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Abstract: We study the links between reflected backward stochastic differential equations (reflected BSDEs) with jumps and partial integro-differential variational inequalities (PIDVIs). In a Markovian framework, we show that the solution of the reflected BSDE corresponds to the unique viscosity solution of the PIDVI. We apply these results to an optimal stopping problem for dynamic risk measures induced by BSDEs with jumps.

Key-words: Reflected backward stochastic differential equations with jumps,viscosity solution, partial integro-differential variational inequality, optimal stopping, dynamic risk-measures

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Equations différentielles stochastiques rétrogrades réfléchies avec sauts et inequations variationnelles integro-différentielles

Résumé : On étudie le lien entre les équations différentielles stochastiques rétrogrades réfléchies avec sauts (EDSR réfléchies) et les inequations variationnelles integro-différentielles (IVID). Dans un cadre markovien, on montre que la solution de l'EDSR réfléchie avec sauts correspond à l'unique solution de viscosité de l'IVID. On applique ces résultats à l'étude d'un problème d'arrêt optimal pour les mesures de risque dynamiques induites par des EDSR avec sauts.

Mots-clés : Equations différentielles stochastiques rétrogrades réfléchies avec sauts, solution de viscosité, inegalités variationnelles integro-différentielles, arrêt optimal, mesures de risque dynamiques

1 Introduction

Backward stochastic differential equations were introduced, in the case of a Brownian filtration, by Bismut (1976), then generalized by Pardoux and Peng ([13]). They represent a useful tool in mathematical finance and stochastic control. Some studies have been done in the case of jumps (see among others [2], [3], [17], [18]).

Reflected BSDEs (RBSDEs) have been introduced by N. El Karoui et al. (1997) (see [10]) in the case of a Brownian filtration. The solutions of such equations are constrained to be greater then given processes called obstacles. In [7, 8, 9], Essaky, Hamadène, Ouknine have studied RBSDEs with jumps. In a recent paper, Quenez and Sulem ([15]) have provided existence and uniqueness results for RBSDEs with jumps, as well as comparison theorems and additional properties, when the obstacle is RCLL, which completes the previous works.

In this paper, we focus on RBSDEs with jumps in the Markovian case and we extend the results obtained in [1]. Our main contribution consists in establishing the link between RBSDEs with jumps, and parabolic partial integro-differential variational inequalities (PIDVI). We show that the solution of a given RBSDE corresponds to a viscosity solution of a PIDVI, which provides an existence result for this obstacle problem. Under additional assumptions, we prove a comparison theorem, which allows us to obtain the uniqueness of the viscosity solution in a certain class of functions. In order to prove these results, we establish new a priori estimates for RBSDEs with jumps.

We illustrate these results on an optimal stopping problem for dynamic risk measures induced by BSDEs with jumps. By [15], the value function, which represents the minimal risk measure, coincides with the solution of an RBSDE with jumps. In the Markovian case, it thus corresponds to the unique viscosity solution of the associated PIDVI.

The paper is organized as follows: In section 2 we introduce the notation and the definitions. In Section 3, we study the relation between an RBSDE with jumps and a PIDVI. We start by proving the continuity and polynomial growth of the function $u(t,x)$ defined via the solution $Y_t^{t,x}$ of the reflected BSDE, using the a priori estimates proved in appendix. We show that the solution of an RBSDE corresponds to a solution of the PIDVI in the viscosity sense. Under additional assumptions, we establish an uniqueness result in the class of continuous functions, with polynomial growth. In Section 4, we apply our results to the optimal stopping problem for dynamic risk measures induced by BSDEs with jumps.

2 Notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt, du)$ be a Poisson random measure with intensity $\nu(du)dt$ such that ν is a σ finite measure on $\mathbb{R}^*$ with $\int_{\mathbb{R}^*}(1 \wedge u^2)\nu(du) < \infty$. Let $\widetilde{N}(dt, du)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N. Fix $T > 0$. Here $\mathcal{P}$ denotes the so-called progressive σ-algebra on $\Omega \times [0,T]$ and $\mathcal{B}(\mathbb{R}^2)$ (resp $\mathcal{B}(\mathcal{L}^2(\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*), \nu; \mathbb{R})))$ is the Borelian σ-algebra on $\mathbb{R}^2$ (resp. on $\mathcal{L}^2(\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*), \nu; \mathbb{R})$).

We adopt the following notation:

- $\mathcal{L}^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and *square*-integrable.

- $\mathbb{H}^2$ is the set of real-values predictable processes ϕ such that

$$\|\phi\|^2_{\mathbb{H}^2} := E[(\int_0^T \phi_t^2 dt)] < \infty. \tag{1}$$

- $\mathcal{S}^2$ is the set of real-values RCLL adapted processes ϕ such that:

$$\|\phi\|^2_{\mathcal{S}^2} := E[\sup_{0\le t\le T} |\phi_t|^2] < \infty. \tag{2}$$

- $\mathcal{L}^2_\nu$ is the set of Borelian functions $l : \mathbb{R}^* \to \mathbb{R}$ such that

$$||l||^2_\nu := \int_{\mathbb{R}^*} |l(u)|^2 \nu(du) < \infty \tag{3}$$

$\mathcal{L}^2_\nu$ is a Hilbert space equipped with the scalar product:

$$< \delta, l >_\nu := \int_{\mathbb{R}^*} \delta(u) l(u) \nu(du), \text{ for all } \delta, l \in \mathcal{L}^2_\nu \times \mathcal{L}^2_\nu.$$

- $\mathcal{H}^2_\nu$ is the set of *predictable* (i.e. measurable) processes

$$l : ([0,T] \times \Omega \times \mathbb{R}^*, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})); (\omega, t, u) \to l_t(\omega, u)$$

such that

$$\|l\|^2_{\mathbb{H}^2_\nu} := E[(\int_0^T ||l_t||^2_\nu dt)] < \infty. \tag{4}$$

- $\mathcal{T}_0$ is the set of stopping times θ such that $\theta \in [0,T]$ a.s.
- For S in $\mathcal{T}_0, \mathcal{T}_S$ is the set of stopping times θ such that $S \le \theta \le T$ a.s.

Definition 2.1 (Driver, Lipschitz driver)**.** *A function f is said to be a **driver** if*

- $f : [0,T] \times \Omega \times \mathbb{R}^2 \times \mathcal{L}^2_\nu \to \mathbb{R}$
  $(\omega, t, x, \pi, l(\cdot)) \to f(\omega, t, x, \pi, l(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathcal{L}^2_\nu)$ *- measurable, and*
- $f(\cdot, 0, 0, 0) \in \mathbb{H}^2$.

*A driver f is called a **Lipschitz driver** if moreover there exists a constant $C \ge 0$ such that $dP \otimes dt$ - a.s., for each (x_1, π_1, l_1), (x_2, π_2, l_2),*

$$|f(\omega, t, x_1, \pi_1, l_1) - f(\omega, t, x_2, \pi_2, l_2)| \le C(|x_1 - x_2| + |\pi_1 - \pi_2| + ||l_1 - l_2||_\nu).$$

3 Relation between a RBSDE with jumps and RCLL obstacle and a partial integro-differential variational inequality (PIDVI)

Let $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ be continuous mappings, globally Lipschitz and $\beta : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ a measurable function and such that for some real K, and for all $e \in \mathbb{R}$

$$|\beta(x,e)| \le K(1 \wedge |e|), x \in \mathbb{R}$$
$$|\beta(x,e) - \beta(x',e)| \le K|x - x'|(1 \wedge |e|), x, x' \in \mathbb{R}.$$

For each $(t,x) \in [0,T] \times \mathbb{R}$, let $\{X_s^{t,x}, t \leq s \leq T\}$ be the unique $\mathbb{R}$-valued solution of the SDE with jumps:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \int_{\mathbb{R}^*} \beta(X_{r^-}, e)\tilde{N}(dr, de)$$

We consider a RBSDE with obstacle ξ and driver f of the following form:

$$\begin{cases} \xi_s = h(s, X_s^{t,x}), t \leq s < T \\ \xi_T = g(X_T) \\ f(s,\omega,y,z,k) = \psi(s, X_s^{t,x}(\omega), y, z, \int_{\mathbb{R}^*} k(e)\gamma(x,e)\nu(de))\mathbf{1}_{s \geq t} \end{cases}$$

where g, ψ, h and γ are as follows.

- $g \in \mathcal{C}(\mathbb{R})$ and has at most polynomial growth at infinity.
- $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous in t and x and satisfies

$$|h(t,x)| \leq K(1 + |x|^p), \forall t \in [0,T], x \in \mathbb{R} \tag{5}$$

- $h(T,x) \leq g(x), \forall x \in \mathbb{R}$.
- $\gamma : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^*)$-measurable and $\exists$ C such that:
  $|\gamma(x,e) - \gamma(x',e)| < C|x - x'|(1 \wedge |e|), x, x' \in \mathbb{R}, e \in \mathbb{R}^*$
  $0 \leq \gamma(x,e) \leq C(1 \wedge |e|)$, $e \in \mathbb{R}^*$
- $\psi : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous in t , uniformly continuous in x with respect to y, z, k.
  (i) $|\psi(t,x,0,0,0)| \leq K(1 + |x|^p), x \in \mathbb{R}$
  (ii) $\exists C \geq 0$ such that $|\psi(t,x,y,z,q) - \psi(t,x',y',z',q')| \leq C(|y - y'| + |z - z'| + |q - q'|)$, $\forall 0 \leq t \leq T$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}$, $q, q' \in \mathbb{R}$
  (iii) $q \to \psi(t,x,y,z,q)$ is non-decreasing, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^3$

The assumptions made on h imply that the obstacle $\xi_s = h(s, X_s^{t,x})$ is left-upper semicontinuous along stopping times that is, for all $\tau \in \tau_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s. , $\xi_\tau \geq \limsup_{n\to\infty} \xi_{\tau_n}$ a.s.

By Theorem 3.3 in [15], there exists a unique quadruple $(Y^{t,x}, Z^{t,x}, K^{t,x}, A^{t,x}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$ of progressively measurable processes, which solves the following RBSDE:

$$\begin{cases} Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r, K_r(\cdot)) + A_T - A_s \\ \qquad - \int_s^T Z_r dW_r - \int_s^T \int_{\mathbb{R}^*} K(r,e)\tilde{N}(dr, de) \\ Y_s \geq h(s, X_s), 0 \leq s \leq T \ a.s., \\ A \text{ is a nondecreasing, continuous predictable process with} \\ \qquad A_0 = 0 \text{ and such that} \\ \int_0^T (Y_t - \xi_t) dA_t = 0 \text{ a.s.} \end{cases} \tag{6}$$

The non-decreasing property of ψ and the assumption made on γ ensure that Assumption 47 (see Appendix) holds, and hence the comparison theorem holds for BSDEs and RBSDEs with jumps associated with driver f (see in [15] and [16]).

Finally, we define:

$$u(t,x) = Y_t^{t,x}, \quad t \in [0,T], x \in \mathbb{R}. \tag{7}$$

which is a deterministic quantity.

3.1 Some properties of the solution of the RBSDE

In this section, we prove the continuity and polynomial growth of the function u defined by (7).

Lemma 3.1. *The function u is continuous in (t,x).*

Proof. We define $Y_s^{t,x}$ for all $s \in [0,T]$ by choosing $Y_s^{t,x} = Y_t^{t,x}$ for $0 \leq s \leq t$. It suffices to show that whenever $(t_n, x_n) \to (t,x)$,

$$|Y_0^{t_n,x_n} - Y_0^{t,x}|^2 \to 0.$$

By applying Prop 48 with $X_s^1 = X_s^{t_n,x_n}$, $X_s^2 = X_s^{t,x}$, $f^1(s,\omega,y,z,q) := \mathbf{1}_{[t,T]}(s) f(s, X_s^{t,x}(\omega), y, z, q)$ and $f^2(s,\omega,y,z,q) := \mathbf{1}_{[t_n,T]}(s) f(s, X_s^{t_n,x_n}(\omega), y, z, q)$ we obtain for $t = 0$:

$$\overline{Y}_0^2 \leq K_{C,T} \mathbb{E}[\sup_{t \geq 0} \overline{\xi}_t^2] + \mathbb{E}[\int_0^T (\overline{f}_s^n)^2] \tag{8}$$

where

$$\begin{cases} K_{C,T} := e^{(3C^2+2C)T} \max(1, \dfrac{1}{C^2}) \\ \overline{f}_s^n(\omega) := \sup_{y,z,q} |\mathbf{1}_{[t,T]} f(s, X_s^{t,x}(\omega), y, z, q) - \mathbf{1}_{[t_n,T]} f(s, X_s^{t_n,x_n}(\omega), y, z, q)|. \end{cases}$$

Consequently, we have:

$$\begin{aligned} |Y_t^t(x) - Y_{t_n}^{t_n}(x_n)|^2 &= |Y_0^t(x) - Y_0^{t_n}(x_n)|^2 \\ &\leq K_{C,T} \mathbb{E}[\sup_{0 \leq s \leq T} |h(s, X_s^{t_n,x_n}) - h(s, X_s^{t,x})|^2 + \int_0^T (\overline{f}_s^n)^2]. \end{aligned} \tag{9}$$

The continuity of u is a consequence of the following convergences as $n \to \infty$:

$$\begin{aligned} &\mathbb{E}(\sup_{0 \leq s \leq T} |h(s, X_s^{t,x}) - h(s, X_s^{t_n}(x_n))|^2) \to 0 \\ &\mathbb{E}[\int_0^T (\overline{f}_s^n)^2 ds] \to 0. \end{aligned}$$

which follow from the Lebesgue's theorem, using the continuity assumptions and polynomial growth of f and h .

Lemma 3.2. *The function u has at most polynomial growth at infinity.*

Proof. By applying Prop. $A.4$, we obtain the following relation:

$$|Y_t^{t,x}|^2 = |Y_0^{t,x}|^2 \leq K_{C,T}(\mathbb{E}(\int_0^T f(s, X_s^{t,x}, 0, 0, 0)^2 ds + \sup_{0 \leq s \leq T} h(s, X_s^{t,x})^2). \tag{10}$$

Using now the hypothesis of polynomial growth on f, h and the standard estimate:

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_s^{t,x}|^2] \leq C(1 + x^2),$$

we obtain that there exist $C \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $|u(t,x)| \leq C(1 + |x|^p), \quad \forall t \in [0,T], \forall x \in \mathbb{R}$.

3.2 Existence of a viscosity solution

We now consider the related obstacle problem for a parabolic PIDE. Roughly speaking, a solution of the obstacle problem is a function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ which satisfies:

$$\begin{cases} \min(u(t,x) - h(t,x), \\ \quad -\dfrac{\partial u}{\partial t}(t,x) - Lu(t,x) - f(t,x,u(t,x), (\sigma \dfrac{\partial u}{\partial x})(t,x), Bu(t,x)) = 0, (t,x) \in [0,T) \times \mathbb{R} \\ u(T,x) = g(x), x \in \mathbb{R} \end{cases} \tag{11}$$

where:

- $L = A + K$
- $A\phi(x) = \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 \phi}{\partial x^2}(x) + b(x)\dfrac{\partial \phi}{\partial x}(x)$, $\phi \in C^2(\mathbb{R})$
- $K\phi(x) = \int_{\mathbb{R}^*} \left(\phi(x + \beta(x,e)) - \phi(x) - \dfrac{\partial \phi}{\partial x}(x)\beta(x,e) \right) \nu(de)$, $\phi \in C^2(\mathbb{R})$
- $B\phi(x) = \int_{\mathbb{R}^*} (\phi(x + \beta(x,e)) - \phi(x))\gamma(x,e)\nu(de)$

We prove that the solution of the reflected BSDE is solution of an obstacle problem by using the classical definition of viscosity solutions ([2]).

Definition 3.3. a) A continuous function u is said to be a viscosity subsolution of (11) if $u(T,x) \leq g(x), x \in \mathbb{R}$, and if for any point $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and for any $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its minimum at (t_0, x_0), then

$$\begin{aligned} &\min(u(t_0,x_0) - h(t_0,x_0), \\ &-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0), (\sigma \frac{\partial \phi}{\partial x})(t,x))(t_0,x_0), B\phi(t_0,x_0)) \leq 0. \end{aligned}$$

In other words, if $u(t_0, x_0) > h(t_0, x_0)$,

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0), (\sigma \frac{\partial \phi}{\partial x})(t,x))(t_0,x_0), B\phi(t_0,x_0)) \leq 0$$

b) A continuous function u is said to be a viscosity supersolution of (11) if $u(T,x) \geq g(x), x \in \mathbb{R}$, and if for any point $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and for any $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), then

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\ &-\frac{\partial}{\partial t}\phi(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \geq 0.\end{aligned}$$

In other words, we have both $u(t_0,x_0) \geq h(t_0,x_0)$,

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \geq 0$$

Theorem 3.4. *The function u, defined by* (7) *is a viscosity solution of the obstacle problem* (11).

Proof

I. We prove that u is a subsolution of (11).
Let $(t_0,x_0) \in (0,T)\times\mathbb{R}$ and $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \geq u(t,x)$, $\forall (t,x) \in [0,T]\times\mathbb{R}$
We suppose that $u(t_0,x_0) > h(t_0,x_0)$ and that

$$-\frac{\partial}{\partial t}\phi(t,x) - L\phi(t,x) - f(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x)) > 0$$

By continuity (the continuity of $K\phi$ and $B\phi$ can be shown using Lebesgue's theorem), we can suppose that there exists $\epsilon > 0$ and $\eta_\epsilon > 0$ such that: $\forall (t,x)$ such that $t_0 \leq t \leq t+\eta_\epsilon < T$ and $|x-x_0| \leq \eta_\epsilon$, we have: $u(t,x) \geq h(t,x)+\epsilon$ and

$$-\frac{\partial}{\partial t}\phi(t,x) - L\phi(t,x) - f(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x)) > \epsilon \tag{12}$$

We define the stopping time θ as:

$$\theta := (t_0+\eta_\epsilon)\wedge \inf\{s \geq t_0 / |X_s^{t_0,x_0} - x_0| > \eta_\epsilon\} \tag{13}$$

By definition of the stopping time,

$$u(s,X_s^{t_0,x_0}) \geq h(s,X_s^{t_0,x_0}) + \epsilon > h(s,X_s^{t_0,x_0}), t_0 \leq s < \theta \ a.s.$$

This means that for a.e. ω the processes $(Y_s^{t_0,x_0}, s \in [t_0,\theta(\omega)[)$ rests above the barrier. It follows that for a.e. ω, the function $s \to A_s^c(\omega)$ is constant on $[t,\theta(\omega)]$. In other words, $Y_s^{t_0,x_0} = X_s(Y_\theta,\theta)$, $t_0 \leq s \leq \theta$ $a.s$ which means that $(Y_s^{t_0,x_0}, s\in[t_0,\theta])$ is solution of the classical BSDE associated with the driver f and the terminal value $Y_\theta^{t_0,x_0}$. In order to apply the comparison theorem for classical BSDE with jumps, we apply Itô's lemma to $\phi(t,X_t^{t_0,x_0})$ and we obtain the following equation:

$$\begin{aligned}\phi(t,X_t^{t_0,x_0}) &= \phi(\theta,X_\theta^{t_0,x_0}) \\ &\quad - \int_t^\theta \psi(s,X_s^{t_0,x_0})ds - \int_t^\theta (\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0})dW_s - \\ &\quad - \int_t^\theta\int_{\mathbb{R}^*}\Phi(s,X_{s^-}^{t_0,x_0},e)\tilde{N}(ds,de)\end{aligned} \tag{14}$$

where:

- $\psi(s,y) := \frac{\partial}{\partial s}\phi(s,y) + L\phi(s,y)$
- $\Phi(s,y,e) := \phi(s,y+\beta(y,e)) - \phi(s,y)$

We can remark that $\left(\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),\Phi(s,X_{s^-}^{t_0,x_0},\cdot); s\in[t_0,\theta]\right)$ is solution of the BSDE corresponding to the terminal value $\phi(\theta,X_\theta^{t_0,x_0})$ and driver $-\psi(s,X_s^{t_0,x_0})$.

By assumption (12) and by definition of the stopping time, we have $\forall s \in [t_0,\theta]$:

$$-\frac{\partial\phi}{\partial t}(s,X_s^{t_0,x_0}) - L\phi(s,X_s^{t_0,x_0}) - f\left(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})\right) > \epsilon. \tag{15}$$

Using the definition on the function ψ, we can rewrite the equation (15) as follows:

$$-\psi(s,X_s^{t_0,x_0})-f\left(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})\right)>\epsilon,\quad\forall s\in[t_0,\theta].$$

This equation gives a relation between the drivers of the two BSDEs . Also, $\phi(\theta,X_\theta^{t_0,x_0})\geq u(\theta,X_\theta^{t_0,x_0})=Y_\theta^{t_0,x_0}$. Consequently, the comparison theorem for classical BSDE implies that:

$$\phi(t_0,x_0)>\phi(t_0,X_{t_0}^{t_0,x_0})-\epsilon(\theta-t_0)\geq Y_{t_0}^{t_0,x_0}=u(t_0,x_0),$$

which leads to a contradiction.

II. We prove that u is a viscosity supersolution of (11).

Let $(t_0,x_0)\in(0,T)\times\mathbb{R}$ and $\phi\in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\phi(t_0,x_0)=u(t_0,x_0)$ and $\phi(t,x)\leq u(t,x)$, $\forall(t,x)\in[0,T]\times\mathbb{R}$. Since the solution $(Y_s^{t_0,x_0})$ stays above the obstacle, we have:

$$u(t_0,x_0)\geq h(t_0,x_0)$$

We have to show that:

$$-\frac{\partial}{\partial t}\phi(t_0,x_0)-L\phi(t_0,x_0)-f\left(t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)\right)\geq 0$$

We suppose that:

$$-\frac{\partial}{\partial t}\phi(t_0,x_0)-L\phi(t_0,x_0)-f\left(t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)\right)<0$$

By continuity (the continuity of $K\phi$ and $B\phi$ can be shown using Lebesgue's theorem) , we can suppose that there exists $\epsilon>0$ and $\eta_\epsilon>0$ such that: $\forall(t,x)$ such that $t_0\leq t\leq t+\eta_\epsilon<T$ and $|x-x_0|\leq\eta_\epsilon$, we have:

$$-\frac{\partial}{\partial t}\phi(t,x)-L\phi(t,x)-f\left(t,x,\phi(t,x),(\sigma\frac{\partial\phi}{\partial x})(t,x),B\phi(t,x)\right)\leq-\epsilon\tag{16}$$

We define the stopping time θ such that:

$$\theta:=(t_0+\eta_\epsilon)\wedge\inf\{s\geq t_0/|X_s^{t_0,x_0}-x_0|>\eta_\epsilon\}$$

Our aim now is to use the comparison theorem. We apply as in the case of subsolution Ito's lemma to $\phi(s,X_s^{t_0,x_0})$ and we obtain, as seen above, that $\left(\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),\Phi(s,X_{s^-}^{t_0,x_0},\cdot);s\in[t_0,\theta]\right)$ is solution of the BSDE associated to the terminal value $\phi(\theta,X_\theta^{t_0,x_0})$ and driver $-\psi(s,X_s^{t_0,x_0})$.

The process $(Y^{t_0,x_0},s\in[t_0,\theta])$ is solution of the classical BSDE associated with terminal condition $Y_\theta^{t_0,x_0}=u(\theta,X_\theta^{t_0,x_0})$ and to generalized driver

$$f(s,X_s^{t_0,x_0},y,z,q)ds+dA_s^{t_0,x_0}$$

By assumption (16) and the definition of the stopping time, we have :

$$(-\frac{\partial}{\partial t}\phi(s,X_s^{t_0,x_0})-L\phi(s,X_s^{t_0,x_0})-f(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),$$
$$(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})))ds-dA_s^{t_0,x_0}\leq-\epsilon\ ds,\quad\forall s\in[t_0,\theta]$$

and, equivalently,

$$-\psi(s,X_s^{t_0,x_0})ds\leq(f(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),$$
$$(\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})))ds+dA_s^{t_0,x_0}-\epsilon\ ds,\quad\forall s\in[t_0,\theta]$$

The above equation gives a relation between the drivers of the two BSDEs. Also, $\phi(\theta,X_\theta^{t_0,x_0})\leq u(\theta,X_\theta^{t_0,x_0})=Y_\theta^{t_0,x_0}$. Consequently, the comparison theorem for classical BSDEs implies that:

$$\phi(t_0,x_0)<\phi(t_0,X_{t_0}^{t_0,x_0})+\epsilon(\theta-t_0)\leq Y_{t_0}^{t_0,x_0}=u(t_0,x_0),$$

which leads to a contradiction.

3.3 Uniqueness of the viscosity solution

Now we give a uniqueness result for (11). This result is obtained under more restrictive assumptions that the existence one. Namely we need the following additional assumptions:

Assumption 3.5. *1. For each $R>0$, there exists a continuous function $m_R:\mathbb{R}_+\to\mathbb{R}_+$ such that $m_R(0)=0$ and*

$$|f(t,x,r,p,q)-f(t,y,r,p,q)|\leq m_R(|x-y|(1+|p|)),\forall t\in[0,T],|x|,|y|\leq R,|r|\leq R,p\in\mathbb{R},q\in\mathbb{R}.$$

2. $|\gamma(x,e)-\gamma(y,e)|\leq C|x-y|(1\wedge|e|^2),x,y\in\mathbb{R},e\in\mathbb{R}^*$.

3. There exists $\gamma>0$ such that for any $x\in\mathbb{R}$, $t\in[0,T]$, $u,v\in\mathbb{R}$, $p\in\mathbb{R}$, $l\in\mathbb{R}$:

$$f(t,x,v,p,l)-f(t,x,u,p,l)\geq\gamma(u-v)\ \text{when}\ u\geq v.$$

Theorem 3.6 (Comparison principle)**.** *Under the above hypothesis, if U is a viscosity subsolution and V is a viscosity supersolution of the obstacle problem (11), then $U(t,x)\leq V(t,x),\forall(t,x)\in[0,T]\times\mathbb{R}$.*

Corollary 3.7 (Uniqueness)**.** *Under the above hypothesis, there exists a unique solution of the obstacle problem (11) in the class of continuous functions with polynomial growth.*

Proof of Theorem 3.6 The proof consists in showing that, for a fixed $K>0$, $M_K=\sup_{x\in[-K,K],t\in[0,T]}(U-V)$, is negative. Let $K>0$. To simplify notation, M_K is denoted by M.

We approximate M by dedoubling the variables. We consider the following function:

$$\psi^{\epsilon,\eta}(t,s,x,y):=U(t,x)-V(s,y)-\frac{|x-y|^2}{\epsilon^2}-\frac{|t-s|^2}{\epsilon^2}-\eta^2(|x|^2+|y|^2).$$

where ϵ,η are small parameters devoted to tend to 0, for x,y in $[-K,K]$. Let $M^{\epsilon,\eta}$ be a maximum of $\psi^{\epsilon,\eta}(t,s,x,y)$. This maximum is reached at some point $(t^{\epsilon,\eta},s^{\epsilon,\eta},x^{\epsilon,\eta},y^{\epsilon,\eta})$ in the compact set

$([0,T]^2 \times \overline{B_{R_\eta}}^2)$, where B_{R_η} is a ball with a large radius R_η.
For ϵ, η small enough, we have:

$$0 < \frac{M}{2} \leq M^{\epsilon,\eta} \leq U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}). \tag{17}$$

We define:

$$\Psi_1(t,x) := V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) + \frac{|x - y^{\epsilon,\eta}|^2}{\epsilon^2} + \frac{|t - s^{\epsilon,\eta}|^2}{\epsilon^2} + \eta^2(|x|^2 + |y^{\epsilon,\eta}|^2);$$

$$\Psi_2(s,y) := U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - \frac{|x^{\epsilon,\eta} - y|^2}{\epsilon^2} - \frac{|t^{\epsilon,\eta} - s|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y|^2).$$

As $(t,x) \to (U - \Psi_1)(t,x)$ reaches its maximum at $(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and U is a subsolution we have the two following cases:

- $t^{\epsilon,\eta} = T$ and then $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq g(x^{\epsilon,\eta})$,
- $t^{\epsilon,\eta} \neq T$ and then

$$\begin{aligned} &\min(U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h(t^{\epsilon,\eta}, x^{\epsilon,\eta}), \\ &\frac{\partial \Psi_1}{\partial t}(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - L\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - f(t^{\epsilon,\eta}, x^{\epsilon,\eta}, U(t^{\epsilon,\eta}, x^{\epsilon,\eta}), \\ &\quad (\sigma \frac{\partial \Psi_1}{\partial x})(t^{\epsilon,\eta}, x^{\epsilon,\eta}))), B\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})) \leq 0. \end{aligned} \tag{18}$$

As $(s,y) \to (\Psi_2 - V)(s,y)$ reaches its maximum at $(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and V is a super-solution we have the two following cases:

- $s^{\epsilon,\eta} = T$ and then $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq g(y^{\epsilon,\eta})$,
- $t^{\epsilon,\eta} \neq T$ and then

$$\begin{aligned} &\min(V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - h(s^{\epsilon,\eta}, y^{\epsilon,\eta}), \\ &\frac{\partial \Psi_2}{\partial t}(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - L\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - f(s^{\epsilon,\eta}, y^{\epsilon,\eta}, V(s^{\epsilon,\eta}, y^{\epsilon,\eta}), \\ &\quad (\sigma \frac{\partial \Psi_2}{\partial x})(s^{\epsilon,\eta}, y^{\epsilon,\eta}))), B\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})) \geq 0. \end{aligned} \tag{19}$$

We have:

$$\lim_{\eta \to 0} \lim_{\epsilon \to 0} M^{\epsilon,\eta} = M. \tag{20}$$

and the properties:

$$\frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} \xrightarrow{\epsilon \to 0} 0 \tag{21}$$

$$\frac{|t^{\epsilon,\eta} - s^{\epsilon,\eta}|^2}{\epsilon^2} \xrightarrow{\epsilon \to 0} 0 \tag{22}$$

Using the fact that $\psi^{\epsilon,\eta}(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq \psi^{\epsilon,\eta}(0,0,0,0)$, we obtain:

$$\begin{aligned} U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - \frac{|t^{\epsilon,\eta} - s^{\epsilon,\eta}|^2}{\epsilon^2} - \frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y^{\epsilon,\eta}|^2) \\ \geq U(0,0) - V(0,0). \end{aligned} \tag{23}$$

and, equivalently,

$$\frac{|t^{\epsilon,\eta}-s^{\epsilon,\eta}|^2}{\epsilon^2}+\frac{|x^{\epsilon,\eta}-y^{\epsilon,\eta}|^2}{\epsilon^2}+\eta^2(|x^{\epsilon,\eta}|^2+|y^{\epsilon,\eta}|^2)\leq$$
$$\leq ||U||_\infty+||V||_\infty-U(0,0)-V(0,0). \tag{24}$$

Consequently, we can find a constant C such that:

$$|x^{\epsilon,\eta}-y^{\epsilon,\eta}|+|t^{\epsilon,\eta}-s^{\epsilon,\eta}|\leq C\epsilon, |x^{\epsilon,\eta}|, |y^{\epsilon,\eta}|\leq\frac{C}{\eta}. \tag{25}$$

As $[0,T]$ is bounded and by (25), extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ converge to a common limit t^η, and from (25) we may also suppose , extracting again, that for each η, the sequences $(x_\epsilon^{\epsilon,\eta})$ and $(y_\epsilon^{\epsilon,\eta})$ converge to a common limit x^η.

1st case: there exists a subsequence of (t^η) such that $t^\eta = T$ for all η (of this subsequence). As U is upper semi continous, for all η and for ϵ small enough

$$U(t^{\epsilon,\eta},x^{\epsilon,\eta})\leq U(t^\eta,x^\eta)+\eta\leq g(x^\eta)+\eta,$$

and as V is lower semicontinuous, for all η and for ϵ small enough

$$V(s^{\epsilon,\eta},y^{\epsilon,\eta})\geq V(t^\eta,x^\eta)-\eta\geq g(x^\eta)-\eta,$$

hence

$$U(t^{\epsilon,\eta},x^{\epsilon,\eta})-V(s^{\epsilon,\eta},y^{\epsilon,\eta})\leq 2\eta$$

and

$$M^{\epsilon,\eta}=U(t^{\epsilon,\eta},x^{\epsilon,\eta})-V(s^{\epsilon,\eta},y^{\epsilon,\eta})-\frac{|x^{\epsilon,\eta}-y^{\epsilon,\eta}|^2}{\epsilon^2}-\frac{|t^{\epsilon,\eta}-s^{\epsilon,\eta}|^2}{\epsilon^2}-$$
$$-\eta^2(|x^{\epsilon,\eta}|^2+|y^{\epsilon,\eta}|^2)\leq U(t^{\epsilon,\eta},x^{\epsilon,\eta})-V(s^{\epsilon,\eta},y^{\epsilon,\eta})\leq 2\eta.$$

Letting $\epsilon\to 0$ and then $\eta\to 0$ one gets, using (20), that $M\leq 0$.

2nd case: there exists a subsequence such that $t^\eta\neq T$, and for all η belonging to this subsequence, there exists a subsequence of $(x^{\epsilon,\eta})_\eta$ such that

$$U(t^{\epsilon,\eta},x^{\epsilon,\eta})-h(t^{\epsilon,\eta},x^{\epsilon,\eta})\leq 0.$$

As from (19) one has

$$V(s^{\epsilon,\eta},y^{\epsilon,\eta})-h(s^{\epsilon,\eta},y^{\epsilon,\eta})\geq 0,$$

it comes that

$$M^{\epsilon,\eta}\leq U(t^{\epsilon,\eta},x^{\epsilon,\eta})-V(s^{\epsilon,\eta},y^{\epsilon,\eta})\leq h(t^{\epsilon,\eta},x^{\epsilon,\eta})-h(s^{\epsilon,\eta},y^{\epsilon,\eta}).$$

Letting $\epsilon\to 0$ and then $\eta\to 0$ one gets, using (20), that $M\leq 0$.

Last case: we are left with the case when, for a subsequence of η, we have $t^\eta\neq T$ and for all η belonging to this subsequence there exists a subsequence of $(x^{\epsilon,\eta})_\epsilon$ such that:

$$U(t^{\epsilon,\eta},x^{\epsilon,\eta})-h(t^{\epsilon,\eta},x^{\epsilon,\eta})>0.$$

We argue by contradiction by assuming that $M > 0$.

We set

$$\varphi(t,s,x,y) := \frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2} + \eta^2(|x|^2+|y|^2) \tag{26}$$

We know the maximum of the function $\psi_{\varepsilon,\eta} := U(t,x) - V(s,y) - \varphi(t,s,x,y)$ is reached at the point $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$. Consequently, we apply the non-local version of Jensen Ishii's lemma [2] and we obtain that there exist:

$$(a,\overline{p},X) \in \mathcal{P}^{2,+}U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),\ (b,\overline{q},Y) \in \mathcal{P}^{2,-}V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})$$

such that

$$\begin{cases} \overline{p} = p + 2\eta^2 x^{\varepsilon,\eta} \\ \overline{q} = p - 2\eta^2 y^{\varepsilon,\eta} \\ p = \frac{2(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})}{\varepsilon^2} \\ a = b = \frac{2(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})}{\varepsilon^2} \\ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{2}{\varepsilon^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2\eta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

Using now the fact that U is a subsolution and V is supersolution, we obtain the two following inequalities:

$$\begin{cases} F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}), a, \overline{p}, X, I_1^{1,\delta}[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]+ \\ \qquad +I_1^{2,\delta}[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\overline{p},U], I_2^{1,\delta}[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]+ \\ \qquad +I_2^{2,\delta}[t^{\varepsilon,\eta},x^{\varepsilon,\eta},U] \leq 0 \\ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}), a, \overline{q}, Y, I_1^{1,\delta}[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]+ \\ \qquad +I_1^{2,\delta}[s^{\varepsilon,\eta},y^{\varepsilon,\eta},\overline{q},V], I_2^{1,\delta}[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y] + I_2^{1,\delta}[s^{\varepsilon,\eta},y^{\varepsilon,\eta},V]) \geq 0 \end{cases} \tag{27}$$

where

$$F(t,x,u,a,p,X,l_1,l_2) := -a - \frac{1}{2}\sigma^2(x)X - b(x)*p - l_1 - f(t,x,u,p\sigma(x),l_2) \tag{28}$$

and we denote by φ_x the function $(t,x) \mapsto \varphi(t,x,s^{\varepsilon,\eta},y^{\varepsilon,\eta})$ and by φ_y the function $(s,y) \mapsto \varphi(t^{\varepsilon,\eta},x^{\varepsilon,\eta},s,y)$.

In order to give an estimation of the difference between integro-differential terms, we use the fact that $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ is a global maximum of $\psi_{\varepsilon,\eta}$. Consequently, we have the inequality:

$$
\begin{aligned}
&\psi_{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e), y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\
&\qquad \leq \psi_{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \\
&\qquad \Leftrightarrow U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\
&\qquad - \frac{|x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e) - y^{\varepsilon,\eta} - \beta(y^{\varepsilon,\eta}, e)|^2}{\varepsilon^2} - \\
&\qquad - \frac{|t^{\varepsilon,\eta} - s^{\varepsilon,\eta}|^2}{\varepsilon^2} - \eta^2(|x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)|^2 + |y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)|^2) \\
&\qquad \leq U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2} - \\
&\qquad - \frac{|t^{\varepsilon,\eta} - s^{\varepsilon,\eta}|^2}{\varepsilon^2} - \eta^2(|x^{\varepsilon,\eta}|^2 + |y^{\varepsilon,\eta}|^2) \\
&\qquad \Leftrightarrow U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leq \\
&\qquad \leq V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \\
&\qquad + \frac{|\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)|^2}{\varepsilon^2} + p(\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)) \\
&\qquad + \eta^2(\beta^2(x^{\varepsilon,\eta}, e) + 2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e) + 2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e) + \beta^2(y^{\varepsilon,\eta}, e)).
\end{aligned}
\tag{29}
$$

We give now the estimation of the difference between the two integro-differential terms:

- We set

$$
\begin{cases}
l_1 := I_1^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + I_1^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] \\
l_1' := I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + I_1^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V]
\end{cases}
\tag{30}
$$

where

$$
\begin{cases}
I_1^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] := \displaystyle\int_{B(0,\delta)} (\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e), y^{\varepsilon,\eta}) - \\
-\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \dfrac{\partial}{\partial x}\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \cdot \beta(x^{\varepsilon,\eta}, e))\nu(de) \\
I_1^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] := \displaystyle\int_{B^c(0,\delta)} (U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - \\
-U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - \overline{p}\beta \cdot (x^{\varepsilon,\eta}, e))\nu(de)
\end{cases}
$$

and

$$
\begin{cases}
I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] := \displaystyle\int_{B(0,\delta)} (-\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\
+\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \dfrac{\partial\varphi}{\partial y}(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})\beta(y^{\varepsilon,\eta}, e))\nu(de) \\
I_1^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V] := \displaystyle\int_{B^c(0,\delta)} (V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - \\
-V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \overline{q}\beta(y^{\varepsilon,\eta}, e))\nu(de).
\end{cases}
$$

We compute now each integral-differential term:

$$\begin{cases} I_1^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] = \int_{B(0,\delta)} (\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e), y^{\varepsilon,\eta}) \\ -\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \frac{\partial}{\partial y}\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})\beta(x^{\varepsilon,\eta}, e))\nu(de) = \\ = \int_{B(0,\delta)} (\frac{\beta^2(x^{\varepsilon,\eta}, e)}{\varepsilon^2} + \eta^2\beta^2(x^{\varepsilon,\eta}, e))\nu(de) = \\ = \int_{B(0,\delta)} (\frac{1}{\varepsilon^2} + \eta^2)\beta^2(x^{\varepsilon,\eta}, e)\nu(de) \\ I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] = \int_{B(0,\delta)} (-\varphi(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\ +\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \frac{\partial\varphi}{\partial y}\varphi(s^{\varepsilon,\eta}, t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \cdot \beta(y^{\varepsilon,\eta}, e))\nu(de) \\ = \int_{B(0,\delta)} (-\frac{1}{\varepsilon^2} - \eta^2)\beta^2(y^{\varepsilon,\eta}, e)\nu(de). \end{cases}$$

We obtain:

$$I_1^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] \leq I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + (\frac{1}{\varepsilon^2} + \eta^2) \int_{B(0,\delta)} \beta^2(y^{\varepsilon,\eta}, e)\nu(de) +$$
$$+ (\frac{1}{\varepsilon^2} + \eta^2) \int_{B(0,\delta)} \beta^2(x^{\varepsilon,\eta}, e)\nu(de) \leq I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + (\frac{1}{\varepsilon^2} + \eta^2)o_\delta(1).$$

and consequently,

$$I_1^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] \leq I_1^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + (\frac{1}{\varepsilon^2} + \eta^2)o_\delta(1) \tag{31}$$

Using the inequality (27) and integrating (for δ small enough) on $\mathbb{R}^d/B(0,\delta)$, we obtain:

$$\begin{aligned} I_1^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] :=& \int_{B^c(0,\delta)} (U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) \\ &- U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - (p + 2\eta^2 x^{\varepsilon,\eta})\beta(x^{\varepsilon,\eta}, e)\nu(de) \\ &\leq \int_{B^c(0,\delta)} (V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \\ &- (p - 2\eta^2 y^{\varepsilon,\eta})\beta(y^{\varepsilon,\eta}, e))\nu(de) \\ &+ \int_{B^c(0,\delta)} \frac{|\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)|^2}{\varepsilon^2}\nu(de) + \\ &+ \eta^2 \int_{B^c(0,\delta)} (\beta^2(x^{\varepsilon,\eta}, e) + \beta^2(y^{\varepsilon,\eta}, e))\nu(de). \end{aligned}$$

Consequently, we have

$$\begin{aligned} I_1^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] \leq& I_1^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V] \\ &+ O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + o_{\eta^2}(1) \end{aligned} \tag{32}$$

From (30), (31) and (32), we obtain:

$$l_1 \leq l_1' + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + o_{\eta_2}(1) + (\frac{1}{\varepsilon^2} + \eta^2)o_\delta(1). \tag{33}$$

- We set

$$\begin{cases} l_2 := I_2^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + I_2^{2,\delta}[x^{\varepsilon,\eta}, t^{\varepsilon,\eta}, U] \\ l_2' := I_2^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + I_2^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V] \end{cases} \tag{34}$$

where

$$\begin{cases} I_2^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] := \int_{B(0,\delta)} (\varphi(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e), y^{\varepsilon,\eta}) - \\ \quad -\varphi(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}))\gamma(x^{\varepsilon,\eta}, e)\nu(de) \\ I_2^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U] := \int_{B(0,\delta)} (U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) \\ \quad -U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}))\gamma(x^{\varepsilon,\eta}, e)\nu(de) \end{cases} \tag{35}$$

and

$$\begin{cases} I_2^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] := \int_{B^c(0,\delta)} (-\varphi(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) + \\ \quad +\varphi(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}))\gamma(y^{\varepsilon,\eta}, e)\nu(de) \\ I_2^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V] := \int_{B^c(0,\delta)} (V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\ \quad -V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}))\gamma(y^{\varepsilon,\eta}, e)\nu(de) \end{cases} \tag{36}$$

We compute now each integral-differential term. From (29) we obtain:

$$\begin{aligned} &[U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})]\gamma(x^{\varepsilon,\eta}, e) \\ &\quad \leq [V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \\ &\quad + \frac{|\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)|^2}{\varepsilon^2} + p(\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)) \\ &\quad + \eta^2(\beta^2(x^{\varepsilon,\eta}, e) + 2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e) + 2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e) + \\ &\quad + \beta^2(y^{\varepsilon,\eta}, e)]\gamma(x^{\varepsilon,\eta}, e) \\ &\quad = (V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}))\gamma(y^{\varepsilon,\eta}, e) \\ &\quad + (V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}))(\gamma(x^{\varepsilon,\eta}, e) - \\ &\quad - \gamma(y^{\varepsilon,\eta}, e)) + \frac{|\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)|^2}{\varepsilon^2}\gamma(x^{\varepsilon,\eta}, e) + \\ &\quad + p \cdot (\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e))\gamma(x^{\varepsilon,\eta}, e) + \\ &\quad + \eta^2(\beta^2(x^{\varepsilon,\eta}, e) + 2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e) + 2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e) + \\ &\quad + \beta^2(y^{\varepsilon,\eta}, e))\gamma(x^{\varepsilon,\eta}, e) \end{aligned}$$

Consequently, using the fact that V is continuous (and therefore locally bounded) and the hypothesis on β, γ we obtain:

$$\begin{aligned} I_2^{2,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U] &\leq I_2^{2,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V] \\ &+ O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + o_{\eta^2}(1). \end{aligned} \tag{37}$$

$$\begin{cases} I_2^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] = \int_{B(0,\delta)} [(\eta^2 + \frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta}, e) \\ \qquad + \frac{2\beta(x^{\varepsilon,\eta}, e)}{\varepsilon^2}|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}| + 2\eta^2 x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e)]\gamma(x^{\varepsilon,\eta}, e) \\ I_2^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] = \int_{B(0,\delta)} [(-\eta^2 - \frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta}, e) \\ \qquad + \frac{2\beta(y^{\varepsilon,\eta}, e)}{\varepsilon^2}|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}| - 2\eta^2 y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e)]\gamma(y^{\varepsilon,\eta}, e). \end{cases}$$

We have:

$$\begin{aligned} &[(\eta^2 + \frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta}, e) + \frac{2\beta(x^{\varepsilon,\eta}, e)}{\varepsilon^2}|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}| + \\ &\quad + 2\eta^2 x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e)]\gamma(x^{\varepsilon,\eta}, e) = \\ &\quad = (-\eta^2 - \frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta}, e)\gamma(y^{\varepsilon,\eta}, e) + \frac{2\beta(y^{\varepsilon,\eta}, e)}{\varepsilon^2}|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|\gamma(y^{\varepsilon,\eta}, e) - \\ &\quad - 2\eta^2 y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e)\gamma(y^{\varepsilon,\eta}, e) + \\ &\quad + (\eta^2 + \frac{1}{\varepsilon^2})[\beta^2(y^{\varepsilon,\eta}, e)\gamma(y^{\varepsilon,\eta}, e) + \beta^2(x^{\varepsilon,\eta}, e)\gamma(x^{\varepsilon,\eta}, e)] \\ &\quad + \frac{2}{\varepsilon^2}|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|[\beta(x^{\varepsilon,\eta}, e)\gamma(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)\gamma(y^{\varepsilon,\eta}, e)] \\ &\quad + 2\eta^2[x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e)\gamma(x^{\varepsilon,\eta}, e) + y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e)\gamma(y^{\varepsilon,\eta}, e)]. \end{aligned}$$

Using the hypothesis on β and γ, we obtain:

$$\begin{aligned} I_2^{1,\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] \le I_2^{1,\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \\ (\eta^2 + \frac{1}{\varepsilon^2})o_\delta(1) + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + o_{\eta^2}(1). \end{aligned} \tag{38}$$

Finally, from (34), (37) and (38), we obtain:

$$l_2 \le l_2' + (\eta^2 + \frac{1}{\varepsilon^2})o_\delta(1) + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + o_{\eta^2}(1). \tag{39}$$

We have supposed that $M > 0$. We use now (17) and the assumptions 3.5 and we obtain:

$$\begin{aligned}
0 < \frac{\gamma}{2}M \leq \gamma M_{\varepsilon,\eta} \leq \gamma(U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}))\\
\leq F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2') -\\
- F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2')\\
= F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2') -\\
- F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2') +\\
+ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2') -\\
- F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1, l_2) +\\
+ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1, l_2) -\\
- F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, l_1, l_2) +\\
+ F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, l_1, l_2) -\\
- F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1', l_2') \leq\\
\leq K|U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})| +\\
+ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1, l_2) -\\
- F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, X^{\varepsilon,\eta}), a, \overline{p}, X, l_1, l_2)\\
+ (\eta^2 + \frac{1}{\varepsilon^2})o_\delta(1) + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + o_{\eta^2}(1).
\end{aligned} \tag{40}$$

We have used the (nonlocal) ellipticity of F, Lipschitz property of F and the estimates (33) and (39). We give the following estimates:

$$\sigma^2(x^{\varepsilon,\eta})(X) - \sigma^2(y^{\varepsilon,\eta})Y \leq \frac{C|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2} + o_{\eta^2}(1),$$
$$b(x^{\varepsilon,\eta})\overline{p} - b(y^{\varepsilon,\eta})\overline{q} \leq \frac{C|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|}{\varepsilon^2} + o_{\eta^2}(1)$$

and we obtain the inequality:

$$\begin{aligned}
F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_1, l_2)\\
- F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, l_1, l_2) \leq\\
\leq \frac{C|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2} + o_{\eta^2}(1) +
\end{aligned} \tag{41}$$

$$\begin{aligned}
&+ f(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), (p + 2\eta^2)\sigma(x^{\varepsilon,\eta}), l_2)\\
&- f(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), (p - 2\eta^2)\sigma(y^{\varepsilon,\eta}), l_2)\\
&\leq \rho(|t^{\varepsilon,\eta} - s^{\varepsilon,\eta}|) + m_R(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|(1 + (p + 2\eta^2)\sigma(x^{\varepsilon,\eta})))\\
&+ K|U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})| + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + o_{\eta^2}(1).
\end{aligned}$$

From (40), (41) we obtain finally:

$$\begin{aligned}0 < \frac{\gamma}{2}M \leq \gamma M^{\varepsilon,\eta} \leq \rho(|t^{\varepsilon,\eta} - s^{\varepsilon,\eta}|) + \\ + m_R(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|(1 + (p + 2\eta^2)\sigma(x^{\varepsilon,\eta})) + \\ + K|U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})| + \\ + O(\frac{|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + (\eta^2 + \frac{1}{\varepsilon^2})o_\delta(1) + o_{\eta^2}(1).\end{aligned} \tag{42}$$

We use now the fact that $\rho(s) \to 0$ when $s \to 0^+$ (property which comes from the continuity of f in t) and the assumptions 3.5. Letting successively δ, ε and η tend to 0 in the relation (42) we obtain a contradiction.

4 Application to optimal stopping for dynamic risk measures induced by BSDEs with jumps

In the framework of risk measures, the state process X defined above may be interpreted for example as an index, an interest rate process, an economic factor, an indicator of the market, the value of a portfolio, which has an influence on the risk measure and the position.

The driver f is defined as above, via a function ψ. We define the following functional: for each $S \in [0, T]$ and $\zeta \in L^2(\mathcal{F}_S)$, set

$$\rho_t(\zeta, S) := -\mathcal{X}(\zeta, S), 0 \leq t \leq S, \tag{43}$$

where $\mathcal{X}_t(\zeta, S) = \mathcal{X}_t$ denotes the solution in $\mathcal{S}^2$ of the BSDE with terminal condition ζ, terminal time S and with driver f, that is, satisfying

$$-d\mathcal{X}_t = f(t, \mathcal{X}_t, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbb{R}^*} l_t(u)\tilde{N}(dt, du)\,;\ \mathcal{X}_S = \zeta, \tag{44}$$

where π_t, l_t are the associated processes, which belong to $\mathbb{H}^2$. If S represents a given maturity and ζ a financial position at time S, then $\rho_t(\zeta, S)$ will be interpreted as the risk measure of ζ at time t. The functional $\rho : (\zeta, S) \to \rho(\zeta, S)$ defines then a dynamic risk measure induced by the BSDE with driver f. Properties of such risk measures are given in [15].

We consider the following optimal stopping problem for dynamic risk measures. The dynamic financial position is given by the process $\{\xi_t, 0 \leq t \leq T\}$, defined as in the previous sections via the state process $X_t := X_t^{0,x}$.

Let $S \in \mathcal{T}_0$ be the initial time. The aim is to stop at a (stopping) time τ greater than S, so that it minimizes the risk measure of position ξ_τ. The minimal risk measure at time S is thus given by:

$$v(S) := ess \inf_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau) \tag{45}$$

where:

$$\mathcal{T}_S := \{\,\text{stopping times with values in } [S, T]\,\}$$

Let us denote by Y the solution (denoted above by $Y^{0,x}$) of RBSDE (6) with $t = 0$.

Proposition 4.1. *Let $S \in \mathcal{T}_0$. The minimal risk measure at time S satisfies*

$$v(S) = -Y_S = -u(S, X_S) \quad a.s. \tag{46}$$

where u is the unique viscosity solution of the PIDIV (11).

Moreover, the stopping time τ_S^* defined by

$$\tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\} = \inf\{t \geq S,\ u(t, X_t) = \bar{h}(t, X_t)\}$$

is optimal for (45), that is $v(S) = \rho_S(\xi_{\tau_S^*}, \tau_S^*)$ a.s.

Here, the function $\bar{h}$ is defined by $\bar{h}(t,x) = h(t,x)$ for $t < T$ and $\bar{h}(T,x) = g(x)$, so that $\xi_t = \bar{h}(t, X_t)$, $0 \leq t \leq T$ a.s.

PROOF: Since by definition, $\rho_S(\xi_\tau, \tau) = -\mathcal{X}_S(\xi_\tau, \tau)$, we have that for each stopping time $S \in \mathcal{T}_0$:

$$v(S) = ess \inf_{\tau \in \mathcal{T}_S} -\mathcal{X}_S(\xi_\tau, \tau) = -ess \sup_{\tau \in \mathcal{T}_S} \mathcal{X}_S(\xi_\tau, \tau)$$

Using Theorem 4.1 in [16], we derive that for each stopping time $S \in \mathcal{T}_0$, we have $v(S) = -Y_S$ a.s. Now, by using the Markov property of X, one can show that $Y_S = -u(S, X_S)$, which gives equalities (46).

By Theorem 4.6 in [16], the stopping time τ_S^* is optimal for (46). The second assertion follows.

A Appendix (A priori estimates)

Let f be a Lipschitz driver, satisfying the following assumption:

Assumption A.1. *$dP \otimes dt$-a.s for each $(x, \pi, l_1, l_2) \in [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$f(t, x, \pi, l_1) - f(t, x, \pi, l_2) \geq \langle \theta_t^{x,\pi,l_1,l_2}\,,\, l_1 - l_2 \rangle_\nu,$$

with

$$\theta : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \mapsto L^2_\nu\,;\ (\omega, t, x, \pi, l_1, l_2) \mapsto \theta_t^{x,\pi,l_1,l_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s. , for each $(x, \pi, l_1, l_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{x,\pi,l_1,l_2}(u) \geq -1 \quad \text{and} \quad |\theta_t^{x,\pi,l_1,l_2}(u)| \leq \psi(u),$$

where $\psi \in L^2_\nu$.

Under this assumption the comparison theorem for the associated BSDEs and RBSDEs with jumps holds (see [17]). We provide the following estimates.

Proposition A.2. *Let $\xi_t^1, \xi_t^2 \in \mathcal{S}^2$. Let f^1, f^2 be Lipschitz drivers satisfying assumption 47 with Lipchitz constant $C > 0$. For $i = 1, 2$, let Y^i be the solution of the RBSDE associated with driver f^i, terminal time T and obstacle ξ_t^i. For $s \in [0,T]$, denote $\overline{Y}_s = Y_s^1 - Y_s^2$, $\overline{\xi}_s = \xi_s^1 - \xi_s^2$ and $\overline{f}_s = \sup_{y,z,k} |f^1(s,y,z,k) - f^2(s,y,z,k)|$. Let $\eta, \beta > 0$ be such that $\beta \geq \dfrac{3}{\eta} + 2C$ and $\eta \leq \dfrac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t}\overline{Y}_t^2 \leq e^{\beta T}(\mathbb{E}[\sup_{s \geq t} \overline{\xi}_s^{\,2} | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T \overline{f}_s^2 ds | \mathcal{F}_t]) \ a.s. \tag{47}$$

Remark A.3. *Note that η and β are universal constants, i.e. they do not depend on T, $\xi_t^1, \xi_t^2, f^1, f^2$. This was not the case for the estimates given in the previous literature (see El Karoui et al. [10])*

Proof. For $i = 1, 2$ and for each $\tau \in \tau_0$, let $(X^{i,\tau}, \pi_s^{i,\tau}, l_s^{i,\tau})$ be the solution of the BSDE associated with driver f^i, terminal time τ and terminal condition ξ_τ^i. Set $\overline{X}_s^\tau = X_s^{1,\tau} - X_s^{2,\tau}$.

By a priori estimate on BSDE (see Proposition $A.4$ in [16]), we have a.s.:

$$e^{\beta t}(\overline{X}_t^\tau)^2 \leq e^{\beta T}\mathbb{E}[\overline{\xi}_\tau^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T e^{\beta s}(f^1(s, X_s^{2,\tau}, \pi_s^{2,\tau}, l_s^{2,\tau}) - f^2(s, X_s^{2,\tau}, \pi_s^{2,\tau}, l_s^{2,\tau}))^2 ds|\mathcal{F}_t] \quad a.s. \tag{48}$$

from which we obtain:

$$e^{\beta t}(\overline{X}_t^\tau)^2 \leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t} \overline{\xi}_s^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T \overline{f}_s^2 ds|\mathcal{F}_t]). \tag{49}$$

Now, by Theorem 4.1 in [16], we have $Y_t^i = ess\sup_{\tau\geq t} X_t^{i,\tau}$ a.s. for $i = 1, 2$. We thus get $|\overline{Y}_t| \leq ess\sup_{\tau\geq t} |\overline{X}_t^\tau|$ a.s.

The result follows.

Proposition A.4. *Let $\xi_t \in \mathcal{S}^2$. Let f be Lipschitz driver satisfying Assumption 47 with Lipchitz constant $C > 0$. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t}Y_t^2 \leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t} \overline{\xi_s}^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T f(s, 0, 0, 0)^2 ds|\mathcal{F}_t]) \ a.s. \tag{50}$$

Proof. Let X_t^τ be the solution of the BSDE associated with driver f, terminal time τ and terminal condition ξ_τ. By applying inequality $(A.2)$ with $f^1 = f$, $\xi_1 = \xi$, $f^2 = 0$ and $\xi^2 = 0$, we get:

$$e^{\beta t}(X_t^\tau)^2 \leq e^{\beta T}\mathbb{E}[\xi_\tau^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T e^{\beta s}(f(s, 0, 0, 0))^2|\mathcal{F}_t] \tag{51}$$

The result follows.

References

[1] Bally,V., Caballero,M.E., Fernandez, B. and N.El-Karoui, Reflected BSDE's, PDE's and Variational Inequalities, INRIA Research report, 2002

[2] Barles, G. and Imbert, C. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567– 585, 2008.

[3] Barles G., Buckdahn R., Pardoux, E., Backward stochastic differential equations and integral-partial differential equations, *Stochastics and Stochastics Reports*, 1997, 60(1-2), 57-83.

[4] Barles G. and Souganidis P.E., Convergence of approximation schemes for fully nonlinear second order equations., *Asymptot. Anal., 4:271-283*, 1991.

[5] Cont R. and Tankov P., FINANCIAL MODELLING WITH JUMP PROCESSES. Chapman & Hall / CRC Press, 2003

[6] Crandall, M., Ishii, H., Lions P-L., USER'S GUIDE TO VISCOSITY SOLUTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS, *American Mathematical Society*, 27:1-67, 1992.

[7] Essaky, E. H. , REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND RCLL OBSTACLE. *Bulletin des Sciences Mathématiques 2008*, 132, 690710

[8] Hamadène, S. and Ouknine, Y., REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND RANDOM OBSTACLE, *Electronic Journal of Probability*, Vol. 8 (2003), Paper no.2, pages 1-20.

[9] Hamadène S. and Y. Ouknine: REFLECTED BACKWARD SDEs WITH GENERAL JUMPS. Manuscript 2007.

[10] Karoui, N.El , Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M-C. REFLECTED SOLUTIONS OF BACKWARD SDE'S, AND RELATED OBSTACLE PROBLEMS FOR PDE'S, *The annals of Probability*, Vol.25, No.2, 702-737, 1997.

[11] Kobylanski,M. , Lepeltier, J.P. , Quenez, M-C. and Torres, S. REFLECTED BSDE WITH SUPER-LINEAR QUADRATIC COEFFICIENT, *Probability and Mathematical Statistics*, Vol.22, pp.51-83.

[12] Ouknine, Y. , REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS, *Stochastic and Stoch. Reports*, 65, 111-125, 1998.

[13] Pardoux E. and S.Peng (1992), Backward Stochastic Differential equations and Quasilinear Parabolic Partial Differential equations, *Lect.Notes in CIS,* 176,200-217.

[14] Peng S. (2004), Nonlinear expectations, nonlinear evaluations and risk measures, 165-253, *Lecture Notes in Math.*, 1856, Springer, Berlin.

[15] Quenez M.-C. and Sulem A., BSDEs WITH JUMPS, OPTIMIZATION AND APPLICATIONS TO DYNAMIC RISK MEASURES, INRIA Research report, June 2012.

[16] Quenez, M.C. and Sulem A., RBSDEs AND OPTIMAL STOPPING FOR DYNAMIC RISK MEASURES WITH JUMPS, 2012.

[17] Royer,M. , BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS AND RELATED NON-LINEAR EXPECTATIONS, *Stochastic Processes and Their Applications* 116 (2006), 1358-1376.

[18] Tang S.H. and X. Li: Necessary conditions for optimal control of stochastic systems with random jumps, *SIAM J. Cont. and Optim.* 32, (1994), 1447–1475.

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