Applications of an existence result for the Coulomb friction problem

Vincent Acary, Florent Cadoux

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Abstract. In a recent paper [2], we prove an abstract existence result for the Coulomb friction problem in discrete time. This problem must be solved at each time step when performing a simulation of the dynamics of a mechanical system involving unilateral contact and Coulomb friction (expressed here at the level of velocities). In this paper, we only recall this result and the gist of its proof and then give an overview of its range of applicability to show the power of our existence criterion. By considering several mechanical systems (Painlevé's example, granular material on a plan or in a drum) and several particular cases (cases with no moving external objects, cases without friction), we demonstrate the broad range of use-cases to which the criterion can be applied by pure abstract reasoning, without any computations. We also show counter-examples where the criterion does not apply. We then turn to more complicated situations where the existence result cannot be used trivially, and discuss the computational methods that are available to check the criterion in practice using optimization software. It turns out that in suffices to solve a linear program (LP) when the problem is bi-dimensional, and a second order cone program (SOCP) when the problem is tri-dimensional.

4.1 Introduction, Motivations

In this paper, the problem of the existence of solution for the Coulomb friction problem is addressed from a very practical point of view. The goal is to show how the proposed existence criterion can be used in practice on several applications before starting to perform a numerical evaluation of the solution. Various numerical algorithms are available for computing solutions for the Coulomb friction problem, but there are few convergence proofs in

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general cases. When an algorithm fails, it is therefore very difficult to know if a convergence problem occurred or if the problem has no solution. By giving a simple but quite general sufficient condition for existence that can be numerically checked in polynomial time, the problem of existence of solutions is partly circumvented and we can decide to change or improve existing solvers.

Under the quasi-static assumption, numerous paper discuss the existence of solutions for the Coulomb friction problem. A bunch of papers has been devoted to the space continuous problem extending the seminal work of Duvaut and Lions [10]. In this paper, we focus on the discrete problem arising, for instance, in the quasi-static case from a finite–element space discretization. As we mention in Section 4.4, Coulomb friction law is usually written in terms of displacements rather than in velocities in a quasi-static analysis. If this problem has a poor physical significance from the engineering point of view, it appears to be valid in a time-incremental approach of the problem. In [12, 13], the existence of solutions of the two-dimensional problem with linear elasticity is proved for any friction coefficient. Note that the elasticity operator is assumed to be coercive which yields a positive definite stiffness matrix. In [3], the copositive LCP theory is used to prove the existence of solutions. Our existence result extends this results for a tridimensional Coulomb cone. In [19], the semi-coercive case is studied where the stiffness matrix is only semi-definite positive. The existence of solutions is proved under the assumption that the data of the problem are included in a specific cone. In [25], the latter result is extended to the fully nonlinear case where the constraints and the equilibrium equation is nonlinear. In latter case, there is no condition for the existence of solutions and this is mainly due to the particular form of the constraints which are only depending on the displacements or the velocities.

For the discrete dynamical problem, an existence result for the incremental problem can be found in [29, 5] which is based on faceting the three-dimensional Coulomb cone and the use of the copositive LCP theory. Note that the mathematical analysis of the incremental problem in dynamics is very similar to those studied in quasi-statics, therefore the results in [19, 25] can be applied to the dynamical case providing some care is taken when formulating the problem. Section 4.4 discusses the link between the quasi-static problem and the dynamical one.

Numerical algorithms for solving the discrete incremental problem are numerous and can be interpreted as extensions of main classes of algorithms that can be found in the mathematical programming theory. To cite a few of them, the numerical algorithms for solving LCP have been extensively used when the Coulomb cone is polyhedral (two-dimensional case or cone faceting approach). In [17, 18, 3, 28, 29, 26], the pivoting method such as Lemke's method are used to solve the LCP. This is the only example of numerical algorithms that is proved to compute a solution when an existence criterion is satisfied [29, 5]. For the second order Coulomb cone, the projection/splitting method for finite dimensional variational inequalities [21, 22, 8, 11, 9, 15, 14, 16] and the nonsmooth (semi-smooth or generalized) Newton methods [7, 4, 6, 27, 20] are the most widespread methods for solving the incremental problem. Unfortunately, there is no general proof of convergence for such methods and therefore the knowledge of existence of solutions is crucial to adapt the numerical strategy if some numerical issues are encountered.

4.2 The Coulomb Friction Incremental Problem

We consider a mechanical system in a d-dimensional space identified to $\mathbb{R}^d$ (in practice, $d = 2$ or $d = 3$) with a finite number m of degrees of freedom. We assume the system is discretized in time, and focus on one moment of the evolution. Unilateral contact is assumed to occur in a finite number n of points in the system. At the i-th contact point, labeling arbitrarily the contacting bodies by A^i and B^i, define a unit normal vector e^i from B^i towards A^i, the discretized relative velocity $u^i \in \mathbb{R}^d$ of A^i with respect to B^i and the discretized impulse r^i exerted by B^i on A^i over the current time-step. Assuming linear discretized kinematics, the generalized velocities $v \in \mathbb{R}^m$

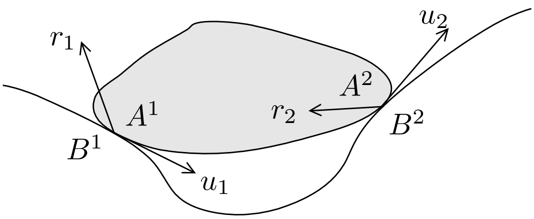

Fig. 4.1 Unknowns u and r

are related to the relative velocities at contact points $u := (u^1, \ldots u^n) \in \mathbb{R}^{nd}$ and to the discretized impulses $r := (r^1, \ldots r^n) \in \mathbb{R}^{nd}$ by affine equations. Specifically, (u, v, r) are related by the kinematic relation

$$u = H\,v + w \tag{4.1}$$

where $H \in \mathbb{R}^{nd \times m}$ and $w \in \mathbb{R}^{nd}$ are known, and by a dynamical equation

$$M\,v + f = H^\top r \tag{4.2}$$

where $M \in \mathbb{R}^{m \times m}$ and $f \in \mathbb{R}^m$ are known. In the sequel, we will make the standard assumption that matrix M is symmetric positive definite.

Assumption 1

$$M \in S_m^{++}.$$

The contact at the point i gives additional coupling constraints. Here, we model friction using Coulomb's law, for which we need the following definition.

Definition 1 (Second order cone). *Let $e \in \mathbb{R}^d$ be a unit vector defining the normal direction, and $x \in \mathbb{R}^d$. The normal and tangential parts of x are defined respectively by*

$$x_N := x \cdot e \in \mathbb{R} \quad \text{and} \quad x_T := x - x_N e \in \mathbb{R}^d.$$

The second order cone $K_{e,\mu}$ with coefficient $\mu \in]0, \infty[$ and direction e is defined by

$$K_{e,\mu} := \{x \in \mathbb{R}^n \; : \; \|x_T\| \leq \mu x_N\}. \tag{4.3}$$

We generalize this definition to $\mu = 0$ by

$$K_{e,0} := \{x \in \mathbb{R}^n \; : \; x_T = 0, \; 0 \leq x_N\}$$

and to $\mu = \infty$ by

$$K_{e,\infty} := \{x \in \mathbb{R}^n \; : \; 0 \leq x_N\}.$$

The velocity u^i and impulse r^i are assumed to satisfy Coulomb's law, which states that $(u^i, r^i) \in \mathcal{C}(e^i, \mu^i)$ where the set $\mathcal{C}(e^i, \mu^i)$ is defined by the following disjunctive constraint.

Definition 2. *Let $(u, r) \in \mathbb{R}^{d \times d}$, $e \in \mathbb{R}^d$ and $\mu \in [0, \infty[$. The set $\mathcal{C}(e, \mu)$ is defined by*

$$(u, r) \in \mathcal{C}(e, \mu) \iff \begin{cases} \text{either: } r = 0 \text{ and } u_N \geq 0 \text{ (take off)} \\ \text{or: } r \in K_{e,\mu} \text{ and } u = 0 \text{ (sticking)} \\ \text{or: } r \in \partial K_{e,\mu} \setminus 0, \; u_N = 0, \exists\, \alpha > 0, \; r_T = -\alpha\, u_T \\ \qquad \text{(sliding).} \end{cases} \tag{4.4}$$

The take-off case occurs when the normal velocity is non-negative and the contact force is zero, which means that there is no attractive force (no adherence, this models dry friction) nor repulsive force when the bodies separate. The sticking case occurs when the relative velocity is zero, then the contact force can lie anywhere in its cone. Finally, the sliding case occurs when the two bodies are moving tangentially one with respect to each other. In this case, the contact force must be "as opposed as possible" to the relative velocity (this is often called the maximum dissipation principle). Altogether, the incremental problem we focus on is

$$\begin{cases} M\, v + f = H^\top r \\ u = H\, v + w \\ (u^i, r^i) \in \mathcal{C}(e^i, \mu^i) \text{ for all } i \in 1, \ldots n \end{cases} \tag{4.5}$$

under Assumption 1 and with $\mathcal{C}(\cdot, \cdot)$ defined by (4.4) for $\mu^i \in [0, \infty[$.

4.3 Existence Criterion

4.3.1 Statement

We state here our main result and provide its mechanical interpretation. Let us first define the main assumption under which Theorem 1 below holds.

$$\exists v \in \mathbb{R}^m \ : \ u := Hv + w \quad \text{satisfies} \quad u^i \in K_{e^i, \frac{1}{\mu^i}} \ (\forall i) \tag{A}$$

where, by convention, $1/0 = \infty$. The following existence results holds.

Theorem 1. *Assume that M is symmetric positive definite and that assumption* (A) *holds. Then the incremental problem* (4.5) *has a solution.*

The mechanical interpretation of assumption (A) is the following; we require that the kinematics of the system allow every pair of contacting bodies to separate with a relative velocity lying in $K_{e,1/\mu}$. Note that when the friction coefficient becomes larger, this condition gets more demanding. Eventually, if μ gets very large, the condition is that it must be kinematically possible to take-off vertically at each contact: the geometry of the system must allow each pair of contacting bodies to separate with a purely normal relative velocity; of course, stating that such a normal separation is possible does not mean that the actual solution of (4.5) will have a normal relative velocity.

Note that this is more demanding than the following assumption

$$\exists v \ : \ u := Hv + w \quad \text{satisfies} \quad u^i_{\mathrm{N}} \geq 0 \quad (\forall i) \tag{A'}$$

which requires that it must be kinematically possible to have a relative velocity whose normal part is non-negative at every contact. Clearly, if this is not verified, then the incremental problem has no solution; Figure 4.2 shows such an example, where a rigid ball is crushed between the motionless ground and a rigid plane with imposed velocity u_0). Theorem 1 requires a little bit more

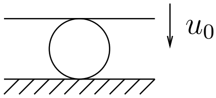

Fig. 4.2 Penetration cannot be prevented

than that (except in the frictionless case , see Subsection 4.5.1, where the sufficient condition (A) and the necessary condition (A') are actually equivalent). Also note that, if vertical take-off is possible at all contact points, then a solution exists for every value of the friction coefficients μ^i.

Remark 1. Condition (A) is purely kinematic and does not use the dynamic information M, f. In addition, (A) is intrinsic: it does not depend on the particular value of H and w (which depend on the chosen reference frame and one the choice of the parameters use to describe the state of the system) but only on the kinematics.

4.3.2 Gist of the Proof

This paper is devoted to showing the usefulness of the existence criterion for practitioners, therefore it is completely out of our scope to prove it rigorously. For this matter, we refer to [2]. However, we provide a sample of the proof with the main ideas.

The first step consists in replacing the incremental problem (4.5) by the equivalent reformulation

$$\begin{cases} M\,v + f = H^\top r \\ \tilde{u} = H\,v + w + E\,s \\ K_{e^i,\frac{1}{\mu^i}} \ni \tilde{u}^i \perp r^i \in K_{e^i,\mu^i} \\ s^i = \|\tilde{u}^i_{\mathrm{T}}\|, \quad \text{for all} \quad i \in 1,\dots n. \end{cases} \tag{4.6}$$

In (4.6), $E := \mathrm{Diag}(\mu^i e^i)$ and instead of $i = 1,\dots n$ one may consider only

$$i \in I := \{i \;:\; \mu^i \neq 0\}$$

otherwise the corresponding i-th column of E is zero and s^i vanishes from the problem. In particular, when all friction coefficients are zero, the whole variable s vanishes (see Subsection 4.5.1). Isolating the first three lines of (4.6) where s is considered as a parameter, we consider

$$\begin{cases} M\,v + f = H^\top r \\ \tilde{u} = H\,v + w + E\,s \\ K_{e^i,\frac{1}{\mu^i}} \ni \tilde{u}^i \perp r^i \in K_{e^i,\mu^i} \end{cases} \tag{4.7}$$

which turns out to be exactly the KKT (Karush-Kuhn-Tucker) conditions of the following optimization problem

$$\begin{cases} \min \quad J(v) := \frac{1}{2} v^\top M v + f^\top v \\ (Hv + w + Es)^i \in K_{e^i,\frac{1}{\mu^i}}. \end{cases} \tag{4.8}$$

Note that the optimization problem (4.8) is parametric: it depends on the value of s. It can be shown that, under assumption (A), the argmin of problem (4.8) (that is to say, the application which maps s to the optimal solution $v(s)$ of (4.8)) is well-defined and is continuous and bounded over $s \in \mathbb{R}^n_+$. Then the remaining equation (the fourth line of (4.6)) defines a fixed-point equation

$$F(s) = s \tag{4.9}$$

where function F is defined by

$$F^i(s) = \|[Hv(s) + w]^i_{\mathrm{T}}\|. \tag{4.10}$$

Said otherwise, $F^i(s)$ is the sliding velocity (the norm of the tangential part of the relative velocity) at the i-th contact point. Since v is a continuous and bounded function of s, F is also bounded and continuous over $\mathbb{R}^n_+$. A direct application of Brouwer's fixed point theorem shows that F admits at least one fixed-point. Therefore, the incremental problem (4.5) has a solution.

4.3.3 Stability

The proof of Theorem 1 shows that it is reassuring to actually have

$$u^i \in \operatorname{int} K_{e^i, \frac{1}{\mu^i}} \quad (\forall i)$$

in Assumption (A), since it ensures stability of the problem: when this assumption (which is obviously stronger than (A)) is satisfied, the existence result remains under a sufficiently small perturbation of the data. This is not the case under the weaker assumption (A): in this case, it may happen that the incremental problem (4.5) has a solution, but that arbitrarily small changes in the data suffice to produce an inconsistent problem which has no solution.

4.4 Instances of the Incremental Problem

In this section, some insights are given on two instances of the incremental problem (4.5). The aim is to motivate the incremental problem studied in this paper by giving some details on how to obtain such a problem. The first one is obtained by the time–discretization of the dynamics of rigid or flexible bodies with unilateral contact impact and friction. The second one is given by the quasi-static problem of flexible bodies.

4.4.1 Time-Discretized Dynamics of Rigid and Flexible Bodies

Let us consider a system of bodies parameterized by a set of generalized coordinates $q(t) \in \mathbb{R}^m$, whose motion is defined on a time interval $[0, T]$, $T > 0$. The generalized velocities $v(t) \in \mathbb{R}^n$ are usually defined as the derivative with respect to time of these generalized coordinates:

$$v(t) = \frac{dq}{dt}(t).$$

The equation of motion is written as

$$M(q(t))\frac{dv}{dt}(t) = F(t, q(t), v(t)) + R(t), \tag{4.11}$$

where

- the matrix $M(q)$, called the mass matrix contains all the masses and the moments of inertia, in most applications one has $M(q) \in S_m^{++}$,
- the vector $F : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ collects the internal and external applied forces,
- the vector $R : \mathbb{R} \to \mathbb{R}^n$ is the generalized reaction force involved in the Coulomb friction contact model.

Note that the equation of motion (4.11) can include the dynamics of continuum media discretized, for instance, by a finite element procedure. The generalized coordinates are then the positions or the displacements of the mesh nodes. Let us assume that there is a finite number n of contacting points for which the unilateral constraints are written such that

$$h^i(t, q(t)) \geq 0, \text{ for all } i \in 1, \ldots n, \tag{4.12}$$

where h^i are assumed to be smooth functions with non-vanishing gradients. This condition can be equivalently written at the velocity level [24]. By denoting the Jacobian of the constraints by

$$H^i(t, q(t)) := \nabla_q^T h^i(t, q(t))$$

and the partial derivative of the constraints with respect to time by

$$w^i(t, q(t)) := \frac{\partial h^i(t, q(t))}{\partial t},$$

the constraints on the relative normal velocity denoted by u_{N}^i is given by

$$u_{\mathrm{N}}^i(t) = H^i(t, q(t))v + w^i(t, q(t)) \geq 0, \text{ if } h^i(t, q(t)) \leq 0 \text{ for all } i \in 1, \ldots n. \tag{4.13}$$

More generally, by defining a local frame at the contact points and collecting the local variables into u the relation between the generalized velocity v and the relative velocities at contact can be written as

$$u(t) = H(t, q(t))v + w(t, q(t)), \tag{4.14}$$

and by duality the generalized reaction forces are expressed as

$$R(t) = H^T(t, q(t))r(t) \tag{4.15}$$

where $r(t)$ are the local reaction forces. The complete dynamics with Coulomb's friction is therefore given by

$$\begin{cases} M(q(t))\dfrac{dv}{dt}(t) = F(t, q(t), v(t)) + H^T(t, q(t))r(t), \\ \dfrac{dq}{dt}(t) = v(t), \\ u(t) = H(t, q(t))v + w(t, q(t)), \\ (u^i(t), r^i(t)) \in \mathcal{C}(e^i(t), \mu^i) \;\text{ if } h^i(t, q(t)) \leq 0 \text{ for all } i \in 1, \dots n. \end{cases} \tag{4.16}$$

It is well-known that the dynamics of such systems may be nonsmooth, that is to say, may exhibit some jumps in velocity. In such cases, the system has to be written in terms of measures and the time-discretization must take care about the possible non-smoothness of the evolution. Without entering into further details, the Moreau's time stepping scheme [24, 23] for a finite time-step $h > 0$ can be written as

$$\begin{cases} M(q_k)(v_{k+1} - v_k) = hF(t_k, q_k, v_k) + H^T(t_k, q_k)r_{k+1}, \\ q_{k+1} = q_k + hv_{k+1}, \\ u_{k+1} = H(t_k, q_k)v_{k+1} + w(t_k, q_k), \\ (u^i_{k+1}, r^i_{k+1}) \in \mathcal{C}(e^i_k, \mu^i) \text{ if } h^i(t_k, q_k(t)) \leq 0 \text{ for all } i \in 1, \dots n. \end{cases} \tag{4.17}$$

In this time–stepping method, the value r_{k+1} plays the role of an impulse and the Coulomb friction law is written in terms of velocity and impulses. Some variants of this time-stepping scheme can be proposed. For instance, a θ–method can be used for the evaluation of the time–integral of the forces F yielding a fully implicit scheme for $\theta \in [1/2, 1]$ and calling for a Newton procedure at each time-step. The non linearity in H can be also included by an implicit discretization and the prediction of the active constraints given by $h^i(t_k, q_k(t)) \leq 0$ can also be improved. For more details on these aspects, we refer the reader to [1]. By identifying the data of (4.5) such that

$$\begin{array}{l} M = M(q_k), \\ f = -hF(t_k, q_k, v_k) - M(q_k)v_k, \\ H = H(t_k, q_k), \end{array} \tag{4.18}$$

the incremental problem (4.5) must be solved at each time step for v_{k+1}, u_{k+1} and r_{k+1}.

4.4.2 Quasi-statics of Flexible Bodies

The quasi-static case of the previous dynamical one can be written as

$$\begin{cases} 0 = F(t, q(t), v(t)) + H^T(t, q(t))r(t), \\ \dfrac{dq}{dt}(t) = v(t), \\ u(t) = H(t, q(t))v + w(t, q(t)), \\ (u^i(t), r^i(t)) \in \mathcal{C}(e^i(t), \mu^i) \;\text{ if } h^i(t, q(t)) \leq 0 \text{ for all } i \in 1, \dots n. \end{cases} \tag{4.19}$$

Usually, the nonlinear behavior of the first and the second equations are taken into account through a Newton method. For the sake of readability, we will consider a linear time invariant behavior law (linear visco-elasticity) and the mapping H is assumed to be independent of q. With these assumptions, the problem (4.19) is

$$\begin{cases} 0 = -Kq(t) - Cv(t) + f(t) + H^T(t)r(t), \\ \dfrac{dq}{dt}(t) = v(t), \\ u(t) = H(t)v(t) + w(t), \\ (u^i(t), r^i(t)) \in \mathcal{C}(e^i(t), \mu^i) \ \text{ if } h^i(t, q(t)) \leq 0 \text{ for all } i \in 1, \dots n, \end{cases} \tag{4.20}$$

where K is the stiffness matrix, C the viscosity matrix and $w(t) = \dfrac{\partial h(t)}{\partial t}$. Considering an Euler backward method for the time–integration of the velocity, we obtain the following time discretized system

$$\begin{cases} (C + hK)v_{k+1} = -Kq_k + f(t_{k+1}) + H^T(t_{k+1})r_{k+1}, \\ u_{k+1} = H(t_{k+1})v_{k+1} + w(t_{k+1}), \\ (u^i_{k+1}, r^i_{k+1}) \in \mathcal{C}(e^i_{k+1}, \mu^i) \ \text{ if } h^i(t_k, q_k) \leq 0 \text{ for all } i \in 1, \dots n. \end{cases} \tag{4.21}$$

The incremental problem (4.5) can be identified with the following data

$$\begin{array}{l} M = (C + hK), \\ f = Kq_k - f(t_{k+1}), \\ H = H(t_{k+1}), \end{array} \tag{4.22}$$

and the existence theorem can be used if $C + hK \in S^{++}_m$. This assumption is satisfied if the stiffness K is at least positive definite. If the boundary conditions on the bodies are prescribed such that any rigid body motion is possible, as it is usual in quasi-static analysis, the resulting stiffness matrix is positive definite.

The use of unilateral constraints at the velocity level is mandatory in the dynamical analysis, however in quasi-static analysis, it is usual to describe constraints at the position level to avoid interpenetration. For the sake of simplicity, let us assume that the local position at contacts, denoted by

$$g(t) = H(t)q(t) + b(t) \tag{4.23}$$

is linear with the respect to $q(t)$. The following definition defines the Coulomb's friction with the unilateral constraints on the position level.

Definition 3 (Coulomb's friction with unilateral contact on position level). *Let* $(g, u, r) \in \mathbb{R}^{d\times d\times d}$, $e \in \mathbb{R}^d$ *and* $\mu \in [0, \infty[$. *The set* $\mathcal{C}_g(e, \mu)$ *is defined by*

$$(g, u, r) \in \mathcal{C}_g(e, \mu) \iff \begin{cases} \textit{either: } r = 0 \textit{ and } g_N(t) \geq 0 \textit{ (no contact)} \\ \textit{or: } r \in K_{e,\mu} \textit{ and } g_N = 0, u_T = 0 \textit{ (sticking)} \\ \textit{or: } r \in \partial K_{e,\mu} \setminus 0,\ g_N = 0, \exists\, \alpha > 0,\ r_T = -\alpha\, u_T \\ \quad \textit{(sliding).} \end{cases} \tag{4.24}$$

The corresponding quasi-static problem is written

$$\begin{cases} 0 = -Kq(t) - Cv(t) + f(t) + H^T(t)r(t) \\ \dfrac{dq}{dt}(t) = v(t) \\ g(t) = H(t)q(t) + b(t) \\ u(t) = H(t)v(t) + w(t) \\ (g^i(t), u^i(t), r^i(t)) \in \mathcal{C}_g(e^i(t), \mu^i). \end{cases} \tag{4.25}$$

Using a backward Euler scheme for the integration of the velocities $v(t)$ and $u(t)$, we obtain

$$\begin{cases} (K + C/h)q_{k+1} = \dfrac{1}{h}Cq_k + f(t_{k+1}) + H^T(t_{k+1})r_{k+1} \\ g_{k+1} = H(t_{k+1})q_{k+1} + b(t_{k+1}) \\ u_{k+1} = H(t_{k+1})\dfrac{q_{k+1} - q_k}{h} + w(t_{k+1}) \\ (g^i_{k+1}, u^i_{k+1}, r_{k+1}) \in \mathcal{C}_g(e^i_{k+1}, \mu^i). \end{cases} \tag{4.26}$$

This problem can be simplified by writing the Coulomb law at the position level in an incremental way on the tangential part. Let us introduce the modified incremental gap function as

$$\tilde{g}_{k+1} = g_{k+1} - (1 - e_k^T e_k)g_k = H(t_{k+1})q_{k+1} + b(t_{k+1}) - (I - e_k^T e_k)g_k. \tag{4.27}$$

With this new notation, we obtain the incremental problem (4.5)

$$\begin{cases} (K + C/h)q_{k+1} = \dfrac{1}{h}Cq_k + f(t_{k+1}) + H^T(t_{k+1})r_{k+1} \\ \tilde{g}_{k+1} = H(t_{k+1})q_{k+1} + b(t_{k+1}) - (I - e_k^T e_k)g_k \\ (\tilde{g}^i_{k+1}, r_{k+1}) \in \mathcal{C}(e^i_{k+1}, \mu^i) \end{cases} \tag{4.28}$$

with $M = K + C/h$, $f = -\dfrac{1}{h}Cq_k - f(t_{k+1})$, $H = H(t_{k+1})$, $w = b(t_{k+1}) - (I - e_k^T e_k)g_k$. Once again, the existence theorem can be used if $C + hK \in S_m^{++}$. This assumption is satisfied if the stiffness K is at least positive definite.

4.5 Checking the Criterion by Hand

In some situations, it is possible to check the criterion without any computation, either because $v = 0$ is an obvious solution to (A) or because one may disregard H and w and consider only the geometry of the system. Such situations are described here. We also treat a few counter-examples, and the case without friction ($\mu = 0$).

4.5.1 *Frictionless Case*

When all the friction coefficients are zero, the matrix E is empty and the variables s and $\tilde{u}$ vanish as well from (4.6). In addition, the sufficient condition (A) for existence is actually exactly the same as the necessary assumption (A') that penetration can be avoided

$$\text{(A)} \iff \text{(A')}.$$

This shows that (A) is actually necessary and sufficient for the frictionless case.

As a side note, this case is much easier than the general case where some of the friction coefficients are nonzero: indeed, since the variable s vanishes, the fixed point problem vanishes as well and it suffices to solve the convex minimization problem (4.8) once to get the solution; in addition, the friction cone $K_{e^i,0}$ reduces to a half-line and non-linearities disappear from the constraints. Hence, when $\mu = 0$, solving the incremental problem (4.5) amounts to solving a quadratic program under linear constraints (QP).

4.5.2 *A Painlevé–like Example*

In this subsection, we describe a toy problem which shows how (4.5) can sometimes be solved by hand. In addition to the illustrative interest, it will be used as a use-case for our existence criterion.

The following very simple example is inspired by the so-called paradox of Painlevé. It has only one degree of freedom and one contact, in dimension 2, and shows that problem (4.5) may have no solution, or a single one, or several (Subsection 4.5.7). Consider the situation depicted on Figure 4.3. The point A is moving along the axis Ox with fixed velocity u_0 (possibly, $u_0 < 0$, in which case the point A is moving leftwards). A rigid rod of length l holding a mass m at its lower end B is articulated with A by a perfect pivot joint. The end B of the bar is subject to unilateral contact with the ground : it can either touch the ground as on Figure 4.3, or take off. In case of contact, the ground applies a force[1] λ onto the bar at B. The only degree of freedom of this system is parameterized by the angle θ, and it is subject to the gravity

[1] Or impulse, to allow for impacts.

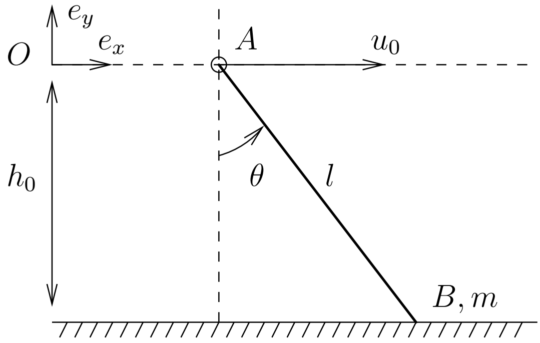

Fig. 4.3 A very simple contact problem

field g along Oy (with $g > 0$ meaning that the gravity is directed *upwards*, and $g < 0$ that it is directed downwards).

The evolution of the system is governed by the equation

$$ml^2\ddot{\theta} = mgl\sin(\theta) + l(\cos(\theta)\lambda_x + \sin(\theta)\lambda_y). \tag{4.29}$$

Let us discretize this equation using a finite time step h. The discrete generalized velocity v approximates $\dot{\theta}$ over the current time step, and v_0 its value at the previous time step. The generalized acceleration $\ddot{\theta}$ is replaced by $(v - v_0)/h$ and the discrete impulse r approximates λh. We obtain the incremental problem (4.5) with

$$M = ml^2,\ f = -mglh\sin\theta - ml^2 v_0,\ H = l\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix},\ w = \begin{bmatrix}u_0\\ 0\end{bmatrix}. \tag{4.30}$$

Remark 2. This mechanical system is not exactly the original problem of Painlevé: in the original problem, one considers a free bar which is not bound to an external body at point A. The bar therefore has its three degrees of freedom, not only one like in our system. However, the original example of Painlevé exhibits a "paradoxical" behavior (namely, non-existence of solutions) only in *continuous time*: indeed, in this problem, the only external object is the ground and it is motionless; Subsection 4.5.4 shows that a solution always exists to the discrete-time problem (the incremental problem) in this case.

4.5.3 Non-existence

Let us take the following values in (4.30) : $m = 1$, $l = 1$, $g = -1$, $h = 1$ and $v_0 = 0$. We do not fix the value of u_0, μ and θ at the moment, and assume that $h_0 < l$ so that the contact can be active, with $\theta \in]0, \pi/2[$. The data in (4.30) become

$$M = 1,\ f = \sin\theta,\ H = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},\ w = \begin{bmatrix} 1 \\ u_0 \end{bmatrix} \tag{4.31}$$

and the incremental problem is

$$\begin{cases} v = \cos(\theta) r_x + \sin(\theta) r_y - \sin\theta \\ u_x = \cos(\theta) v + u_0 \\ u_y = \sin(\theta) v \\ (u, r) \in \mathcal{C}(u_y, \mu). \end{cases} \tag{4.32}$$

Since $d = 2$, the second order cone is polyhedral so that all constraints and equations are linear; in addition, there is a single contact so that there are only three possible cases to check in (4.4) (there would be 3^n cases for n contact points). As a consequence, it is easy to solve problem (4.32) by inspection.

- Take off : $r = 0$ implies $v = -\sin\theta$, so that $u_{\mathrm{N}} = u_y = -\sin(\theta)^2 < 0$, this is impossible.
- Sticking : $u = 0$. If $u_0 \neq 0$, this is impossible. If $u_0 = 0$, then $v = 0$ and r can take any value such that $\cos(\theta) r_x + \sin(\theta) r_y = \sin\theta$ and r in $K_{e_y,\mu}$ (and the set of such r is non-empty since $r = (0, 1)$ is a solution).
- Sliding : $u_{\mathrm{N}} = u_y = 0$ implies $v = 0$ and $u_{\mathrm{T}} = u_0 \neq 0$.

If $u_0 < 0$, then $u_{\mathrm{T}} < 0$ so that r_{T} must be positive and lie on the boundary of $K_{e_y,\mu}$. We obtain the linear system

$$\begin{cases} \cos(\theta) r_x + \sin(\theta) r_y = \sin(\theta) \\ -r_x + \mu r_y = 0 \end{cases} \tag{4.33}$$

associated with the condition $r_y \geq 0$. The only solution is $r_y = \frac{\tan(\theta)}{\tan(\theta)+\mu} \geq 0$. If $u_0 > 0$, then $u_{\mathrm{T}} > 0$ so that r_{T} must be negative and lie on the boundary of $K_{e_y,\mu}$. We obtain the linear system

$$\begin{cases} \cos(\theta) r_x + \sin(\theta) r_y = \sin(\theta) \\ r_x + \mu r_y = 0 \end{cases} \tag{4.34}$$

associated with the condition $r_y \geq 0$. The solution of this system is $r_y = \frac{\tan(\theta)}{\tan(\theta)-\mu}$ for $\tan\theta \neq \mu$ (otherwise, no solution exists). This value for r_y is acceptable if and only if it is positive ; said otherwise, the following lemma holds.

Lemma 1. *Problem* (4.32) *has a solution if and only if*

$$u_0 \leq 0 \ \textit{or} \ [u_0 > 0 \ \textit{and} \ \tan\theta > \mu]. \tag{4.35}$$

Remark 3. This is coherent with intuition: when $\tan\theta > \mu$, the torque applied by the friction force r acts on the bar counter-clockwise, and allows to compensate the effect of gravity which tends to drive the bar downwards, towards the ground. If $\tan\theta = \mu$, the friction force exerts no torque at all and plays no role. Finally, if $\tan\theta < \mu$, the torque applied by the friction force r acts clockwise and *increases* the effect of gravity by driving B towards the ground as well. The friction force being unable to compensate gravity, nothing prevents the bar from penetrating the ground and the unilateral constraint *has* to be violated, therefore no physical solution exists.

Now that we know that problem (4.32) has a solution if and only if the condition (4.35) on u_0, θ and μ is satisfied, let us see what assumption (A) means on this particular problem. It requires that

$$\exists v \in \mathbb{R} \; ; \; (u_0, 0) + (\cos\theta, \sin\theta)\, v \in K_{e_y, \frac{1}{\mu}}. \tag{4.36}$$

To lighten notations, denote by $K := K_{e_y,\frac{1}{\mu}}$ the friction cone and by (Δ) the line

$$(\Delta) := \{(x, y) = (u_0, 0) + (\cos\theta, \sin\theta)\, v \quad \text{for} \quad v \in \mathbb{R}\}.$$

(Δ) is the line passing through $(u_0, 0)$ which makes an oriented angle θ with the x axis. The question is then to determine whether the intersection of the cone K and the line (Δ) is empty or not. Figure 4.4 shows the situation in the three following cases.

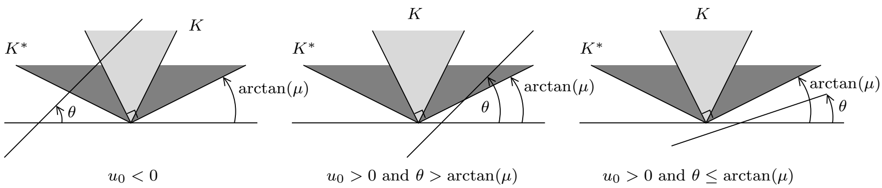

Fig. 4.4 Application of our criterion to Painlevé's example

- If $u_0 < 0$, then the point $(u_0, 0)$ lies to the left of the origin on the x axis and the line (Δ) must intersect the cone K for any value of $\mu \geq 0$ and $\theta \in]0, \frac{\pi}{2}[$ (we could add 0 and $\frac{\pi}{2}$ but these values were not considered since the mechanical problem makes little sense in this case).
- If $u_0 > 0$ and $\theta > \arctan(\mu)$ then the point $(u_0, 0)$ lies to the right of the origin and the oriented angle θ between Ox and the line (Δ) is strictly larger than the angle $\arctan(\mu)$ between the x axis and the boundary of K. This means that (Δ) and K intersect.
- If $u_0 > 0$ and $\theta \leq \arctan(\mu)$, then the point $(u_0, 0)$ lies to the right of the origin and two cases occur; if $\theta = \arctan(\mu)$ then the line (Δ) is parallel to the boundary of K and they do not intersect; if $\theta < \arctan(\mu)$ then (Δ) and the boundary of K are not parallel but they do not intersect either.

The limit case where $u_0 = 0$ is obvious and is not depicted here: in this case, the origin lies in both sets (the line and the cone) and $v = 0$ is a solution to (4.36). Said otherwise, we see that the sufficient condition for existence (A) (which takes the form (4.36) on this example) is equivalent to the necessary and sufficient condition (4.35). For this example, the converse of Theorem 1 is actually true: if a solution exists to the incremental problem, then condition (A) is satisfied.

Remark 4. The example developed in this subsection, together with the frictionless case (Subsection 4.5.1), may lead to the idea that the converse of Theorem 1 is true in general and that (A) is actually a necessary and sufficient condition. This is not true, however: consider the example of Painlevé with $u_0 > 0$ and $\theta < \arctan(\mu)$ (so that (A) is *not* satisfied) and change the sign of gravity by imposing $g = +1$ instead of $g = -1$. The weight of the bar is now directed upwards. Easy computations show that, in accordance with intuition, a solution exists and that no contact force is needed at all to prevent penetration since gravity already tends to separate the bar from the ground. This shows that condition (A) is not necessary for a solution to the incremental problem to exist.

4.5.4 External Objects with Rigid Motion

Thanks to the intrinsic character (see Remark 1) of the criterion, we are able to show that for a large class of systems, the incremental problem (4.5) always has a solution. Suppose that the external objects, if any, move as a single rigid body. Then, applying this same field of velocity to all the internal objects of the system yields zero relative velocity at all contact points (since the whole system is moving as a rigid object), which means that

$$\exists v \in \mathbb{R}^m \ : \ Hv + w = 0$$

so that (A) is satisfied.

In particular, when there are no external objects or when the external objects are motionless, then $w = 0$ in general (this is true for usual parameterizations, but may be false if one uses a time-dependent parameterization or a moving reference frame). In this case, taking $v = 0$ suffices to verify (A).

As an illustration, all systems pictured on Figure 4.5 have a solution to the incremental problem at each time step. On this figure, the first picture

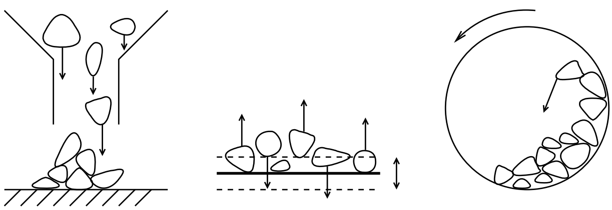

Fig. 4.5 Three classical situations where the criterion applies

represents a very classical situation where (usually rigid) bodies fall from a funnel-shaped tank under gravity and pile on the ground, eventually producing a static stack. Since all external objects (the ground and the tank) are motionless, a solution always exists. The second picture represents an experiment where bodies are piled on a vertically vibrating plane under gravity; since the only external object (the plane) is moving as a rigid body, a solution always exists. The third example consists in a rotating drum filled with bodies; once again, the only external object (the drum) is moving as a rigid body and a solution exists.

As a counter-example, on Figure 4.3, there are two external objects with imposed motion: the ground, which is fixed, and the upper end of the bar (point A) which moves with velocity u_0).

4.5.5 Deformable Solids

Assume that the mechanical system is composed of a deformable solid whose degrees of freedom correspond to the positions of a set of nodes on a mesh. Assume, in addition, that each node is involved in at most one contact, and that contacts occur only at nodes (and not on facets, for instance). Then it suffices to give to each node a velocity which is purely normal to ensure that (A) is satisfied. This shows that a solution exists to the incremental problem for *any value of* μ.

More generally, when a system has enough degrees of freedom so that we are able to give a purely normal velocity to all the contact points by setting the generalized velocities to a chosen value, then the incremental problem of this system has a solution at all time steps for any value of μ.

4.5.6 When the Criterion Does Not (Obviously) Apply

When the mechanical system is more complex, for instance if it contains several external objects with different velocities, it is not obvious to check the assumption (A). For instance, on Figure 4.6, one cannot check the criterion directly: indeed, the meaningless situation of Figure 4.2 could very well happen and (A) would not be satisfied. As a consequence, the criterion cannot be checked "once for all" but we have to consider the actual values of H, w, e^i and μ^i at each time step. In this case, one must rely on numerical algorithms run on a computer to check (A). Section 4.6 explains how this idea can be used with the help of existing optimization software. Before turning to this question, let us consider the related problem of uniqueness of a solution to the incremental problem.

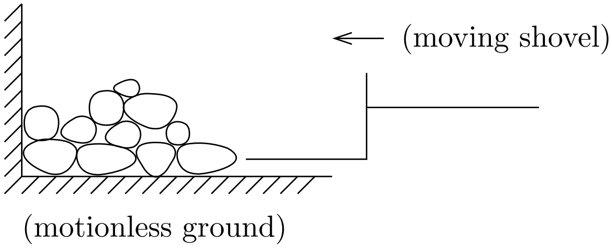

Fig. 4.6 Two external objects with different motion

4.5.7 Non-uniqueness

Related to the question of existence is the question of uniqueness. We show here on an example that the solution of the incremental problem should not be expected to be unique. It is clear that, when rigid solids are used, contact forces may be non-unique. This example shows a worse situation: not only are the contact forces different in the two solutions, but also the dynamical behavior is not the same.

Consider the situation of Figure 4.3, except that $g = 1$ (gravity is now directed upwards) and we set $u_0 = 1$. The incremental problem becomes

$$\begin{cases} v = \cos(\theta) r_x + \sin(\theta) r_y + \sin\theta \\ u_x = \cos(\theta) v + 1 \\ u_y = \sin(\theta) v \\ (u, r) \in \mathcal{C}(u_y, \mu). \end{cases} \tag{4.37}$$

Again, we can solve the incremental problem (4.37) by inspection.

- Take off : $r = 0$ implies $v = \sin\theta$, so that $u_{\mathrm{N}} = u_y = \sin(\theta)^2 > 0$, this is a solution.
- Sticking : $u = 0$ is impossible since $u_0 \neq 0$.
- Sliding : $u_{\mathrm{N}} = u_y = 0$ implies $v = 0$ and $u_{\mathrm{T}} = 1 > 0$, so that r_{T} must be negative and lie on the boundary of $K_{e_y,\mu}$. We obtain a linear system

$$\begin{cases} \cos(\theta) r_x + \sin(\theta) r_y = -\sin(\theta) \\ r_x + \mu r_y = 0 \end{cases} \tag{4.38}$$

  associated with the condition $r_y \geq 0$.

The solution of this system is $r_y = -\frac{\tan(\theta)}{\tan(\theta) - \mu}$ for $\tan\theta \neq \mu$ (otherwise, no solution exists). This value for r_y is acceptable if and only if it is non-negative; said otherwise, problem (4.32) has exactly one solution solution if $\tan\theta > \mu$, and exactly two solutions otherwise.

4.6 Checking the Criterion Computationally

Sometimes, it is not obvious to check the criterion by hand. In this case, we can rely on optimization software to find out whether assumption (A) is satisfied or not. Checking (A) is a problem of *feasibility*: we are trying to find out whether a given set is empty or not. We are going to replace it by an *optimization* problem which can be solved using existing software. In addition to a certificate showing that a solution exists, this optimization problem will provide an idea of the robustness of the problem.

4.6.1 Optimization Problem

Consider the vector c_e defined by

$$c_e := \begin{bmatrix} e^1 \\ \vdots \\ e^n \end{bmatrix}$$

and the problem

$$\begin{cases} \max s \\ (Hv + w - c_e s)^i \in K_{e^i, \frac{1}{\mu^i}} \ (\forall i) \end{cases} \tag{4.39}$$

where we introduced an auxiliary variable $s \in \mathbb{R}$. If a non-negative value is obtained in this problem, then (A) is satisfied and the problem has a solution. If, in addition, s is (strictly) positive, then we know that the problem is robust: a small change in the data cannot turn it into an inconsistent problem with no solution (see Subsection 4.3.3).

4.6.2 Solvers

In 2D, problem (4.39) is fully linear: the constraints are linear since $K_{e^i,1/\mu^i}$ is a polyhedral cone. In other words, (4.39) is a linear program (LP) and it can be solved very efficiently using any of the many LP solvers that are available on the market. Thanks to the extreme speed and robustness of today's LP solvers, it is conceivable to practically check the criterion (A) through the optimization problem (4.39) for systems having tens or hundreds of thousand of variables (if the data is sparse enough).

In 3D, the situations is less comfortable: problem (4.39) is a second-order cone program (SOCP). The SOCP problem is significantly more difficult that the LP problem, less solvers are available and they are far less effective. According to our experiments, the problem (4.39) can be solved quickly up to a few hundreds or thousands of variables, if the data is sparse enough.

Remark 5. The optimization approach proposed here is not the only way to tackle the feasibility problem of checking (A); its main interest is to use only available solvers. Dedicated approaches could be used and would be potentially faster than SOCP solvers in the 3D case.

4.7 Conclusion

By reformulating the incremental problem (4.5) as (4.6), we divide it into an "easy" part which exhibits convexity (the optimization problem), and a smaller part which concentrates all the difficulty (non-smoothness and non-convexity). By doing so, we obtain an existence proof under the assumption (A); the assumption is not very restrictive, in view of the numerous examples which can be dealt with and considering that it is actually a necessary and sufficient condition in several particular cases. In addition, the proof of the existence criterion is reasonably simple and intuitive.

In this paper, we are only interested in the theoretical interest of the reformulation (4.6). However, the fixed-point equation can be tackled numerically and, due to the fact that the problem is now split into a "large easy part" (the convex optimization problem) and a "small difficult part" (the fixed point equation), we expect

- a gain in robustness (since the part which can fail has reduced in size)
- and a gain in speed (since a large part of the problem can now be tackled by specific efficient algorithms).

It would be very interesting to compare this approach (both in terms of speed and robustness) with existing algorithms such as the method of Alart and Curnier (or more generally, any method based on applying Newton's method to a functional reformulation of the constraints (4.5)). The so-called "Gauss-Seidel" algorithm, which turns the multiple contact problem into a sequence of small problems involving only one contact (and which are usually solved easily) is also a good challenger since it is often considered as very robust. These numerical aspects are kept as a direction for future work.

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