Joint Spectral Radius, Dilation Equations, and Asymptotic Behavior of Radix-Rational Sequences

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Abstract

Radix-rational sequences are solutions of systems of recurrence equations based on the radix representation of the index. For each radix-rational sequence with complex values we provide an asymptotic expansion, essentially in the scale $N^\alpha \log^\ell N$. The precision of the asymptotic expansion depends on the joint spectral radius of the linear representation of the sequence of first-order differences. The coefficients are Hölderian functions obtained through some dilation equations, which are usual in the domains of wavelets and refinement schemes. The proofs are ultimately based on elementary linear algebra.

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1. Introduction

This article shows a new domain of application for tools of linear algebra, namely joint spectral radius and dilation equations. Its topic is the asymptotic study of a class of sequences, which fall into the class of divided-and-conquer sequences. The sequences under consideration satisfy a linear recursion with constant coefficients that essentially links the value for the index n to the value for the index $n/2$.

Example 1 (Odd numbers in Pascal triangle). We take a simple example with the sequence u_n which counts the number of odd numbers in the first n rows (that is to say rows number k with $0 \leq k < n$) of the Pascal triangle [20, 26]. This sequence satisfies $u_n = 2u_{\lfloor n/2 \rfloor} + u_{\lceil n/2 \rceil}$, that is

$$u_{2n} = 3u_n, \qquad u_{2n+1} = 2u_n + u_{n+1}, \tag{1}$$

and the initial conditions $u_0 = 0$, $u_1 = 1$. △

A more formal definition is the following. Let us assume that A_0 and A_1 are square matrices and C is a column vector. We define a vector-valued sequence U_n by its first term $U_0 = C$ and the recurrence relations $U_{2n} = A_0 U_n$ and $U_{2n+1} = A_1 U_n$. Next let us add to the data a row vector L from which we derive a scalar sequence $u_n = LU_n$. The value of the latter for the index n can be computed as follows: we make explicit the binary representation of n as a binary word $w = w_{\ell-1} \ldots w_1 w_0$ and we compute the product LA_wC with $A_w = A_{w_{\ell-1}} \cdots A_{w_1} A_{w_0}$. This process parallels the computation of a classical rational sequence r_n, that is a sequence whose ordinary generating function is a rational function. For such a sequence the term of index n is given by $r_n = LA_0^n C$ (for some matrices L, A_0, C, which are respectively row, square, column matrices), and we can see the word $w = 0 \ldots 0$, with the symbol 0 repeated n times, as the writing of n with n sticks (or rings). This leads us to call the sequence u_n a rational sequence with respect to radix 2, or a radix-2 rational sequence, or simply a 2-rational sequence. Obviously, the idea generalizes to B-rational sequences,

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where B is an integer larger than 1, using the radix-B expansion of integers and a family $(A_b)_{0\leq b<B}$ of B square matrices. The data L, $(A_b)_{0\leq b<B}$, C is called a *linear representation* of the B-rational sequence under consideration.

Example 2 (Odd numbers in Pascal triangle, continued). Before we go further, it is worth showing how we find such a linear representation. A simple example, which avoid heavy computations, is provided by the sequence u_n of Example 1. We start with the empty set $\mathcal{B} = \{\ \}$ and will iteratively saturate $\mathcal{B}$ so that we obtain a basis. Because the sequence u_n is non-zero, we add it to $\mathcal{B}$, which becomes $\mathcal{B} = \{u_n\}$. Next we consider the sections u_{2n} and u_{2n+1}. According to (1), u_{2n} is only a multiple of u_n. On the contrary, the sequence u_{2n+1} is linearly independent of u_n, as it may be seen by looking at their first two values $(1, 5)$ and $(0, 1)$ respectively. We add it to $\mathcal{B}$, which is now $\mathcal{B} = \{u_n, u_{2n+1}\}$. We consider the sections u_{4n+1} and u_{4n+3} of u_{2n+1}. We find $u_{4n+1} = 6u_{2n} + u_{2n+1}$ and $u_{4n+3} = 2u_{2n+1} + 3u_{n+1} = 5u_{2n+1} - 6u_n$. Therefore the 2-dimensional space with basis $\mathcal{B}$ is left stable by the section operators $S_0 : v_n \mapsto v_{2n}$ and $S_1 : v_n \mapsto v_{2n+1}$. We have proved that u_n is a radix-2 rational sequence and that it admits the representation

$$L = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2}$$

because u_n is the first vector of the basis, hence C, and the initial values of the sequences in $\mathcal{B}$ are 0 and 1, hence L.

The matrices A_0 and A_1 in (2) express the section operators and the linear representation allows us to compute the values of the sequence u_n. For example, let us consider the integer $13 = (1101)_2$ and its radix-2 expansion 1101. Because C is the coordinates vector of u_n in the basis, the vector A_1C gives the coordinates of u_{2n+1}, the vector A_0A_1C gives the coordinates of u_{4n+1}, the vector $A_1A_0A_1C$ ives the coordinates of u_{8n+5}, the vector $A_1A_1A_0A_1C$ gives the coordinates of u_{16n+13}. Moreover L is the row vector of the evaluation at $n = 0$ and $u_{13} = LA_1A_1A_0A_1C$. $\triangle$

If the ring of scalars is the field of complex numbers, it makes sense to ask about the asymptotic behavior of a B-rational sequence. It will be shown in this article, by elementary linear algebra, that there exists an asymptotic expansion

$$u_n \underset{n\to+\infty}{=} \sum_{\substack{\alpha>\alpha_* \\ \ell\geq 0,\,\vartheta}} n^\alpha \binom{\log_B n}{\ell} \exp(i\vartheta \log_B n)\Psi_{\alpha,\ell,\vartheta}(\log_B n) \; + \; O(n^{\alpha_*}), \tag{3}$$

where exponents α, angular variables ϑ are real numbers, while the ℓ's are nonnegative integers, and the Ψ's are 1-periodic functions. This is our main theoretical result, Theorem 3 below.

The algebraic framework of sequences that are rational with respect to a radix is not new. It has been defined by Allouche and Shallit [1, 2], under the name *k-regular sequences*. Besides, many authors have dealt with the asymptotic behavior of sequences among which are some B-rational sequences. Roughly speaking, the earliest works used elementary arguments, which come from analysis [13] or combinatorics of words [18]. More recent works employ sophisticated arguments, based on analytical number theory, and more precisely on the use of some meromorphic Dirichlet series [19, 20]. This approach has been considerably developed in the last twenty years [16] and is summarized in [14]. The present article uses an approach that is different and new, based on linear algebra. It emphasizes the occurrence of dilation equations that determine the functions Ψ in the asymptotic expansion, and explain their often fractal nature. Moreover, the joint spectral radius provides a natural bound to the precision of the expansion, while the precision based on analytic number theory arguments seems to be limited only by the ability of the user.

The structure of the article is the following. Having qualitatively but clearly stated the theorem that we have in mind in Section 2, we proceed to the proof in the following sections. First, in Section 3 we establish a recursion formula, which is the basis of our analysis, for a sum directly related to the partial sum s_N. Next in Section 4 a Jordan reduction breaks the problem into sub-problems. They are not really simpler but more structured and some dilation equations appear. As a consequence, the notion of joint spectral radius enters naturally into the computation. This leads us to asymptotic expansions for each of the sub-problems. In Section 5 we merge these partial asymptotic expansions into a global asymptotic expansion and we obtain a quantitative version of our theorem. The asymptotic expansion it announces appears as a result of a process of computation. We conclude the section by showing that the coefficients of the expansion have some regularity. The last section, Section 6, presents some examples.

2. Linear representation

In the sequel, we will constantly consider a B-rational sequence u_n, $n \geq 0$ defined by a linear representation L, $(A_b)_{0\leq b<B}$, C of dimension d. We will estimate the asymptotic behavior of the partial sum

$$s_N = \sum_{0\leq n\leq N} u_n. \tag{4}$$

In other words, we have a linear representation for the sequence of backward differences $u_n = \nabla s_n$ and we want an asymptotic expansion for the sequence s_n. As a matter of fact, it is equivalent to start with u_n or s_n, at least theoretically. This is obvious from the standpoint of sequences (each sequence determines the other), but also true from the standpoint of radix-rational sequences and their linear representations, as it is shown in the next lemma.

Lemma 1. *Let u_n be a B-rational sequence that admits the linear representation L, $(A_b)_{0\leq b<B}$, C of dimension d. Then the sequence s_n of its partial sums is B-rational too, and admits the linear representation of size $2d$*

$$L' = \begin{pmatrix} L & L \end{pmatrix}, \qquad A'_b = \begin{pmatrix} B_0 & 0 \\ -B_{b+1} & A_b \end{pmatrix} \quad \textit{for } 0 \leq b < B, \qquad C' = \begin{pmatrix} C \\ 0 \end{pmatrix},$$

with

$$B_b = \sum_{c=b}^{B-1} A_c \quad \textit{for } 0 \leq b \leq B.$$

Conversely, let s_n be a B-rational sequence that admits the linear representation L, $(A_b)_{0\leq b<B}$, C of dimension d. Then the sequence u_n of its backward differences is B-rational too, and admits the linear representation of size $2d$

$$L' = \begin{pmatrix} L & 0 \end{pmatrix}, \quad A'_0 = \begin{pmatrix} A_0 & 0 \\ 0 & A_{B-1} \end{pmatrix}, \quad A'_b = \begin{pmatrix} A_b & A_{b-1} \\ 0 & 0 \end{pmatrix} \quad \textit{for } 0 < b < B, \quad C' = \begin{pmatrix} C \\ -C \end{pmatrix},$$

with $A_{-1} = 0$.

Proof. We sketch the proof for the second assertion. The existence of the linear representation means that there exists a family $s_n^1, \ldots, s_n^d$ of sequences which generates a space left stable by the section operators $S_b : v_n \mapsto v_{Bn+b}$. We consider the family of sequences $t_n^1, \ldots, t_n^d$ defined by $t_n^j = s_{n-1}^j$, with $s_{-1}^j = 0$. On one hand we have

$$S_b s_n^j = s_{Bn+b}^j = \sum_{i=1}^{d} A_{b,i,j} s_n^i.$$

On the other hand, we have for $b > 0$

$$S_b t_n^j = s_{Bn+b-1}^j = \sum_{i=1}^{d} A_{b-1,i,j} s_n^i$$

Using the whole family $s_n^1, \ldots, s_n^d, t_n^1, \ldots, t_n^d$ as a generating set, we obtain the above representation. The first assertion pertains to a similar proof [1, 2]. $\square$

The essential result of this article can be qualitatively expressed by the following assertion, and Sections 3 to 5 will be devoted to its proof. The notions of joint spectral radius and of dilation equation will be recalled in Section 4.2.

Theorem 1. *Let L, $(A_b)_{0\leq b<B}$, C be a linear representation of a complex radix-rational sequence u_n. The partial sum*

$$s_N = \sum_{0\leq n\leq N} u_n \tag{5}$$

admits an asymptotic expansion with variable coefficients of the form (3). Its error term is $O(N^{\log_B r})$ for every $r > \rho_*$, where ρ_* is the joint spectral radius of the family $(A_b)_{0\leq b<B}$. The used asymptotic scale is the family of sequences $N^\alpha\binom{\log_B N}{\ell}$, $\alpha \in \mathbb{R}$, $\ell \in \mathbb{N}_{\geq 0}$. The exponents α which appear in the expansion are the logarithms in base B of the moduli, larger than ρ_*, of the eigenvalues of a matrix Q, which is the sum of the square matrices of the representation. The numbers ϑ are the arguments of the eigenvalues. The coefficients $\Psi(\log_B N)$ are Hölderian functions and defined through some dilation equations.

As a consequence, s_N behaves at most as $N^{\log_B \rho(Q)}$ where $\rho(Q)$ is the spectral radius of Q, times perhaps some $\log_B^\ell N$.

3. Recursion equation

The figures of radix-B expansions of the integers may be viewed as elements of an alphabet and the linear representation defines not only the radix-rational sequence u_n but also a rational formal power series S, by

$$S(w) = LA_wC \tag{6}$$

with $A_w = A_{w_1}A_{w_2}\cdots A_{w_K}$ for a word $w = w_1w_2\ldots w_K$ of length K. Let us recall that a formal power series [5, 29] is a map which associates to each word w a value $S(w)$ in the ring of scalars. The basic case is the case where there is only one letter x, hence a power series $\sum_{n\in\mathbb{N}} S_n x^n$. The sequence u_n is nothing but the restriction of the formal power series S to the encoding of integers in the radix-B numeration system.

3.1. Numeration system

We cut the sum s_N into two pieces, the cutting point being the largest power of B not greater than N, say B^K. In other words, we use the following notation: The integer N writes $N = B^{K+t}$, where K is the integer part of $\log_B N$ and t is its fractional part, that is

$$K = \lfloor \log_B N \rfloor, \qquad t = \{\log_B N\}. \tag{7}$$

For the sum from 0 to $B^K - 1$, the radix-B expansion of the index n is a word w of length not greater than K, but whose first figure is not 0. For the complementary sum from B^K to B^{K+t}, the length is $K+1$ and the same remark is valid. We obtain

$$\sum_{n\leq B^{K+t}} u_n = \sum_{0\leq k\leq K}\Big(\sum_{|w|=k} LA_wC - \sum_{|w'|=k-1} LA_0A_{w'}C\Big) + \Big(\sum_{\substack{|w|=K+1\\ (w)_B\leq B^{K+1}B^{t-1}}} LA_wC - \sum_{|w'|=K} LA_0A_{w'}C\Big),$$

that is

$$\sum_{n\leq B^{K+t}} u_n = L(\mathrm{I}_d - A_0)\sum_{0\leq k\leq K} Q^kC + \sum_{\substack{|w|=K+1\\ (w)_B\leq B^{K+1}B^{t-1}}} LA_wC. \tag{8}$$

The matrix Q which appears in the formula is the sum of the square matrices of the representation,

$$Q = A_0 + A_1 + \cdots + A_{B-1}. \tag{9}$$

It will prove to be the main ingredient that governs the asymptotic behavior of s_N.

The first term in the right hand side of (8) is not mysterious, for it is a classical rational sequence with respect to K. It remains to understand the last one. To this end, we define the vector valued partial sum, associated to the formal power series S,

$$\mathbf{S}_K(x) = \sum_{\substack{|w|=K \\ (0.w)_B \leq x}} A_w C \tag{10}$$

where x is a real number in the segment $[0,1]$. As a matter of fact, s_N and $\mathbf{S}_K(x)$ are related by

$$s_N = L(\mathrm{I}_d - A_0) \sum_{0 \leq k \leq K} Q^k C + L\mathbf{S}_{K+1}(B^{t-1}). \tag{11}$$

This is the key for the asymptotic of s_N. But, it needs the study of $\mathbf{S}_K(x)$ first.

3.2. Basic recursion

Let $x = (0.x_1x_2\cdots)_B$ be the radix-B expansion of the real number x taken in $[0,1)$. The B-adic numbers located in the segment $[0,x]$ with a radix-B expansion $(0.w)_B$ and a length K mantissa w fall in exactly one of the following intervals: $[0,(0.x_1)_B)$, $[(0.x_1)_B,(0.x_1x_2)_B)$, $[(0.x_1x_2)_B,(0.x_1x_2x_3)_B)$, and so on. In other words, we are sorting the words according to their first figures/letters. This remark leads to the formula

$$\begin{aligned}\mathbf{S}_K(x) = \sum_{b_1<x_1} A_{b_1} Q^{K-1} C + \sum_{b_2<x_2} A_{x_1} A_{b_2} Q^{K-2} C + \sum_{b_3<x_3} A_{x_1} A_{x_2} A_{b_3} Q^{K-3} C \\ + \cdots + \sum_{b_K \leq x_K} A_{x_1} A_{x_2} \cdots A_{b_K} C,\end{aligned} \tag{12}$$

which renders the next lemma obvious.

Lemma 2. *The sequence of partial sums $\mathbf{S}_K(x)$ satisfies the recursion*

$$\mathbf{S}_{K+1}(x) = \sum_{b_1<x_1} A_{b_1} Q^K C + A_{x_1} \mathbf{S}_K(Bx - x_1), \tag{13}$$

where x_1 is the first digit in the radix-B expansion of x in $[0,1)$, with $\mathbf{S}_0(x) = C$.

Lemma 2 is our starting point in the asymptotic study of $\mathbf{S}_K(x)$.

4. Dilation equation and joint spectral radius

4.1. Jordan reduction

We introduce a Jordan basis for Q and we expand C on this basis. As a consequence, we have to study the asymptotic behavior of $\mathbf{S}_K(x)$ for a generalized eigenvector. More precisely, we consider an eigenvalue $\rho\omega$ of Q, with $\rho > 0$ and $|\omega| = 1$, and a free family $(V^{(j)})_{0\leq j<\nu}$, with $\nu \geq 1$, which satisfies $QV^{(0)} = \rho\omega V^{(0)}$ and $QV^{(j)} = \rho\omega V^{(j)} + V^{(j-1)}$ for $0 < j < \nu$. Let us assume that the partial sum associated to the vector $V^{(\nu-1)}$

$$\mathbf{S}_K(x) = \sum_{\substack{|w|=K \\ (0.w)_B \leq x}} A_w V^{(\nu-1)} \tag{14}$$

admits an asymptotic expansion with variable coefficients [7, Chap. V] of the form

$$\begin{aligned}\mathbf{S}_K(x) \underset{K\to+\infty}{=} \binom{K}{\nu-1} (\rho\omega)^{K-\nu+1} \mathbf{F}^{(0)}(x) + \binom{K}{\nu-2} (\rho\omega)^{K-\nu+2} \mathbf{F}^{(1)}(x) + \cdots \\ + \binom{K}{1} (\rho\omega)^{K-1} \mathbf{F}^{(\nu-2)}(x) + (\rho\omega)^K \mathbf{F}^{(\nu-1)}(x) + \mathrm{error}_K(x),\end{aligned} \tag{15}$$

where the $\mathbf{F}^{(j)}$'s are continuous functions from $[0,1]$ into $\mathbb{C}^d$, and with $\mathrm{error}_K(x) = o(\rho^K)$ say. This expansion must be compared with

$$Q^K V^{(\nu-1)} = \binom{K}{\nu-1}(\rho\omega)^{K-\nu+1}V^{(0)} + \binom{K}{\nu-2}(\rho\omega)^{K-\nu+2}V^{(1)} + \cdots \\ + \binom{K}{1}(\rho\omega)^{K-1}V^{(\nu-2)} + (\rho\omega)^K V^{(\nu-1)}. \quad (16)$$

The polynomial function $K \mapsto (\rho\omega)^{-K}Q^K V^{(\nu-1)}$ has a unique representation on the basis $\binom{K}{j}$, $0 \le j < \nu$, and we have necessarily $\mathbf{F}^{(j)}(1) = V^{(j)}$ for $0 \le j < \nu$. In the same manner the uniqueness of asymptotic expansions shows that the family $(\mathbf{F}^{(j)})_{0\le j<\nu}$ must satisfy the system of equations,

$$\left\{\begin{array}{rcl} \rho\omega\mathbf{F}^{(0)}(x) & = & \sum_{b<x_1} A_b V^{(0)} + A_{x_1}\mathbf{F}^{(0)}(Bx - x_1), \\ \mathbf{F}^{(0)}(x) + \rho\omega\mathbf{F}^{(1)}(x) & = & \sum_{b<x_1} A_b V^{(1)} + A_{x_1}\mathbf{F}^{(1)}(Bx - x_1), \\ & \vdots & \\ \mathbf{F}^{(\nu-2)}(x) + \rho\omega\mathbf{F}^{(\nu-1)}(x) & = & \sum_{b<x_1} A_b V^{(\nu-1)} + A_{x_1}\mathbf{F}^{(\nu-1)}(Bx - x_1). \end{array}\right. \quad (17)$$

We obtain these formulæ by substituting the asymptotic expansion into the functional equation of Lemma 2.

With the use of the usual Jordan block of size ν,

$$J = J_{\nu,\rho\omega} = \begin{pmatrix} \rho\omega & 1 & & & \\ & \rho\omega & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \rho\omega \end{pmatrix} \quad (18)$$

the previous system becomes a single equation

$$\mathbf{F}(x)J = \sum_{b<x_1} A_b V + A_{x_1}\mathbf{F}(Bx - x_1), \quad (19)$$

where the unknown $\mathbf{F}(x)$ and V are matrix-valued, with values in $\mathbb{C}^{[1,d]\times[0,\nu)}$. Anyway, the existence of the expansion (15) implies that the coefficients are solutions of the functional equation (19).

4.2. Dilation equation and joint spectral radius

Equation (19) is a dilation equation or two-scale difference equation, or refinement equation [24, 11, 12], because it uses x and Bx. It has not the usual aspect of dilation equations for it is inhomogeneous and works by cases, according to the first digit of the real x. But, if we decide to extend the function $\mathbf{F}(x)$ on the left of 0, respectively on the right of 1, by the value 0, respectively by the value V, the equation becomes

$$\mathbf{F}(x)J = \sum_{0\le b<B} A_b\mathbf{F}(Bx - b), \quad (20)$$

which is a more standard form, however with additional boundary conditions

$$\mathbf{F}(x) = 0 \quad \text{for } x \le 0, \qquad \mathbf{F}(x) = V \quad \text{for } x \ge 1. \quad (21)$$

A simple assumption which ensures that this equation has a unique solution is related to the joint spectral radius of the family $(A_b)_{0\le b<B}$. Let us recall its definition. We equip $\mathbb{C}^d$ with a norm and $\mathbb{C}^{d\times d}$ with the

induced norm. We consider all the products A_w, for words w of a given length T, together with their norms. With the notation

$$\rho_T = \max_{|w|=T} \|A_w\|^{1/T}, \tag{22}$$

the joint spectral radius [6, 28] of the set A_b, $0 \le b < B$, is the number

$$\rho_* = \lim_{T\to+\infty} \rho_T. \tag{23}$$

It is known that the joint spectral radius is not greater than any of the numbers ρ_T. Moreover ρ_* is independent of the induced norm used.

Lemma 3. *Let us assume that the joint spectral radius of the family $(A_b)_{0\le b<B}$ is smaller than the eigenvalue modulus, that is $\rho_* < \rho$. Then the dilation system (20) has a unique solution $\mathbf{F}$ in the space $\mathcal{C}$ of continuous function Φ from $\mathbb{R}$ into $\mathbb{C}^{d\times[0,\nu)}$ such that $\Phi(x) = 0$ for $x \le 0$ and $\Phi(x) = V$ for $x \ge 1$.*

Proof. Because of the positivity of ρ, the matrix J is invertible and the system turns out to be a fixed point equation $\mathbf{F} = \mathcal{L}\mathbf{F}$ for some operator $\mathcal{L}$, namely

$$\mathcal{L}\Phi(x) = \sum_{0\le b<B} A_b \Phi(Bx - b) J^{-1}. \tag{24}$$

Let μ lie between ρ_* and ρ, that is $\rho_* < \mu < \rho$. Because the spectral radius of J^{-1} is $1/\rho$, there exists a norm on $\mathbb{C}^{[0,\nu)}$ whose induced norm provides $1/\rho \le \|J^{-1}\| < 1/\mu$. We choose a norm on $\mathbb{C}^d$ and use the induced norms on $\mathbb{C}^{d\times d}$, $\mathbb{C}^{d\times[0,\nu)}$, $\mathbb{C}^{[0,\nu)\times[0,\nu)}$. The space of bounded continuous functions from $\mathbb{R}$ into the space $\mathbb{C}^{d\times[0,\nu)}$, equipped with the norm of the maximum $\|\Phi\|_\infty = \max_x \|\Phi(x)\|$, is a complete normed space. It is not difficult to verify that the space $\mathcal{C}$ is left stable by $\mathcal{L}$ using the relation $QV = VJ$. Next there exists an integer T which gives $\rho_* \le \rho_T < \mu$. For a word w of length T, we have $\|A_w\|\|J^{-T}\| < \mu^T \times 1/\mu^T = 1$. As a consequence $\mathcal{L}^T$ is a contracting operator, for there is only a finite number of length T words. It has a unique fixed point in $\mathcal{C}$, say $\mathbf{F}^0$. Moreover if $(\mathbf{F}_k)$ is the sequence defined by $\mathbf{F}_0 = 0$ (the null matrix in $\mathbb{C}^{d\times[0,\nu)}$) and $\mathbf{F}_{k+1} = \mathcal{L}\mathbf{F}_k$, then the subsequence $\mathbf{F}_{kT}$ converges toward $\mathbf{F}^0$. The operator $\mathcal{L}$ is continuous and each of the sequence $\mathbf{F}_{kT+s}$ converges towards say $\mathbf{F}^s = \mathcal{L}^s\mathbf{F}^0$. But each of these functions satisfies $\mathbf{F}^s = \mathcal{L}^T\mathbf{F}^s$ because $\mathbf{F}^0$ is a fixed point of $\mathcal{L}^T$. Since the unique solution of the equation $\Phi = \mathcal{L}^T\Phi$ is $\mathbf{F}^0$, all these functions are equal. Because of the equality $F^0 = F^1$, the function F^0 is a solution of the equation $\Phi = \mathcal{L}\Phi$. Moreover it is the only one for it is continuous and the equation determines it on the dense set of B-adic numbers, by the so called cascade algorithm (quickly recalled in § 5.3). □

4.3. Elementary expansions

The joint spectral radius is not only useful in determining the terms of the asymptotic development but also in quantifying the error term. Specifically, it is necessary to distinguish two cases. In the first case, there exists a norm and an integer T such that $\rho_* = \rho_T$, that is the set of matrices A_b, $0 \le b < B$, has the finiteness property [23, 22]. We also say that the linear representation has the finiteness property. When this property is satisfied the error term will essentially be $O(\rho_*^K)$, while in the contrary it will be only $O(r^K)$ for every $r > \rho_*$.

Lemma 4. *Let $\mathbf{A}_K$ be the function from $[0,1]$ into $\mathbb{C}^d$ defined by*

$$\mathbf{A}_K(x) = \binom{K}{\nu-1}(\rho\omega)^{K-\nu+1}\mathbf{F}^{(0)}(x) + \binom{K}{\nu-2}(\rho\omega)^{K-\nu+2}\mathbf{F}^{(1)}(x) + \cdots$$
$$+ \binom{K}{1}(\rho\omega)^{K-1}\mathbf{F}^{(\nu-2)}(x) + (\rho\omega)^K\mathbf{F}^{(\nu-1)}(x) \tag{25}$$

for each nonnegative integer K, where $\mathbf{F}^{(0)}$, …, $\mathbf{F}^{(\nu-1)}$ are the functions whose existence and uniqueness are asserted by Lemma 3. With the assumption $\rho_* < \rho$, the partial sum $\mathbf{S}_K$ associated to $V^{(\nu-1)}$ writes $\mathbf{S}_K(x) = \mathbf{A}_K(x) + O(r^K)$ for every $r > \rho_*$ and the big oh is uniform with respect to x in $[0,1]$. Moreover if the linear representation has the finiteness property we may replace r by ρ_*.

Despite of its technical aspect, the result is very simple: to obtain the expansion of $\mathbf{S}_K(x)$, each term of the expansion of $Q^K V^{(\nu-1)} = \mathbf{S}_K(1)$ is shaken by a vector-valued function which goes from 0 to the vector which appears at the same place in the expansion.

Proof. With $\mathrm{error}_K(x) = \mathbf{S}_K(x) - \mathbf{A}_K(x)$ it is not difficult to obtain, by substitution, the recursion

$$\mathrm{error}_{K+1}(x) = A_{x_1}\,\mathrm{error}_K(Bx - x_1).$$

More precisely, we use the recursion of Lemma 2, System (17), and Pascal's recurrence for binomial coefficients. For $r > \rho_*$ we choose an integer T such that $\rho_* \le \rho_T < r$. We readily obtain $\|\mathrm{error}_{KT}\|_\infty = O(\rho_T^{KT})$. Next the recursion shows that we have $\|\mathrm{error}_{KT+s}\|_\infty = O(\rho_T^{KT+s})$ for $0 \le s < T$, hence the result. If ρ_* is some ρ_T (that is the linear representation has the finiteness property), it is useless to consider a $r > \rho_*$. $\square$

It remains to take into account the case $\rho \le \rho_*$. We content ourselves with exhibiting a natural bound.

Lemma 5. *Under assumption $0 \le \rho \le \rho_*$, the sequence of partial sums $\mathbf{S}_K(x)$ associated to $V^{(\nu-1)}$ satisfies $\|\mathbf{S}_K\|_\infty = O(r^K)$ for every $r > \rho_*$. Moreover if the linear representation has the finiteness property, we may replace $O(r^K)$ by $O(\rho_*^K)$ in case $\rho < \rho_*$ and by $O(\rho_*^K K^\nu)$ in case $\rho = \rho_*$.*

Proof. Let us assume first that we have a r such that $\rho \le r$ and $\|A_b\| \le r$ for $0 \le b < B$ and for some induced norm. The generalized eigenvector $V^{(\nu-1)}$ satisfies $Q^K V^{(\nu-1)} = O(\rho^K K^{\nu-1})$. The recursion formula of Lemma 2 gives, for some positive constant c, $\|\mathbf{S}_{K+1}\|_\infty \le c\rho^K K^{\nu-1} + r\,\|\mathbf{S}_K\|_\infty$. From this follows $\|\mathbf{S}_K\|_\infty = O(r^K)$ in case $\rho < r$ and $\|\mathbf{S}_K\|_\infty = O(r^K K^\nu)$ in case $\rho = r$.

Second we consider the general case, which divides into two sub-cases. Let us assume that the joint spectral radius is smaller that all ρ_T, for every induced norm. For any $r > \rho_*$, we may find, for a given induced norm, a T such that $\rho_* < \rho_T \le r$. We apply the previous argument to the sequence $\mathbf{S}_{KT}$. This gives $\|\mathbf{S}_{KT}\|_\infty = O(\rho_T^{KT})$ because we have $\rho \le \rho_* < \rho_T$. Next, Lemma 2 gives $\|\mathbf{S}_{KT+s}\|_\infty = O(\rho_T^{KT+s})$ for $0 \le s < T$, because there is a finite number of s between 0 and T. In this way, we obtain $\|\mathbf{S}_K\|_\infty = O(\rho_T^K)$, and we conclude $\|\mathbf{S}_K\|_\infty = O(r^K)$ for every $r > \rho_*$.

Let us assume now that ρ_* is some ρ_T for some induced norm. In case $0 \le \rho < \rho_* = \rho_T$ the same reasoning as above, with ρ_* in place of r, gives $\|\mathbf{S}_K\|_\infty = O(\rho_*^K)$. In case $\rho = \rho_*$ we find $\|\mathbf{S}_{KT}\|_\infty = O(K^\nu \rho_*^{KT})$. As above , we obtain $\|\mathbf{S}_K\|_\infty = O(K^\nu \rho_*^K)$. $\square$

5. Asymptotic expansions

5.1. Formal power series

Gathering previous lemmata, we arrive at the following result.

Theorem 2. *Let L, $(A_b)_{0\le b<B}$, C be a linear representation of a rational formal power series. The sequence of partial sums*

$$\mathbf{S}_K(x) = \sum_{\substack{|w|=K \\ (0.w)_B \le x}} A_w C \tag{26}$$

admits an asymptotic expansion

$$\mathbf{S}_K(x) \underset{K\to+\infty}{=} \sum_{\substack{\rho>r \\ \ell\ge 0,\omega}} \rho^K \binom{K}{\ell} \omega^K G(x) + O(r^K), \tag{27}$$

which is a sum of elementary expansions $\mathbf{A}_K(x)$ provided by Lemma 4 and an error term $O(r^K)$ for every $r > \rho_*$, where ρ_* is the joint spectral radius of the family $(A_b)_{0\le b<B}$. The used asymptotic scale is the family of sequences $\rho^K\binom{K}{\ell}$, $\rho > 0$, $\ell \in \mathbb{N}_{\ge 0}$. The ρ's which appear in the asymptotic expansion are the moduli greater than r of the eigenvalues $\rho\omega$ of matrix $Q = A_0 + A_1 + \cdots + A_{B-1}$. The coefficients are functions defined through some dilation equations (20). The error term is uniform with respect to x in $[0,1]$.

Practically, the elementary expansions (25) are obtained through Lemma 4 and the solution of the dilation equation (20) with the boundary conditions (21) ensured by Lemma 3. It is worthwhile that the error term can be made more precise in case where the linear representation has the finiteness property (Lemma 5).

5.2. Translation for the radix-rational sequence

With Formula (11), Theorem 2 translates immediately into Theorem 1. Practically, we replace K by $K+1$ and x by B^{t-1}; we write $\omega = \exp(i\vartheta)$ for some real ϑ; every occurrence of $K+1 = (K+t)+(1-t)$ is replaced by $(\log_B N) + (1 - \{\log_B N\})$. Since this is an essential step of the calculation, we focus on the transformation and provide to the reader a more precise assertion. In the next assertion, we use the decomposition $J = \rho e^{i\vartheta}\,\mathrm{I}_{[0,\nu)} + Z$ and Z is a nilpotent matrix of index ν.

Lemma 6. *The contribution of the Jordan cell $J = J_{\nu,\rho e^{i\vartheta}}$, with $\rho > \rho_*$, to the asymptotic expansion of s_N writes $L\mathbf{E}^{(\nu-1)}(\log_B N)$ where $\mathbf{E}^{(\nu-1)}(\log_B N)$ is the last column of a $d \times \nu$ matrix $\mathbf{E}(\log_B N)$. In case $\rho e^{i\vartheta} \neq 1$, this matrix reads*

$$\mathbf{E}(\log_B N) = \mathit{\Gamma} + N^{\log_B \rho}(\rho^{-1}e^{-i\vartheta}J)^{\log_B N} \times e^{i\vartheta \log_B N} \times \mathbf{\Phi}(\log_B N), \tag{28}$$

where $\mathit{\Gamma}$ is a constant matrix

$$\mathit{\Gamma} = (\mathrm{I}_d - A_0)V(\mathrm{I}_{[0,\nu)} - J)^{-1} \tag{29}$$

and $\mathbf{\Phi}(t)$ is a 1-periodic function, expressed with the function $\mathbf{F}(x)$ defined in Lemma 3,

$$\mathbf{\Phi}(t) = \Big(-(\mathrm{I}_d - A_0)V(\mathrm{I}_{[0,\nu)} - J)^{-1} + \mathbf{F}(B^{\{t\}-1})\Big)\, J^{1-\{t\}}. \tag{30}$$

In case $\rho e^{i\vartheta} = 1$, the matrix $\mathbf{E}(\log_B N)$ reads

$$\mathbf{E}(\log_B N) = \mathbf{P}(\log_B N) + \mathbf{\Phi}_1(\log_B N) J^{\log_B N}, \tag{31}$$

with $\mathbf{P}(t)$ a polynomial function

$$\mathbf{P}(t) = -(\mathrm{I}_d - A_0)V \sum_{\ell=0}^{\nu-1} \binom{t}{\ell+1} Z^\ell \sum_{\ell=0}^{\nu-1} \binom{t}{\ell} Z^\ell \tag{32}$$

and $\mathbf{\Phi}_1(t)$ a 1-periodic function

$$\mathbf{\Phi}_1(t) = (\mathrm{I}_d - A_0)V \sum_{\ell=0}^{\nu-1} \binom{1-\{t\}}{\ell+1} Z^\ell + \mathbf{F}(B^{\{t\}-1}) J^{1-\{t\}}. \tag{33}$$

Note that we have slightly changed the meaning of the variable t, which is no more the fractional part of $\log_B N$, but $\log_B N$ itself. The powers of $\mathrm{I}_{[0,\nu)} + \mu Z$ with real exponents α write [21, Chap. 5]

$$(\mathrm{I}_{[0,\nu)} + \mu Z)^\alpha = \sum_{\ell=0}^{\nu-1} \binom{\alpha}{\ell} \mu^\ell Z^\ell, \tag{34}$$

and in particular, with $J = \rho e^{i\vartheta}\,\mathrm{I}_{[0,\nu)} + Z$,

$$(\rho^{-1}e^{-i\vartheta}J)^{\log_B N} = \sum_{\ell=0}^{\nu} \binom{\log_B N}{\ell} \rho^{-\ell} e^{-i\ell\vartheta} Z^\ell, \quad J^{1-\{t\}} = \rho^{1-\{t\}} e^{i\vartheta(1-\{t\})} \sum_{\ell=0}^{\nu} \binom{1-\{t\}}{\ell} \rho^{-\ell} e^{-i\ell\vartheta} Z^\ell. \tag{35}$$

As a consequence, $(\rho^{-1}e^{-i\vartheta}J)^{\log_B N}$ appears as a polynomial in $\log_B N$. In case $\rho e^{i\vartheta} \neq 1$, the constant term $\mathit{\Gamma}$ is to be taken into account only if the joint spectral radius ρ_* is smaller than 1. Apart this term, $\mathbf{E}(t)$ proves to be the product of three terms:

– the power $N^{\log_B \rho}$ times a polynomial in $\log_B N$, and this leads to the asymptotic scale $N^\alpha \log_B^\ell N$ or, more naturally in the problem, $N^\alpha \binom{\log_B N}{\ell}$;

– a periodic function $e^{i\vartheta \log_B N}$, frequently hidden in practice where the eigenvalue $\rho e^{i\vartheta}$ is a positive number;

– a 1-periodic function $\mathbf{\Phi}(t)$ with respect to $t = \log_B N$, which can be a constant. This often happens in the dominant term of the asymptotic expansion.

The case $\rho e^{i\vartheta} = 1$ appears as a degenerate case: the power $N^{\log_B \rho}$ disappears and the oscillating function $e^{i\vartheta t}$ too. It remains only a logarithmic term (with degree in $\log_B N$ less than 2ν) plus a logarithmic term (with degree in $\log_B N$ less than ν) times a 1-periodic function with respect to $\log_B N$.

Proof. The link between the formal power series and the sequence is given by Formula (11)

$$s_N = L(\mathrm{I}_d - A_0) \sum_{0\le k\le K} Q^k C + L\mathbf{S}_{K+1}(B^{t-1}). \tag{36}$$

As indicated above, we change the meaning of t, and consequently we replace t by $\{t\}$ in this formula. We are dealing with the contribution of the vector $V^{(\nu-1)}$ and this vector is the last column of the matrix V used in the boundary condition (21) of the dilation equation. Moreover we have found in Lemma 4 the asymptotic expansion of $\mathbf{S}_K(x)$ associated to $V^{(\nu-1)}$. It is not difficult to see that the expression $\mathbf{A}_K(x)$ of Formula (25) is the last column of the product $\mathbf{F}(x)J^K$. It therefore appears that the contribution of the vector $V^{(\nu-1)}$ in the asymptotic expansion of s_N is obtained as the product $L\mathbf{E}^{(\nu-1)}$, where $\mathbf{E}^{(\nu-1)}$ is the last column of the matrix

$$\mathbf{E}(\log_B N) = (\mathrm{I}_d - A_0) \sum_{k=0}^{K} Q^k V + \mathbf{F}(B^{\{t\}-1})J^{K+1}. \tag{37}$$

But the assumption about $V^{(\nu-1)}$, as a generalized eigenvector of Q, translates into the equality $QV = VJ$. Hence we have to deal with the expression

$$\mathbf{E}(\log_B N) = (\mathrm{I}_d - A_0)V \sum_{k=0}^{K} J^k + \mathbf{F}(B^{\{t\}-1})J^{K+1}. \tag{38}$$

In case $\rho e^{i\vartheta} \neq 1$, the matrix $\mathrm{I}_{[0,\nu)} - J$ is invertible and we find readily

$$\begin{aligned}\mathbf{E}(\log_B N) &= (\mathrm{I}_d - A_0)V(\mathrm{I}_{[0,\nu)} - J)^{-1}(\mathrm{I}_{[0,\nu)} - J^{K+1}) + \mathbf{F}(B^{\{t\}-1})J^{K+1} \\ &= (\mathrm{I}_d - A_0)V(\mathrm{I}_{[0,\nu)} - J)^{-1} + \Big(-(\mathrm{I}_d - A_0)V(\mathrm{I}_{[0,\nu)} - J)^{-1}\mathbf{F}(B^{\{t\}-1})\Big) \times \\ &\qquad \rho^{\log_B N} e^{i\vartheta \log_B N} (\rho^{-1}e^{-i\vartheta}J)^{\log_B N} J^{1-\{\log_B N\}}\end{aligned} \tag{39}$$

by rewriting the factor J^{K+1} and immediately the expected result.

The case $\rho e^{i\vartheta} = 1$ is a little bit more tedious. We write, with $t = \log_B N$,

$$\mathbf{E}(t) = \left((\mathrm{I}_d - A_0)V \sum_{k=0}^{K} J^{-k-\{t\}} + \mathbf{F}(B^{\{t\}-1})J^{1-\{t\}} \right) J^t. \tag{40}$$

We have changed the index k of summation into $K - k$, pulled out the common factor J^K, and written it in the form $J^{-\{t\}} \times J^t$. The sum of the powers of J can be expressed as follows

$$\sum_{k=0}^{K} J^{-k-\{t\}} = \sum_{k=0}^{K} \sum_{\ell=0}^{\nu-1} \binom{-k-\{t\}}{\ell} Z^\ell = \sum_{\ell=0}^{\nu-1} \left(\binom{1-\{t\}}{\ell+1} - \binom{-t}{\ell+1} \right) Z^\ell. \tag{41}$$

By substitution, we obtain

$$\mathbf{E}(t) = \left((\mathrm{I}_d - A_0)V \sum_{\ell=0}^{\nu-1} \binom{1-\{t\}}{\ell+1} Z^\ell + \mathbf{F}(B^{\{t\}-1}) J^{1-\{t\}} \right) \sum_{\ell=0}^{\nu-1} \binom{t}{\ell} Z^\ell \\ - (\mathrm{I}_d - A_0)V \sum_{\ell=0}^{\nu-1} \binom{t}{\ell+1} Z^\ell \sum_{\ell=0}^{\nu-1} \binom{t}{\ell} Z^\ell \quad (42)$$

and the expected result. □

Let us summarize the obtained results.

Theorem 3. *Let L, $(A_b)_{0\le b<B}$, C be a linear representation of a complex radix-rational sequence u_n. The partial sum*

$$s_N = \sum_{0\le n\le N} u_n \quad (43)$$

admits an asymptotic expansion obtained by the following process:

- *compute the joint spectral radius ρ_*,*
- *find a Jordan basis for $Q = A_0 + \cdots + A_{B-1}$,*
- *decompose the coordinates vector C over this basis, and keep the generalized eigenvectors V of the basis which appear in the writing of C and which are associated to an eigenvalue larger than ρ_*,*
- *for each of these generalized eigenvectors, solve the associated dilation equation (20) with boundary conditions (21),*
- *for each of these generalized eigenvectors, write its contribution to the asymptotic expansion, according to Lemma 6,*
- *if the linear representation has not the finiteness property, write an error term $O(N^{\log_B r})$ for an arbitrary r between ρ_* and the smallest modulus of eigenvalue larger than ρ_*; else use Lemma 5,*
- *sum all the elementary expansions and the error term.*

5.3. Dilation equation, regularity and cascade algorithm

Theorem 1 is almost proved, but it lacks the statement on the regularity of the coefficients. As a matter of fact, dilation equations are usual in the domains of wavelets and refinement schemes. The study of the regularity of their solutions has led to a huge amount of publications, and because the result is common [12, Eq. (7) of Th. 2.2], [27, 8], we take the following assertion for granted.

Lemma 7. *The matrix-valued function $\mathbf{F}(t)$, defined in Lemma 3, is Hölder with exponent $\log_B(\rho/r)$ for every $r > \rho_*$. If the linear representation has the finiteness property, $\mathbf{F}(t)$ is Hölder with exponent $\log_B(\rho/\rho_*)$.*

This result translates immediately in a result about the matrix-valued functions $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}_1(t)$ of Lemma 6.

Theorem 4. *The matrix-valued function $\mathbf{\Phi}(t)$ defined in Lemma 6 is Hölder with exponent $\log_B(\rho/r)$ for every $r > \rho_*$. If the linear representation has the finiteness property, $\mathbf{F}(t)$ is Hölder with exponent $\log_B(\rho/\rho_*)$. The matrix-valued function $\mathbf{\Phi}_1(t)$ has the same property with $\rho = 1$.*

Proof. The only point which is to be verified is the continuity at integers, because up to this property the Hölderian character is evident: the coefficients are expressed as combinations of solutions of dilation equations and functions which are smooth except perhaps to integers.

Let us verify the continuity in case $\rho e^{i\vartheta} \neq 1$. For this we are considering the matrix-valued function $\mathbf{\Phi}(t)$ of Eq. (30),

$$\mathbf{\Phi}(t) = \left(-(\mathrm{I}_d - A_0)V(Id_{[0,\nu)} - J)^{-1} + \mathbf{F}(B^{\{t\}-1})\right) J^{1-\{t\}}. \tag{44}$$

With $F(1/B) = A_0 V J^{-1}$ and $F(1) = V$, according to the dilation equation and its boundary conditions, we find

$$\mathbf{\Phi}(1^-) - \mathbf{\Phi}(0) = -(\mathrm{I}_d - A_0)V(Id_{[0,\nu)} - J)^{-1}(Id_{[0,\nu)} - J) + V - A_0 V J^{-1} J = 0. \tag{45}$$

In the same way, we find $\mathbf{\Phi}_1(1^-) = \mathbf{\Phi}_1(0) = V$ and the same conclusion. □

In case where the modulus 1 complex number $\omega = e^{i\vartheta}$ is a root of unity of order q, it would have been tempting in the transformation of Lemma 6 to retain the term ω^K, which highlights a q-periodic function of $t = \log_B N$. But doing so, we lost the continuity of the 1-periodic cofactor because $e^{i\vartheta\lfloor t\rfloor}$ is not continuous.

Often the self-similarity is taken as an argument for a chaotic behavior, but a solution of a dilation equation may be very regular, for example a constant or a polynomial function. Our framework is not the same as in wavelets theory or in refinement schemes, but the same ideas work [11, 24], and it is certainly possible to identify conditions for explicit solutions. For example, a solution of the dilation system (20) is differentiable only if the same system with J replaced by J/B (and with a small change ρ by ρ/B) has a solution. We do not pursue this path.

Whether they are explicit or implicit, we can visualize the solutions of the dilation system (20) by the cascade algorithm [10, § 6.5]. In case $B = 2$, it calculates with the help of the system the values of the solution for $1/2$, $1/4$, $3/4$, and all dyadic numbers, and more generally for all B-adic numbers. In this paper all drawings of solutions of dilation equation or associated periodic functions are computed in this way.

6. Examples and comments

The steps of the above proof can be turned into an algorithm. In each example, the starting point is a linear representation of the sequence. We showed how to obtain such a representation in Example 2 and more generally the process of Example 2 is guaranteed [17] for all sequences u_n that satisfy a recurrence of divide-and-conquer type, where the $u_{B^K+\ell}$, $0 \le \ell < B^K$, are expressed linearly as a function of the u_{B^k+j} with $k < K$. We first exemplify the computation of the asymptotic expansion on a simple case.

Example 3 (Coquet sequence). The Coquet sequence [9] is defined as $u(n) = (-1)^{s_2(3n)}$. (Recall that $s_2(n)$ is the sum of the bits in the binary expansion of n.) It is 4-rational and admits (with $(u(n), u(4n+2), u(4n+3))$ as a basis) the representation

$$A_0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

The eigenvalues of $Q = A_0 + A_1 + A_2 + A_3$ are 3, which is double, and 0, which is simple. The joint spectral radius of the family $(A_b)_{0\le b<4}$ is $\rho_* = 1$, because all square matrices have a maximum sum column norm equal to 1 and 1 is an eigenvalue of each matrix A_b, $0 \le b < 4$. The column vector C decomposes as $C = V_3 + V_0$, with

$$V_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} 1/3 \\ -1/3 \\ -1/3 \end{pmatrix}$$

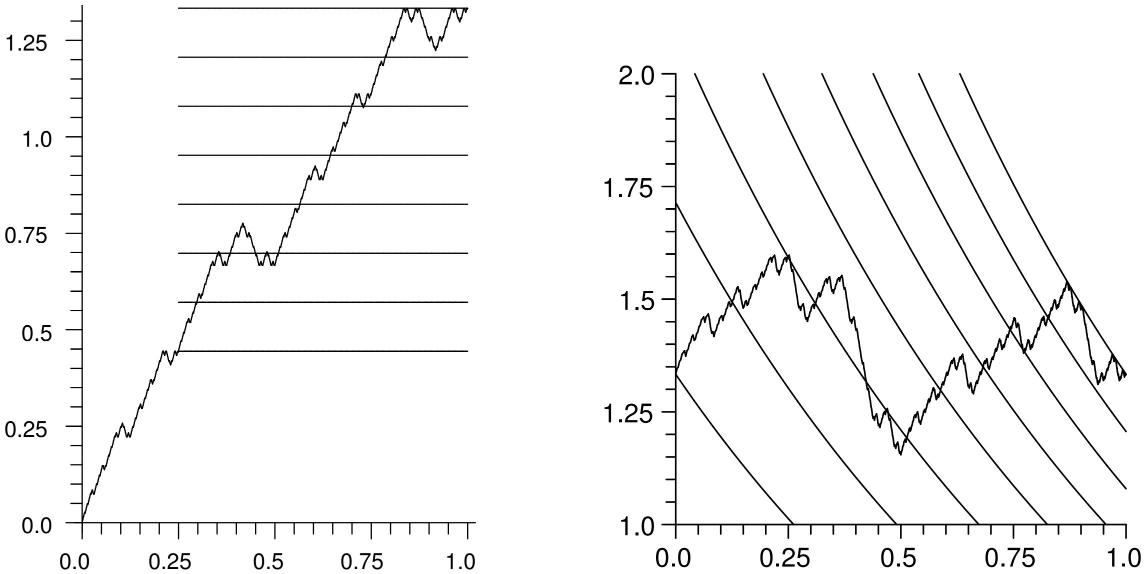

Figure 1: The change from the function $F(x)$ (left) to the periodic function $\Phi(t)$ (right) for the Coquet sequence. In processing, the first quarter of the graph and thus its self-similarity are lost.

and V_ρ is an eigenvector for $\rho = 3,\, 0$. Our method applies and we obtain [9, 20]

$$\sum_{n \le N} (-1)^{s_2(3n)} \underset{N\to+\infty}{=} N^{\log_4 3}\, 3^{1-\{t\}} F(4^{\{t\}-1}) + O(1),$$

with $t = \log_4 N$. The function F is the sum $F = F_1 + F_2 + F_3$, where $\mathbf{F} = (F_1, F_2, F_3)$ is the unique solution of the dilation equation

$$\begin{cases} F_1(x) = \dfrac{1}{3}F_1(4x) + \dfrac{1}{3}F_2(4x) + \dfrac{1}{3}F_3(4x) + \dfrac{1}{3}F_1(4x-1), \\ F_2(x) = \dfrac{1}{3}F_2(4x-1) - \dfrac{1}{3}F_3(4x-1) + \dfrac{1}{3}F_1(4x-2) + \dfrac{1}{3}F_2(4x-2), \\ F_3(x) = \dfrac{1}{3}F_3(4x-2) + \dfrac{1}{3}F_1(4x-3) - \dfrac{1}{3}F_2(4x-3) + \dfrac{1}{3}F_3(4x-3), \end{cases}$$

with the conditions $F_1(x) = F_2(x) = F_3(x) = 0$ for $x \le 0$, and $F_1(x) = 2/3$, $F_2(x) = F_3(x) = 1/3$ for $x \ge 1$. The function $F(x)$ and the periodic function $\Phi(t) = 3^{1-\{t\}} F(4^{\{t\}-1})$ are illustrated in Figure 1 respectively on the left and on the right (the latter appears for example in [1, p. 99]). Both functions are Hölder with exponent $\log_4 3 \simeq 0.795$. In the translation from F to Φ, the first quarter of F is lost and Φ has not the auto-similar character of F. This explains why the self-similar nature of such functions has not been seen before. Yet this is what justifies the appellation fractal, often used in the treatment of such examples. △

The next example is a little more intricate, because it uses a Jordan cell of size 2.

Example 4 (Discrepancy of the van der Corput sequence). The (binary) van der Corput sequence is defined as follows: for an integer n, we write its binary expansion $(n_{K-1} \dots n_1 n_0)_2$; we reverse it and we place it after the binary dot. The real number $(0.n_0 n_1 \dots n_{K-1})_2$ is the value x_n of the van der Corput sequence for the integer n. The extreme discrepancy of the sequence is

$$D(n) = \sup_{0 \le \alpha < \beta \le 1} \left| \frac{\nu(n, \alpha, \beta)}{n} - (\beta - \alpha) \right|,$$

where $\nu(n, \alpha, \beta)$ is the number of terms x_k, $1 \le k \le n$, which fall in the interval $[\alpha, \beta)$. It measures the deviation from the uniform distribution for the sequence x_n [25]. Béjian and Faure [3, 4] showed that the sequence $\Delta(n) = nD(n)$

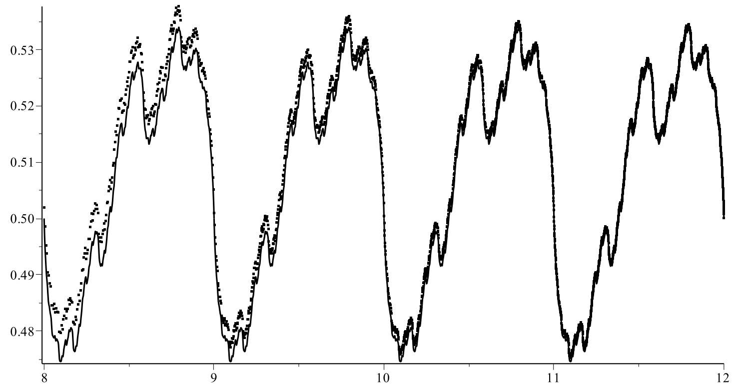

Figure 2: The comparison between the empirical periodic function (dotted line) of Ex. 4 for the mean of the sequence $\Delta(n) = nD(n)$, where $D(n)$ is the extreme discrepancy of the van der Corput sequence, and the theoretical periodic function Φ (solid line) computed by the cascade algorithm.

satisfies

$$\Delta(1) = 1, \qquad \Delta(2n) = \Delta(n), \qquad \Delta(2n+1) = \frac{1}{2}(\Delta(n) + \Delta(n+1) + 1)$$

and we add $\Delta(0) = 0$.

We want to evaluate more precisely the mean value of the sequence $\Delta(n)$ [4, Th. 3]

$$\frac{1}{N}\sum_{n=1}^{N} \Delta(n) \underset{N\to+\infty}{=} \frac{1}{4}\log_2 N + O(1).$$

The sequence $\Delta(n)$ is 2-rational and admits the following linear representation

$$L = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

with respect to the generating family $(\Delta(n), \Delta(n+1), 1)$. The joint spectral radius is $\rho_* = 1$ and it it has the finiteness property, as may be seen by using the change of coordinate matrix

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which renders both matrices A_0 and A_1 upper triangular. Because 1 is a simple eigenvalue of $Q = A_0 + A_1$, the error term of the asymptotic expansion for the partial sum is $O(\log N)$.

Using the basis (V_1, V_2^0, V_2^1) with

$$V_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}, \qquad V_2^0 = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}, \qquad V_2^1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix},$$

the matrix Q takes the Jordan form

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The vector C expands as $C = V_1 + V_2^1$, and because V_1 is related to the eigenvalue 1 it may be neglected. We find

$$\mathbf{S}_K(x) \underset{K\to+\infty}{=} \frac{1}{2}2^K\, K\,\mathbf{F}^0(x) + 2^K\,\mathbf{F}^1(x) + O(K)$$

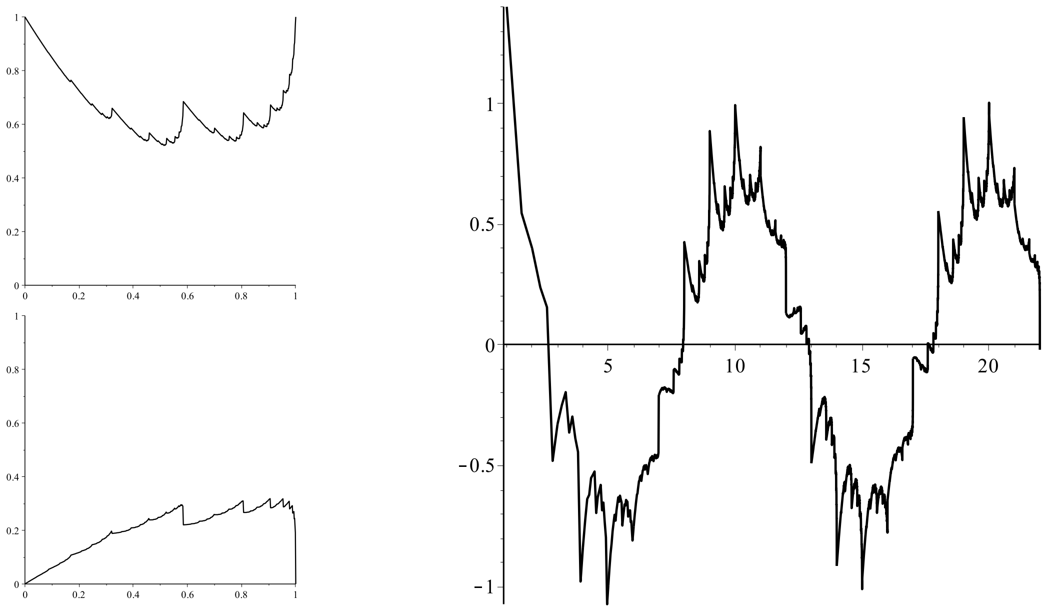

Figure 3: The occurrence of an eigenvalue whose argument is incommensurable with π reveals a pseudo-periodic function. Here are, on the right, the empirical pseudo-periodic function of Ex. 5, and on the left the patterns of 1-periodic functions A and B such that the theoretical pseudo-periodic function writes $\cos(\vartheta\log_2 N)A(\log_2 N)+\sin(\vartheta\log_2 N)B(\log_2 N)$ with $\cos\vartheta = 4/5$, $\sin\vartheta = 3/5$.

where $\mathbf{F}^0$ and $\mathbf{F}^1$ are solutions of the dilation equations

$$\mathbf{F}^0(x) = \frac{1}{2}A_0\mathbf{F}^0(2x) + \frac{1}{2}A_1\mathbf{F}^0(2x-1),$$

$$\mathbf{F}^1(x) = -\frac{1}{2}\mathbf{F}^0(x) + \frac{1}{2}A_0\mathbf{F}^1(2x) + \frac{1}{2}A_1\mathbf{F}^1(2x-1),$$

with the boundary conditions $\mathbf{F}^0(x)=0$, $\mathbf{F}^1(x)=0$ for $x\le 0$, $\mathbf{F}^0(x)=V_2^0$, $\mathbf{F}^1(x)=V_2^1$ for $x\ge 1$. It is readily seen that $\mathbf{F}^0(x)=xV_2^0$ for $0\le x\le 1$, but $\mathbf{F}^1(x)$ is not explicit. Eventually we arrive at the formula

$$\frac{1}{N}\sum_{n=1}^{N}\Delta(n) \underset{N\to+\infty}{=} \frac{1}{4}\log_2 N + \Phi(t) + O\left(\frac{\log N}{N}\right),$$

with

$$\Phi(t) = \frac{1}{4}\left(1-\{t\}+2^{3-\{t\}}\left(F_2^1(2^{\{t\}-1})+F_3^1(2^{\{t\}-1})\right)\right).$$

Figure 2 shows the empirical periodic function and the comparison between the empirical function and the theoretical $\Phi(t)$ periodic function (with a logarithmic scale for the abscissæ). △

Often the authors who study the asymptotic behavior of radix-rational sequences emphasize the occurrence of 1-periodic functions. If this is true of the usual examples, yet it is not the general phenomenon. It depends on the arguments of the eigenvalues of Q. Usually they are commensurable with π and can be easily hidden by changing the radix. For example, the Coquet sequence (Ex. 3) is 2-rational and the eigenvalues are 0 and $\pm\sqrt{3}$, but almost all people who study this sequence [2, 9, 18] view it as a 4-rational sequence and use the squares of the previous eigenvalues, namely 0 and 3. More broadly if all the eigenvalues larger than the joint spectral radius have an argument commensurable with π, it is possible to change the radix into a power of it and to use only positive eigenvalues. Nevertheless, the following example shows that the use of modulus 1 complex numbers is unavoidable in full generality and can lead to pseudo-periodic functions.

Example 5 (Pseudo-periodicity). The linear representation

$$L = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{46}$$

provides a mere example where an angle incommensurable with π occurs. The matrix

$$Q = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} \tag{47}$$

has eigenvalues $\lambda_\pm = 4 \pm 3i$, with modulus 5. The joint spectral radius has value $3\sqrt{2} \simeq 4.24$. It is merely the spectral radius of A_1. All computations made, we arrive at the formula

$$\sum_{n=0}^{N} u_n \underset{N\to+\infty}{=} N^{\log_2 5} \times (\cos(\vartheta \log_2 N) A(\log_2 N) + \sin(\vartheta \log_2 N) B(\log_2 N)) + O(N^{\log_2 3\sqrt{2}}), \tag{48}$$

where $A(t)$, $B(t)$ are 1-periodic functions defined by

$$A(t) = 5^{1-\{t\}}(\cos((1-\{t\})\vartheta) U^1(2^{\{t\}-1}) + \sin((1-\{t\})\vartheta) V^1(2^{\{t\}-1})) \tag{49}$$

$$B(t) = 5^{1-\{t\}}(\cos((1-\{t\})\vartheta) V^1(2^{\{t\}-1}) - \sin((1-\{t\})\vartheta) U^1(2^{\{t\}-1})), \tag{50}$$

with $\vartheta = \arccos(4/5)$, which is the argument of $4+3i$. Moreover the solution $F_+(x)$ of the dilation equation associated to λ_+ and the eigenvector $(1, -i)^{\mathrm{tr}}$ writes $(U^1(x) + iV^1(x), -V^1(x) + iU^1(x))$ with real valued functions $U^1(x)$ and $V^1(x)$. Figure 3 shows the empirical quotient of the partial sums by $N^{\log_2 5}$ (with logarithmic scale for the abscissæ). The reader can think that he sees a 10-periodic function, but this is only an artifact of which the cause is the good approximation of $\vartheta/(2\pi) \simeq 0.1024$ by $1/10$. All functions involved are Hölder with exponent $\log_2(5/3\sqrt{2}) \simeq 0.24$. △

The last example asks us about the limit of the method.

Example 6 (Discrepancy, continued). Of course it is tempting to apply our method not to $\Delta(n)$, as in Example 4, but to the sequence of backward differences $\delta(n) = \Delta(n) - \Delta(n-1)$, to obtain information on the discrepancy $D(n) = \Delta(n)/n$ itself. The sequence $\delta(n)$ admits the 4-dimensional linear representation, associated to the basis $(\delta(n), \varepsilon(n), \delta(n+1), 1)$ with $\varepsilon(n) = 0$ if $n = 0$ and $\varepsilon(n) = 1$ if $n > 0$,

$$L = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 \\ B_0 & B_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with

$$B_0 = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}.$$

As for both matrices A_0 and A_1 the maximum absolute column sum is 1 and 1 is an eigenvalue, the joint spectral radius of the representation is $\rho_* = 1$. The matrix $Q = A_0 + A_1$ is lower triangular and diagonalizable with eigenvalues 2, 1 (double), and 1/2. The vector C has no component on the eigenspace relative to 2, so that the largest eigenvalue involved is 1, which is exactly the joint spectral radius. It turns out that our theorem does not provide an asymptotic equivalent. More precisely, as the linear representation has the finiteness property, we know (Lemma 5) that $\Delta(n) = O(\log n)$, but no more.

Nevertheless, we may consider the dilation system associated to the eigenvalue 1 and the component V_1 of C over the eigenspace relative to 1. (We have $C = V_1 + V_{1/2}$ with natural notations.) We readily find

$$F_1(x) = 0, \qquad F_2(x) = \begin{cases} 0 & \text{if } x \le 0, \\ -1 & \text{if } x > 0, \end{cases} \qquad F_3(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 < x \le 1, \\ 1 & \text{if } 1 < x. \end{cases}$$

But the function $F_4(x)$ remains unknown. The graph of $LF(x) = F_3(x) + F_4(x)$, obtained by the cascade algorithm, and the graph of the partial sum $L\mathbf{S}_K(x)$, below, show a striking similarity (Fig. 4). It is natural to ask for a formula which relates both functions.

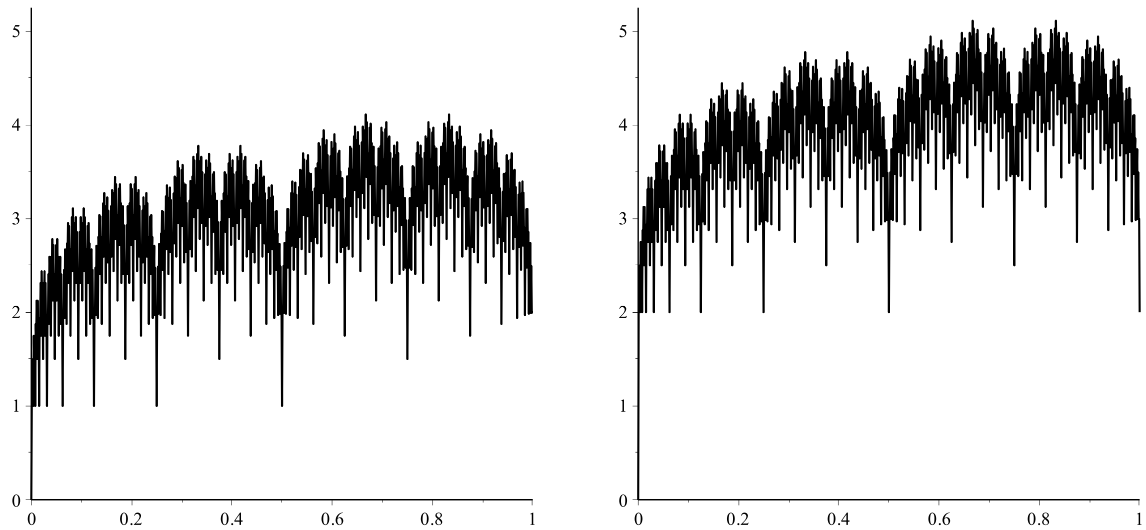

Figure 4: On the left the empirical partial sum $LS_{10}(x)$ associated to the discrepancy function $\Delta(n)$ of the van der Corput sequence, and on the right the function $LF(x)$, with $F(x)$ the solution of the dilation equation, computed by the cascade algorithm for dyadic numbers whose binary mantissa is not longer than 10. The similarity of both drawings raises a question.

The only stable point of reference we have is the recursion formula (13) of Lemma 2 for the sums $\mathbf{S}_K(x)$. An immediate consequence is $\|\mathbf{S}_{K+1}(x)\|_1 \le \|A_0\|_1 \left\|Q^K C\right\|_1 + \|A_1\| \, \|\mathbf{S}_K(2x - x_1)\|_1$. But because in this example we have $\|A_0\|_1 = \|A_1\|_1 = 1$ and $\left\|Q^K C\right\|_1 \le \|C\|_1$, we obtain right away $\|\mathbf{S}_K(x)\|_1 \le (\ell + 1)\,\|C\|_1$, if ℓ is the mantissa length of the dyadic number $x = j/2^K$ with $0 \le j < 2^K$. The integer ℓ lies between 0 and K and we conclude

$$\|\mathbf{S}_K(x)\| \underset{K \to +\infty}{=} O(K). \tag{51}$$

This bound is optimal for it is reached with $x_k = (0.(10)^{k-1}1^2)_2$ for $K = 2k$. Anew this is proved with the recursion formula (13) of Lemma 2, which provides $L\mathbf{S}_K(x_k) \sim K/3$. This formula is consistent with [4, Th. 3].

With regard to the function $F_4(x)$, its drawing is an illusion, because the dilation equation has no solution. This is readily seen for instance with $x = 1/3 = (0.111\ldots)_2$ which provides us with the absurd formula

$$F_4\left(\frac{1}{3}\right) = \frac{1}{3} + F_4\left(\frac{2}{3}\right) = \frac{2}{3} + F_4\left(\frac{1}{3}\right).$$

This is not really an obstacle, because when we substitute $2^{\{t\}-1}$ for x in $F_4(x)$ with $t = \log_2 N$ we obtain $F_4(N/2^{K+1})$ with K the integer part of $\log_2 N$. Hence only the values of $F_4(x)$ for x a dyadic number are useful. So we consider that $F_4(x)$ is defined on the dyadic numbers, thanks to the dilation equation and the cascade algorithm.

The function $F_4(x)$ is not so unknown that it can seem. Let us introduce the function $H(x)$ defined by

$$H(x) = \sum_{k=0}^{+\infty} h(2^k x),$$

where $h(x)$ is the distance from x to the set of integers, also denoted by $\|x\|$ in number theory. This function is not defined for all real numbers but only for dyadic numbers. It is not difficult to realize that it is an even 1-periodic function which satisfies $H(x) = h(x) + H(2x)$ for $0 \le x \le \frac{1}{2}$. Contrary to the others functions for which we have considered dilation equations, the function $H(x)$ is not constant on the left of 0 and on the right of 1, so we will take care to write the equation by distinguishing the case $0 \le x < 1/2$ and the case $1/2 \le x < 1$. So, we have

$$H(x) = \begin{cases} h(x) + H(2x) & \text{for } 0 \le x < 1/2, \\ h(x) + H(2x-1) & \text{for } 1/2 \le x < 1 \end{cases} \tag{52}$$

with $H(0) = H(1) = 0$. Besides, according to the dilation system for the vector valued function $F(x)$, the function $F_4(x)$ satisfies

$$F_4(x) = \begin{cases} h(x) + F_4(2x) & \text{for } 0 \le x \le 1/2, \\ h(x) + F_4(2x-1) & \text{for } 1/2 < x \le 1, \end{cases} \tag{53}$$

with $F_4(0) = 0$, $F_4(1) = 1$, essentially because $F_3(2x) - F_3(2x-1) = 2h(x)$ for $0 \le x \le 1$. As a consequence we find

$$F_4(x) = H(x) + 1 \qquad \text{for } 0 < x \le 1. \tag{54}$$

Let us compare the functional equations satisfied on one side by $F(x)$,

$$F(x) = \begin{cases} A_0 F(2x) & \text{for } 0 \le x < 1/2, \\ A_0 V_1 + A_1 F(2x-1) & \text{for } 1/2 \le x < 1 \end{cases}$$

with the boundary conditions $F(x) = 0$ for $x \le 0$ and $F(x) = V_1$ for $x \ge 1$, and on other side by $\mathbf{S}_K(x)$

$$\mathbf{S}_{K+1}(x) = \begin{cases} A_0 \mathbf{S}_K(2x) & \text{for } 0 \le x < 1/2, \\ A_0 Q^K C + A_1 \mathbf{S}_K(2x-1) & \text{for } 1/2 \le x < 1. \end{cases}$$

with the initial conditions $S_0(x) = C$ for $0 \le x \le 1$. With $C = V_1 + V_{1/2}$ and $Q^K C = V_1 + V_{1/2}/2^K$, we obtain for $E_K(x) = F(x) - \mathbf{S}_K(x)$

$$E_{K+1}(x) = \begin{cases} A_0 E_K(2x) & \text{for } 0 \le x < 1/2, \\ -A_0 V_{1/2}/2^K + A_1 E_K(2x-1) & \text{for } 1/2 \le x < 1, \end{cases}$$

with the initial conditions $E_0(0) = -C = -V_1 - V_{1/2}$ and $E_0(1) = -V_{1/2}$. With some elementary linear algebra we arrive at

$$L\mathbf{S}_K(x) = LF\left(\frac{j}{2^K}\right) - 1 \qquad \text{for } \frac{j}{2^K} \le x < \frac{j+1}{2^K} \text{ with } 0 < j < 2^K.$$

It is worthwhile to note that $L\mathbf{S}_K(x)$ and $LF(x) = F_3(x) + F_4(x)$ are not of the same nature. The function $L\mathbf{S}_K(x)$ is a step function of one real variable, while $LF(x)$ is defined only on dyadic numbers.

Merging the previous formulæ, we obtain for $2^K \le N < 2^{K+1}$

$$\Delta(N) = L\mathbf{S}_{K+1}(2^{\{\log_2 N\}-1}) = F_3\left(\frac{N}{2^{K+1}}\right) + F_4\left(\frac{N}{2^{K+1}}\right) - 1 = \frac{N}{2^{K+1}} + H\left(\frac{N}{2^{K+1}}\right) \tag{55}$$

or equivalently

$$D(N) = \frac{1}{N} H\left(2^{\{\log_2 N\}-1}\right) + \frac{1}{2^{\lfloor \log_2 N \rfloor + 1}}. \tag{56}$$

The last term is $O(1/N)$, while we know that the first has size $\log(N)/N$, because the order of growth of $\mathbf{S}_K(x)$ is $O(K)$. Formula (55) is nothing but Theorem 1 in [4] or Proposition 2 in [15]

For what we are concerned, we see that we do not obtain a formula $D(N) \sim \log N/N \times \Psi(\log_2 N)$ with $\Psi(t)$ a 1-periodic function, as Figure 4 could let us hope, and such a formula cannot exist. The situation is not akin the frame of Theorems 2 and 3. Nevertheless the computation has emphasized something similar with the formula $D(N) \sim H(2^{\{\log_2 N\}-1})/N$, which catches a little the behavior of the sequence, but the information is hidden in function $H(x)$. Moreover the recursion formula for the partial sums $\mathbf{S}_K(x)$ is valid in every case and permits us to estimate the order of growth of these sums, and of the sequence $s_N = \Delta(N)$. △

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