The lovebirds problem: why solve Hamilton-Jacobi-Bellman equations matters in love affairs

Thierry Goudon, Pauline Lafitte

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Abstract

The lovebirds problem consists in finding the compromise between well–being and efforts that are necessary to sustain a sentimental relationship. According to a modeling introduced by J.-M. Rey, the problem can be described as finding the initial data for a certain dynamical system, guaranteeing that the associated trajectory belongs to the stable manifold. We further discuss this interpretation by means of the Dynamical Programming Principle and the Hamilton-Jacobi-Bellman framework. It allows us to propose an algorithm that determines numerically the solution of the lovebirds problem.

1 Introduction

In a recent paper [9], J.-M. Rey introduces a convincing mathematical description of the "sentimental dynamics". The couple is seen as an entity, and the model is based on a contest between the quest of a common well-being, embodied into a quantity $t \mapsto x(t)$, depending only on the time variable, and the cost $t \mapsto c(t)$ of the efforts necessary to maintain a satisfactory well-being. Indeed, the well–being has a natural tendency to fade, and efforts can counter-balance the erosion and sustain the well-being. The model is completed by a utility structure which accounts for the balance between the valuation of the well-being and sacrifices induced by the efforts. Too small values of the well-being as well as too high values of the efforts are not tolerable, which leads to the dissolution of the relationship. Despite its simplicity, the model proposed in [9] provides interesting information on the dynamics of sentimental partnerships. In

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particular it brings an explanation to the so–called "failure paradox": while partners consistently plan a firm and eternal sentimental relationship, a large proportion of unions are eventually going to break up. As we shall recall below, according to J.-M. Rey's modeling, the sentimental dynamics can be mathematically understood in terms of dynamical systems and analysis of the phase portrait $t \mapsto (x(t), c(t))$. It turns out that the underlying dynamical system has a single equilibrium point, denoted hereafter $(\bar{x}, \bar{c})$, and the corresponding linearized system admits eigenvalues of opposite signs. The instability of the equilibrium state explains the difficulty in maintaining a durable relationship. Overall, the partners have to determine their efforts policy so that the trajectory $t \mapsto (x(t), c(t))$ remains on the stable manifold of the dynamical system. In this paper we are thus interested in the "lovebirds problem": given an initial well–being state x_0, which is hopefully supposed to be quite high, how can we find the corresponding effort rate c_0 which guarantees an unbreakable relationship, with a loving trajectory staying on the stable manifold? We shall see that the problem can be interpreted by means of the Dynamic Programming Principle that leads to the Hamilton-Jacobi-Bellman formalism. We describe a numerical procedure based on this interpretation which allows to find a fair approximation of the searched initial state.

The paper is organized as follows. At first we review the basis of J.-M. Rey's modeling. In particular we make the dynamical system which governs the sentimental dynamics appear. In Section 3 we detail several questions of mathematical analysis on the model to make the role of the stable manifold clear. Finally, we discuss the problem in the framework of Dynamic Programming, which allows us to design a numerical method that gives an approximate solution to the lovebirds problem.

2 The dynamics of sentimental relationships

Let us remind the basis of J.-M. Rey's model. Readers interested in further mathematical modeling of sentimental dynamics, based on different ideas, can consult [6] or [8, Chapter 5]. Also an interesting model of parental care relying on game theory and with common technical features with the problem under consideration here can be found in [5]. The evolution of the well–being is governed by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -rx + c, \qquad x(0) = x_0. \tag{1}$$

Given x_0 and $t \mapsto c(t)$, we denote by $X_{x_0,c}$ the corresponding solution of (1):

$$X_{x_0,c} : t \longmapsto X_{x_0,c}(t) = x_0 e^{-rt} + \int_0^t e^{-r(t-s)} c(s)\,\mathrm{d}s.$$

The coefficient $r > 0$ contains the natural erosion of the sentimental feeling. The value of the erosion parameter r depends on the considered relationship, as well as the threshold $x_{\min}$ below which the relationship becomes unsatisfactory and cannot survive. Next, we introduce the utility structure. A satisfaction function $x \mapsto \mathcal{U}(x)$ returns a valuation of the feeling state x. While the shape of the function obeys general rules, that are listed below, the details depend again on the individual situation: different couples might give different valuations to the same common feeling. Producing efforts at a certain level reduces the satisfaction: the function $c \mapsto \mathcal{D}(c)$ evaluates the dissatisfaction due to the effort rate c. The two functions $\mathcal{U}$ (utility) and $\mathcal{D}$ (disutility) are defined on $[0, \infty)$, and they are required to satisfy the following properties:

i) $\mathcal{U}$ and $\mathcal{D}$ are (at least) C^2 non negative functions,

ii) $\mathcal{U}'(x) > 0$ for any $x \geq 0$ and $\lim_{x\to\infty} \mathcal{U}'(x) = 0$,

iii) $\mathcal{U}''(x) \leq 0$ for any $x \geq 0$,

iv) $0 < \alpha \leq \mathcal{D}''(c) \leq A < \infty$ for any $c \geq 0$ and $\lim_{c\to\infty} \mathcal{D}'(c) = \infty$,

v) there exists $c_\star \geq 0$ such that $\mathcal{D}(c_\star) \geq 0$, $\mathcal{D}'(c_\star) = 0$.

The quest for happiness then results in a competition between feeling and efforts, which can be expressed as the optimization of the following cost function, the so–called "total satisfaction"

$$\mathbf{J} : (x, c) \longmapsto \int_0^\infty e^{-\rho t}(\mathcal{U}(x(t)) - \mathcal{D}(c(t)))\,\mathrm{d}t,$$

the functional framework being clarified later on. This quantity involves the so–called impatience parameter $\rho > 0$, which enters in the definition of the weight in the history of the relationship evaluated with $\mathbf{J}(x, c)$. We refer to [9] for further details on the modeling issues. We seek to maximize $\mathbf{J}$ under the constraint (1). Therefore we wish to define the mapping

$$\mathbb{J} : x_0 \longmapsto \max_c\{\mathcal{J}(x_0, c)\}$$

where

$$\mathcal{J} : (x_0, c) \longmapsto \int_0^\infty e^{-\rho t}(\mathcal{U}(X_{x_0,c}(t)) - \mathcal{D}(c(t))\,\mathrm{d}t.$$

As a matter of fact, we deduce from i)-v) that $\mathcal{D}$ is non–negative, convex and satisfies

$$0 \leq \mathcal{D}(c_\star) + \alpha|c - c_\star|^2 \leq \mathcal{D}(c) \leq \mathcal{D}(c_\star) + A|c - c_\star|^2$$

while $\mathcal{U}$ is concave and satisfies, for any $x \geq 0$

$$0 \leq \mathcal{U}(x) \leq \mathcal{U}(0) + \mathcal{U}'(0)x.$$

This remark allows to make the functional framework precise: given $x_0 > 0$, $\mathcal{J}(x_0, \cdot)$ makes sense on the set

$$L^2_{\rho,+} = \Big\{c \geq 0,\ \int_0^\infty e^{-\rho t}|c(t)|^2\,\mathrm{d}t < \infty\Big\}.$$

Indeed, it is clear that, for any $x_0 > 0$ and $c \in L^2_{\rho,+}$, we have $X_{x_0,c} \in L^1_{\rho,+}$, and $X_{x_0,c} \in L^2_{\kappa,+}$ for any $\kappa > \rho$. Therefore $\mathcal{J}(x_0, \cdot)$ is $\alpha-$concave on $L^2_{\rho,+}$, which is a weakly closed convex subset of a Banach space. Computing the derivative with respect to c (denoted here with the prime symbol) we get

$$\mathcal{J}'(x_0, c)(h) = \int_0^\infty e^{-\rho t}\Big(\mathcal{U}'(X_{x_0,c}(t))e^{-rt}\int_0^t e^{rs}h(s)\,\mathrm{d}s - \mathcal{D}'(c(t))h(t)\Big)\,\mathrm{d}t.$$

Integrating by parts yields

$$\mathcal{J}'(x_0, c)(h) = \int_0^\infty h(t)\Big(e^{rt}\int_t^\infty \mathcal{U}'(X_{x_0,c}(s))e^{-(r+\rho)s}\,\mathrm{d}s - e^{-\rho t}\mathcal{D}'(c(t))\Big)\,\mathrm{d}t.$$

We conclude with the following statement.

Proposition 2.1 *The functional $\mathcal{J}(x_0,\cdot)$ admits a unique maximizer $t \mapsto c_\heartsuit(t)$ in $L^2_{\rho,+}$, characterized by the property:*

$$\text{For any } h \in L^2_{\rho,+}, \text{ we have } \mathcal{J}'(x_0, c_\heartsuit)(c_\heartsuit - h) \geq 0.$$

Assuming that $c_\heartsuit$ remains positive, we can choose the trial function $h = c_\heartsuit \pm \zeta$, with $\zeta \geq 0$ in $C_c^\infty((0,\infty))$ and we end up with the following relation, which holds for any $t \geq 0$,

$$c_\heartsuit(t) = (\mathcal{D}')^{-1}\left(e^{(\rho+r)t}\int_t^\infty e^{-(r+\rho)s}\mathcal{U}'\left(x_0e^{-rs} + \int_0^s e^{-r(s-\sigma)}c_\heartsuit(\sigma)d\sigma\right)\mathrm{d}s\right). \quad (2)$$

In turns out that the optimal solution can be investigated by means of dynamical systems, as a consequence of the following facts (which are already presented in [9]).

Theorem 2.2 *Let $x_0 \geq 0$. Let $c_\heartsuit$ be the maximizer of $\mathcal{J}(x_0,\cdot)$. The following assertions hold true:*

a) The function $t \mapsto c_\heartsuit(t)$ does not vanish and remains positive a. e..

b) It is thus characterized by (2).

c) The pair $(x_\heartsuit = X_{x_0,c_\heartsuit}, c_\heartsuit)$ is a solution of the differential system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}x = -rx + c, \\ &\mathcal{D}''(c)\frac{\mathrm{d}}{\mathrm{d}t}c = -\mathcal{U}'(x) + (r+\rho)\mathcal{D}'(c). \end{aligned} \quad (3)$$

d) Furthermore, we have $\lim_{t\to\infty}(x_\heartsuit(t), c_\heartsuit(t)) = (\bar{x},\bar{c})$, where $(\bar{x},\bar{c})$ is the (unique) equilibrium of system (3); in other words $(x_\heartsuit, c_\heartsuit)$ belongs to the stable manifold of (3).

For the forthcoming discussion, it can be helpful to have in mind the typical phase portrait of the dynamical system (3), as displayed in Figure 1. Before detailing the analysis, it is worth explaining the question by means of the Lagrangian framework. We denote by $\mathbf{L}$ the following Lagrangian functional

$$\mathbf{L} : (x,c;\lambda) \longmapsto \mathbf{J}(x,c) - \int_0^\infty \lambda(t)e^{-\rho t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}x + rx - c\Big)(t)\,\mathrm{d}t.$$

For the functional framework, $\mathbf{L}(x,c;\lambda)$ is well defined with $c \in L^2_\rho$, $x \in L^2_\kappa$ with $\frac{\mathrm{d}}{\mathrm{d}t}x \in L^2_\kappa$, where $\rho < \kappa < 2\rho$ so that $L^2_\kappa \subset L^1_\rho$, and $\lambda \in L^2_{2\rho-\kappa}$.

Theorem 2.3 *If $(x_\heartsuit, c_\heartsuit, \lambda_\heartsuit)$ is a saddle point of $\mathbf{L}$ then we have*

$$\mathbf{J}(x_\heartsuit, c_\heartsuit) = \max\Big\{\mathbf{J}(x,c),\ \frac{\mathrm{d}}{\mathrm{d}t}x = -rx + c,\ x(0) = x_0\Big\} = \max_c \mathcal{J}(x_0,c) = \mathbb{J}(x_0). \quad (4)$$

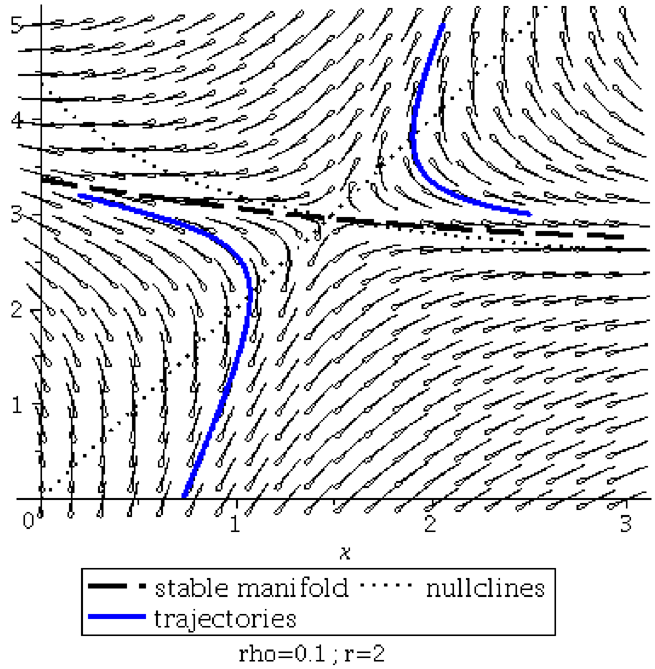

Figure 1: Typical phase portrait of the sentimental dynamics.

Furthermore, as long as $c_\heartsuit(t) > 0$, $(x_\heartsuit, c_\heartsuit, \lambda_\heartsuit)$ *is a solution of the following differential system*

$$\frac{\mathrm{d}}{\mathrm{d}t} x_\heartsuit(t) = -r x_\heartsuit(t) + c_\heartsuit(t), \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} c_\heartsuit(t) = \frac{(r+\rho)\mathcal{D}'(c_\heartsuit(t)) - \mathcal{U}'(x_\heartsuit(t))}{\mathcal{D}''(c_\heartsuit(t))}, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \lambda_\heartsuit(t) = (r+\rho)\lambda_\heartsuit(t) - \mathcal{U}'(x_\heartsuit(t)). \tag{7}$$

Proof. For any trial function $t \mapsto \mu(t)$ we observe that

$$\mathbf{L}(x_\heartsuit, c_\heartsuit; \lambda_\heartsuit + \mu) - \mathbf{L}(x_\heartsuit, c_\heartsuit; \lambda_\heartsuit) = -\int_0^\infty e^{-\rho t} \mu(t) \Big(\frac{\mathrm{d}}{\mathrm{d}t} x_\heartsuit(t) + r x_\heartsuit(t) - c_\heartsuit(t) \Big) \,\mathrm{d}t \geq 0$$

holds, which allows to justify (5). Then, we deduce (4). By the same token, we study the behavior of $\mathbf{L}$ with respect to perturbations of $x_\heartsuit$ and $c_\heartsuit$. Owing to the convexity assumptions on $\mathcal{U}$ and $\mathcal{D}$ we show that for any $t \mapsto \xi(t)$ and $t \mapsto \kappa(t)$ such that $c_\heartsuit(t) + \kappa(t) \geq 0$, the quantity

$$\begin{aligned} &\int_0^\infty e^{-\rho t} \Big[U'(x_\heartsuit(t))\xi(t) - D'(c_\heartsuit(t))\kappa(t) - \lambda_\heartsuit(t) \Big(\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) + r\xi(t) - \kappa(t) \Big) \Big] \,\mathrm{d}t \\ &= \int_0^\infty e^{-\rho t} \Big[\xi(t) \Big(U'(x_\heartsuit(t)) - (\rho + r)\lambda_\heartsuit(t) + \frac{\mathrm{d}}{\mathrm{d}t}\lambda_\heartsuit(t) \Big) - \kappa(t) \Big(D'(c_\heartsuit(t)) - \lambda_\heartsuit(t) \Big) \Big] \,\mathrm{d}t \end{aligned}$$

is non positive. It implies the relations (6) and (7). ∎

Theorem 2.2 establishes the connection between the optimization problem and the determination of the stable manifold of the associated differential system: given $x_0 \geq 0$, finding the optimal strategy $c_\heartsuit$ reduces to finding $c_0 \geq 0$ such that (x_0, c_0) lies in the stable manifold of (3). However, this observation, which has a key role in the discussion in [9], left open how the optimal state c_0 could be found. Formula (2) defines c_0 and the trajectory $c_\heartsuit$ implicitly, but the formula is of reduced practical use: how to derive a numerical method based on (2) to find the optimal strategy is far from clear. We shall see that c_0 can be obtained as a function of x_0 by means of the resolution of a HJB equation. This viewpoint which involves now the resolution of a PDE might look complicated. However, it turns out that efficient numerical procedures can be used to solve this PDE, thus determining the stable manifold of (3).

3 Proof of Theorem 2.2

We start by discussing the existence-uniqueness of the equilibrium point of (3). For $x \geq 0$ given, since $\mathcal{D}'$ and $\mathcal{U}'$ are monotone, their asymptotic behavior being prescribed in Assumptions ii) and iv) above, there exists a unique $\gamma(x) \in [c_\star, +\infty)$ such that $(r+\rho)\mathcal{D}'(\gamma(x)) = \mathcal{U}'(x)$. We note that $\gamma'(x) = \frac{\mathcal{U}''(x)}{(r+\rho)\mathcal{D}''(\gamma(x))} < 0$ and $\lim_{x\to\infty} \gamma(x) = c_\star$. Hence, in the phase plane (x, c), the curve $x \mapsto \gamma(x)$ intersects the line $c = rx$ at a unique point $(\bar{x}, \bar{c}) \in (0, \infty) \times (c_\star, \infty)$. It defines the unique equilibrium point of (3). As observed in [9], there is an effort gap: the rate of effort $\bar{c}$ at the equilibrium is larger than the rate of least effort $c_\star$. Note also that the equilibrium values $\bar{x}$ and $\bar{c} = r\bar{x}$ are non increasing functions of ρ, because

$$r + \rho = \frac{\mathcal{U}'(\bar{x})}{\mathcal{D}'(r\bar{x})}$$

where the right-hand side is a non increasing function of x, which means that the longer the memory lasts, the better the feeling.

The goal is to show the existence of a curve $c_0 = \varphi(x_0)$ such that the points $(x_0, \varphi(x_0))$ are zeroes of the mapping

$$\Phi(x_0, c_0) = \mathcal{D}'(c_0) - \int_0^\infty e^{-(r+\rho)s}\mathcal{U}'(x(x_0, c_0; s))\,\mathrm{d}s$$

with $t \mapsto (x, c)(x_0, c_0; t)$ solution of (3) associated to the initial data (x_0, c_0). The mapping under consideration corresponds to the evaluation of (2) at time $t = 0$. To start with, we observe that $(\bar{x}, \bar{c})$ is such a zero:

$$\Phi(\bar{x}, \bar{c}) = \mathcal{D}'(\bar{c}) - \frac{\mathcal{U}'(\bar{x})}{r + \rho} = 0.$$

We are going to consider Φ over certain perturbations of this specific solution.

To this end, we need to introduce a couple of definitions. The linearization of (3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}y = Ay$$

with

$$A = \begin{pmatrix} -r & 1 \\ -\dfrac{\mathcal{U}''(\bar{x})}{\mathcal{D}''(\bar{c})} & r+\rho \end{pmatrix}.$$

The characteristic polynomial reads

$$P_A(\lambda) = \lambda^2 - \rho\lambda + \frac{\mathcal{U}''(\bar{x})}{\mathcal{D}''(\bar{c})} - r(r+\rho) = (\lambda - \lambda_+)(\lambda - \lambda_-).$$

The discriminant $\Delta = (2r+\rho)^2 - 4\frac{U''(\bar{x})}{\mathcal{D}''(\bar{c})}$ is positive. Hence the system admits two eigenvalues of opposite signs

$$\lambda_\pm = \frac{1}{2}(\rho \pm \sqrt{\Delta}). \tag{8}$$

Some associated eigenvectors are

$$V_\pm = \frac{1}{N_\pm}\begin{pmatrix} 1 \\ r+\lambda_\pm \end{pmatrix}, \qquad N_\pm = \sqrt{1+(r+\lambda_\pm)^2}.$$

For further purposes we remark that

$$\begin{aligned} &r+\lambda_- = r+\rho-\lambda_+ = \frac{2r+\rho}{2}\left(1-\sqrt{1-\frac{4\mathcal{U}''(\bar{x})}{(2r+\rho)^2\mathcal{D}''(\bar{c})}}\right) < 0, \\ &\lambda_+ - \lambda_- = \sqrt{\Delta} > 0, \end{aligned} \tag{9}$$

holds. Let $P = (V_+ \quad V_-)$ be the passage matrix from the canonical basis (e_1, e_2) to the eigenbasis (V_+, V_-). For the inverse matrix we find

$$P^{-1} = \frac{N_- N_+}{\lambda_- - \lambda_+}\begin{pmatrix} \dfrac{r+\lambda_-}{N_-} & -\dfrac{1}{N_-} \\ -\dfrac{r+\lambda_+}{N_+} & \dfrac{1}{N_+} \end{pmatrix}.$$

We introduce the projection Π_- on the negative eigenspace Span$\{V_-\}$ following the direction V_+, namely in the canonical basis

$$\Pi_- = P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}.$$

We set

$$\ell(\tilde{x},\tilde{c}) = \begin{pmatrix} \bar{x} \\ \bar{c} \end{pmatrix} + \Pi_-\begin{pmatrix} \tilde{x} \\ \tilde{c} \end{pmatrix}$$

and

$$\Psi(\tilde{x},\tilde{c}) = \mathcal{D}'(\ell(\tilde{x},\tilde{c})\cdot e_2) - \int_0^\infty e^{-(r+\rho)s}\mathcal{U}'(x(\ell(\tilde{x},\tilde{c});s))\,\mathrm{d}s = \Phi(\ell(\tilde{x},\tilde{c})).$$

Of course, we have $\Psi(0,0) = 0$. We rephrase the problem of finding zeroes (x_0, c_0) of Φ as searching for perturbations $\tilde{y} = (\tilde{x},\tilde{c})$ that make Ψ vanish. To this end, we are going to use the Implicit Function Theorem.

We compute the derivative

$$\nabla_{\tilde{y}}\Psi(0)^T = \mathcal{D}''(\bar{c})(0 \quad 1)\Pi_- - \int_0^\infty e^{-(\rho+r)s}\mathcal{U}''(\bar{x})\tilde{X}(s)\,\mathrm{d}s$$

where $s \mapsto \tilde{X}(s)$ is given by the first row of the matrix Y solution of the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}Y = AY, \qquad Y(0) = \Pi_-.$$

In other words, since $A = P\ \mathrm{diag}(\lambda_+, \lambda_-)\ P^{-1}$, we have

$$Y(t) = P\begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}P^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda_- t}\end{pmatrix}P^{-1}.$$

It follows that

$$\begin{aligned}
\nabla_{\tilde{y}}\Psi(0)^T &= \Big(\mathcal{D}''(\bar{c})(0 \quad 1) - \int_0^\infty e^{-(\rho+r)s}e^{\lambda_- s}\mathcal{U}''(\bar{x})(1 \quad 0)\,\mathrm{d}s\Big)\ P\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}P^{-1} \\
&= \Big(\mathcal{D}''(\bar{c})(0 \quad 1) - \frac{\mathcal{U}''(\bar{x})}{\rho + r - \lambda_-}(1 \quad 0)\,\mathrm{d}s\Big)\ P\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}P^{-1} \\
&= (\partial_{\tilde{x}}\Psi(0) \qquad \partial_{\tilde{c}}\Psi(0)).
\end{aligned}$$

We compute

$$P\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}P^{-1} = \begin{pmatrix} \dfrac{r+\lambda_+}{\lambda_+ - \lambda_-} & \dfrac{N_+}{N_-(\lambda_+ - \lambda_-)} \\ -\dfrac{(r+\lambda_+)(r+\lambda_-)}{\lambda_+ - \lambda_-} & -\dfrac{r+\lambda_-}{\lambda_+ - \lambda_-}\end{pmatrix}.$$

Then, coming back to (9), we obtain

$$\partial_{\tilde{c}}\Psi(0) = \frac{-(r+\lambda_-)}{\lambda_+ - \lambda_-}\mathcal{D}''(\bar{c}) + \frac{-\mathcal{U}''(\bar{x})}{r + \rho - \lambda_-}\frac{N_+}{N_-(\lambda_+ - \lambda_-)} > 0.$$

We can thus apply the Implicit Function Theorem and we are led to the following statement.

Lemma 3.1 *There exists a neighborhood $\mathcal{V}$ of $(0,0)$ and an application $\tilde{x} \mapsto \psi(\tilde{x})$ such that $(\tilde{x}, \psi(\tilde{x})) \in \mathcal{V}$ and $\Psi(\tilde{x}, \psi(\tilde{x})) = 0 = \Phi(\ell(\tilde{x}, \psi(\tilde{x})))$. The set $\{(\tilde{x}, \psi(\tilde{x}))\}$ characterizes the zeroes of Ψ in $\mathcal{V}$.*

It remains to establish clearly the connection with the optimization problem and claims in Theorem 2.2. This is the purpose of the following sequence of lemmas.

Lemma 3.2 *Assume that $c_\heartsuit$ satisfies (2). Then, $(x_\heartsuit = X_{x_0, c_\heartsuit}, c_\heartsuit)$ fulfills (3), $c_\heartsuit(t) > 0$ for any $t \geq 0$ and $t \mapsto c_\heartsuit(t)$ is uniformly bounded.*

Proof. By definition $x_\heartsuit$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x_\heartsuit = -rx_\heartsuit + c_\heartsuit$. Next, we differentiate (2) written as

$$D'(c_\heartsuit(t)) = e^{(r+\rho)t}\int_t^\infty \mathcal{U}'(x_\heartsuit(s))e^{(r+\rho)s}\,\mathrm{d}s.$$

Finally, (2) implies that $\mathcal{D}'(c_\heartsuit(t)) > 0 = \mathcal{D}'(c_\star)$. Since $\mathcal{D}'$ is increasing, we conclude that $c_\heartsuit(t) > c_\star \geq 0$. We also observe that

$$0 \leq e^{(r+\rho)t}\int_t^\infty \mathcal{U}'(x_\heartsuit(s))e^{(r+\rho)s}\,\mathrm{d}s \leq \frac{\mathcal{U}'(0)}{r+\rho},$$

since $\mathcal{U}'$ is non increasing. Thus, (2) leads to $c_\heartsuit(t) \leq \mathcal{D}'(\frac{\mathcal{U}'(0)}{r+\rho})$. ■

Lemma 3.3 *Assume that (x, c) is a solution of (3) associated to an initial data (x_0, c_0) satisfying $\Phi(x_0, c_0) = 0$. Then, $t \mapsto c(t)$ fulfills (2).*

Proof. We have $x = X_{x_0,c}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-(\rho+r)t}\mathcal{D}'(c(t))\Big) = e^{-(\rho+r)t}\mathcal{U}'(X_{x_0,c}(t)).$$

Integrating this relation and using $\Phi(x_0, c_0) = 0$ yield

$$\begin{aligned}
\mathcal{D}'(c(t)) &= e^{(\rho+r)t}\Big(\mathcal{D}'(c_0) - \int_0^t e^{-(\rho+r)s}\mathcal{U}'(X_{x_0,c}(s))\,\mathrm{d}s\Big)\\
&= e^{(\rho+r)t}\Big(\int_0^\infty e^{-(\rho+r)s}\mathcal{U}'(X_{x_0,c}(s))\,\mathrm{d}s - \int_0^t e^{-(\rho+r)s}\mathcal{U}'(X_{x_0,c}(s))\,\mathrm{d}s\Big)\\
&= e^{(\rho+r)t}\int_t^\infty e^{-(\rho+r)s}\mathcal{U}'(X_{x_0,c}(s))\,\mathrm{d}s
\end{aligned}$$

which is (2). ■

Lemma 3.4 *The set $\{(x, c) \in (0,\infty) \times (0,\infty),\ \Phi(x, c) = 0\}$ is invariant under the flow associated to (3).*

Proof. This is a consequence of the fact that (3) is autonomous. Let us denote $(x(x_0, c_0; t), c(x_0, c_0; t))$ the evaluation at time $t \geq 0$ of the solution of (3) issued from (x_0, c_0) at time 0. Therefore we have

$$(x(x_0, c_0; t+s), c(x_0, c_0; t+s)) = (x(x(x_0, c_0; s), c(x_0, c_0; s); t), c(x(x_0, c_0; s), c(x_0, c_0; s); t)).$$

By Lemma 3.3, when $\Phi(x_0, c_0) = 0$, we can write

$$\begin{aligned}
c(x_0, c_0; s) &= \mathcal{D}'^{-1}\Big(e^{(\rho+r)s}\int_s^\infty e^{-(\rho+r)\sigma}\mathcal{U}'(x(x_0, c_0; \sigma))\,\mathrm{d}\sigma\Big)\\
&= \mathcal{D}'^{-1}\Big(\int_0^\infty e^{-(\rho+r)\tau}\mathcal{U}'(x(x_0, c_0; s+\tau))\,\mathrm{d}\tau\Big)\\
&= \mathcal{D}'^{-1}\Big(\int_0^\infty e^{-(\rho+r)\tau}\mathcal{U}'(x(x(x_0, c_0; s), c(x_0, c_0; s); \tau))\,\mathrm{d}\tau\Big).
\end{aligned}$$

We deduce that $\Phi(x(x_0, c_0; s), c(x_0, c_0; s)) = 0$.

Lemma 3.5 *Assume that (x, c) is a solution of* (3) *associated to an initial data* (x_0, c_0) *satisfying* $\Phi(x_0, c_0) = 0$. *Then, for any* $t \geq 0$, $c(t)$ *lies in the interval defined by* $\bar{c}$ *and the nullcline* $\{(r + \rho)\mathcal{D}'(c) = \mathcal{U}'(x)\}$.

Proof. The result follows from the analysis of the phase portrait of (3). We justify the result for c_0, the general case being a consequence of Lemma 3.4. Assume that $x_0 > \bar{x}$. If $c_0 > \bar{c}$, then $\lim_{t\to\infty} c(t) = \infty$; if c_0 is below the nullcline, then $t \mapsto c(t)$ vanishes in finite time. In both case it contradicts Lemma 3.2. Similar reasoning applies when $0 < x_0 < \bar{x}$. In particular, it means that if either the initial well–being or the initial effort is too low a lasting relationship is hopeless. ■

By Lemma 3.1, we have constructed a solution of (3), associated to an initial data which satisfies $\Phi(x_0, c_0) = 0$. Then, by Lemma 3.3, $c(t)$ satisfies (2) for any $t \geq 0$ and we have $c(t) > 0$. Therefore the obtained c realizes the optimum of the functional $\mathcal{J}(x_0, \cdot)$. By Lemma 3.5 we deduce that $t \mapsto c(t)$ and $t \mapsto x(t)$ are monotone (for instance $t \mapsto c(t)$ is increasing and $t \mapsto x(t)$ is decreasing when $x_0 > \bar{x}$). Accordingly $x(t)$ and $c(t)$ admit limits, denoted x_∞ and c_∞ respectively, as time goes to ∞. Now letting $t \to \infty$ in (2) we are led to

$$c(t) \underset{t\to\infty}{\longrightarrow} c_\infty = (\mathcal{D}')^{-1}\left(\frac{\mathcal{U}'(x_\infty)}{r + \rho}\right).$$

The cluster point is nothing but the expected equilibrium point $(\bar{x}, \bar{c})$. It shows that the obtained trajectory lies in the stable manifold of (3). ■

4 Relation with the HJB equation; numerical approach of the lovebirds problem

We shall reinterpret the problem by means of optimal control theory, which in turn will lead to define the stable manifold through the resolution of the Hamilton-Jacobi-Bellman equation. Let us denote

$$f : (x, c) \longmapsto -rx + c, \qquad g : (x, c) \longmapsto \mathcal{U}(x) - \mathcal{D}(c).$$

Given the function c, the evolution equation for $X_{x_0,c}$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{x_0,c} = f(X_{x_0,c}, c) \qquad X_{x_0,c}(0) = x_0.$$

We want to find

$$\mathbb{J} : x_0 \longmapsto \sup_c \mathcal{J}(x_0, c) = \sup_c \int_0^\infty e^{-\rho t} g\big(X_{x_0,c}(t), c(t)\big)\,\mathrm{d}t.$$

Let us apply the Dynamic Programming Principle [1, Section 3.1.2, Th. 3.1]: for any $t > 0$, we have

$$\mathbb{J}(x_0) = \sup_c \Big\{ \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t}\mathbb{J}(X_{x_0,c}(t)) \Big\}. \tag{10}$$

Accordingly, we deduce that $\mathbb{J}$ solves the following stationary HJB equation [1, Section 3.1.4, Th. 3.3]

$$\rho\mathbb{J}(x_0) = H(x_0, \mathbb{J}'(x_0)) \tag{11}$$

with

$$H : (x_0, p) \mapsto \sup_{\kappa\geq 0} \{g(x_0, \kappa) + pf(x_0, \kappa)\} = g(x_0, (\mathcal{D}')^{-1}(p)) + pf(x_0, (\mathcal{D}')^{-1}(p))$$

since $p = \mathcal{D}'(\kappa)$ at the optimum. For the sake of completeness, the proof of (10) and (11) is sketched in the Appendix. We can therefore rewrite (11) as follows

$$\rho\mathbb{J}(x_0) = g(x_0, (\mathcal{D}')^{-1}(\mathbb{J}'(x_0)) + \mathbb{J}'(x_0)f(x_0, (\mathcal{D}')^{-1}(\mathbb{J}'(x_0))).$$

Equation (11) is completed by $\mathbb{J}(\bar{x}) = \bar{c}$. Unfortunately solving (11) is at least as complicated as solving the integral relation (2). Nevertheless, we bear in mind that what we are really interested in is to determine the initial control $c_0(x_0)$ such that the trajectory issued from $(x_0, c_0(x_0))$ lies on the stable manifold. Knowing $(x_0, c_0(x_0))$, the whole trajectory can be obtained by coming back to the differential system (3), which can be solved by standard numerical routines for ODEs. To this end we go back to the Dynamic Programming Principle. We shall use (10) to find an approximation of the initial control. The method we propose works as follows.

i) We need a discrete set of initial well-beings $\mathcal{P}_{\text{wb}} = \{x_0^1, ..., x_0^J\}$, with step size $0 < \Delta x = x_0^{j+1} - x_0^j \ll 1$.

ii) We consider a discrete set of controls $\mathcal{P}_{\text{control}} = \{c_0^1, ..., c_0^K\}$, with $K \gg J$ and step size $0 < \Delta c = c_0^{k+1} - c_0^k \ll 1$. Of course, we use as far as possible the a priori knowledge of the dynamical system (e. g. provided by statements like Lemma 3.2, Lemma 3.5 and the phase portrait analysis) so that the initial controls associated to the elements of $\mathcal{P}_{\text{wb}}$ likely belong to $[c_0^1, c_0^K]$.

iii) We pick $0 < \tau \ll 1$, which plays the role of time step. The controls are assumed to remain constant on the time interval $[0, \tau]$.

iv) Given c_0^k and x_0^j, we approximate the well being at time τ by using the simple forward Euler approximation of (1); it defines

$$X_{j,k}(\tau) = x_0^j + \tau(-rx_0^j + c_0^k).$$

v) Starting from $\mathbb{J}^0 = 0$, we construct a sequence of functions $(\mathbb{J}^m)_{m\geq 1}$, defined on $\mathcal{P}_{\text{wb}}$ as follows

$$\mathbb{J}^{m+1}(x_0^j) = \max_{k\in\{1,...,K\}} \Big\{\frac{1}{2}\tau(g((x_0^j, c_0^k) + e^{-\rho\tau}g(X_{j,k}(\tau), c_0^k)) + e^{-\rho\tau}\mathbb{J}^m(X_{j,k}(\tau))\Big\}. \tag{12}$$

Namely, we use the trapezoidal rule for replacing the integral in the right hand side of (10). The definition of the last term in (12) is slightly misleading because $\mathbb{J}^m$ is not defined on a continuous set, but on the discrete set $\mathcal{P}_{\text{wb}}$ only. Hence, $\mathbb{J}^m(X_{j,k}(\tau))$ needs to be defined through a suitable interpolation procedure using the available values of $\mathbb{J}^m$. This is the reason why we cannot consider a single point in $\mathcal{P}_{\text{wb}}$. We denote by $c_0^{k_j^{m+1}}$ the obtained maximizers, $k_j^{m+1} \in \{1, \dots, K\}$ being the index of the discrete control that solves (12).

vi) We wish that $\mathbb{J}^m(x_0^j)$ resembles $\mathbb{J}(x_0^j)$ as $m \to \infty$, while the $c_0^{k_j^m}$'s approach the controls $c_0(x_0^j)$ that we seek. In practice we stop the algorithm when the relative error between two consecutive iterates remains under a (small) given threshold.

vii) This procedure allows to associate to each $x_0^j \in \mathcal{P}_{\mathrm{wb}}$ a control $\tilde{c}_0^{k_j}$, depending on $\Delta x, \Delta c$, which is intended to approximate $c_0(x_0^j)$ with $(x_0^j, c_0(x_0^j))$ on the stable manifold. Having at hand the pairs $(x_0^j, \tilde{c}_0^{k_j})$ we solve the dynamical system (3) with a standard high–order scheme, for instance the Runge–Kutta algorithm. Actually we can complete this approach with a dichotomy procedure from the guess $\tilde{c}_0^{k_j}$ in order to determine a fair evaluation of the control $c_0(x_0^j)$. We point out that the computational effort essentially relies on the iteration procedure in Steps i)–vi), while in comparison the resolution of the differential system is almost cost-free. Then, having obtained a fair estimate of c_0 in Steps i)–vi), it is worth refining the result by computing a few trajectories.

We illustrate the behavior of the method with the following simple example:

$$\mathcal{U} : x \mapsto 5\log(1+x) \quad \text{and} \quad \mathcal{D} : c \mapsto \frac{(c-0.2)^2}{2}, \quad \rho = 0.1, \quad r = 2.$$

The equilibrium point is $\bar{x} \simeq 0.772$, $\bar{c} \simeq 1.544$ and the phase portrait can be found in Figure 1. We work with 256 points equidistributed in the range $0 \leq x \leq 3$; it defines $\mathcal{P}_{\mathrm{wb}}$. For the set of controls, we work with 4×128 points equidistributed in the range $1 \leq c \leq 5$; it defines $\mathcal{P}_{\mathrm{control}}$ while the time step is $\tau = 0.01$. The dynamical system (3) is approached with the 4th order Runge–Kutta scheme with the constant time step $\Delta t = 10^{-2}$. In the figures below, the trajectories are all represented up to the same final time $T = 2$.

In Figure 2, the snapshot represents the trajectories computed for the initial state $x_0 = 3$ for several values of initial control. The algorithm described above returns $\tilde{c}_0 \simeq 2.750$ and we compute trajectories for a couple of controls around this value. Based on the numerical experiment represented on the picture, we assert that we have found c_0 up to 10^{-3}. In Figure 3, the snapshot corresponds to the same data with the initial state $x_0 = 2.53$. We obtain $\tilde{c}_0 \simeq 2.81$. Based on the numerical experiment represented on the picture, we assert that we have found c_0 up to 10^{-3}. In Figure 4, we change the value of the impatience factor: $\rho = 1$ here. The positive eigenvalue λ_+ defined in (8) is a non decreasing function of ρ and thus, the larger ρ, the more unstable the system. The problem thus should be more demanding for numerics. However, the method is still able to produce a decent approximation of the control. The snapshot corresponds to the initial state $x_0 = 2.53$ and we find $\tilde{c}_0 \simeq 2.57$. Based on the numerical experiment represented on the picture, we assert that we have found c_0 up to 10^{-2}.

Remark 4.1 *The discussion uses the fact that the dynamical system is autonomous. This is a questionable aspect of the model which does not take into account external events which can impact, positively or negatively, either the well–being or the effort policy. As mentioned in [9] it could be relevant also to insert in the modeling alerts and corrections strategies intended to restore from time to time a deteriorated effort policy at a safe level.*

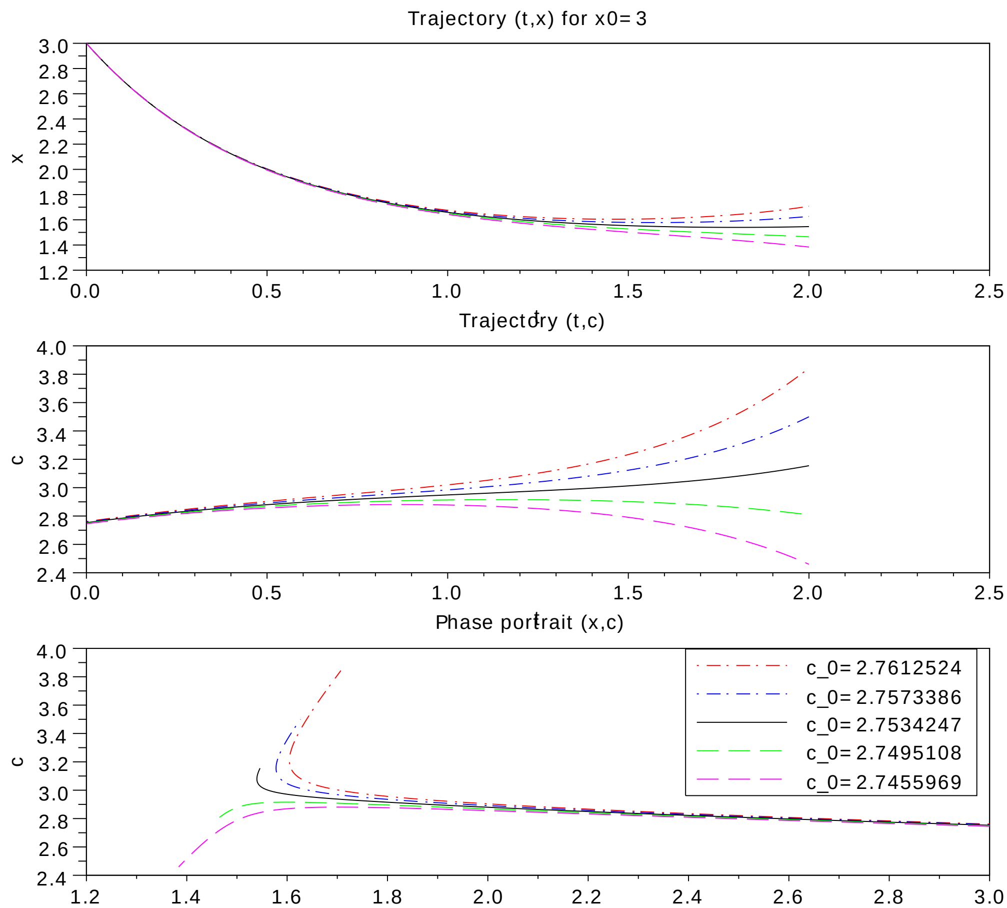

Figure 2: Trajectories issued from $x_0 = 3$ and several values of c_0 around the estimated value $\widetilde{c}_0 = 2.75$.

Remark 4.2 *The model can be slightly generalized. For instance it can be relevant to deal with a non linear well–being equation, generalizing* (1) *to* $\frac{\mathrm{d}}{\mathrm{d}t}x = \phi(x) + c$, *with* ϕ *a suitable non increasing function. On the mathematical viewpoint it could be interesting also to think the other way around: given a* 2×2 *differential system with stable and unstable directions at equilibrium, can we design a numerical method for finding points on the stable manifold? The difficulty to handle such a general case relies on the definition of the cost function, analog of* $\mathbb{J}$.

A Proof of (10) and (11)

We follow [1] and we also refer to [4, Chap. 10] for further details and different arguments. Let us denote

$$\mathbb{K}(x_0, t) = \sup_c \Big\{ \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t}\mathbb{J}(X_{x_0,c}(t)) \Big\}$$

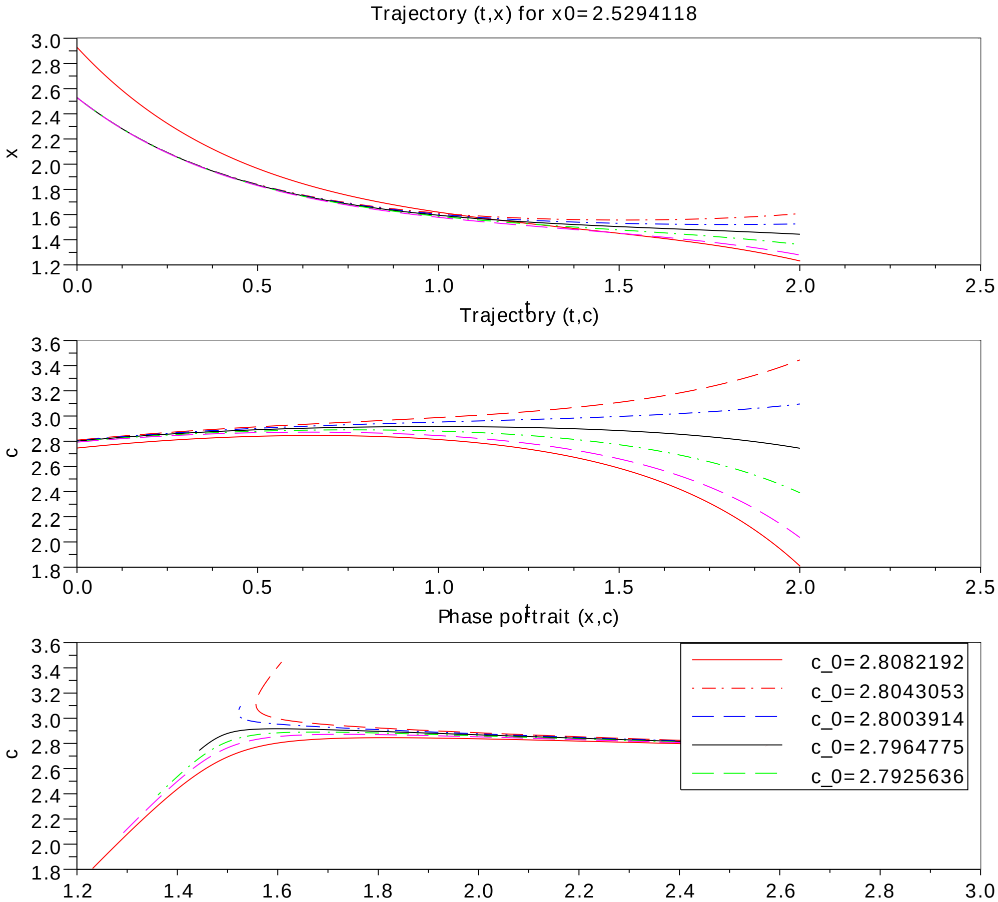

Figure 3: Trajectories issued from $x_0 = 2.53$ and several values of c_0 around the estimated value $\tilde{c}_0 = 2.81$.

the rhs of (10). A priori this quantity depends on $0 \le t < \infty$. Let c be a fixed function and denote $\bar{X} = X_{x_0,c}(t)$. By definition of $\mathbb{J}(X_{x_0,c}(t)) = \mathbb{J}(\bar{X})$ as a supremum, for any $\epsilon > 0$, there exists $t \mapsto c_\epsilon(t)$ such that

$$\mathbb{J}(\bar{X}) - \epsilon \le \int_0^\infty e^{-\rho s} g(X_{\bar{X},c_\epsilon}(s), c_\epsilon(s))\,\mathrm{d}s = \mathcal{J}(\bar{X}, c_\epsilon) \le \mathbb{J}(\bar{X}).$$

We set $\tilde{c}(s) = c(s)\mathbf{1}_{0\le s\le t} + c_\epsilon(s-t)\mathbf{1}_{s>t}$. We have

$$\begin{aligned}
\mathbb{J}(x_0) &\ge \int_0^\infty e^{-\rho s} g(X_{x_0,\tilde{c}}(s), \tilde{c}(s))\,\mathrm{d}s \\
&\ge \int_0^t e^{-\rho s} g(X_{x_0,\tilde{c}}(s), c(s))\,\mathrm{d}s + \int_t^\infty e^{-\rho s} g(X_{x_0,\tilde{c}}(s), c_\epsilon(s-t))\,\mathrm{d}s \\
&\ge \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t}\int_0^\infty e^{-\rho s} g(X_{x_0,\tilde{c}}(t+s), c_\epsilon(s))\,\mathrm{d}s.
\end{aligned}$$

because for $0 \le s \le t$, $\tilde{c}(s) = c(s)$ and thus $X_{x_0,c}$, $X_{x_0,\tilde{c}}$ satisfy the same ODE and are issued form the same initial state x_0. Furthermore, we get $\bar{X} = X_{x_0,c}(t) = X_{x_0,\tilde{c}}(t)$,

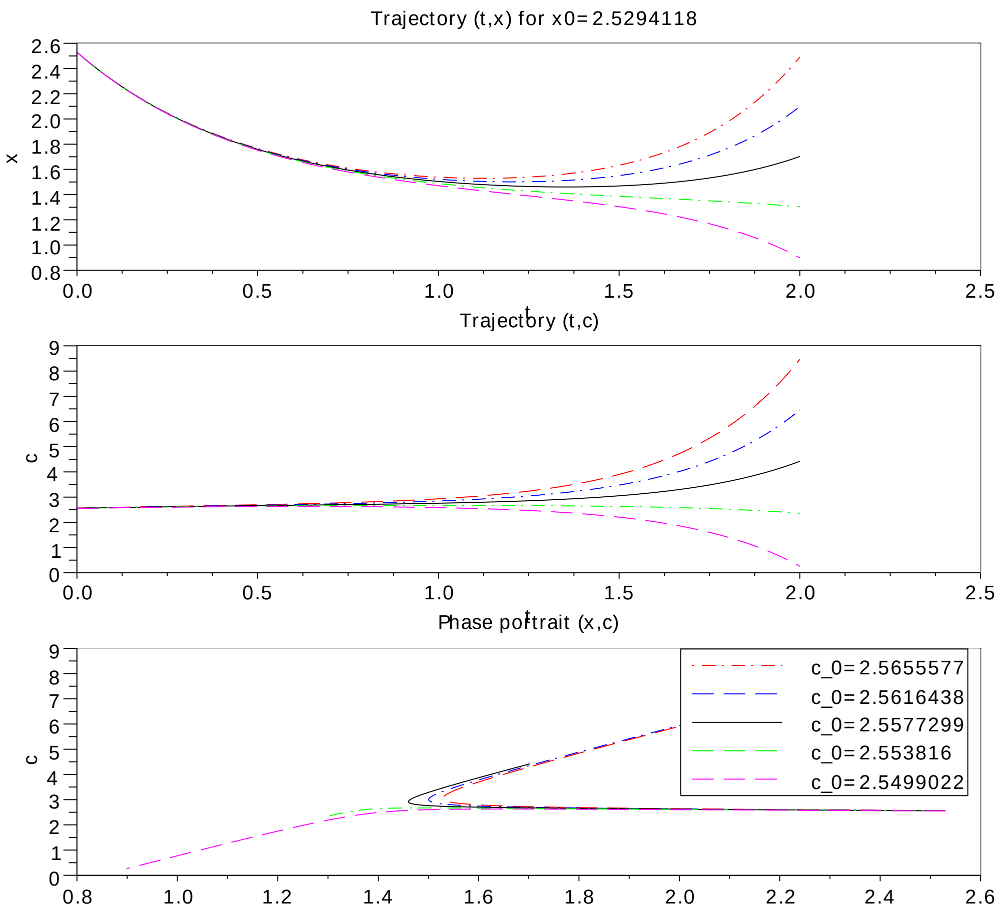

Figure 4: Trajectories issued from $x_0 = 2.53$ and several values of c_0 around the estimated value $\widetilde{c}_0 = 2.57$.

which is the data at $s = 0$ of $s \mapsto X_{x_0,\tilde{c}}(t+s)$ and $s \mapsto X_{\bar{X},c_\epsilon}(s)$, both being solution of the ODE $\frac{\mathrm{d}}{\mathrm{d}s}y(s) = f(y(s), c_\epsilon(s))$ for $s \geq 0$. We deduce that $X_{x_0,\tilde{c}}(t+s) = X_{\bar{X},c_\epsilon}(s)$ for any $s \geq 0$. We thus arrive at

$$\begin{aligned}
\mathbb{J}(x_0) &\geq \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t} \int_0^\infty e^{-\rho s} g(X_{\bar{X},c_\epsilon}(s), c_\epsilon(s))\,\mathrm{d}s \\
&\geq \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t} \mathcal{J}(\bar{X}, c_\epsilon) \\
&\geq \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t} (\mathbb{J}(\bar{X}) - \epsilon).
\end{aligned}$$

It holds for any $\epsilon > 0$ and any function c so that we infer $\mathbb{J}(x_0) \geq \mathbb{K}(x_0, t)$.

Similarly, we can find $t \mapsto c_\epsilon(t)$ such that

$$\mathbb{J}(x_0) - \epsilon \leq \int_0^\infty e^{-\rho s} g(X_{x_0,c_\epsilon}(s), c_\epsilon(s))\,\mathrm{d}s = \mathcal{J}(x_0, c_\epsilon) \leq \mathbb{J}(x_0).$$

Proceeding the same way, we are led to

$$\mathbb{J}(x_0) - \epsilon \leq \int_0^t e^{-\rho s} g(X_{x_0,c_\epsilon}(s), c_\epsilon(s))\,\mathrm{d}s + e^{-\rho t} \int_0^\infty e^{-\rho s} g(X_{x_0,c_\epsilon}(t+s), c_\epsilon(s))\,\mathrm{d}s.$$

The last integral is dominated by $\mathbb{J}(X_{x_0,c_\epsilon}(t))$ and thus we obtain

$$\mathbb{J}(x_0) - \epsilon \leq \int_0^t e^{-\rho s} g(X_{x_0,c_\epsilon}(s), c_\epsilon(s))\,\mathrm{d}s + e^{-\rho t}\mathbb{J}(X_{x_0,c_\epsilon}(t)) \leq \mathbb{K}(x_0, t).$$

Letting ϵ go to 0 ends the proof of (10).

We turn to the proof of (11). A crucial ingredient is the following simple remark: for any $\varphi \in C^1(\mathbb{R})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-\rho t}\varphi(X_{x_0,c}(t))\Big) = e^{-\rho t}\ \Big(- \rho\varphi(X_{x_0,c}(t)) + \varphi'(X_{x_0,c}(t)) f(X_{x_0,c}(t), c(t))\Big) \qquad (13)$$

owing to the chain rule. Let us explain formally the formula (11): assume that the supremum in (10) is attained by a certain function $t \mapsto c_\star(t)$ and that $\mathbb{J}$ is smooth. Then, for any $t > 0$ and $\kappa \geq 0$, (10) implies

$$\begin{aligned} \frac{1}{t}\int_0^t e^{\rho s} g(X_{x_0,c_\star}(s), c_\star(s))\,\mathrm{d}s + \frac{e^{-\rho t}\mathbb{J}(X_{x_0,c_\star}(t)) - \mathbb{J}(x_0)}{t} &= 0 \\ \geq \frac{1}{t}\int_0^t e^{\rho s} g(X_{x_0,\kappa}(s), \kappa)\,\mathrm{d}s + \frac{e^{-\rho t}\mathbb{J}(X_{x_0,\kappa}(t)) - \mathbb{J}(x_0)}{t}&. \end{aligned}$$

Letting $t \to 0$ yields

$$g(x_0, c_\star(0)) + \mathbb{J}'(x_0) f(x_0, c_\star(0)) - \rho\mathbb{J}(x_0) = 0 \geq g(x_0, \kappa) + \mathbb{J}'(x_0) f(x_0, \kappa) - \rho\mathbb{J}(x_0).$$

by using (13). It proves that $\rho\mathbb{J}(x_0) = H(x_0, \mathbb{J}'(x_0))$ holds.

The technical difficulty is related to a lack of regularity of the function $\mathbb{J}$, which make these manipulations questionable. To circumvent the difficulty, one needs to introduce the notion of viscosity solutions of (11).

Definition A.1 *We say that $\mathbb{J}$ is a viscosity solution of* (11) *if for any trial function $\varphi \in C^1$ such that x_0 is a local minimum (resp. maximum) of $\mathbb{J} - \varphi$, we have $H(x_0, \phi'(x_0)) - \rho\phi(x_0) \leq 0$ (resp. ≥ 0).*

Clearly if $\mathbb{J}$ is a viscosity solution which is known to be smooth at x_0, it suffices to set $\varphi = \mathbb{J}$ to obtain (11). Conversely, as far $\mathbb{J}$ is smooth, when x_0 is an extremum of $\mathbb{J} - \varphi$ we have $\mathbb{J}'(x_0) = \varphi'(x_0)$ so that a classical solution of (11) is a viscosity solution. The name "viscosity solution" comes from the possible approximation of such solutions by the solutions u_ϵ of the elliptic equations $\rho u_\epsilon(x_0) - H(x_0, u'_\epsilon(x_0)) = \epsilon u''_\epsilon(x_0)$ as the regularizing parameter $\epsilon > 0$ goes to 0, and the analogies with the theory of conservation laws in gas dynamics. We refer on these aspects to the seminal works [2, 3, 7].

Let us discuss (11) in the framework of viscosity solutions. Let us first assume that x_0 is a local minimum of $x \mapsto \mathbb{J}(x) - \varphi(x)$. Adding a constant to φ if necessary, we thus have:

$$\mathbb{J}(x_0) = \varphi(x_0), \qquad \text{for some } \delta > 0,\ \mathbb{J}(x) \geq \varphi(x) \text{ holds for any } |x - x_0| \leq \delta.$$

Let $\kappa > 0$; using (10) yields

$$\varphi(x_0) = \mathbb{J}(x_0) \geq \int_0^t e^{-\rho s} g(X_{x_0,\kappa}(s), \kappa)\,\mathrm{d}s + e^{-\rho t}\mathbb{J}(X_{x_0,\kappa}(t)).$$

By continuity, as far as $t > 0$ is small enough, $|X_{x_0,\kappa}(t) - x_0| \leq \delta$ so that $\mathbb{J}(X_{x_0,\kappa}(t)) \geq \varphi(X_{x_0,\kappa}(t))$ and therefore

$$\frac{\varphi(x_0) - e^{-\rho t}\varphi(X_{x_0,\kappa}(t))}{t} \geq \frac{1}{t}\int_0^t e^{-\rho s} g(X_{x_0,\kappa}(s), \kappa)\,\mathrm{d}s.$$

By using (13), letting $t \to 0$ leads to

$$+\rho\varphi(x_0) - \varphi'(x_0) f(x_0, \kappa) \geq g(x_0, \kappa).$$

This relation holds for any $\kappa \geq 0$ and we conclude that $\rho\varphi(x_0) - H(x_0, \varphi'(x_0)) \geq 0$ holds.

Next, we assume that x_0 is a local maximum of $x \mapsto \mathbb{J}(x) - \varphi(x)$:

$$\mathbb{J}(x_0) = \varphi(x_0), \qquad \text{for some } \delta > 0,\ \mathbb{J}(x) \leq \varphi(x) \text{ holds for any } |x - x_0| \leq \delta.$$

By using (10) as above, we obtain

$$\varphi(x_0) = \mathbb{J}(x_0) \leq \sup_c \left\{ \int_0^t e^{-\rho s} g(X_{x_0,c}(s), c(s))\,\mathrm{d}s + e^{-\rho t}\varphi(X_{x_0,c}(t)) \right\}$$

But (13) allows us to write

$$e^{-\rho t}\varphi(X_{x_0,c}(t)) - \varphi(x_0) = \int_0^t e^{-\rho t} \Big(-\rho\varphi(X_{x_0,c}(s)) + \varphi'(X_{x_0,c}(s)) f(X_{x_0,c}(s), c(s)) \Big)\,\mathrm{d}s$$

which leads to

$$\begin{aligned}
0 &\leq \sup_c \left\{ \int_0^t e^{-\rho s} \Big(g(X_{x_0,c}(s), c(s)) - \rho\varphi(X_{x_0,c}(s)) + \varphi'(X_{x_0,c}(s)) f(X_{x_0,c}(s), c(s)) \Big)\,\mathrm{d}s \right\} \\
&\leq \sup_c \left\{ \int_0^t e^{-\rho s} \sup_{\kappa \geq 0} \Big(g(X_{x_0,c}(s), \kappa) - \rho\varphi(X_{x_0,c}(s)) + \varphi'(X_{x_0,c}(s)) f(X_{x_0,c}(s), \kappa) \Big)\,\mathrm{d}s \right\} \\
&\leq \sup_c \left\{ \int_0^t e^{-\rho s} \Big(-\rho\varphi(X_{x_0,c}(s)) + H\big(X_{x_0,c}(s), \varphi'(X_{x_0,c}(s))\big) \Big)\,\mathrm{d}s \right\}.
\end{aligned}$$

Finally, we divide by $t > 0$ and let t go to 0; by using a continuity argument we arrive at $0 \leq -\rho\varphi(x_0) + H(x_0, \varphi'(x_0))$.

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