtained by orienting all the equalities defining the algebraic equivalence from left to right, except for (5c), without formally reproducing such development. We rather slightly extend our notion of equality of terms by considering a new quotient set of raw terms in a way such that the order of summands in a $\sum_{i=1}^{n} L_i$ no longer matters. Afterwards, we give a mutually inductive definition of the syntax founded on *base terms* and *canonical terms* which is proved to match the quotient set $\Lambda_{\mathsf{R}}^1/\triangleq$ of algebraic λ-terms.

Definition 7. *Permutative equality* $\equiv \subseteq \Lambda_{\mathsf{R}}^1 \times \Lambda_{\mathsf{R}}^1$ is the least contextual equivalence relation such that $\sum_{i=1}^{n} L_i \equiv \sum_{i=1}^{n} L_{\sigma(i)}$ holds, for all $L_1, \ldots, L_n \in \Lambda_{\mathsf{R}}^1$ and all permutations σ of $\{1, \ldots, n\}$.

Since free variables of a sum do not depend on the order of the summands, $\equiv$ preserves free variables.

Definition 8. We write Λ_{R} for the quotient set $\Lambda_{\mathsf{R}}^1/\equiv$, and we call *permutative terms* the elements of Λ_{R}.

Proposition 9. Substitution is well defined on Λ_{R}: if $L, L' \in \Lambda_{\mathsf{R}}^1$ are such that $L \equiv L'$ and, for all $i \in \{1, \ldots, n\}$ are such that $M_i \equiv M'_i$, then $L[M_1, \ldots, M_n/x_1, \ldots, x_n] \equiv L'[M'_1, \ldots, M'_n/x_1, \ldots, x_n]$ for all pairwise distinct variables $x_1, \ldots, x_n$.

Except when stated otherwise, we use the same notation for a raw term L and its $\equiv$-class, and use them interchangeably. This is in general harmless, since the properties we consider are all invariant under $\equiv$.

Of course algebraic equality already subsumes permutative equality on raw terms, so that $\triangleq$ is well defined on Λ_{R} and $(\Lambda_{\mathsf{R}}^1/\triangleq) = (\Lambda_{\mathsf{R}}/\triangleq)$.

2.3. Canonical Forms

We give now the definition of canonical forms of raw terms as particular permutative terms such that every class in $\Lambda_{\mathsf{R}}/\triangleq$ contains exactly one canonical element.

Definition 10. We define the set $\mathsf{C}_{\mathsf{R}} \subset \Lambda_{\mathsf{R}}$ of *canonical terms* (denoted by capital letters S, T, U, V, W) and the set $\mathsf{B}_{\mathsf{R}} \subset \Lambda_{\mathsf{R}}$ of *base terms* (denoted by small letters s, t, u, v, w) by mutual induction as follows:

- any variable x is a base term;
- let $x \in \mathcal{V}$ and s be a base term, then $\lambda x\, s$ is a base term;
- let s be a base term and T a canonical term, then $(s)T$ is a base term;
- let $a_1, \ldots, a_n \in \mathsf{R}^\bullet$ and $s_1, \ldots, s_n$ be pairwise distinct base terms, then $\sum_{i=1}^{n} a_i s_i$ is a canonical term.

The reader should get the intuition about each canonical form being the most simplified version of an entire $\triangleq$-class of raw terms. Mapping s to the "singleton" $1s$ defines an injection from base terms into canonical ones.

We give now some definitions and intermediate results useful to prove the uniqueness of each canonical term as representative of an entire $\Lambda_{\mathsf{R}}/\triangleq$-class of terms. First of all, we define the function can as the function taking a permutative term in input and returning its canonical form as result. To define it properly, we need also to provide a way to canonize a sum term.

Definition 11. Let $L = \sum_{i=1}^{n} a_i s_i \in \Lambda_{\mathsf{R}}$ be a linear combination of base terms, not necessarily canonical. For all base term s, we call *coefficient of s in L* the scalar $\sum_{1 \leq i \leq n,\ s_i = s} a_i$ (the sum of those a_i's such that $s_i = s$), which we denote by $L_{(s)}$. Then we define $\mathsf{cansum}(L) \in \mathsf{C}_{\mathsf{R}}$ by:

$$\mathsf{cansum}(L) = \sum_{j=1}^{p} L_{(t_j)} t_j$$

where $\{t_1, \ldots, t_p\}$ is the set of those s_i's with non-zero coefficient in L.

Definition 12. Canonization of terms $\mathsf{can} : \Lambda_{\mathsf{R}} \to \mathsf{C}_{\mathsf{R}}$ is given by:

- $\mathsf{can}(x) = 1x$;
- if $\mathsf{can}(M) = \sum_{i=1}^{n} a_i s_i$ then $\mathsf{can}(\lambda x\, M) = \sum_{i=1}^{n} a_i(\lambda x\, s_i)$;
- if $\mathsf{can}(M) = \sum_{i=1}^{n} a_i s_i$ and $\mathsf{can}(N) = T$ then $\mathsf{can}((M)N) = \sum_{i=1}^{n} a_i(s_i)T$;
- $\mathsf{can}(\mathbf{0}) = \mathbf{0}$;
- if $\mathsf{can}(M) = \sum_{i=1}^{n} a_i s_i$ then $\mathsf{can}(aM) = \sum_{aa_i \neq 0}(aa_i)s_i$;
- if $\mathsf{can}(M) = \sum_{i=1}^{n} a_i s_i$ and $\mathsf{can}(N) = \sum_{i=n+1}^{n+p} a_i s_i$ then $\mathsf{can}(M+N) = \mathsf{cansum}\left(\sum_{i=1}^{n+p} a_i s_i\right)$.

The following result clarifies the role of can function and canonical forms on permutative terms. We do not report here the relative proof.

Theorem 13. Algebraic equality is equality of canonical forms: for all $M, N \in \Lambda_{\mathsf{R}}$, $M \triangleq N$ if and only if $\mathsf{can}(M) = \mathsf{can}(N)$.

Corollary 14. For all $S, T \in \mathsf{C}_{\mathsf{R}}$, $S \triangleq T$ if and only if $S = T$.

Proof. This is a direct consequence of the previous theorem and the fact that $\mathsf{can}(S) = S$, for all canonical terms S. $\square$

Corollary 15. Substitution is well defined on $\Lambda_{\mathsf{R}}/\triangleq$: if $L, L' \in \Lambda_{\mathsf{R}}$ are such that $L \triangleq L'$ and, for all $i \in \{1, \ldots, n\}$, $M_i, M_i' \in \Lambda_{\mathsf{R}}$ are such that $M_i \triangleq M_i'$, then $L[M_1, \ldots, M_n/x_1, \ldots, x_n] \triangleq L'[M_1', \ldots, M_n'/x_1, \ldots, x_n]$ for all pairwise distinct variables $x_1, \ldots, x_n$.

In his work, Vaux [8] proves that can function is an isomorphism of R-modules from $\Lambda_{\mathsf{R}}/\triangleq$ to C_{R}. This result formally confirms, along with the fact that $\triangleq$ is contextual, that the quotient structure of algebraic terms is subsumed by the mutually inductive structure of base and canonical terms. We recall that if $L \in \Lambda_{\mathsf{R}}$, then with the notation $\underline{L}$ we refer to its $\triangleq$-class. Therefore, we write $\underline{\mathcal{C}} = \{\underline{S} \mid S \in \mathcal{C}\}$ in the case $\mathcal{C}$ is a set of canonical terms; then $(\Lambda_{\mathsf{R}}/\triangleq) = \underline{\mathsf{C}_{\mathsf{R}}}$. Thus, from now on, we will define functions and prove properties on algebraic terms using induction on base terms and canonical terms, implying the use of can function (typically, obvious). Moreover, we denote Δ_{R} the set $\underline{\mathsf{B}_{\mathsf{R}}}$ and $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ the set $\underline{\mathsf{C}_{\mathsf{R}}}$:

Definition 16. We define *simple terms* as the $\triangleq$-classes of base terms. We write Δ_{R} for the set of simple terms and $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ for the set of algebraic terms, which we may just call *terms*.

When we write a simple term (resp. a term) as $\underline{s}$, $\underline{t}$, $\underline{u}$, $\underline{v}$ or $\underline{w}$ (resp. $\underline{S}$, $\underline{T}$, $\underline{U}$, $\underline{V}$ or $\underline{W}$), we mean that s, t, u, v or w is a base term (resp. S, T, U, V or W is a canonical term). When we do not make this assumption, we use greek letters σ, τ, ρ. We will often use the notation $\lambda x\, \sigma$, $(\sigma)\tau$, $a\sigma$, $\sigma + \tau$ with the obvious sense: in general, actual terms are not canonical forms but they are well defined by contextuality of $\triangleq$.

3. Reduction

In this section, we define two reduction notions. The first one captures the definition of reduction introduced by Ehrhard and Regnier [1], minus differentiation, using (3) and (4) as key reduction rules.

The second reduction notion is a variant of the first one, tailored to fit the case of canonical terms only. In particular, the latter relation is contained in the former one. Surprisingly, we show a counterexample to local confluence of the second reduction, while a proof of the general property has been provided by Vaux [8] for the first reduction.

3.1. Reduction of Linear Combinations of Terms

We call *relation from simple terms to terms* any subset of $\Delta_{\mathsf{R}} \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, and we call *relation from terms to terms* any subset of $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. Given a relation r from simple terms to terms, we define the new relation $\widetilde{\mathsf{r}}$ from terms to terms by:

$$\sigma \;\widetilde{\mathsf{r}}\; \sigma' \text{ if } \sigma = \underline{as + T} \text{ and } \sigma' = \underline{aS + T} \text{ where } a \neq 0 \text{ and } \underline{s} \;\mathsf{r}\; \underline{S}. \tag{8}$$

Note that we do not require $as+T$ to be a canonical term, implying the fact that such reduction works modulo $\triangleq$-equivalence. In other words, a sequence of reductions may alternate algebraic calculations.

We now introduce the definition of one-step β-reduction $\to$ as a relation from simple term to terms, so that the actual reduction relation on terms is obtained as $\widetilde{\to}$. According to what has been done for the differential calculus [1], we define simple term reduction $\to$ by induction on the depth of the fired redex.

Definition 17. We define an increasing sequence of relations from simple terms to terms by the following statements. Let $\to_0$ be the empty relation $\emptyset \subseteq \Delta_{\mathsf{R}} \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. Assume $\to_k$ is defined. Then we set $\sigma \to_{k+1} \sigma'$ as soon as one of the following holds:

- $\sigma = \underline{\lambda x\, s}$ and $\sigma' = \underline{\lambda x\, S}$ with $\underline{s} \to_k \underline{S}$;
- $\sigma = \underline{(s)T}$ and $\sigma' = \underline{(S)T}$ with $\underline{s} \to_k \underline{S}$, or $\sigma' = \underline{(s)T'}$ with $\underline{T} \,\widetilde{\to}_k\, \underline{T'}$;
- $\sigma = \underline{(\lambda x\, s)T}$ and $\sigma' = \underline{s[T/x]}$.

Let $\to = \bigcup_{k\in\mathsf{N}} \to_k$. We call *one-step reduction* or simply *reduction*, the relation $\widetilde{\to}$. We denote with $\widetilde{\to}^*$ the reflexive and transitive closure of $\widetilde{\to}$.

Such reduction is confluent as provable using a natural variant of Tait-Martin-Löf technique to the algebraic case: introduce a parallel extension of reduction, in which redexes can be semplified simultaneously, and prove it enjoys the diamond property (i.e. strong confluence).

As far as normalization properties are concerned, it is not difficult to prove that, in the case R is the set of natural numbers, strongly normalizing terms are the linear combinations of strongly normalizing simple terms. In particular, coefficients cannot be the cause of infinite sequences of $\widetilde{\to}$-reduction.

One might wonder if such a result holds even in the case of negative coefficients. Vaux [8] has shown that it is not the case. In fact, although rule (8) extends $\to$ to the case of sum in a crucial way needed to prove confluence, it is also the point of failure for normalization in presence of negative coefficients.

Lemma 18. If R is not positive (or, at least, 1 has an opposite, i.e. $-1 \in \mathsf{R}$ with $1 + (-1) = 0$), then for all terms σ and τ, $\sigma \,\widetilde{\to}^*\, \tau$.

Proof. Take a fixpoint operator Y of the λ-calculus such that $(Y)s \to^* (s)(Y)s$, for any λ-term s. Let $\infty_\sigma = (\underline{Y})\lambda x\, (\sigma + \underline{x})$ which reduces as $\infty_\sigma \to^* \sigma + \infty_\sigma$. Then we get:

$$\begin{aligned}
\sigma &= \sigma + \infty_\sigma - \infty_\sigma + \infty_\tau - \infty_\tau \\
&\widetilde{\to}\; \sigma + \infty_\sigma - \sigma - \infty_\sigma + \infty_\tau - \infty_\tau \\
&\widetilde{\to}\; \sigma + \infty_\sigma - \sigma - \infty_\sigma + \tau + \infty_\tau - \infty_\tau \\
&= \sigma - \sigma + \tau = \tau.
\end{aligned}$$

□

In particular, even the zero term $\underline{\mathbf{0}}$ can reduce, implying that there is no irreducible term. Hence, it does not even make sense to talk about normal form of terms as notion. Moreover, it is straightforward that the reductional equivalence relation defined from $\widetilde{\to}$ as its reflexive, symmetric and transitive closure is fatally unsound: it identifies terms which bear no relationship with each other.

Since lemma 18 involves fixpoints, a first approach to the problem might be a type system. For sure, typing is a well known technique to prevent fixpoints and infinite computations in general. Indeed, it is a solution which avoids such *collapse*, but it is not sufficient at all to prevent exploitation of negative coefficients to construct infinite sequences of reduction. Actually, it is worse than that.

Although not with the same consequences, the notion of normalization might be affected by a similar kind of scalar manipulation even in the case R is positive. In particular, normalization may be trivial, making sense to talk about normalizability only in the case of normal terms. In fact, assume R to be the positive rig $\mathbb{Q}^+$ of non-negative rational numbers, and $\sigma \to \sigma'$. Then, there is an infinite sequence of reductions from σ:

$$\sigma = \frac{1}{2}\sigma + \frac{1}{2}\sigma \,\widetilde{\to}\, \frac{1}{2}\sigma + \frac{1}{2}\sigma' = \frac{1}{4}\sigma + \frac{1}{4}\sigma + \frac{1}{2}\sigma' \,\widetilde{\to}\, \frac{1}{4}\sigma + \frac{1}{4}\sigma' + \frac{1}{2}\sigma' = \frac{1}{4}\sigma + \frac{3}{4}\sigma' = \dots$$

It has been proved that strong normalization results, for the untyped and typed algebraic calculus, hold when R is *finitely splitting* and an *integral domain*. Intuitively, the former property ensures that each element of R can only be written as a finite sum of elements of $\mathsf{R}^\bullet$. The latter property guarantees that the result of multiplying elements of R can be the zero element of R only when the zero element is one of the factors.

Theorem 19. Let R to be finitely splitting and to satisfy an integral domain property. Then a term is strongly normalizing iff it is a linear combination of strongly normalizing simple terms.

Proof. The developments needed to prove the theorem and the proof itself are treated in section 5, and in particular subsection 5.1, in Vaux's work [8]. □

At the least, these conditions imply the positivity of R, which already is a strong constraint. Then, one might think about some fine tuning to the present setting restraining the syntax of terms from the above coefficients manipulations, while allowing R to be a richer set. What follows is an attempt in this direction.

3.2. Reduction of Canonical Terms

We consider a variant of reduction defined on canonical forms only, which immediately prevents the problematic examples just discussed. More formally, rather than (4), extend reduction from simple terms to terms as follows:

$$\sigma \,\widehat{\to}\, \sigma' \text{ if } \sigma = \underline{as + T} \text{ and } \sigma' = \underline{aS + T}, \text{ with } as + T \in \mathsf{C}_\mathsf{R} \text{ and } \underline{s} \to \underline{S}. \tag{9}$$

Notice that such reduction is not contextual in the sense of definition 3 and it does not permit the interleaving of algebraic calculations. In particular, since defined on canonical terms only, the result of a $\widehat{\to}$-reduction on every simple term does not depend on coefficient handlings but only on the fired redex. The important consequence is that the tricks involving ∞_σ and rational coefficients are no longer possible, since such manipulations make canonical terms into non-canonical ones.

Lemma 20. The two reduction notions given by (8) and (9) are related as follows: $\widehat{\to} \subset \widetilde{\to}$.

As far as normalization is concerned, $\widehat{\to}$-reduction shares many properties with $\widetilde{\to}$. Indeed, in the case of natural numbers as rig, it enjoys all the results proved by Vaux [8], both in the untyped and typed setting. Moreover, since algebraic calculations are possible only applying $\mathsf{can}(\cdot)$ function, $\widehat{\to}$ is well founded for the simple typed version of the calculus even in the case R would potentially allow infinite reductions based on coefficients manipulations.

Quite strikingly, the current reduction is not confluent, since it is not even locally confluent. In fact, the following is a counterexample to the latter:

Counterexample 21. Let $\underline{s}$ and $\underline{t}$ be simple terms such that $\underline{s} \mathrel{\widehat{\rightarrow}} \underline{t}$ and $\underline{t} \mathrel{\widehat{\rightarrow}} \underline{s + y}$. Then the canonical term $\underline{s + t}$ is not locally confluent.

Proof. Reducing $\underline{s}$ in $\underline{s + t}$ gives as result the term $\underline{2t}$. On the contrary, reducing $\underline{t}$ gives as result the term $\underline{2s + y}$. Now, it is easy to see that proceeding to reduce the first reduct gives an even amount of $\underline{y}$ variable, while the second one gives an odd amount of it. $\square$

Of course, such $\underline{s}$, $\underline{t}$ are constructible simple terms and, moreover, we need to formalize only the first one: set $\underline{I} = \underline{\lambda x\, x}$ and $\underline{D} = \underline{\lambda x\, (I)((x)x + y)}$, then $\underline{s} = \underline{(D)D}$. This result prevents a proper notion of normal form of a term to be defined.

The current reduction has another flaw. As in the case of $\widetilde{\rightarrow}$, the reductional equivalence relation defined as the reflexive, symmetric and transitive closure of $\widehat{\rightarrow}$ is unsound: one can reproduce that argument replacing terms like $\infty_\sigma - \infty_\sigma$ with $\infty_\sigma - (\underline{I})\infty_\sigma$. Indeed, the latter is a canonical term which reduces both to σ and to $\underline{\mathbf{0}}$.

Even if the current attempt fails in being useful for a proper definition of normal form and term equivalence, the reduction $\widehat{\rightarrow}$ is a step forward to a solution, since it is not inconsistent in the sense of lemma 18. In particular, there are irreducible terms: for instance, $\underline{\mathbf{0}}$.

4. A rig of polynomials

Previous failed attempts to define a good reduction notion remark the complexity of the problem. We do not deny the possibility to define a working reduction notion, but it would be cumbersome and technical (e.g. treating corner cases). Most likely, we might be forced to rethink $\triangleq$-equivalence as we defined it in definition 5. We would like to avoid it.

While $\widetilde{\rightarrow}$-reduction notion makes the algebraic λ-calculus collapse, $\widehat{\rightarrow}$-reduction notion is not confluent. The latter is a crucial property if one wants to talk about normal forms of terms. Moreover, theorem 19 explains why the first reduction notion works fine when coefficients are taken in $\mathbb{N}$. Indeed, such a rig is finitely splitting and has no zero divisor. One more notable case is the rig of polynomials, over a set of variables, with non-negative integer coefficients.

Considering the $\widetilde{\rightarrow}$-reduction notion, the idea then is to exploit this last algebraic structure to define a notion of normal form even in the case of modules of terms over richer rigs, admitting negative coefficients too. In fact, as an example, let us consider the module of terms with coefficients in $\mathbb{Z}$. Despite the fact that the associated algebraic calculus is not consistent, we are able to figure out which terms are normalizing: indeed, modulo coefficients manipulations, the normalizing terms are again the linear combinations of normalizing simple terms. Hence, in some sense, this means that forgetting coefficients and the related rewriting dynamics, we can determine the subset of normalizing terms and their normal form. This is precisely what the following construction is all about.

The solution we are going to use was firstly mentioned by Ehrhard and Regnier [1] during the work on normalization properties concerning their differential λ-calculus. The technique permits to deactivate coefficients and tame $\triangleq$ during reduction. Roughly speaking, it consists in replacing the coefficients of a term with formal variables, reducing some steps and, at last, replacing the variables with their values. Notice that such a clever solution implements what we have just discussed above.

Summing up, we formalize the terms obtained by replacing coefficients with formal variables as the terms of the algebraic λ-calculus defined over the rig of polynomials with non-negative integer coefficients. Hence, we use the reduction $\widetilde{\rightarrow}$ because confluent in this setting.

4.1. Terms with polynomials as coefficients

Let R be any rig and $\Xi = \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \ldots\}$ any denumerable set of variables which we consider as *indeterminates.*

Definition 22. We define $\mathsf{P} = \mathbb{N}[\Xi]$ as the rig of polynomials with non-negative integer coefficients over variables in Ξ.

Definition 23. We call *variable assignment* any function $f : \Xi \to \mathsf{R}$ assigning an element of R to each indeterminate of Ξ.

Every variable assignment with values in R extends naturally to a function evaluating polynomials into elements of R.

Definition 24. Let f be a variable assignment. We call *polynomial evaluation* the rig morphism, parametrised over f and denoted $[\![\cdot]\!]_f : \mathsf{P} \to \mathsf{R}$, returning the value of a given polynomial of P in R. In particular, if $P \in \mathsf{P}$ then $[\![P]\!]_f$ is its *value* calculated in R once each variable in P has been replaced with the respective value in R by means of f.

Such an evaluation extends to the one defined on terms of $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ and returning its corresponding term in $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ by replacing each polynomial coefficient with its value. We call it *term evaluation.*

Definition 25. A *term evaluation* is a module morphism $[\![\cdot]\!]_f : \mathsf{P}\langle\Delta_\mathsf{P}\rangle \to \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ defined by induction on terms as follows:

$$\begin{aligned}
[\![\underline{x}]\!]_f &= x \\
[\![\underline{\lambda x\, s}]\!]_f &= \lambda x\, [\![\underline{s}]\!]_f \\
\left[\!\!\left[\underline{(s)T}\right]\!\!\right]_f &= ([\![\underline{s}]\!]_f)[\![\underline{T}]\!]_f \\
\left[\!\!\left[\underline{\sum_{i=1}^{n} P_i s_i}\right]\!\!\right]_f &= \sum_{i=1}^{n} [\![P_i]\!]_f\, [\![\underline{s_i}]\!]_f .
\end{aligned}$$

Given a term evaluation, we naturally associate to a term in $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ its corresponding term in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$.

Definition 26. Let f be a variable assignment and $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$. We say that a term $\dot{\sigma} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ is a *notation for* σ if $[\![\dot{\sigma}]\!]_f = \sigma$. Two, or more, different notations for σ are said *sibling notations.*

Note that whenever R is equipped with an additive inverse which is image element for some indeterminate in Ξ, then to a term in $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ we can associate an infinite set of terms in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ corresponding to it. In particular, under these conditions, to an irreducible term in $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ we can associate a term in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ which is not, or even worse, that might reduce endlessly. In the following, we use dotted notation only when demanded by an unclear context.

Depending on a term evaluation, for every term σ in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$, we define the set of terms of $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ which are notations for the same term of $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$.

Definition 27. Let f be a variable assignment. For all $\sigma \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$, we define the set of *sibling notations of* σ as $\nabla_f(\sigma) = \{\tau \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle \mid [\![\sigma]\!]_f = [\![\tau]\!]_f\}$.

Preliminary Lemmas. Let us fix, once and for all, one particular variable assignment and write $[\![\cdot]\!]$ the related term evaluation morphism. Moreover, we write $\nabla(\cdot)$ the related set of sibling notations.

From the fact that every term either belongs to the set of sibling notations of a simple term or not, the following is a self-evident truth.

Fact 28. Let $\underline{s} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ be a simple term. For all $\sigma \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ of the form $\sigma = \underline{\sum_{i=1}^{n} P_i t_i}$, σ can be written as $\sigma = \underline{\sum_{j\in J} P_j t_j + \sum_{k\in K} P_k t_k}$ with $|J| + |K| = n$, $\underline{t_j} \in \nabla(\underline{s})$ and $\underline{t_k} \notin \nabla(\underline{s})$, for all $j \in J$ and $k \in K$.

We prove some lemmas explaining the reason why two terms σ_1, $\sigma_2 \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ are related by means of a term evaluation. In particular, if they are sibling notations, $[\![\sigma_1]\!] = [\![\sigma_2]\!]$, then they can be seen as composed of two summands $\sigma_i = \sigma_{i1} + \sigma_{i2}$ satisfying $[\![\sigma_{1j}]\!] = [\![\sigma_{2j}]\!]$, with $i, j \in \{1,\ 2\}$.

Lemma 29. Let $\sigma \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ of the form $\sigma = \underline{Ps + Z}$ and $\underline{Z} \notin \nabla(\underline{s})$. For all $\tau \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if $\tau \in \nabla(\sigma)$, then $\tau = \underline{\sum_{i\in I} P_i t_i + Z'}$ such that:

1. $\sum_{i\in I} [\![P_i]\!] = [\![P]\!]$ and, for all $i \in I$, $[\![\underline{t_i}]\!] = [\![\underline{s}]\!]$;
2. $\underline{Z'} \notin \nabla(\underline{s})$ and $[\![\underline{Z'}]\!] = [\![\underline{Z}]\!]$.

Proof. By definitions 27 and 25, $\tau \in \nabla(\sigma)$ implies $[\![\tau]\!] = [\![\sigma]\!] = [\![P]\!]\,[\![\underline{s}]\!] + [\![\underline{Z}]\!]$. Then, according to fact 28, $\tau = \underline{\sum_{i\in I} P_i t_i + \sum_{j\in J} P_j t_j}$ with $\underline{t_i} \in \nabla(\underline{s})$ and $\underline{t_j} \notin \nabla(\underline{s})$, for all $i \in I$ and $j \in J$. Hence, take $\underline{Z'} = \underline{\sum_{j\in J} P_j t_j}$ and deduce that $\underline{Z'} \notin \nabla(\underline{s})$. By the fact that, for all $i \in I$, $\underline{t_i} \in \nabla(\underline{s})$, we deduce $[\![\tau]\!] = \sum_{i\in I} [\![P_i]\!]\,[\![\underline{t_i}]\!] + [\![\underline{Z'}]\!] = \sum_{i\in I} [\![P_i]\!]\,[\![\underline{s}]\!] + [\![\underline{Z'}]\!]$. Therefore, to be identifiable as the hypothesis $\tau \in \nabla(\underline{s})$ says, $\sum_{i\in I} [\![P_i]\!] = [\![P]\!]$ and $[\![\underline{Z'}]\!] = [\![\underline{Z}]\!]$. ☐

In particular, if we consider a non-sum term σ of $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, then we are able to show that every sibling notation of σ can be seen as composed of two summands: the first one is a notation for the same term in $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ as σ, while the second one is a notation for the zero sum term.

Corollary 30. Let $\underline{s} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ be a simple term and $P \in \mathsf{P}$ be a polynomial. For all $\tau \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ of the form $\tau = \underline{\sum_{i=1}^{n} P_i t_i}$, if $\tau \in \nabla(\underline{Ps})$ then $\tau = \underline{\sum_{j\in J} P_j t_j + Z'}$ such that:

1. $\sum_{j\in J} [\![P_j]\!] = [\![P]\!]$ and, for all $j \in J$, $\left[\!\!\left[\underline{t_j}\right]\!\!\right] = [\![\underline{s}]\!]$;
2. $Z' \notin \nabla(\underline{Ps})$ and $[\![\underline{Z'}]\!] = \underline{\mathbf{0}}$.

Proof. This is a straightforward consequence of lemma 29 taking $\underline{Z} = \underline{\mathbf{0}}$. ☐

We prove now that term evaluation is compatible with term substitution.

Lemma 31. For all $\sigma, \tau \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, for all $x \notin \mathsf{FV}(\tau)$, $[\![\sigma[\tau/x]]\!] = [\![\sigma]\!][[\![\tau]\!]/x]$.

Proof. By mutual induction on the definitions of simple terms and terms.

In particular, the following are the cases involving σ as a simple term:

- Let σ be a variable. If $\sigma = \underline{x}$, then $[\![\underline{x}[\tau/x]]\!] = [\![\tau]\!] = \underline{x}[[\![\tau]\!]/x] = [\![\underline{x}]\!][[\![\tau]\!]/x]$. Otherwise, for all $y \in \mathcal{V}$ and $\underline{y} \neq \underline{x}$, $[\![\underline{y}[\tau/x]]\!] = [\![\underline{y}]\!] = \underline{y} = \underline{y}[[\![\tau]\!]/x] = [\![\underline{y}]\!][[\![\tau]\!]/x]$.
- Let $\sigma = \underline{\lambda x\, s}$. Thus, $\left[\!\!\left[\underline{(\lambda x\, s)}[\tau/x]\right]\!\!\right] = \lambda x\,[\![\underline{s}[\tau/x]]\!]$. By induction hypothesis, $[\![\underline{s}[\tau/x]]\!] = [\![\underline{s}]\!][[\![\tau]\!]/x]$. Then, $\lambda x\,[\![\underline{s}[\tau/x]]\!] = \lambda x\,[\![\underline{s}]\!][[\![\tau]\!]/x] = \left[\!\!\left[\underline{(\lambda x\, s)}\right]\!\!\right][[\![\tau]\!]/x]$.
- Let $\sigma = \underline{(s)T}$. Thus, $\left[\!\!\left[\underline{((s)T)}[\tau/x]\right]\!\!\right] = ([\![\underline{s}[\tau/x]]\!])[\![\underline{T}[\tau/x]]\!]$. By induction hypothesis, $[\![\underline{s}[\tau/x]]\!] = [\![\underline{s}]\!][[\![\tau]\!]/x]$ and $[\![\underline{T}[\tau/x]]\!] = [\![\underline{T}]\!][[\![\tau]\!]/x]$. Then, $([\![\underline{s}[\tau/x]]\!])[\![\underline{T}[\tau/x]]\!] = ([\![\underline{s}]\!][[\![\tau]\!]/x])[\![\underline{T}]\!][[\![\tau]\!]/x] = (([\![\underline{s}]\!])[\![\underline{T}]\!])[[\![\tau]\!]/x] = \left[\!\!\left[\underline{((s)T)}\right]\!\!\right][[\![\tau]\!]/x]$.

In the general case, let $\sigma = \underline{\sum_{i=1}^{n} P_i s_i}$. If $n = 0$, then $\sigma = \underline{\mathbf{0}}$ and the result trivially holds. Therefore, suppose $n \neq 0$. Thus, $\left[\!\left[\underline{(\sum_{i=1}^{n} P_i s_i)[\tau/x]}\right]\!\right] = \sum_{i=1}^{n} [\![P_i]\!] \, [\![\underline{s_i}[\tau/x]]\!]$. By induction hypothesis, for all $i \in \{1, \ldots, n\}$, $[\![\underline{s_i}[\tau/x]]\!] = [\![\underline{s_i}]\!][[\![\tau]\!]/x]$. Then, $\sum_{i=1}^{n} [\![P_i]\!] \, [\![\underline{s_i}[\tau/x]]\!] = \sum_{i=1}^{n} [\![P_i]\!] \, ([\![\underline{s_i}]\!][[\![\tau]\!]/x]) = (\sum_{i=1}^{n} [\![P_i]\!] \, [\![\underline{s_i}]\!])[[\![\tau]\!]/x] = \left[\!\left[\underline{\sum_{i=1}^{n} P_i s_i}\right]\!\right][[\![\tau]\!]/x]$. □

4.2. Normal forms for strongly normalizable terms

What follows is the key result of the current work which concerns a normalization property for the terms of the algebraic λ-calculus. In particular, theorem 32 asserts that normal forms of two strongly normalizable terms in $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, notations for the same noted term in $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, still note a same term in $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. Notice that, since a term evaluation does not change the actual structure of a term, the term evaluation of a normal form in $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ is an irreducible term in $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ (actually, it is a redex-free term). Roughly speaking, we prove a theorem in the setting of $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ which permits to characterize a notion of normal form of terms in $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, for any R.

Recall that, since polynomials are defined over non-negative integer coefficients, the current setting is a conservative extension of the pure λ-calculus. Moreover, $\widetilde{\rightarrow}$-reduction is also confluent in the current setting. This permits to talk about the notion of normal form of terms and, given a term $\sigma \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, uniquely define its normal form, noted $\mathsf{NF}(\sigma)$. In particular, we use the common notions and terminology of the classical λ-calculus, naturally extended to the present scenario.

Theorem 32. For all strongly normalizable $\sigma, \tau \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if $[\![\sigma]\!] = [\![\tau]\!]$ then $[\![\mathsf{NF}(\sigma)]\!] = [\![\mathsf{NF}(\tau)]\!]$.

Again, the above theorem does not suppose anything on the module $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ in which terms of $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ are evaluated to. Moreover, it does not consider at all the actual reduction notion defined on $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$.

We will show that this theorem is quite a direct consequence of the following lemma.

Lemma 33. For all $\sigma, \sigma' \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ with $\sigma \widetilde{\rightarrow} \sigma'$, there exists $\sigma_0 \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ such that $\sigma' \widetilde{\rightarrow}^* \sigma_0$ and, for all $\tau \in \nabla(\sigma)$, there exists $\tau_0 \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ such that $\tau \widetilde{\rightarrow}^* \tau_0$ and $\tau_0 \in \nabla(\sigma_0)$.

Proof. We prove it by induction on the definition of $\sigma \widetilde{\rightarrow} \sigma'$. More precisely, we will reason by induction on k as $\sigma \widetilde{\rightarrow}_k \sigma'$.

The case by which $k = 0$ is vacuously true. Suppose the result holds for some k, then we extend it to $k+1$ by inspecting the possible cases about $\sigma \widetilde{\rightarrow}_{k+1} \sigma'$. We will firstly address the ones where σ is a simple term thus implying $\sigma \rightarrow_{k+1} \sigma'$. Then one of the following applies:

- $\sigma = \underline{\lambda x\, u}$ and $\sigma' = \underline{\lambda x\, U}$ with $\underline{u} \rightarrow_k \underline{U}$. Hence, by induction hypothesis, there exists $\underline{U_0} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ such that $\underline{U} \widetilde{\rightarrow}^* \underline{U_0}$ and for all $\underline{W} \in \nabla(\underline{u})$, there exists $\underline{W_0} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ such that $\underline{W} \widetilde{\rightarrow}^* \underline{W_0}$ and $\underline{W_0} \in \nabla(\underline{U_0})$. Then take $\sigma_0 = \underline{\lambda x\, U_0}$ which obviously has the property that $\sigma' \widetilde{\rightarrow}^* \sigma_0$.

  Let $\tau \in \nabla(\sigma)$, that is $[\![\tau]\!] = [\![\underline{\lambda x\, u}]\!]$. By lemma 30, this implies $\tau = \underline{\sum_{i \in I} P_i t_i + Z'}$ with $\left[\!\left[\sum_{i \in I} P_i\right]\!\right] = \sum_{i \in I} [\![P_i]\!] = 1$ and $[\![\underline{t_i}]\!] = [\![\underline{\lambda x\, s}]\!]$ for all $i \in I$, while $\underline{Z'} \notin \nabla(\sigma)$ and $[\![\underline{Z'}]\!] = \underline{\mathbf{0}}$. By definition 25, for all $i \in I$, $\underline{t_i} = \underline{\lambda x\, w_i}$ with $\underline{w_i} \in \nabla(\underline{u})$.

  Since each $\underline{w_i} \in \nabla(\underline{u})$, we get that, for all $i \in I$, there exists $\underline{W_{0_i}} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ such that $\underline{w_i} \widetilde{\rightarrow}^* \underline{W_{0_i}}$ and $\underline{W_{0_i}} \in \nabla(\underline{U_0})$. Then take $\tau_0 = \underline{\sum_{i \in I} P_i \lambda x\, W_{0_i} + Z'}$ and easily check that $\tau \widetilde{\rightarrow}^* \tau_0$. Moreover, $[\![\tau_0]\!] = \left[\!\left[\underline{\sum_{i \in I} P_i \lambda x\, W_{0_i} + Z'}\right]\!\right] = \sum_{i \in I} [\![P_i]\!] \left[\!\left[\underline{\lambda x\, W_{0_i}}\right]\!\right] + [\![\underline{Z'}]\!] = 1 \lambda x\, [\![\underline{U_0}]\!] + \underline{\mathbf{0}} = \lambda x\, [\![\underline{U_0}]\!] = [\![\sigma_0]\!]$ which implies $\tau_0 \in \nabla(\sigma_0)$.

- $\sigma = \underline{(u)V}$ and $\sigma' = \underline{(U)V}$ with $\underline{u} \rightarrow_k \underline{U}$, or $\sigma' = \underline{(u)V'}$ with $\underline{V} \widetilde{\rightarrow}_k \underline{V'}$.

In the first case, by induction hypothesis, there exists $\underline{U_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{U} \stackrel{\sim}{\to}^* \underline{U_0}$ and for all $\underline{W} \in \nabla(\underline{u})$, there exists $\underline{W_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{W} \stackrel{\sim}{\to}^* \underline{W_0}$ and $\underline{W_0} \in \nabla(\underline{U_0})$. Then take $\sigma_0 = \underline{(U_0)V}$ which obviously has the property that $\sigma' \stackrel{\sim}{\to}^* \sigma_0$.

Let $\tau \in \nabla(\sigma)$, that is $\llbracket\tau\rrbracket = \left\llbracket\underline{(u)V}\right\rrbracket$. By lemma 30, this implies $\tau = \underline{\sum_{i\in I} P_i t_i + Z'}$ with $\left\llbracket\sum_{i\in I} P_i\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket = 1$ and $\llbracket\underline{t_i}\rrbracket = \left\llbracket\underline{(u)V}\right\rrbracket$ for all $i \in I$, while $\underline{Z'} \notin \nabla(\sigma)$ and $\llbracket\underline{Z'}\rrbracket = \underline{\mathbf{0}}$. By definition 25, for all $i \in I$, $\underline{t_i} = \underline{(w_i)V_i}$ with $\underline{w_i} \in \nabla(\underline{u})$ and $\underline{V_i} \in \nabla(\underline{V})$.

Since each $\underline{w_i} \in \nabla(\underline{u})$, we get that, for all $i \in I$, there exists $\underline{W_{0_i}} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{w_i} \stackrel{\sim}{\to}^* \underline{W_{0_i}}$ and $\underline{W_{0_i}} \in \nabla(\underline{U_0})$. Then take $\tau_0 = \underline{\sum_{i\in I} P_i(W_{0_i})V_i + Z'}$ and easily check that $\tau \stackrel{\sim}{\to}^* \tau_0$. Moreover, $\llbracket\tau_0\rrbracket = \left\llbracket\underline{\sum_{i\in I} P_i(W_{0_i})V_i + Z'}\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket \left\llbracket\underline{(W_{0_i})V_i}\right\rrbracket + \llbracket\underline{Z'}\rrbracket = 1(\left\llbracket\underline{U_0}\right\rrbracket)\llbracket\underline{V}\rrbracket + \underline{\mathbf{0}} = (\left\llbracket\underline{U_0}\right\rrbracket)\llbracket\underline{V}\rrbracket = \llbracket\sigma_0\rrbracket$ which implies $\tau_0 \in \nabla(\sigma_0)$.

In the second case, by induction hypothesis, there exists $\underline{V_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{V'} \stackrel{\sim}{\to}^* \underline{V_0}$ and for all $\underline{W} \in \nabla(\underline{V})$, there exists $\underline{W_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{W} \stackrel{\sim}{\to}^* \underline{W_0}$ and $\underline{W_0} \in \nabla(\underline{V_0})$. Then take $\sigma_0 = \underline{(u)V_0}$ which obviously has the property that $\sigma' \stackrel{\sim}{\to}^* \sigma_0$.

Let $\tau \in \nabla(\sigma)$, that is $\llbracket\tau\rrbracket = \left\llbracket\underline{(u)V}\right\rrbracket$. By lemma 30, this implies $\tau = \underline{\sum_{i\in I} P_i t_i + Z'}$ with $\left\llbracket\sum_{i\in I} P_i\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket = 1$ and $\llbracket\underline{t_i}\rrbracket = \left\llbracket\underline{(u)V}\right\rrbracket$ for all $i \in I$, while $\underline{Z'} \notin \nabla(\sigma)$ and $\llbracket\underline{Z'}\rrbracket = \underline{\mathbf{0}}$. By definition 25, for all $i \in I$, $\underline{t_i} = \underline{(w_i)V_i}$ with $\underline{w_i} \in \nabla(\underline{u})$ and $\underline{V_i} \in \nabla(\underline{V})$.

Since each $\underline{V_i} \in \nabla(\underline{V})$, we get that, for all $i \in I$, there exists $\underline{V_{0_i}} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{V_i} \stackrel{\sim}{\to}^* \underline{V_{0_i}}$ and $\underline{V_{0_i}} \in \nabla(\underline{V_0})$. Then take $\tau_0 = \underline{\sum_{i\in I} P_i(w_i)V_{0_i} + Z'}$ and easily check that $\tau \stackrel{\sim}{\to}^* \tau_0$. Moreover, $\llbracket\tau_0\rrbracket = \left\llbracket\underline{\sum_{i\in I} P_i(w_i)V_{0_i} + Z'}\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket \left\llbracket\underline{(w_i)V_{0_i}}\right\rrbracket + \llbracket\underline{Z'}\rrbracket = 1(\llbracket\underline{u}\rrbracket)\left\llbracket\underline{V_0}\right\rrbracket + \underline{\mathbf{0}} = (\llbracket\underline{u}\rrbracket)\left\llbracket\underline{V_0}\right\rrbracket = \llbracket\sigma_0\rrbracket$ which implies $\tau_0 \in \nabla(\sigma_0)$.

- $\sigma = \underline{(\lambda x\, u)V}$ and $\sigma' = \underline{u[V/x]}$. Then take σ' as σ_0.

  Let $\tau \in \nabla(\sigma)$, that is $\llbracket\tau\rrbracket = \left\llbracket\underline{(\lambda x\, u)V}\right\rrbracket$. By lemma 30, this implies $\tau = \underline{\sum_{i\in I} P_i t_i + Z'}$ with $\left\llbracket\sum_{i\in I} P_i\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket = 1$ and $\llbracket\underline{t_i}\rrbracket = \left\llbracket\underline{(\lambda x\, u)V}\right\rrbracket$ for all $i \in I$, while $\underline{Z'} \notin \nabla(\sigma)$ and $\llbracket\underline{Z'}\rrbracket = \underline{\mathbf{0}}$. By definition 25, for all $i \in I$, $\underline{t_i} = \underline{(\lambda x\, w_i)V_i}$ with $\underline{w_i} \in \nabla(\underline{u})$ and $\underline{V_i} \in \nabla(\underline{V})$.

  Then take $\tau_0 = \underline{\sum_{i\in I} P_i w_i[V_i/x] + Z'}$ and easily check that $\tau \stackrel{\sim}{\to}^* \tau_0$. Moreover, $\llbracket\tau_0\rrbracket = \left\llbracket\underline{\sum_{i\in I} P_i w_i[V_i/x] + Z'}\right\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket \left\llbracket\underline{w_i[V_i/x]}\right\rrbracket + \llbracket\underline{Z'}\rrbracket$. By lemma 31, $\sum_{i\in I}\llbracket P_i\rrbracket \left\llbracket\underline{w_i[V_i/x]}\right\rrbracket + \llbracket\underline{Z'}\rrbracket = \sum_{i\in I}\llbracket P_i\rrbracket \llbracket\underline{w_i}\rrbracket[\llbracket\underline{V_i}\rrbracket/x] + \llbracket\underline{Z'}\rrbracket = 1\llbracket\underline{u}\rrbracket[\llbracket\underline{V}\rrbracket/x] + \underline{\mathbf{0}} = \llbracket\underline{u}\rrbracket[\llbracket\underline{V}\rrbracket/x] = \llbracket\sigma_0\rrbracket$ which implies $\tau_0 \in \nabla(\sigma_0)$.

Now we will address the case where σ is a term, in particular a sum, and $\sigma \stackrel{\sim}{\to}_{k+1} \sigma'$. Then, by definition of $\stackrel{\sim}{\to}$, $\sigma = \underline{Pu + Z}$ and $\sigma' = \underline{PU + Z}$ with $\underline{u} \to_{k+1} \underline{U}$. Hence, from the fact that $\underline{u}$ is a simple term and by what we have just shown for simple terms, we get that there exists $\underline{U_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{U} \stackrel{\sim}{\to}^* \underline{U_0}$ and for all $\underline{W} \in \nabla(\underline{u})$, there exists $\underline{W_0} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{W} \stackrel{\sim}{\to}^* \underline{W_0}$ and $\underline{W_0} \in \nabla(\underline{U_0})$. By fact 28, $\underline{Z}$ is in general a sum term that can be written as $\underline{Z} = \underline{\sum_{i\in I} P_i v_i + \sum_{j\in J} P_j v_j}$ with, for all $i \in I$, $j \in J$, $\underline{v_i} \in \nabla(\underline{u})$ and $\underline{v_j} \notin \nabla(\underline{u})$. By the remark we have just made, for all $i \in I$, there exists $\underline{V_{0_i}} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{v_i} \stackrel{\sim}{\to}^* \underline{V_{0_i}}$ and $\underline{V_{0_i}} \in \nabla(\underline{U_0})$. Hence, take $\sigma_0 = \underline{PU_0 + \sum_{i\in I} P_i V_{0_i} + \sum_{j\in J} P_j v_j}$ about which one can easily check that $\sigma' \stackrel{\sim}{\to}^* \sigma_0$.

Let $\tau \in \nabla(\sigma)$, that is $\llbracket\tau\rrbracket = \llbracket\underline{Pu + Z}\rrbracket = \left\llbracket\underline{Pu + \sum_{i\in I} P_i v_i + \sum_{j\in J} P_j v_j}\right\rrbracket$. Since for all $i \in I$, $\llbracket\underline{v_i}\rrbracket = \llbracket\underline{u}\rrbracket$, we get that $\left\llbracket\underline{Pu + \sum_{i\in I} P_i v_i + \sum_{j\in J} P_j v_j}\right\rrbracket = \left\llbracket\underline{(P + \sum_{i\in I} P_i)u + \sum_{j\in J} P_j v_j}\right\rrbracket$. By lemma 29, this implies $\tau = \underline{\sum_{k\in K} P_k t_k + Z'}$ with the following properties:

- $[\![\sum_{k\in K} P_k]\!] = \sum_{k\in K} [\![P_k]\!] = [\![P + \sum_{i\in I} P_i]\!]$ and, for all $k \in K$, $[\![\underline{t_k}]\!] = [\![\underline{u}]\!]$;
- $\underline{Z'} \notin \nabla(\underline{u})$ and $[\![\underline{Z'}]\!] = \left[\!\!\left[\underline{\sum_{j\in J} P_j v_j}\right]\!\!\right]$.

Since each $\underline{t_k} \in \nabla(\underline{u})$, we get that, for all $k \in K$, there exists $\underline{T_{0_k}} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\underline{t_k} \stackrel{\sim}{\rightarrow}^* \underline{T_{0_k}}$ and $\underline{T_{0_k}} \in \nabla(\underline{U_0})$. Then take $\tau_0 = \underline{\sum_{k\in K} P_k T_{0_k} + Z'}$ and easily check that $\tau \stackrel{\sim}{\rightarrow}^* \tau_0$. Moreover, $[\![\tau_0]\!] = \left[\!\!\left[\underline{\sum_{k\in K} P_k T_{0_k} + Z'}\right]\!\!\right] = \sum_{k\in K} [\![P_k]\!] \left[\!\!\left[\underline{T_{0_k}}\right]\!\!\right] + [\![\underline{Z'}]\!] = [\![P + \sum_{i\in I} P_i]\!]\, [\![\underline{U_0}]\!] + \left[\!\!\left[\underline{\sum_{j\in J} P_j v_j}\right]\!\!\right] = [\![P]\!]\, [\![\underline{U_0}]\!] + [\![\sum_{i\in I} P_i]\!]\, [\![\underline{U_0}]\!] + \left[\!\!\left[\underline{\sum_{j\in J} P_j v_j}\right]\!\!\right] = [\![P]\!]\, [\![\underline{U_0}]\!] + [\![\sum_{i\in I} P_i]\!] \left[\!\!\left[\underline{V_{0_i}}\right]\!\!\right] + \left[\!\!\left[\underline{\sum_{j\in J} P_j v_j}\right]\!\!\right] = [\![\sigma_0]\!]$ which implies $\tau_0 \in \nabla(\sigma_0)$. □

We now proceed to prove theorem 32.

Proof. Since both σ, τ are strongly normalizable terms, let us consider the longest $\stackrel{\sim}{\rightarrow}$-reduction sequences to their normal forms:

(1a) $\sigma \stackrel{\sim}{\rightarrow}_1 \sigma_1 \stackrel{\sim}{\rightarrow}_2 \ldots \stackrel{\sim}{\rightarrow}_m \sigma_m = \mathsf{NF}(\sigma)$

(2a) $\tau \stackrel{\sim}{\rightarrow}_1 \tau_1 \stackrel{\sim}{\rightarrow}_2 \ldots \stackrel{\sim}{\rightarrow}_n \tau_n = \mathsf{NF}(\tau)$

for some $m, n \in \mathbb{N}$.

We prove the result by induction on $m+n$. If $m+n = 0$, then both terms σ, τ are in normal form and the result trivially holds. Otherwise, let us suppose (at least) σ to be a reducible term, that is $m \neq 0$. Then, since $\tau \in \nabla(\sigma)$, we can apply lemma 33 on $\sigma \stackrel{\sim}{\rightarrow}_1 \sigma_1$: there exist σ', $\tau' \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $\sigma_1 \stackrel{\sim}{\rightarrow}^* \sigma'$, $\tau \stackrel{\sim}{\rightarrow}^* \tau'$ and $\tau' \in \nabla(\sigma')$. The latter proves $[\![\sigma']\!] = [\![\tau']\!]$.

The hypothesis on σ, τ implies σ' and τ' to be strongly normalizable terms. Moreover, the fact that $\stackrel{\sim}{\rightarrow}$-reduction is confluent assures that the normal forms of the latter are the same of the former. Therefore, as before, let us consider their longest $\stackrel{\sim}{\rightarrow}$-reduction sequences to normal forms:

(1b) $\sigma' \stackrel{\sim}{\rightarrow}_1 \sigma'_1 \stackrel{\sim}{\rightarrow}_2 \ldots \stackrel{\sim}{\rightarrow}_p \mathsf{NF}(\sigma)$

(2b) $\tau' \stackrel{\sim}{\rightarrow}_1 \tau'_1 \stackrel{\sim}{\rightarrow}_2 \ldots \stackrel{\sim}{\rightarrow}_q \mathsf{NF}(\tau)$

for some $p, q \in \mathbb{N}$.

Since $\sigma \stackrel{\sim}{\rightarrow}_1 \sigma_1 \stackrel{\sim}{\rightarrow}^* \sigma'$ and from the fact that $(1a)$ is the longest $\stackrel{\sim}{\rightarrow}$-reduction sequences to $\mathsf{NF}(\sigma)$, it follows $p < m$. Moreover, $\tau \stackrel{\sim}{\rightarrow}^* \tau'$ and from the fact that $(2a)$ is the longest $\stackrel{\sim}{\rightarrow}$-reduction sequences to $\mathsf{NF}(\tau)$, it follows $q \leq n$. The latter implies $p + q < m + n$. Then, the result $[\![\mathsf{NF}(\sigma)]\!] = [\![\mathsf{NF}(\tau)]\!]$ follows by induction hypothesis. □

As mentioned before, such result allows us to give a normal form to every term $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ by assigning to it the normal form computed in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ for its notations. In particular, such normal form is computed as in a "big-step" defined operational semantics.

Definition 34. Let $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ and $\dot{\sigma} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$. Then, we write $\mathsf{NF}(\sigma) = [\![\mathsf{NF}(\dot{\sigma})]\!]$.

Moreover, theorem 32 permits to define a proper reductional equivalence relation.

Definition 35. For all $\sigma, \tau \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$, $\sigma \simeq \tau$ if there exist $\dot{\sigma}, \dot{\tau} \in \mathsf{P}\langle\Delta_\mathsf{P}\rangle$ such that $[\![\mathsf{NF}(\dot{\sigma})]\!] = [\![\mathsf{NF}(\dot{\tau})]\!]$.

Following this definition, we are able to determine that $\underline{\mathbf{0}} \not\simeq \infty_\sigma - (\underline{I})\infty_\sigma$, for all $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$.

5. Conclusion and Further Work

We have studied the algebraic λ-calculus, an extension of the pure λ-calculus, obtained by enriching the set of λ-terms with a structure of module over a semiring.

Afterwards, we have examined two reduction notions, presenting their properties in terms of normalization and confluence. In particular, we have remarked that none of them are suitable to give a proper definition of normal form for terms of the algebraic λ-calculus.

The last part of the current work presents a new result as far as normalization properties of the calculus are concerned. Using a well-known method to tame interaction between the algebraic component of the calculus and the computational one, we have proved a kind of normalization theorem for the algebraic terms whose notations are terms with coefficients over a rig of polynomial. In particular, such a result allows to define, if it exists and always reachable, a normal form for any algebraic term even in the case where actual reduction notions are inconsistent. Another consequence is the definition of a reductional equivalence relation capable of distinguishing the normal form $\underline{\mathbf{0}}$ from the classical counterexample $\infty_\sigma - \infty_\sigma$, or even $\infty_\sigma - (\underline{I})\infty_\sigma$.

The present result is limited by a strong hypothesis, that is, terms in $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ have to be strongly normalizable. Despite the interest of such first result, it would have been more interesting to present a theorem relying on a weaker hypothesis, say, weak normalization. In that case, definitions 34 and 35 extend to every algebraic term having notations with (not always) reachable normal form.

The very next step of our research will be devoted to prove a theorem involving a weak normalization hypothesis. Afterwards, we will investigate on the characterization of finer relations between $\widetilde{\to}$-reduction defined on $\mathsf{P}\langle\Delta_\mathsf{P}\rangle$ and those defined on a general module $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$. Firstly, we will focus on an analogous theorem of that we have proved, but relative to a "small-step" reduction notion: in particular, the reduction $\widehat{\to}$ seems to fit our needs. Moreover, theorem 32 could be used to prove a confluence of the latter by means of normal forms, a property that it does not show *per se*. Then, such reduction notion for $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ will be the basis of our effort towards the study of those problems we find in the classical theory of the pure λ-calculus: for instance, the standardization theorem and the separation theorem (also known as Böhm's theorem).

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