Homogenization of a nonstationary convection-diffusion equation in a thin rod and in a layer

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Abstract

The paper deals with the homogenization of a non-stationary convection-diffusion equation defined in a thin rod or in a layer with Dirichlet boundary condition. Under the assumption that the convection term is large, we describe the evolution of the solution's profile and determine the rate of its decay. The main feature of our analysis is that we make no assumption on the support of the initial data which may touch the domain's boundary. This requires the construction of boundary layer correctors in the homogenization process which, surprisingly, play a crucial role in the definition of the leading order term at the limit. Therefore we have to restrict our attention to simple geometries like a rod or a layer for which the definition of boundary layers is easy and explicit.

Keywords: Homogenization, convection-diffusion, localization, thin cylinder, layer.

1 Introduction

The paper deals with the homogenization of a nonstationary convection-diffusion equation with large convection stated either in a thin rod or in a layer. In the previous work [4] the authors addressed a similar homogenization problem for an

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equation defined in a general bounded domain $\Omega \subset \mathbb{R}^d$. Namely, the following initial-boundary value problem has been considered:

$$\begin{cases} \partial_t u^\varepsilon - \mathrm{div}\big(a\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon\big) + \frac{1}{\varepsilon}\, b\big(\frac{x}{\varepsilon}\big)\cdot \nabla u^\varepsilon = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

with periodic coefficients a_{ij}, b_j and a small parameter ε. Notice that in the case of a solenoidal vector-field $b(y)$ with zero mean-value the problem can be studied by the classical homogenization methods (see, for example, [8], [24]). In particular, the sequence of solutions is bounded in $L^\infty[0,T;L^2(\Omega)]\cap L^2[0,T;H^1(\Omega)]$ and converges, as $\varepsilon\to 0$, to the solution of an effective or homogenized problem in which there is no convective term. For more general vector fields b, a similar behaviour of u^ε is observed if the so-called effective drift (a suitable weighted average of b) is equal to zero. The behaviour of the solution changes essentially if the effective drift is nontrivial. Problem (1.1) with nonzero effective drift has first been considered in the whole space $\mathbb{R}^d$ [3], [12], [18], [21] by using the method of moving coordinates: the solution travels at a large speed (equal to the effective drift divided by ε) and its profile is given by the solution of an homogenized diffusion equation. Recently the authors solved the same problem in a bounded domain Ω under the crucial assumption that the initial function u_0 has a compact support in Ω [4]. In this case the initial profile moves towards the boundary during a time of order ε, and then, upon reaching the boundary, starts dissipating. As a result, the solution is asymptotically small for time $t\gg\varepsilon$ and our paper [4] describes precisely the asymptotics of u^ε, which is quite different from that obtained in the case of $\mathbb{R}^d$.

Without the assumption that u_0 has a compact support in Ω, one faces the necessity to construct boundary layer correctors in the neighbourhood of $\partial\Omega$. It is well known that the construction of boundary layers for general domains is a difficult problem which cannot be expressed in explicit form (see however the recent papers [13], [14]). However, it is a feasible task if the periodic structure agrees with the geometry of the boundary of Ω. In the present paper we consider two types of domains which possess this property. Namely, we study a convection-diffusion models in a thin rod (see Fig. 1) and in a layer (see Fig. 2) in $\mathbb{R}^d$. We emphasize that, unlike in classical homogenization, the boundary layers we shall construct for (1.1) are not just corrector terms but, rather, they play a crucial role in the definition of the leading order term in the asymptotic analysis (for more details, see the discussion after Theorem 2.1).

In the case of a thin rod (Section 2) we impose homogeneous Neumann boundary conditions on the lateral boundary of the rod and homogeneous Dirichlet boundary conditions on its bases. As in the case of a general bounded domain [4], the solution asymptotically vanishes for time $t\gg\varepsilon$. Theorem 2.1 determines the rate of vanishing of the solution and describes the evolution of its profile. If the effective axial drift is not zero (otherwise the problem is trivial), the rescaled solution concentrates in the vicinity of one of the rod ends, and the choice of the end depends on the sign of the effective convection. In order to characterize the rate of decay we introduce a 1-parameter family of auxiliary cell spectral problems, similar to Bloch waves but with real exponential argument (see [8], [9], [11]). The asymptotic behaviour of the solution is then governed by the first eigenpair of the said family of spectral problems and by a one-dimensional homogenized problem with a singular initial data.

In the case of a layer, addressed in Section 3, in addition to the factorization principle, we also have to introduce moving coordinates [3], [12]. More precisely, we use a parameterized cell spectral problem and factorization principle to suppress the normal component of the effective drift (perpendicular to the layer boundary). While, due to the presence of the longitudinal components of the effective convection, we have to introduce moving coordinates (parallel to the layer boundary). The main result in this case is given by Theorem 3.1. The asymptotic behaviour of u_ε is again governed by the first eigenpair of the spectral cell problem and by a homogenized problem with a singular initial data.

In both cases (rod or layer) the initial data of the homogenized problem, and thus the asymptotic behavior of solutions to (1.1), differ from those obtained for the case of a general domain in [4] (see again the discussion after Theorem 2.1). Among the technical tools used in the paper, are factorization principle (see [16], [23], [24], [2], [9]), dimension reduction arguments and qualitative results required for constructing boundary layer correctors.

2 The case of a thin rod

This section is concerned with the homogenization of equation (1.1) stated in a thin rod $G_\varepsilon = (-1,1)\times\varepsilon Q$ (see Figure 1). Here $Q\subset\mathbb{R}^{d-1}$ is a bounded domain with Lipschitz boundary ∂Q, $\varepsilon>0$ is a small parameter. Without loss of generality, we assume that Q has a unit $(d-1)$-dimensional measure, i.e. $|Q|_{d-1}=1$. Throughout this section the points in $\mathbb{R}^d$ are denoted $x=(x_1,x')$ with $x'\in\mathbb{R}^{d-1}$. The lateral boundary of the rod G_ε is denoted $\Sigma_\varepsilon=(-1,1)\times\varepsilon\partial Q$. For $T>0$, we consider the following model:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + A_\varepsilon\, u^\varepsilon(t,x) = 0, & \text{in } (0,T)\times G_\varepsilon, \\ B_\varepsilon u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\Sigma_\varepsilon, \\ u^\varepsilon(t,\pm 1,x') = 0, & \text{on } (0,T)\times\varepsilon Q, \\ u^\varepsilon(0,x) = u_0(x_1), & x\in G_\varepsilon \end{cases} \tag{2.1}$$

with

$$A_\varepsilon u^\varepsilon = -\mathrm{div}\big(a^\varepsilon\nabla u^\varepsilon\big) + \frac{1}{\varepsilon}b^\varepsilon\cdot\nabla u^\varepsilon; \quad B_\varepsilon u^\varepsilon = a^\varepsilon\nabla u^\varepsilon\cdot n.$$

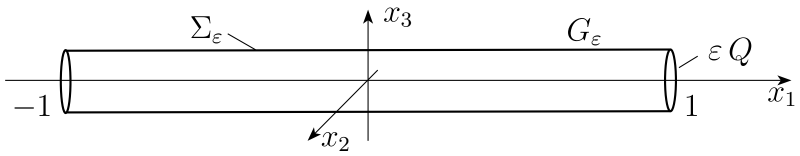

Figure 1: The rod G_ε

The coefficients of the equation are given by

$$a^\varepsilon_{ij} = a_{ij}\big(\frac{x}{\varepsilon}\big), \quad b^\varepsilon_j = b_j\big(\frac{x}{\varepsilon}\big), \qquad 1 \le i\,, j \le d. \tag{2.2}$$

Note that the fixed domain Ω in (1.1) is replaced in (2.1) by G_ε which has a vanishing cross-section and that the Dirichlet boundary conditions are applied merely at the end bases of the thin rod. If the rod had a square cross-section, the problem with the Neumann boundary condition on the lateral boundary Σ_ε could be reduced to a problem with periodic boundary conditions in a cylinder having in the cross-section the square of double size. This gives us an idea that our results can be extended to the case of periodic boundary conditions on the lateral boundary of the rod. Indeed, the arguments used in the paper also apply, with some simplifications, to the case of periodic boundary conditions.

We assume that:

(H1) The coefficients of A_ε are measurable bounded functions, that is $a_{ij}, b_j \in L^\infty(\mathbb{R}\times Q)$. Moreover, $a_{ij}(y_1, y'), b_j(y_1, y')$ are 1-periodic with respect to y_1.

(H2) The $d\times d$ matrix $a(y)$ is symmetric and satisfies the uniform ellipticity condition, that is there exists $\Lambda > 0$ such that

$$a_{ij}(y)\xi_i\xi_j \ge \Lambda|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.$$

(H3) The initial function $u_0(x_1) \in C^1[-1,1]$.

(H4) For simplicity, we assume that $\varepsilon = 1/N$, $N \in \mathbb{Z}_+$.

Remark 2.1. *In assumption* **(H2)** *the Einstein summation convention over repeated indices is used, as well as later in this paper. Assumption* **(H4)** *means that the rod is made up of a number of entire cells which are not cut at both ends.*

Since the rod has a vanishing thickness and u_0 is smooth, there is no fundamental restriction in assuming that u_0 depends only on x_1.

Under the stated assumptions we study the asymptotic behaviour of solutions $u^\varepsilon(t,x)$ of problem (2.1), as $\varepsilon \to 0$.

2.1 Auxiliary spectral problems and main result

In what follows we denote

$$Au = -\mathrm{div}(a\nabla u) + b\cdot\nabla u, \quad Bu = a\nabla u\cdot n; \tag{2.3}$$

$$A^*u = -\mathrm{div}(a\nabla u) - \mathrm{div}(b\,u), \quad B^*u = a\nabla u\cdot n + (b\cdot n)\,u. \tag{2.4}$$

Following [8], [9], for $\theta \in \mathbb{R}$, we introduce two parameterized families of spectral problems (direct and adjoint) which are different from the usual Floquet-Bloch spectral problems because the exponential factor θ is real instead of being purely imaginary. They reads

$$\begin{cases} e^{-\theta y_1}\,A\,e^{\theta y_1}\,p_\theta(y) = \lambda(\theta)\,p_\theta(y), \quad \text{in } Y = \mathbb{T}_1\times Q,\\ e^{-\theta y_1}\,B\,e^{\theta y_1}\,p_\theta(y) = 0, \quad \text{on } \partial Y = \mathbb{T}_1\times\partial Q,\\ y_1 \to p_\theta(y) \quad \text{1-periodic,} \end{cases} \tag{2.5}$$

and

$$\begin{cases} e^{\theta y_1}\,A^*\,e^{-\theta y_1}\,p^*_\theta(y) = \lambda(\theta)\,p^*_\theta(y), \quad \text{in } Y,\\ e^{\theta y_1}\,B^*\,e^{-\theta y_1}\,p^*_\theta(y) = 0, \quad \text{on } \partial Y,\\ y_1 \to p^*_\theta(y) \quad \text{1-periodic.} \end{cases}$$

Here $\mathbb{T}_1$ is the 1-dimensional unit circle. Note that the exponential transform is applied only with respect to the first space component y_1. The next result, based on the Krein-Rutman theorem, has been proved in [9].

Lemma 2.1. *For each $\theta \in \mathbb{R}$, the first eigenvalue $\lambda_1(\theta)$ of problem* (2.5) *is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, $\theta \to \lambda_1(\theta)$ is twice differentiable, strictly concave and admits a maximum which is obtained for a unique $\theta = \Theta$.*

The eigenfunctions p_θ and p^*_θ defined by Lemma 2.1, are normalized by

$$\int_Y |p_\theta(y)|^2\,dy = 1 \quad \text{and} \quad \int_Y p_\theta(y)\,p^*_\theta(y)\,dy = 1. \tag{2.6}$$

Differentiating equation (2.5) with respect to θ, integrating against p^*_θ and writing down the compatibility condition for the resulting equation, yield

$$\frac{d\lambda_1}{d\theta} = \int_Y \big(b_1\,p_\theta\,p^*_\theta + a_{1j}(p_\theta\,\partial_{y_j}p^*_\theta - p^*_\theta\,\partial_{y_j}p_\theta) - 2\,\theta\,p_\theta\,p^*_\theta\,a_{11}\big)\,dy. \tag{2.7}$$

Noticing that $\lambda_1(0) = 0$ and $p_\theta(y)|_{\theta=0} = 1$, one obtains

$$\frac{d\lambda_1}{d\theta}\Big|_{\theta=0} = \int_Y \big(a_{1j}\,\partial_{y_j}p^* + b_1\,p^*\big)\,dy \equiv \bar{b}_1, \tag{2.8}$$

where $p^*(y) = p^*_\theta(y)|_{\theta=0}$. The last expression is the so-called effective axial drift $\bar{b}_1 \in \mathbb{R}$.

In what follows we assume that $\bar{b}_1 > 0$ (which is equivalent to $\Theta > 0$). The case $\bar{b}_1 < 0$ is symmetric and can be considered in the same way.

To avoid excessive technicalities, we first formulate our main result in a loose way.

Theorem 2.1. *Let conditions* (**H1**) – (**H4**) *be fulfilled and* $\bar{b}_1 > 0$ *(see* (2.8)*). Suppose that* $u_0(-1) \neq 0$*. Then there exist constants* a^{eff} *and* M *such that, for* $t > 0$ *and* $x \in G_\varepsilon$*, the asymptotics of the solution* u^ε *of problem* (2.1) *takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2 \, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}} \, e^{\frac{\Theta(x_1+1)}{\varepsilon}} \, p_\Theta\Big(\frac{x}{\varepsilon}\Big) \, \big[u(t,x_1) + r_\varepsilon(t,x)\big],$$

where u *is a solution of the one-dimensional effective problem*

$$\begin{cases} \partial_t u = a^{\text{eff}} \, \partial_{x_1}^2 u, & (t,x_1) \in (0,T) \times (-1,1), \\ u(t,\pm 1) = 0, & t \in (0,T), \\ u(0,x_1) = -M \, u_0(-1) \, \delta'(x_1+1), & x_1 \in (-1,1). \end{cases}$$

Here $r_\varepsilon(t,x)$ *is such that* $|r_\varepsilon(t,\cdot)| \le C\,\varepsilon$ *for* $t \ge t_0 > 0$, $x \in I^+ \times \varepsilon Q$, $I^+ \Subset (-1,1]$, *and the constant* C *depends on* I^+, Λ, Q, d.

A more precise statement of Theorem 2.1 can be found below in Theorems 2.2 and 2.3. The interpretation of Theorem 2.1 is that it is a result of both localization/concentration and homogenization. Indeed, up to a multiplicative constant ε^2, the solution u^ε is asymptotically equal to the product of two exponential terms, a periodically oscillating function $p_\Theta\big(\frac{x}{\varepsilon}\big)$ (which is uniformly positive and bounded) and the homogenized function $u(t,x_1)$ (which is independent of ε). The first exponential term $e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}$ indicates a fast decay in time, uniform in space. The second exponential term, $e^{\frac{\Theta(x_1+1)}{\varepsilon}}$, indicates a localization of the solution in a small neighborhood of the right end of the rod, where the solution attains its maximum; everywhere else in $(-1,1)$ the solution is exponentially smaller. The homogenized solution u depends only on the value of the initial data u_0 at the opposite extremity $x_1 = -1$ and it is proportional to the constant M which depends on some homogenization boundary layers.

The role of boundary layers is thus crucial in the result of Theorem 2.1. Furthermore, if the initial data u_0 had a compact support $[\alpha,\beta] \Subset (-1,1)$ and $u_0(\alpha) \neq 0$, then Theorem 5.2 in [4] gives a similar asymptotic behaviour except for the initial data of the homogenized problem which would be $u(0,x_1) = \tilde{M}\, u_0(\alpha)\, \delta(x_1 - \alpha)$. In other words, the derivative of the Dirac mass would be replaced with the Dirac mass itself.

Remark 2.2. *The error estimate for the remainder term* r_ε *is not precise enough and it shall be improved in Theorem 2.3. Indeed, the homogeneous Dirichlet boundary condition for* $u(t,x_1)$*, together with the exponential* $e^{\frac{\Theta(x_1+1)}{\varepsilon}}$ *shows that* $u^\varepsilon(t,x)$ *attains its maximum at a distance of order* ε *of the end point* $x_1 = 1$*: there, by a Taylor expansion,* $u(t,x_1)$ *is of the order of* ε*, like the remainder term* $r_\varepsilon(t,x)$ *which is thus not negligible. A better ansatz with a better error estimate will be given in Theorem 2.3 (again, boundary layers will be crucial).*

The proof of Theorem 2.1 is performed in several steps. First, we make use of a factorization principle in order to simplify the original problem. Then, we represent the new unknown function in terms of the corresponding Green's function. And, finally, we study the asymptotic behaviour of the mentioned Green's function, as $\varepsilon \to 0$.

2.2 Proof of Theorem 2.1

2.2.1 Factorization

In order to simplify the original problem we perform the change of unknowns, as was suggested in [3], [4], [10], [23].

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, v^\varepsilon(t,x). \tag{2.9}$$

Note that (2.9) is a proper definition of v^ε since p_Θ is a positive function. Substituting (2.9) into (2.1) yields the problem for the new unknown function v^ε

$$\begin{cases} \rho_\Theta\big(\frac{x}{\varepsilon}\big)\,\partial_t v^\varepsilon - \mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v^\varepsilon\big) + \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v^\varepsilon = 0, & \text{in } (0,T)\times G_\varepsilon,\\ a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v^\varepsilon\cdot n = 0, & \text{on } (0,T)\times\Sigma_\varepsilon,\\ v^\varepsilon(t,\pm 1,x') = 0, & x'\in(0,T)\times\varepsilon Q,\\ v^\varepsilon(0,x) = u_0(x_1)\, p_\Theta^{-1}\big(\frac{x}{\varepsilon}\big)\, e^{-\frac{\Theta(x_1+1)}{\varepsilon}}, & x\in G_\varepsilon. \end{cases} \tag{2.10}$$

Here

$$\begin{aligned} &\rho_\Theta(y) = p_\Theta(y)\, p^*_\Theta(y), \quad a^\Theta(y) = p_\Theta(y)\, p^*_\Theta(y)\, a(y),\\ &b^\Theta(y) = p_\Theta(y)\, p^*_\Theta(y)\, b(y) - 2\,\Theta\, p_\Theta(y)\, p^*_\Theta(y)\, a(y) e_1 \\ &+a(y)\,\big[p_\Theta(y)\,\nabla_y p^*_\Theta(y) - p^*_\Theta(y)\,\nabla_y p_\Theta(y)\big], \end{aligned} \tag{2.11}$$

with e_1 the first coordinate vector. For brevity, in what follows we denote

$$\begin{aligned} A^\varepsilon_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) + \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v, \quad B^\varepsilon_\Theta v = a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\cdot n,\\ A_\Theta v = -\mathrm{div}(a^\Theta\nabla v) + b^\Theta\cdot\nabla v, \quad B_\Theta v = a^\Theta\nabla v\cdot n, \end{aligned} \tag{2.12}$$

$$\begin{aligned} A^{*,\varepsilon}_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) - \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v,\\ A^*_\Theta v = -\mathrm{div}(a^\Theta\nabla v) - b^\Theta\cdot\nabla v. \end{aligned} \tag{2.13}$$

Straightforward calculations yield that, for any $\theta\in\mathbb{R}$,

$$\mathrm{div}_y b^\theta(y) = 0 \quad \text{in } Y, \quad b^\theta\cdot n = 0 \quad \text{on } \partial Y. \tag{2.14}$$

Taking into account the fact that Θ is the maximum point of λ_1 and equality (2.7), we obtain that the first component of b^Θ has zero mean:

$$\int_Y b_1^\Theta(y)\, dy = 0. \tag{2.15}$$

Due to (2.14), (2.15), the partial differential equation in (2.10) could be homogenized by standard methods [7], [8] if the initial data were independent of ε. However, the presence of an asymptotically singular initial condition in (2.10) brings some difficulties into the homogenization procedure. In particular, the classical approach of homogenization (based on energy estimates in Sobolev spaces) cannot be applied since the initial data is not uniformly bounded in $L^2(G_\varepsilon)$.

In order to study the asymptotic behaviour of v^ε, following our previous work [4], we use its representation in terms of the corresponding Green's function $K_\varepsilon(t,x,\xi)$

$$v^\varepsilon(t,x) = \int\limits_{G_\varepsilon} K_\varepsilon(t,x,\xi)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi. \tag{2.16}$$

Here K_ε, as a function of t and x, for each $\xi \in G_\varepsilon$, solves the problem

$$\begin{cases} \rho_\Theta^\varepsilon\, \partial_t K_\varepsilon + A_\Theta^\varepsilon K_\varepsilon = 0, & (t,x) \in (0,T) \times G_\varepsilon, \\ B_\Theta^\varepsilon K_\varepsilon = 0, & (t,x) \in (0,T) \times \Sigma_\varepsilon, \\ K_\varepsilon(t,x,\xi)\Big|_{x_1=\pm 1} = 0, & (t,x') \in (0,T) \times \varepsilon Q, \\ K_\varepsilon(0,x,\xi) = \delta(x-\xi), & x \in G_\varepsilon. \end{cases} \tag{2.17}$$

Note that K_ε with respect to (t,ξ) is a solution of the formally adjoint problem, which differs from (2.17) by the sign in front of the first-order terms.

Because of the presence of the delta-function in the initial condition, it is difficult to construct the asymptotics for K_ε directly. Let us introduce a function

$$V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi) - K_\varepsilon(t,x,\xi), \tag{2.18}$$

where Φ_ε stands for the Green function in the infinite cylinder $\mathbb{G}_\varepsilon = \mathbb{R} \times \varepsilon Q$. As a function of t and ξ, it is a solution to the following problem

$$\begin{cases} \rho_\Theta^\varepsilon(\xi)\, \partial_t \Phi_\varepsilon + A_\Theta^{*,\varepsilon} \Phi_\varepsilon = 0, & (t,\xi) \in (0,T) \times \mathbb{G}_\varepsilon, \\ B_\Theta^\varepsilon \Phi_\varepsilon = 0, & (t,\xi) \in (0,T) \times \Gamma_\varepsilon, \\ \Phi_\varepsilon(0,x,\xi) = \delta(x-\xi), & \xi \in \mathbb{G}_\varepsilon. \end{cases} \tag{2.19}$$

By Γ_ε we denote the lateral boundary $\mathbb{R} \times \partial(\varepsilon Q)$ of the cylinder $\mathbb{G}_\varepsilon$. For each $x \in G_\varepsilon$, V_ε as a function of t and ξ solves the problem

$$\begin{cases} \rho_\Theta^\varepsilon(\xi)\, \partial_t V_\varepsilon + A_\Theta^{*,\varepsilon} V_\varepsilon = 0, & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B_\Theta^\varepsilon V_\varepsilon = 0, & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ V_\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm 1} = \Phi_\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm 1}, & (t,\xi) \in (0,T) \times \varepsilon Q, \\ V_\varepsilon(0,x,\xi) = 0, & \xi \in G_\varepsilon. \end{cases} \tag{2.20}$$

In the following subsection we construct an asymptotic expansion for Φ_ε which is a relatively easy task because it is defined in an infinite cylinder (thus not requiring any boundary layers). Subsection 2.2.3 will be devoted to the approximation of V_ε which is delicate because of the necessity of defining boundary layers but still possible since the boundary condition for V_ε is smooth for $x \neq \pm 1$. The final subsection will combine these two results to get an ansatz for K_ε and, using (2.16), to prove Theorem 2.1.

2.2.2 Asymptotics for $\Phi_\varepsilon(t, x, \xi)$

The goal of this section is to compute an asymptotic expansion for the Green function Φ_ε with a bound on the error term (see Lemma 2.3 below). Denote by Φ_0 a fundamental solution of the 1-dimensional homogenized problem

$$\begin{cases} \partial_t \Phi_0 = a^{\text{eff}}\, \partial^2_{\xi_1} \Phi_0(t, x_1, \xi_1), \quad (t, \xi_1) \in (0, T) \times \mathbb{R},\ x_1 \in \mathbb{R}, \\ \Phi_0(0, x_1, \xi_1) = \delta(x_1 - \xi_1), \quad \xi_1, x_1 \in \mathbb{R}. \end{cases} \tag{2.21}$$

Here the effective coefficient a^{eff} is given by one of the two equivalent formulae

$$a^{\text{eff}} = \int\limits_Y (a^\Theta_{11} + a^\Theta_{1j} \partial_{y_j} N - b^\Theta_1\, N)\, dy = \int\limits_Y (a^\Theta_{11} + a^\Theta_{1j} \partial_{y_j} N^* + b^\Theta_1\, N^*)\, dy, \tag{2.22}$$

where the 1-periodic in y_1 functions N and N^* solve the standard cell problems (direct and adjoint, respectively):

$$\begin{cases} A_\Theta N(y) = \partial_{y_j} a^\Theta_{j1}(y) - b^\Theta_1(y), \quad y \in Y, \\ B_\Theta N(y) = 0, \quad y \in \partial Y; \end{cases} \tag{2.23}$$

$$\begin{cases} A^*_\Theta N^*(\eta) = \partial_{\eta_j} a^\Theta_{j1}(\eta) + b^\Theta_1(\eta), \quad \eta \in Y, \\ B_\Theta N^*(\eta) = 0, \quad \eta \in \partial Y. \end{cases} \tag{2.24}$$

Of course, (2.21) is the homogenized problem for (2.19) and it can be shown that $a^{\text{eff}} > 0$. Note that N and N^* are Hölder continuous functions (see [15]). The fundamental solution Φ_0 admits the explicit formula

$$\Phi_0(t, x_1, \xi_1) = \frac{1}{2\sqrt{\pi\, t}}\, \frac{1}{a^{\text{eff}}}\, e^{-\frac{|x_1 - \xi_1|^2}{4 a^{\text{eff}}\, t}}. \tag{2.25}$$

We also introduce the first- and second-order approximations of Φ_ε by

$$\begin{aligned} \Phi^\varepsilon_1(t, x, \xi) \ &= \Phi_0(t, x_1, \xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} \Phi_0(t, x_1, \xi_1) \\ &+ \varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_1} \Phi_0(t, x_1, \xi_1), \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\Phi^\varepsilon_2(t, x, \xi) = \Phi^\varepsilon_1(t, x, \xi) + \varepsilon^2\, N_2\big(\frac{x}{\varepsilon}\big)\, \partial^2_{x_1} \Phi_0(t, x_1, \xi_1) \\ &+ \varepsilon^2\, N^*_2\big(\frac{\xi}{\varepsilon}\big)\, \partial^2_{\xi_1} \Phi_0(t, x_1, \xi_1) + \varepsilon^2\, N\big(\frac{x}{\varepsilon}\big)\, N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{x_1} \partial_{\xi_1} \Phi_0(t, x_1, \xi_1). \end{aligned} \tag{2.27}$$

Our further analysis relies on Aronson type upper bound for Φ_ε. Consider the Green function $\Phi(t, y, \eta)$ of the following initial boundary problem in the infinite rescaled cylinder $\mathbb{G} = \mathbb{R} \times Q$ with lateral boundary Σ:

$$\begin{cases} \rho_\Theta(y)\,\partial_t \Phi + A_\Theta \Phi = 0, & (t,y) \in (0,\infty) \times \mathbb{G}, \\ B_\Theta \Phi = 0, & (t,y) \in (0,\infty) \times \Sigma, \\ \Phi(0,y,\eta) = \delta(y-\eta), & y \in \mathbb{G}. \end{cases} \tag{2.28}$$

Lemma 2.2. *The Green function Φ, solution of* (2.28)*, satisfies the following Aronson type estimate*

$$0 < \Phi(t,y,\eta) \leq C_1 \max\Big(t^{-d/2}, t^{-1/2}\Big) \exp\Big(-c\frac{|y-\eta|^2}{t}\Big). \tag{2.29}$$

with positive constants C_1 and c.

Remark 2.3. *In the right hand side of estimate* (2.29) *the factor $t^{-d/2}$ takes care of the short times (for which there is no difference between the cylinder $\mathbb{G}$ and the full space $\mathbb{R}^d$) while the other factor $t^{-1/2}$ is valid for the longer times (for which the cylinder $\mathbb{G}$ behaves as a 1-d line).*

Proof. We only briefly sketch this proof. The idea is to derive (2.29) from the classical Aronson estimate in $\mathbb{R}^d$ (see [5]) for divergence form operators. Let us check first that the operator A_Θ can be rewritten in divergence form. Since b_Θ is a divergence-free vector field and the average of its first component is zero, there is a skew-symmetric periodic in y_1 matrix $S(y)$ with bounded entries such that $b_\Theta = \mathrm{div} S$ (see e.g. [9]). Then

$$A_\Theta \phi = -\mathrm{div}\big((a^\Theta - S)\nabla\phi\big).$$

Assume for a moment that the cross section Q is the unit cube in $\mathbb{R}^{d-1}$. We duplicate the cube by symmetric reflection of the operator coefficients and the solution $\Phi(t,y,\eta)$ of (2.28) with respect to each direction orthogonal to its faces. The resulting problem is now periodic with period 2 in each coordinate direction. It should be noted that the initial condition on each period is the sum of 2^{d-1} delta functions in y at the point η and its symmetric reflections. We denote these points by $\{\eta_k(\eta)\}_{k=1}^{2^{d-1}}$ with $\eta_1(\eta) = \eta$. Then the solution $\widetilde{\Phi}(t,y,\eta)$ of the introduced above $2Q$-periodic problem coincides with $\Phi(t,y,\eta)$ on Q.

Due to the linearity of the problem

$$\widetilde{\Phi}(t,y,\eta) = \sum_{k=1}^{2^{d-1}} G_\#(t,y,\eta_k(\eta)),$$

where $G_\#(t,y,\eta)$ is the Green function of the corresponding $2Q$-periodic operator. Clearly, $G_\#(t,y,\eta)$ is constructed from the fundamental solution $G(t,y,\eta)$ in the whole space by summing over the square periodic network of period $2Q$. Namely,

$$G_\#(t,y,\eta) = \sum_{n\in\mathbb{Z}^{d-1}} G(t,y,\eta+2n).$$

Making use of the classical Aronson estimate for the fundamental solution $G(t,y,\eta)$ in $\mathbb{R}^d$, we get

$$\begin{aligned} G_{\#}(t,y,\eta) &= \sum_{n\in\mathbb{Z}^{d-1}} G(t,y,\eta+2n) \\ &\le \frac{C}{t^{d/2}} \sum_{n\in\mathbb{Z}^{d-1}} e^{-C_0\frac{|y_1-\eta_1|^2}{t}}\, e^{-C_0\frac{|y'-\eta'-2n|^2}{t}}, \end{aligned} \tag{2.30}$$

for some positive constants C, C_0. For small time the contributions of the distant cells are negligible because of the exponential decay, and the main contribution is given by the term with $n=0$. Consequently, for small time

$$G_{\#}(t,y,\eta) \le \frac{\tilde{C}}{t^{d/2}}\, e^{-\tilde{C}_0\frac{|y-\eta|^2}{t}},$$

with some positive constants $\tilde{C}, \tilde{C}_0$. For large time t all the terms in (2.30) contribute. Indeed, after making the change of variables

$$t = \frac{\tau}{\delta^2},\ y = \frac{\tilde{y}}{\delta},\ \eta = \frac{\tilde{\eta}}{\delta},\ n = \frac{\tilde{n}}{\delta},$$

for small $\delta > 0$, we get

$$\begin{aligned} G_{\#}(\frac{\tau}{\delta^2}, \frac{\tilde{y}}{\delta}, \frac{\tilde{\eta}}{\delta}) &\le \frac{C\,\delta^d}{\tau^{d/2}}\, e^{-C_0\frac{|\tilde{y}_1-\tilde{\eta}_1|^2}{\tau}} \sum_{\tilde{n}\in(\delta\mathbb{Z})^{d-1}} e^{-C_0\frac{|\tilde{y}'-\tilde{\eta}'-\tilde{n}|^2}{\tau}} \\ &\le \frac{C_1\,\delta}{\tau^{d/2}}\, e^{-C_0\frac{|\tilde{y}_1-\tilde{\eta}_1|^2}{\tau}} \int_{\mathbb{R}^{d-1}} e^{-C_0\frac{|\tilde{y}'-\tilde{\eta}'-\tilde{n}|^2}{\tau}}\, d\tilde{n} \le \frac{\tilde{C}_1\,\delta}{\tau^{1/2}}\, e^{-C_0\frac{|\tilde{y}_1-\tilde{\eta}_1|^2}{\tau}}. \end{aligned}$$

Changing back the variables we have

$$G_{\#}(t,y,\eta) \le \frac{C_1}{\sqrt{t}}\, e^{-C_0\frac{|y_1-\eta_1|^2}{t}}$$

for any time t such that $t \ge t_0 > 0$. Thus, estimate (2.29) is satisfied when Q is the unit cube.

Finally, if Q is not a cube, we first map it to the unit cube by a Lipschitz diffeomorphism which preserves the divergence form and elliptic character of the operator with uniformly bounded coefficients. □

Using Lemma 2.2, we can paraphrase the upper bound, announced in [24] (see Chapter II, page 85) and then proved rigorously in [1] (similar results were proved in [6]). The difference is that we address the case of an infinite cylinder instead of the whole space as in these previous references.

Lemma 2.3. *For any $x, \xi \in \mathbb{G}_\varepsilon$ and $t \ge \varepsilon^2$,*

$$|\varepsilon^{d-1}\,\Phi_\varepsilon(t,x,\xi) - \Phi_k^\varepsilon(t,x_1,\xi_1)| \le C\,\frac{\varepsilon^{k+1}}{t^{(k+2)/2}}, \quad k = 0,1,2, \tag{2.31}$$

where $\Phi_0^\varepsilon \equiv \Phi_0$, Φ_1^ε is defined by (2.26) *and Φ_2^ε by* (2.27)*.*

We do not give the details of the proof of Lemma 2.3 which is completely similar to that in [1]. It relies on two arguments. The first one is the Bloch decomposition and m-sectorial property of the decomposition of the operator A_Θ in Y which still holds true in the present case. The second one is the Aronson estimate which is granted by Lemma 2.2. Estimate (2.31) holds true if $|Q|_{d-1} = 1$. Otherwise, the multiplier $|Q|_{d-1}$ appears in front of $\varepsilon^{d-1}\,\Phi_\varepsilon(t,x,\xi)$.

2.2.3 Asymptotics for $V_\varepsilon(t,x,\xi)$

The goal of this section is to construct an asymptotic expansion for the difference V_ε, defined by (2.18), with a bound on the remainder term (see Lemmas 2.4 and 2.5 below). Bearing in mind estimate (2.31), it is $\varepsilon^{d-1}V_\varepsilon$, rather than V_ε, which has a limit. The formal asymptotic expansion for $\varepsilon^{d-1}\,V_\varepsilon$ takes the form (see e.g. [7], [19])

$$
\begin{aligned}
&W_\varepsilon(t,x,\xi) = V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} V_0(t,x_1,\xi_1)\\
&+\varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_1} V_0(t,x_1,\xi_1) + \varepsilon\, V_1(t,x_1,\xi_1) + \varepsilon\, V_{bl}^\varepsilon(t,x,\xi)\\
&+\varepsilon^2\, V_2\big(t,x_1,\xi_1;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big) + \varepsilon^3\, W_{bl}^\varepsilon(t,x,\xi),
\end{aligned}
\tag{2.32}
$$

where V_0, for each x_1, is the solution of the homogenized problem

$$
\begin{cases}
\partial_t V_0 = a^{\rm eff}\,\partial_{\xi_1}^2 V_0, & (t,\xi_1)\in(0,T)\times(-1,1),\\
V_0(t,x_1,\pm 1) = \Phi_0(t,x_1,\pm 1), & t\in(0,T),\\
V_0(0,x_1,\xi_1) = 0, & \xi_1\in(-1,1)
\end{cases}
\tag{2.33}
$$

with the effective coefficient $a^{\rm eff}$ defined by (2.22). Recall that N and N^* are solutions of (2.23) and (2.24), respectively. The other terms in (2.32) are defined as follows.

The function V_2 is defined by

$$
\begin{aligned}
&V_2(t,x_1,\xi_1;y,\eta) = N_2(y)\,\partial_{x_1}^2 V_0(t,x_1,\xi_1)\\
&+N_2^*(\eta)\,\partial_{\xi_1}^2 V_0(t,x_1,\xi_1) + N(y)\,N^*(\eta)\,\partial_{x_1}\partial_{\xi_1} V_0(t,x_1,\xi_1)\\
&+N(y)\,\partial_{x_1} V_1(t,x_1,\xi_1) + N^*(\eta)\,\partial_{\xi_1} V_1(t,x_1,\xi_1)
\end{aligned}
\tag{2.34}
$$

where the functions $N_2(y)$ and $N_2^*(\eta)$ (1-periodic with respect to their first variable) solve the following problems:

$$
\begin{cases}
A_\Theta N_2 = \partial_{y_i}(a_{i1}^\Theta N) + a_{1j}^\Theta \partial_{y_j} N + a_{11}^\Theta - b_1^\Theta\, N - a^{\rm eff}\,\rho_\Theta, & \text{in } Y,\\
B_\Theta N_2 = -a_{i1}^\Theta\, n_i\, N, & \text{on } \partial Y,
\end{cases}
\tag{2.35}
$$

and

$$
\begin{cases}
A_\Theta^* N_2^* = \partial_{\eta_i}(a_{i1}^\Theta N^*) + a_{1j}^\Theta \partial_{\eta_j} N^* + a_{11}^\Theta + b_1^\Theta\, N^* - a^{\rm eff}\,\rho_\Theta, & \text{in } Y,\\
B_\Theta N_2^* = -a_{i1}^\Theta\, n_i\, N^*, & \text{on } \partial Y.
\end{cases}
$$

In order to define V_1 and the boundary layer corrector V_{bl}^ε in (2.32), we introduce two functions $v^\pm$ defined in semi-infinite cylinders, v^- in $\mathbb{G}^+ = (0,+\infty)\times Q$ and v^+ in $\mathbb{G}^- = (-\infty,0)\times Q$:

$$
\begin{cases}
A_\Theta^* v^\pm(\eta) = 0, & \eta\in\mathbb{G}^\mp,\\
B_\Theta v^\pm(\eta) = 0, & \eta\in\Sigma^\mp,\\
v^\pm(0,\eta') = -N^*(0,\eta'),
\end{cases}
\tag{2.36}
$$

where $\Sigma^\pm$ are the lateral boundaries of $\mathbb{G}^\pm$. It has been proved in [20] that bounded solutions $v^\pm$ exist, are uniquely defined and stabilize to some constants $\hat{v}^\pm$ at an exponential rate, as $\eta_1 \to \pm\infty$:

$$\begin{aligned}
&|v^\pm(\eta_1, \eta') - \hat{v}^\pm| \le C_0\, e^{-\gamma|\eta_1|}, \quad C_0, \gamma > 0; \\
&\|\nabla v^-\|_{L^2((n,n+1)\times Q)} \le C\, e^{-\gamma n}, \quad \forall n > 0, \\
&\|\nabla v^+\|_{L^2((-(n+1),-n)\times Q)} \le C\, e^{-\gamma n}, \quad \forall n > 0.
\end{aligned} \tag{2.37}$$

Then the first boundary layer corrector is given by

$$\begin{aligned}
V_{bl}^\varepsilon(t,x,\xi) \;&= \Big[v^-\big(\frac{\xi_1+1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big) - \hat{v}^-\Big]\, \partial_{\xi_1}(V_0 - \Phi_0)(t, x_1, \xi_1 = -1) \\
&+ \Big[v^+\big(\frac{\xi_1-1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big) - \hat{v}^+\Big]\, \partial_{\xi_1}(V_0 - \Phi_0)(t, x_1, \xi_1 = 1),
\end{aligned} \tag{2.38}$$

and V_1, for $x_1 \in (-1,1)$, satisfies the problem

$$\begin{cases}
\partial_t V_1 = a^{\rm eff}\, \partial^2_{\xi_1} V_1 + F(t, x_1, \xi_1), \quad (t,\xi_1) \in (0,T) \times (-1,1), \\
V_1(t, x_1, \pm 1) = \hat{v}^\pm\, \partial_{\xi_1}\, (V_0 - \Phi_0)\Big|_{\xi_1 = \pm 1}, \qquad t \in (0,T), \\
V_1(0, x_1, \xi_1) = 0, \qquad \xi_1 \in (-1,1),
\end{cases} \tag{2.39}$$

where

$$\begin{aligned}
F(t,x_1,\xi_1) = \partial^3_{\xi_1} V_0(t,x_1,\xi_1) \int\limits_Y \Big(a^\Theta_{1j}(\eta)\partial_{\eta_j} N_2^*(\eta) \\
+\, a^\Theta_{11}(\eta) N^*(\eta) + b^\Theta_1(\eta) N^*_2(\eta) - a^{\rm eff}\, \rho_\Theta(\eta)\, N^*(\eta)\Big)\, d\eta.
\end{aligned} \tag{2.40}$$

Finally, the second boundary layer corrector W_{bl}^ε is designed to compensate the time derivative of V_{bl}^ε and is defined by

$$\begin{aligned}
W_{bl}^\varepsilon(t,x,\xi) \;&= \Big[w^-\big(\frac{\xi_1+1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big) - \hat{w}^-\Big]\, \partial_t\partial_{\xi_1}(V_0 - \Phi_0)(t, x_1, \xi_1 = -1) \\
&+ \Big[w^+\big(\frac{\xi_1-1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big) - \hat{w}^+\Big]\, \partial_t\partial_{\xi_1}(V_0 - \Phi_0)(t, x_1, \xi_1 = 1).
\end{aligned}$$

The functions $w^\pm$ solve nonhomogeneous problems

$$\begin{cases}
A^*_\Theta w^\pm(\eta) = (\hat{v}^\pm - v^\pm(\eta))\, \rho_\Theta(\eta), \quad \eta \in \mathbb{G}^\mp, \\
B_\Theta w^\pm(\eta) = 0, \qquad \eta \in \Sigma^\mp, \\
w^\pm(0, \eta') = 0.
\end{cases}$$

Bounded solutions $w^\pm$ exist, are uniquely defined and stabilize to some constants $\hat{w}^\pm$ at an exponential rate, as $\eta_1 \to \pm\infty$ (see [20]).

Using the standard elliptic estimates one can easily show that, for $x_1 \ne \pm 1$, the function V_0 belongs to $C^\infty([0,T] \times (-1,1) \times [-1,1])$, and for $t \in [0,T]$, $x_1 \in I \Subset (-1,1)$, $\xi_1 \in [-1,1]$, we have

$$|\partial^k_t\, \partial^l_{x_1}\, \partial^m_{\xi_1} V_0(t,x_1,\xi_1)| \le \frac{C}{\min\{|x_1 - 1|, |x_1 + 1|\}^{2k+l+m+1}}. \tag{2.41}$$

Then V_1 is also a smooth function of its variables for $x \in I \Subset (-1,1)$. Notice finally that N_2 and N_2^* are Hölder continuous. Indeed, it is straightforward to check that the equation and the boundary conditions in (2.35) can be rewritten in the form

$$\begin{cases} A_\Theta(N_2 + y_1 N + \frac{1}{2}y_1^2) = -a^{\text{eff}}\rho_\Theta, & y \in Y, \\ B_\Theta(N_2 + y_1 N + \frac{1}{2}y_1^2) = 0, & y \in \partial Y. \end{cases}$$

Since $a^{\text{eff}}\rho_\Theta \in L^\infty(Y)$, then it is known that the corresponding solution is Hölder continuous (see [15]). The Hölder continuity of N_2^* can be justified in a similar way.

We denote by V_1^ε the first-order approximation of $\varepsilon^{d-1}V_\varepsilon$

$$\begin{aligned} &V_1^\varepsilon(t,x,\xi) = V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}V_0(t,x_1,\xi_1) \\ &+\varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_1}V_0(t,x_1,\xi_1) + \varepsilon\, V_1(t,x_1,\xi_1) + \varepsilon\, V_{bl}^\varepsilon(t,x,\xi). \end{aligned} \tag{2.42}$$

By construction, its trace at the cylinder ends coincide with that of Φ_1^ε, namely

$$\Big\{V_1^\varepsilon(t,x,\xi)\Big\}\Big|_{\xi_1=\pm 1} = \Phi_1^\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm 1},$$

where Φ_1^ε is defined by (2.26). Of course, V_1^ε is also the first-order approximation of W_ε, defined by (2.32). It turns out that all terms in V_1^ε will contribute to the leading term of the asymptotics of $\varepsilon^{d-1}V_\varepsilon$, while the other terms, V_2 and W_{bl}^ε, in (2.32) are constructed in order to guarantee the required accuracy.

Lemma 2.4. *Let V_ε be defined by* (2.18)*, or equivalently be a solution of* (2.20)*. Let V_1^ε be defined by* (2.42)*. Then, there exists a constant C, depending on I, Λ, Q, d and independent of ε, such that, for $x \in I \times \varepsilon Q$ and $t \geq 0$, $I \Subset (-1,1)$,*

$$\int_{G_\varepsilon} |\varepsilon^{d-1}V_\varepsilon - V_1^\varepsilon|^2\, d\xi \leq C\,\varepsilon^4\,\varepsilon^{d-1}. \tag{2.43}$$

Proof. The strategy of the proof is the following: we plug the difference $(W_\varepsilon - \varepsilon^{d-1}V_\varepsilon)$ into the boundary value problem (2.20) and calculate the right hand sides in the equation and in the boundary condition. The terms of the asymptotic expansion W_ε were designed in a such a way that these right-hand sides are small. Thus, by a priori estimates, the difference $W_\varepsilon - \varepsilon^{d-1}V_\varepsilon$ is small in a appropriate norm. For the sake of clarity, we divide the proof in several steps.

Step 1. We first prove a priori estimates for the following problem:

$$\begin{cases} \rho_\Theta^\varepsilon\partial_t w^\varepsilon + A_\Theta^{*,\varepsilon}\, w^\varepsilon = f(t,x) + \text{div}F(t,x), & \text{in } (0,T)\times G_\varepsilon, \\ B_\Theta^\varepsilon\, w^\varepsilon = \varepsilon g(t,x) - F\cdot n, & \text{on } (0,T)\times \Sigma_\varepsilon, \\ w^\varepsilon(t,\pm 1,x') = 0, & (t,x') \in (0,T)\times Q, \\ w^\varepsilon(0,x) = 0, & x \in G_\varepsilon. \end{cases} \tag{2.44}$$

Since by (2.14) $\operatorname{div} b_\Theta^\varepsilon = 0$ and $b_\Theta^\varepsilon \cdot n = 0$ on the lateral boundary, a priori estimates are obtained in a standard way. Multiplying the equation in (2.44) by w^ε and integrating by parts and exploiting the Cauchy-Bunyakovsky inequality and Grönwall's lemma, we obtain for any $t \le T$

$$\begin{aligned} &\int_{G_\varepsilon} |w^\varepsilon(t)|^2\,dx + \int_0^t \int_{G_\varepsilon} |\nabla w^\varepsilon|^2\,dx\,d\tau \\ &\le C\,e^{C_1 t}\left(\|f\|^2_{L^2((0,T)\times G_\varepsilon)} + \varepsilon^2 \|g\|^2_{L^2((0,T)\times \Sigma_\varepsilon)} + \|F\|^2_{L^2((0,T)\times G_\varepsilon)}\right), \end{aligned} \tag{2.45}$$

where the constants C, C_1 are independent of ε and t.

Step 2. To estimate the $L^2(G_\varepsilon)$ norm of $W_\varepsilon - \varepsilon^{d-1}V_\varepsilon$, we first substitute $W_\varepsilon - \varepsilon^{d-1}V_\varepsilon$ for w^ε in (2.44). This yields

$$\begin{aligned} &\rho_\Theta^\varepsilon\,\partial_t(W_\varepsilon - \varepsilon^{d-1}V_\varepsilon) + A_\Theta^{*\varepsilon}(W_\varepsilon - \varepsilon^{d-1}V_\varepsilon) \\ &\quad = \varepsilon\,R_1(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) + \varepsilon \partial_{\eta_i} \tilde{R}_{1,i}(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) \\ &\quad +\varepsilon^2\,R_2(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) + \varepsilon^3\,R_3^\varepsilon(t, x_1, \xi), \\ &B_\Theta^\varepsilon(W_\varepsilon - \varepsilon^{d-1}V_\varepsilon) = \varepsilon^2\,n_i\,\tilde{R}_{1,i}(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon), \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} &R_1(t, x_1, \xi_1; y, \eta) = \rho_\Theta(\eta) N(y) \partial_t \partial_{x_1} V_0(t, x_1, \xi_1) \\ &+\rho_\Theta(\eta) N^*(\eta) \partial_t \partial_{\xi_1} V_0 + \rho_\Theta(\eta) \partial_t V_1 - a_{11}^\Theta(\eta) N(y) \partial_{\xi_1}^2 \partial_{x_1} V_0(t, x_1, \xi_1) \\ &-a_{11}^\Theta(\eta) N^*(\eta) \partial_{\xi_1}^3 V_0(t, x_1, \xi_1) - a_{11}^\Theta(\eta) \partial_{\xi_1}^2 V_1(t, x_1, \xi_1) \\ &-a_{1j}^\Theta(\eta) \partial_{\xi_1} \partial_{\eta_j} V_2(t, x_1, \xi_1; y, \eta) - b_1^\Theta(\eta) \partial_{\xi_1} V_2(t, x_1, \xi_1; y, \eta), \end{aligned}$$

and

$$\begin{aligned} &\tilde{R}_{1,i}(t, x_1, \xi_1; y, \eta) = a_{i1}^\Theta(\eta) \partial_{\xi_1} V_2(t, x_1, \xi_1; y, \eta), \\ &R_2(t, x_1, \xi_1; y, \eta) = \rho_\Theta\,\partial_t V_2(t, x_1, \xi_1; y, \eta) - a_{11}^\Theta(\eta) \partial_{\xi_1}^2 V_2(t, x_1, \xi_1; y, \eta), \\ &R_3^\varepsilon(t, x_1, \xi) = \rho_\Theta^\varepsilon\,\partial_t W_{bl}^\varepsilon(t, x, \xi). \end{aligned}$$

All cancellations on the right hand side of (2.46) are classical (see e.g. [7]) except for the one due to the additional boundary layer corrector term $\varepsilon^3 W_{bl}^\varepsilon$ in the ansatz (2.32) for W_ε. Indeed, the coefficient ε^3 in front of W_{bl}^ε allows us to cancel the time derivative of V_{bl}^ε. By construction

$$\partial_t V_{bl}^\varepsilon(t, x, \xi) = -\varepsilon^2\,A_\Theta^{*,\varepsilon} W_{bl}^\varepsilon(t, x, \xi)$$

and

$$(\rho_\Theta^\varepsilon\,\partial_t + A_\Theta^{*\varepsilon})\,(\varepsilon\,V_{bl}^\varepsilon(t, x, \xi) + \varepsilon^3\,W_{bl}^\varepsilon(t, x, \xi)) = \varepsilon^3\,\rho_\Theta^\varepsilon\,\partial_t W_{bl}^\varepsilon(t, x, \xi).$$

By linearity, we have $W_\varepsilon - \varepsilon^{d-1}V_\varepsilon = \tilde{V_1^\varepsilon} + \tilde{V_2^\varepsilon}$, where $\tilde{V_1^\varepsilon}$ and $\tilde{V_2^\varepsilon}$, for each $x \in G_\varepsilon$, solve the following problems:

$$\begin{cases} \rho^\varepsilon_\Theta\, \partial_t \tilde{V_1^\varepsilon} + A^{*\varepsilon}_\Theta \tilde{V_1^\varepsilon} = \varepsilon\, R_1(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) + \varepsilon\partial_{\eta_i} \tilde{R}_{1,i}(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) + \\ +\varepsilon^2\, R_2(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon) + \varepsilon^3\, R_3^\varepsilon(t, x_1, \xi), & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B^\varepsilon_\Theta \tilde{V_1^\varepsilon} = \varepsilon^2\, n_i\, \tilde{R}_{1,i}(t, x_1, \xi_1; x/\varepsilon, \xi/\varepsilon), & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ \tilde{V_1^\varepsilon}(t, x, \xi)\Big|_{\xi_1 = \pm 1} = 0, & t \in (0,T) \\ \tilde{V_1^\varepsilon}(0, x, \xi) = 0, & \xi \in G_\varepsilon; \end{cases}$$

$$\begin{cases} \rho^\varepsilon_\Theta\, \partial_t \tilde{V_2^\varepsilon} + A^{*\varepsilon}_\Theta \tilde{V_2^\varepsilon} = 0, & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B^\varepsilon_\Theta \tilde{V_2^\varepsilon} = 0, & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ \tilde{V_2^\varepsilon}(t, x, \xi)\Big|_{\xi_1 = \pm 1} = (W_\varepsilon - \varepsilon^{d-1}\Phi_\varepsilon)(t, x, \xi)\Big|_{\xi_1 = \pm 1}, & t \in (0,T) \\ \tilde{V_2^\varepsilon}(0, x, \xi) = 0, & \xi \in G_\varepsilon. \end{cases}$$

Step 3. We estimate $\tilde{V_1^\varepsilon}$ using the a priori estimates (2.45) obtained in the first step. To this end, we notice that, in view of (2.33) and (2.39),

$$\int_Y R_1(t, x_1, \xi_1; y, \eta)\, d\eta = 0.$$

Thus, there exists a 1-periodic with respect to η_1 vector-function $\chi = \chi(t, x_1, \xi_1; y, \eta)$ such that

$$\begin{cases} -\mathrm{div}_\eta \chi = R_1 & \eta \in Y, \\ \chi \cdot n = 0, & \eta \in \partial Y. \end{cases}$$

Obviously,

$$R_1(t, x_1, \xi_1; y, \eta)\Big|_{\eta = \xi/\varepsilon} = -\varepsilon \mathrm{div}_\xi \left((\chi(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon}) \right) + \varepsilon \partial_{\xi_1} \chi_1(t, x_1, \xi_1; y, \eta)\Big|_{\eta = \xi/\varepsilon},$$

and

$$\partial_{\eta_i} \tilde{R}_{1,i}(t, x_1, \xi_1; y, \eta)\Big|_{\eta = \xi/\varepsilon} = \varepsilon \partial_{\xi_i} \left(\tilde{R}_{1,i}(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon}) \right) - \varepsilon \partial_{\xi_i} \tilde{R}_{1,i}(t, x_1, \xi_1; y, \eta)\Big|_{\eta = \xi/\varepsilon}.$$

Considering (2.34) and (2.41), we see that

$$\int_{G_\varepsilon} |\varepsilon^2 R_2(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon}) + \varepsilon^3\, R_3^\varepsilon(t, x_1, \xi)|^2\, d\xi \le C\, \varepsilon^4\, \varepsilon^{d-1}, \;\; x \in I \times \varepsilon Q. \qquad (2.47)$$

With the help of (2.45) the above relations yield, for $x \in I \times \varepsilon Q$,

$$\int_{G_\varepsilon} |\tilde{V_1^\varepsilon}(t, x, \xi)|^2\, d\xi \le C\, \varepsilon^4\, \varepsilon^{d-1}, \quad t \ge 0, \qquad (2.48)$$

with the constant C depending on I, Λ, Q, d only.

Step 4. We proceed to the estimate of $\tilde{V_2^\varepsilon}$. Due to the presence of the boundary layer corrector V_{bl}^ε, some cancellations occur and the axial boundary conditions read

$$W_\varepsilon(t,x,\xi',\pm 1) - \varepsilon^{d-1}V_\varepsilon(t,x,\xi',\pm 1) = W_\varepsilon(t,x,\xi',\pm 1) - \varepsilon^{d-1}\Phi_\varepsilon(t,x,\xi',\pm 1)$$
$$= \Big(\varepsilon^2 V_2(t,x_1,\xi_1;y,\frac{\xi}{\varepsilon}) + \varepsilon^3 W_{bl}^\varepsilon(t,x,\xi)\Big) + \big(\Phi_1^\varepsilon(t,x,\xi) - \varepsilon^{d-1}\Phi_\varepsilon(t,x,\xi)\big).$$

Taking into account (2.41) and the fact that N, N^*, N_2, N_2^* are Hölder continuous functions, we see that

$$\Big|\varepsilon^2 V_2(t,x_1,\xi_1;y,\frac{\xi}{\varepsilon}) + \varepsilon^3 W_{bl}^\varepsilon(t,x,\xi)\Big| \le C\,\varepsilon^2, \quad t\ge 0,\ \xi\in G_\varepsilon,\ x\in I\times\varepsilon Q, \tag{2.49}$$

where C depends on I,Λ,Q,d only.

To estimate the other term $(\Phi_1^\varepsilon - \varepsilon^{d-1}\Phi_\varepsilon)$ we consider separately small times $t\le\varepsilon^\beta$, $\beta\in(0,2)$, and larger times $t>\varepsilon^\beta$. For $t\le\varepsilon^\beta$ we have

$$|\Phi_1^\varepsilon - \varepsilon^{d-1}\Phi_\varepsilon| \le \Phi_1^\varepsilon + \varepsilon^{d-1}\Phi_\varepsilon.$$

The first term on the right-hand side here is small by its very definition (2.26) while we use Aronson's estimates (see Lemma 2.2) for the second one. Namely, thanks to (2.14)-(2.15), for $x\in I\times\varepsilon Q$ and $t\le\varepsilon^\beta$

$$|\Phi_\varepsilon(t,x,\pm 1,\xi')| \le O(e^{-C/\varepsilon^\beta})$$

with some positive constant C.

For large time $t\ge\varepsilon^\beta$, we use Lemma 2.3. Namely, for $x,\xi\in G_\varepsilon$, the following estimate holds true:

$$|\varepsilon^{d-1}\Phi_\varepsilon(t,x,\xi) - \Phi_2^\varepsilon(t,x,\xi)| \le C\,\varepsilon^{3-3\beta/2}, \quad \forall\beta>0,$$

with the constant C independent of ε. On the other hand, in view of (2.25), for any $t\ge 0$,

$$|\Phi_2^\varepsilon(t,x,\pm 1,\xi') - \Phi_1^\varepsilon(t,x,\pm 1,\xi')| \le C\,\varepsilon^2, \quad \xi'\in\varepsilon Q,\ x\in I\times\varepsilon Q,$$

with some constant $C = C(I,\Lambda,Q,d)$. Finally, choosing small enough β, we obtain that, for any $t\ge 0$,

$$|\varepsilon^{d-1}\Phi_\varepsilon(t,x,\pm 1,\xi') - \Phi_1^\varepsilon(t,x,\pm 1,\xi')| \le C\,\varepsilon^2, \quad \xi'\in\varepsilon Q,\ x\in I\times\varepsilon Q,$$

where C depends on I,Λ,Q,d only.

Combining the last estimate with (2.49), we obtain that the boundary conditions on the bases of the rod are satisfied up to the second order in ε:

$$|W_\varepsilon(t,x,\pm 1,\xi') - \varepsilon^{d-1}\Phi_\varepsilon(t,x,\pm 1,\xi')| \le C\,\varepsilon^2, \quad t\ge 0,\ x\in I\times\varepsilon Q \tag{2.50}$$

where C depends on I,Λ,Q,d. Thus, by the maximum principle, for $x\in I\times\varepsilon Q$,

$$|\tilde{V_2^\varepsilon}(t,x,\xi)| \le C\,\varepsilon^2, \quad t\ge 0,\ \xi\in G_\varepsilon, \tag{2.51}$$

where C depends on I, Λ, Q, d.

Step 5. Recalling that $W_\varepsilon - \varepsilon^{d-1} V_\varepsilon = \tilde{V_1^\varepsilon} + \tilde{V_2^\varepsilon}$, by summing (2.48) and (2.51), for any $t \in [0, T]$, we obtain

$$\int\limits_{G_\varepsilon} |\varepsilon^{d-1} V_\varepsilon - W_\varepsilon|^2 \, dx \leq C \, \varepsilon^4 \, \varepsilon^{d-1}, \quad x \in I \times \varepsilon Q.$$

It is easy to see that for $x \in I \times \varepsilon Q$, $t \geq 0$,

$$\int\limits_{G_\varepsilon} \Big| V_2\Big(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon}\Big) \Big|^2 \, d\xi + \int\limits_{G_\varepsilon} |W_{bl}^\varepsilon(t, x, \xi)|^2 \, d\xi \leq C \, \varepsilon^{d-1}.$$

Consequently, last two estimates yield (2.43). Lemma 2.4 is proved. □

Lemma 2.4 provides an L^2 estimate for the discrepancy. By working harder we can get an L^∞ estimate of the same order. Namely, we prove the following result.

Lemma 2.5. *Let V_ε be a solution of* (2.20) *and V_1^ε be defined by (2.42) as a first-order approximation of $\varepsilon^{d-1} V_\varepsilon$. Then, for $t \geq 0$, $x \in I^+ \times \varepsilon Q$ and $\xi \in I^- \times \varepsilon Q$, the following estimate is valid:*

$$|\varepsilon^{d-1} V_\varepsilon(t, x, \xi) - V_1^\varepsilon(t, x, \xi)| \leq C \, \varepsilon^2 \tag{2.52}$$

where $I^+ \Subset (-1, 1]$, $I^- \Subset [-1, 1)$; the constant C depends on I^+, I^-, Λ, Q, d and is independent of ε.

Remark 2.4. *The same estimate holds if $\xi \in I^+ \times \varepsilon Q$ and $x \in I^- \times \varepsilon Q$.*

Proof. Estimate in Lemma 2.4 is based on two auxiliary bounds, (2.48) and (2.51). Notice that estimate (2.51) gives a bound in L^∞ norm and, thus, need not be improved. Our goad is to modify the ansatz W^ε in order to obtain a greater power of ε on the right-hand side of (2.48). This will allow us to use L^∞ elliptic estimates.

Observe that adding interior higher order terms to the asymptotic expansion (2.32) (without adding additional boundary layer correctors) increases the power of ε in estimate (2.48). More precisely, denote by $W_k^\varepsilon(t, x, \xi)$ the k-order approximation for $\varepsilon^{d-1} V_\varepsilon$

$$W_k^\varepsilon(t, x, \xi) = W_\varepsilon(t, x, \xi) + \sum_{n=3}^{k} \varepsilon^n \, V_n(t, x_1, \xi_1; y, \eta)\Big|_{y = \frac{x}{\varepsilon}, \eta = \frac{\xi}{\varepsilon}},$$

where $V_n(t, x_1, \xi_1; y, \eta)$ are 1-periodic with respect to y_1, η_1. For the sake of brevity, we do not specify the form of functions V_n (for precise formulae see [7], [19]). Let us substitute $W_k^\varepsilon - \varepsilon^{d-1} V_\varepsilon$ into (2.20) and then, represent $W_k^\varepsilon - \varepsilon^{d-1} V_\varepsilon$ as a sum $\tilde{W}_1^\varepsilon + \tilde{W}_2^\varepsilon$, where $\tilde{W}_1^\varepsilon$ solves nonhomogeneous problem with homogeneous Dirichlet boundary conditions at the rod ends (compare with $\tilde{V}_1^\varepsilon$), and $\tilde{W}_2^\varepsilon$ is a solution of a homogeneous problem with nonhomogeneous Dirichlet boundary conditions at $\xi_1 = \pm 1$ (compare with $\tilde{V}_2^\varepsilon$). Arguing exactly like in Lemma 2.4, we see that

$$\int\limits_{G_\varepsilon} |\tilde{W}_1^\varepsilon|^2 \, d\xi \le C_1 \, \varepsilon^{2\,k} \, \varepsilon^{d-1}, \ t \ge 0, \ x \in I \times \varepsilon Q, \tag{2.53}$$

where $I \Subset (-1, 1)$; and by the maximum principle,

$$|\tilde{W}_2^\varepsilon(t, x, \xi)| \le C_2 \, \varepsilon^2, \ t \ge 0, x \in I \times \varepsilon Q, \ \xi \in G_\varepsilon,$$

where C_1, C_2 depend on I, Λ, Q, d.

Notice that V_ε is Hölder continuous, and by the Nash–De Giorgi estimates in the rescaled cylinder, for $\xi, \zeta \in G_\varepsilon$

$$|V_\varepsilon(t, x, \xi) - V_\varepsilon(t, x, \zeta)| \le C \, \varepsilon^{-\alpha} |\xi - \zeta|^\alpha, \ t \ge 0, \ x \in I \times \varepsilon Q, \tag{2.54}$$

where C, α depend on Λ, Q, d and are independent of ε. Indeed, let us change the variables $\tau = t/\varepsilon^2, y = x/\varepsilon, \eta = \xi/\varepsilon$ in (2.20) and denote $\tilde{V}_\varepsilon(\tau, y, \eta) = V_\varepsilon(\varepsilon^2 \tau, \varepsilon y, \varepsilon \eta)$. By the maximum principle,

$$|\tilde{V}_\varepsilon(\tau, y, \eta)| \le C \quad \tau \ge 0, \ \eta \in (-\varepsilon^{-1}, \varepsilon^{-1}) \times Q, \ y \in \varepsilon^{-1} I \times Q,$$

where $I \Subset (-1, 1)$. Due to the local Nash–De Giorgi estimates, for any $n \in \mathbb{Z}$, $\tau \ge 0$, $y \in \varepsilon^{-1} I \times Q$

$$|\tilde{V}_\varepsilon(\tau, y, \eta) - \tilde{V}_\varepsilon(\tau, y, \vartheta)| \le C \, |\eta - \vartheta|^\alpha, \ \eta, \vartheta \in (n, n+1) \times Q,$$

for some $0 < \alpha < 1$ and C depending on Λ, Q, d. Changing back the variables in the last inequality yields (2.54).

Due to the Hölder continuity properties of N, N^*, N_2, N_2^*, regularity of V_0, the function W_ε is uniformly w.r.t. ε Hölder continuous. Indeed, for example, since N^* is Hölder continuous, so is $N^*(\xi/\varepsilon)$ and

$$|N^*\big(\frac{\xi}{\varepsilon}\big) - N^*\big(\frac{\zeta}{\varepsilon}\big)| \le C \, \varepsilon^{-\alpha} \, |\xi - \zeta|^\alpha, \quad \xi_1, \xi_2 \in G_\varepsilon, \ 0 < \alpha < 1.$$

Thus, $\varepsilon N^*(\xi/\varepsilon) \partial_{\xi_1} V_0(t, x_1, \xi_1)$ is Hölder continuous uniformly with respect to ε.

By similar arguments, W_k^ε and $\tilde{W}_2^\varepsilon$ are Hölder continuous functions, so is $\tilde{W}_1^\varepsilon$. By contradiction one can prove that, if (2.53) holds, then for some $\delta \in (0, 1)$

$$|\tilde{W}_1^\varepsilon(t, x, \xi)| \le C \, \varepsilon^{\delta\,(k-\alpha)},$$

where δ depends on Λ, Q, d. Thus, for sufficiently large k,

$$|\varepsilon^{d-1} V_\varepsilon(t, x, \xi) - W_k^\varepsilon(t, x, \xi)| \le C_3 \, \varepsilon^2, \quad t \ge 0, \ \xi \in G_\varepsilon, \ x \in I \times \varepsilon Q,$$

where C_3 depends on I, Λ, Q, d and is independent of ε. Clearly, by regularity of V_0

$$|W_k^\varepsilon(t, x, \xi) - W_\varepsilon(t, x, \xi)| \le C_4 \, \varepsilon^2, \ \xi \in G_\varepsilon, \ x \in I \times \varepsilon Q,$$

with $C_4 = C_4(I, \Lambda, d, Q)$.

Combining the two last estimates implies a similar bound for $(\varepsilon^{d-1}V_\varepsilon - W_\varepsilon)$ with the constant C that depends on I, Λ, Q, d only. Eventually, using (2.49) which proves that $(W_\varepsilon - V_1^\varepsilon)$ is of order ε^2 we obtain (2.52), at least for x_1 in a compact subset of $(-1, 1)$.

Now we extend this estimate to point x and ξ such that for $x \in I^+ \times \varepsilon Q$ and $\xi \in I^- \times \varepsilon Q$ (or $\xi \in I^+ \times \varepsilon Q$ and $x \in I^- \times \varepsilon Q$). To this end, considering $V_\varepsilon(t, x, \xi)$ as a solution of the equation in (t, x) (for fixed ξ), we get a "symmetric" estimate

$$|W_\varepsilon(t, x, \xi) - \varepsilon^{d-1}V_\varepsilon(t, x, \xi)| \le C_5\, \varepsilon^2, \quad t \ge 0,\ x \in G_\varepsilon,\ \xi \in I \times \varepsilon Q,$$

with the constant C_5 depending on I, Λ, Q, d. In particular,

$$|W_\varepsilon(t, x, \xi) - \varepsilon^{d-1}V_\varepsilon(t, x, \xi)|\Big|_{\xi_1=0} \le C\, \varepsilon^2, \quad t \ge 0,\ x \in G_\varepsilon \tag{2.55}$$

with the constant C independent of t, x, ξ, ε. Considering $W_\varepsilon(t, x, \xi) - \varepsilon^{d-1}V_\varepsilon(t, x, \xi)$ as a solution (w.r.t. t, ξ, for fixed x) of a nonhomogeneous initial boundary problem stated first in $I^- \times \varepsilon Q$ and then in $I^+ \times \varepsilon Q$, using estimate (2.55) and arguing as above we obtain, for $x \in I^+ \times \varepsilon Q$ and $\xi \in I^- \times \varepsilon Q$ (or $\xi \in I^+ \times \varepsilon Q$ and $x \in I^- \times \varepsilon Q$),

$$|\varepsilon^{d-1}V_\varepsilon(t, x, \xi) - V_1^\varepsilon(t, x, \xi)| \le C\, \varepsilon^2, \quad t \ge 0,$$

with the constant C depending on I^-, I^+, Λ, d, Q and independent of ε. □

2.2.4 Asymptotics for v^ε and main results

Recalling from (2.18) that $K_\varepsilon = \Phi_\varepsilon - V_\varepsilon$ and using the first order approximations (2.26) and (2.42) obtained in the previous sections, we define a first order approximation of the Green function K_ε

$$\begin{aligned} K_1^\varepsilon(t, x, \xi) &= \Phi_1^\varepsilon(t, x, \xi) - V_1^\varepsilon(t, x, \xi) \\ &= K_0(t, x_1, \xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} K_0(t, x_1, \xi_1) \\ &+ \varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_1} K_0(t, x_1, \xi_1) + \varepsilon\, K_1(t, x_1, \xi_1) - \varepsilon\, V_{bl}^\varepsilon(t, x, \xi), \end{aligned} \tag{2.56}$$

where $K_0 = \Phi_0 - V_0$ is the Green function of the one-dimensional effective problem

$$\begin{cases} \partial_t K_0 = a^{\text{eff}}\, \partial_{\xi_1}^2 K_0, & (t, \xi_1) \in (0, T) \times (-1, 1), \\ K_0(t, x_1, \pm 1) = 0, & t \in (0, T), \\ K_0(0, x_1, \xi_1) = \delta(x_1 - \xi_1), & \xi_1 \in (-1, 1), \end{cases} \tag{2.57}$$

$K_1 = -V_1$ with V_1, the solution of (2.39), and the boundary layer corrector V_{bl}^ε is defined by (2.36) and (2.38). By combining Lemmata 2.3 and 2.5, we immediately obtain the following statement.

Lemma 2.6. *Denote by I^+, I^- compact subsets of $(-1,1]$ and $[-1,1)$, respectively. Let conditions* (**H1**) – (**H4**) *be fulfilled. Then, for each $x \in I^+ \times \varepsilon Q$, $\xi \in I^- \times \varepsilon Q$, and $t \geq t_0 > 0$, there exists a constant C depending on I^+, I^-, Λ, Q, d and independent of ε such that*

$$|\varepsilon^{d-1} K_\varepsilon(t,x,\xi) - K_1^\varepsilon(t,x,\xi)| \leq C\,\varepsilon^2. \tag{2.58}$$

We can now state our main result.

Theorem 2.2. *Let conditions* (**H1**) – (**H4**) *be fulfilled and $\bar{b}_1 > 0$. Let Θ be the maximum point of $\lambda_1(\theta)$ and p_Θ the corresponding eigenfunction defined by Lemma 2.1.*

1. *Suppose $u_0 \in C^1[-1,1]$ is such that $u_0(-1) \neq 0$. The asymptotics of the solution u^ε of problem (2.1), for $t \geq t_0 > 0$ and $x \in G_\varepsilon$, takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, \big[u(t,x_1) + r_\varepsilon(t,x)\big],$$

where u is the solution of the homogenized problem

$$\begin{cases} \partial_t u = a^{\rm eff}\, \partial_{x_1}^2 u, & (t,x_1) \in (0,T)\times(-1,1),\\ u(t,\pm 1) = 0, & t \in (0,T),\\ u(0,x_1) = -M\, u_0(-1)\, \delta'(x_1+1), & x_1 \in (-1,1), \end{cases} \tag{2.59}$$

where the effective coefficient $a^{\rm eff}$ is defined by (2.22), and the constant M is defined by

$$M = \int\limits_0^{+\infty} \int\limits_Q (z_1 + N^*(z) + v^-(z))\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1, \tag{2.60}$$

with N^*, solution of the adjoint cell problem (2.24) and v^-, solution of the boundary layer problem (2.36). For some constant $C = C(I^+, \Lambda, Q, d)$, the remainder term satisfies the estimate

$$|r_\varepsilon(t,x)| \leq C\,\varepsilon,$$

which is uniform for $t \geq t_0 > 0$, $x \in I^+ \times \varepsilon Q$, with $I^+ \Subset (-1,1]$.

2. *If $u_0 \in C^{k+1}(-1,1)$ is such that $u_0^{(l)}(-1) = 0$, $l = 0, \cdots, k-1$, and $u_0^{(k)}(-1) \neq 0$, then*

$$u^\varepsilon(t,x) = \varepsilon^{k+2}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, \big[\widetilde{u}(t,x) + \widetilde{r}^\varepsilon(t,x)\big],$$

where $\widetilde{u}$ is the solution of the homogenized problem

$$\begin{cases} \partial_t \widetilde{u} = a^{\rm eff}\, \partial_{x_1}^2 \widetilde{u}, & (t,x_1) \in (0,T)\times(-1,1),\\ \widetilde{u}(t,\pm 1) = 0, & t \in (0,T),\\ \widetilde{u}(0,x_1) = -M_k\, u_0^{(k)}(-1)\, \delta'(x_1+1), & x_1 \in (-1,1), \end{cases}$$

with the constant M_k given by

$$M_k = \frac{1}{k!} \int_0^{+\infty} \int_Q (z_1)^k \Big(z_1 + N^*(z) + v^-(z) \Big)\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1 .$$

The remainder term satisfies $|\widetilde{r}^\varepsilon(t,x)| \le C\,\varepsilon$, and the estimate is uniform for $t \ge t_0 > 0$, $x \in I^+ \times \varepsilon Q$, with $I^+ \Subset (-1,1]$.

Remark 2.5. *If the initial data u_0 is non-negative, then the effective initial data is non-negative too. Indeed, $-\delta'(x_1+1)$ is non-negative in distributional sense, and M is positive, because by the maximum principle, $(z_1 + N^* + v^-)$ is positive.*

The multiplicative constant M depends explicitly on the boundary layer v^- for the left end point $x_1 = -1$ (see formula (2.60)). It is quite surprinsing that such a boundary layer (which is of lower order in classical homogenization theory) enters the asymptotics of u^ε at the main order.

Note also that, if the initial data u_0 had a compact support, then Theorem 5.2 in [4] gives a similar asymptotic behaviour with a different initial data for the homogenized problem, featuring a Dirac mass instead of the derivative of the Dirac mass as in (2.59).

Remark 2.6. *Theorem 2.2 provides the leading term of the asymptotics of u^ε. But, as already explained in Remark 2.2, the error estimate for the remainder term r_ε is not precise enough in the region of interest where $u^\varepsilon(t,x)$ achieves its maximum. A better ansatz with a better error estimate are given in Theorem 2.3 below (again, boundary layers will be crucial).*

Proof. Based on Lemma 2.6 we can compute the asymptotics of v^ε, given by (2.16) in terms of the corresponding Green function K_ε. Obviously, (2.16) can be rewritten in the following form

$$\begin{aligned} \varepsilon^{d-1} v^\varepsilon(t,x) &= \int_{G_\varepsilon} K_1^\varepsilon(t,x,\xi)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi \\ &+ \int_{G_\varepsilon} (\varepsilon^{d-1} K_\varepsilon(t,x,\xi) - K_1^\varepsilon(t,x,\xi))\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi . \end{aligned} \tag{2.61}$$

Thanks to (2.58), for $x \in I^+ \times \varepsilon Q$, $t \ge t_0 > 0$, we have

$$\begin{aligned} &\Big| \int_{G_\varepsilon} (\varepsilon^{d-1} K_\varepsilon(t,x,\xi) - K_1^\varepsilon(t,x,\xi))\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi \Big| \\ &\le C_1\,\varepsilon^2 \int_{G_\varepsilon} e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi \le C_1\,\varepsilon^2\,\varepsilon^d |Q| \int_0^{+\infty} e^{-\Theta\,\eta_1}\, d\eta_1 \le C\,\varepsilon^{d+2} \end{aligned}$$

with the change of variables $\xi_1 + 1 = \varepsilon z_1$, $\xi' = \varepsilon z'$ and for some constants C, C_1 which do not depend on ε.

We proceed by evaluating the first integral in (2.61). We compute separately the contributions of each summand in (2.56). Expanding K_0 and u_0 into Taylor series in the neighbourhood of $\xi_1 = -1$, and recalling that $K_0(t, x_1, -1) = 0$, we see that, for $t \geq t_0 > 0$,

$$\int\limits_{G_\varepsilon} K_0(t, x_1, \xi_1)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi$$
$$= \Big(u_0(-1)\, \partial_{\xi_1} K_0(t, x_1, -1) + O(\varepsilon)\Big) \int\limits_{G_\varepsilon} (\xi_1 + 1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi.$$

Performing again the change of variables $\xi_1 + 1 = \varepsilon z_1$, $\xi' = \varepsilon z'$ and using the periodicity of p_Θ yields

$$\int\limits_{G_\varepsilon} (\xi_1 + 1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi = \varepsilon^{d+1} \int\limits_0^{+\infty} \int\limits_Q z_1\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1 + O(\varepsilon^{d+2}). \tag{2.62}$$

Recall that, for the simplicity of presentation, we assumed **(H4)**, namely $\varepsilon = 1/N$, $N \in \mathbb{Z}_+$. Similarly, for $t \geq t_0 > 0$,

$$\varepsilon \int\limits_{G_\varepsilon} N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_1} K_0(t, x, \xi)\, \frac{u_0(\xi_1)}{p_\Theta(\xi/\varepsilon)}\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi$$
$$= \varepsilon^{d+1}\, u_0(-1)\, \partial_{\xi_1} K_0(t, x_1, -1) \int\limits_0^{+\infty} \int\limits_Q \frac{N^*(z)}{p_\Theta(z)}\, e^{-\Theta z_1}\, dz' dz_1 + O(\varepsilon^{d+2}).$$

On the contrary, since differentiating (2.57) with respect to x_1 does not affect the homogeneous Dirichlet boundary conditions, we have $\partial_{x_1} K_0(t, x_1, \pm 1) = 0$ and, therefore, the following term can be neglected

$$\varepsilon \int\limits_{G_\varepsilon} N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} K_0(t, x_1, \xi_1)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi = O(\varepsilon^{d+2}).$$

The last summand $(\varepsilon K_1 - \varepsilon V_{bl}^\varepsilon)$ in (2.56) is written as a sum of three terms. The first one, since $K_1(t, x_1, -1) - \hat{v}^- \partial_{\xi_1} K_0(t, x_1, -1) = 0$, gives a negligible contribution

$$\varepsilon \int\limits_{G_\varepsilon} (K_1(t, x_1, \xi_1) - \hat{v}^- \partial_{\xi_1} K_0(t, x_1, -1))\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi = O(\varepsilon^{d+2}).$$

For the second one, performing a change of variables as above and using the periodicity of p_Θ yields

$$\varepsilon \int\limits_{G_\varepsilon} v^-\big(\frac{\xi_1 + 1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big)\, \partial_{\xi_1} K_0(t, x_1, -1)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi$$
$$= \varepsilon^{d+1}\, u_0(-1)\, \partial_{\xi_1} K_0(t, x_1, -1) \int\limits_0^{+\infty} \int\limits_Q \frac{v^-(z)}{p_\Theta(z)}\, e^{-\Theta z_1}\, dz' dz_1 + O(\varepsilon^{d+2}). \tag{2.63}$$

Thanks to (2.37), the third term containing the boundary layer corrector near the right base of the rod $x_1 = 1$ is exponentially small. Combining (2.61)–(2.63) yields

$$v^\varepsilon(t, x) = \varepsilon^2\, (M\, u_0(-1)\, \partial_{\xi_1} K_0(t, x_1, -1) + O(\varepsilon)), \tag{2.64}$$

where $O(\varepsilon)$ is uniform for $t \geq t_0 > 0$ and $x \in I^+ \times \varepsilon Q$.

The second statement of Theorem 2.2 can be proved in the same way as the first one and we safely leave it to the reader. □

Theorem 2.2 provided the leading term of the asymptotics of u^ε. But, as already explained in Remark 2.6, due to the presence of the exponentially large factor $e^{\Theta(x_1+1)/\varepsilon}$, we are mostly interested in the asymptotics of u^ε in a ε-neighbourhood of the right end of the rod, where both, the leading and the corrector terms (together with the boundary layer corrector) are of the same order. Therefore, we can not claim that, in this localization zone, we have $e^{\Theta(x_1+1)/\varepsilon}\, r^\varepsilon(t,x) \ll e^{\Theta(x_1+1)/\varepsilon}\, u(t,x_1)$.

Due to similar reasons, we had to construct extra terms in the asymptotics of the Green function K_ε. Indeed, because of the factor $e^{-\Theta(x_1+1)/\varepsilon}$ in (2.16), only the behaviour of K_ε in a ε-neighbourhood of the left end plays a significant part. To obtain a precise asymptotics near the left end of the rod, we have constructed the corrector terms for K_ε. Notice that the integrals (2.61)–(2.63) are of the same order.

In Theorem 2.3 below we construct the corrector for u^ε, that improves the asymptotics of u^ε near the right end of the rod and, therefore, makes the result of Theorem 2.2 complete.

Theorem 2.3. *Under the same assumptions as in Theorem 2.2, the refined asymptotics of the solution u^ε of problem* (2.1)*, for $t \geq t_0 > 0$ and $x \in G_\varepsilon$, takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, \big[U^\varepsilon(t,x) + R_\varepsilon(t,x)\big],$$

where U^ε is given by

$$\begin{aligned} U^\varepsilon(t,x) &= u(t,x_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u(t,x_1) \\ &+\varepsilon u_1(t,x_1) + \varepsilon\Big[v_*^+\big(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{v}_*^+\Big]\, \partial_{x_1} u(t,1), \end{aligned} \tag{2.65}$$

where $u(t,x_1)$ is the solution of the homogenized problem (2.59)*, N solves* (2.23)*, u_1 and the boundary layer corrector v_*^+ are defined in* (2.70) *and* (2.69)*, respectively. For some constant $C = C(\Lambda, Q, d)$, the remainder term satisfies the estimate*

$$|R_\varepsilon(t,x)| \leq C\, \varepsilon (1 - x_1),$$

which is uniform for $t \geq t_0 > 0$, $x \in G_\varepsilon$.

Proof. In view of the facorization (2.9), it is sufficient to improve the asymptotics of v^ε. Because of (2.64), the function $u(t,x_1)$, solution of (2.59), is in fact the leading term of the asymptotics for $\varepsilon^{-2} v^\varepsilon(t,x)$ for $t \geq t_0 > 0$. Let us construct the corrector for $\varepsilon^{-2}v^\varepsilon(t,x)$. Obviously, due to the semigroup property of the parabolic operator, one can represent $\varepsilon^{-2}v^\varepsilon(t,x)$ as a sum $\tilde v_1^\varepsilon+\tilde v_2^\varepsilon$, where

$$\begin{cases} \rho_\Theta^\varepsilon(x)\,\partial_t\tilde v_1^\varepsilon + A_\Theta^\varepsilon\tilde v_1^\varepsilon = 0, & \text{in } (t_0,T)\times G_\varepsilon,\\ B_\Theta^\varepsilon\tilde v_1^\varepsilon = 0, & \text{on } (t_0,T)\times\Sigma_\varepsilon,\\ \tilde v_1^\varepsilon(t,\pm1,x') = 0, & x'\in(t_0,T)\times\varepsilon Q,\\ \tilde v_1^\varepsilon(t_0,x) = u(t_0,x_1), & x\in G_\varepsilon; \end{cases} \tag{2.66}$$

$$\begin{cases} \rho_\Theta^\varepsilon(x)\,\partial_t\tilde v_2^\varepsilon + A_\Theta^\varepsilon\tilde v_2^\varepsilon = 0, & \text{in } (t_0,T)\times G_\varepsilon,\\ B_\Theta^\varepsilon\tilde v_2^\varepsilon = 0, & \text{on } (t_0,T)\times\Sigma_\varepsilon,\\ \tilde v_2^\varepsilon(t,\pm1,x') = 0, & x'\in(t_0,T)\times\varepsilon Q,\\ \tilde v_2^\varepsilon(t_0,x) = \varepsilon^{-2}v^\varepsilon(t_0,x)-u(t_0,x_1), & x\in G_\varepsilon. \end{cases} \tag{2.67}$$

It is easy to see that the asymptotics of $\tilde v_1^\varepsilon$ takes the form

$$\begin{aligned} &\tilde U^\varepsilon(t,x) = u(t,x_1)+\varepsilon\,N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}u(t,x_1)\\ &+\varepsilon u_1(t,x_1)+\varepsilon\Big[v_*^+\big(\frac{x_1-1}{\varepsilon},\frac{x'}{\varepsilon}\big)-\hat v_*^+\Big]\,\partial_{x_1}u(t,1)\\ &+\varepsilon\Big[v_*^-\big(\frac{1+x_1}{\varepsilon},\frac{x'}{\varepsilon}\big)-\hat v_*^-\Big]\,\partial_{x_1}u(t,-1), \end{aligned} \tag{2.68}$$

where the boundary layer correctors $v_*^\pm(y)$ and their asymptotic limits $\hat v_*^\pm$ are defined similarly to $v^\pm(y)$ and $\hat v^\pm$ in (2.36), except that the adjoint operator and the adjoint cell functions are replaced by the direct ones. In other words, $v_*^\pm$ are solution in the semi-infinite cylinders $\mathbb{G}^-=(-\infty,0)\times Q$ and $\mathbb{G}^+=(0,+\infty)\times Q$ of

$$\begin{cases} A_\Theta v_*^\pm(y)=0, & y\in\mathbb{G}^\mp,\\ B_\Theta v_*^\pm(y)=0, & y\in\Sigma^\mp,\\ v_*^+(0,y')=-N(0,y'). \end{cases} \tag{2.69}$$

The boundary layers $v_*^\pm(y)$ stabilize at infinity to constants $\hat v_*^\pm$ exponentially fast, as in (2.37).

In (2.68) the function u_1 is designed so that $\tilde U^\varepsilon$ satisfy homogeneous boundary conditions at $x_1=\pm1$, namely it solves

$$\begin{cases} \partial_t u_1(t,x_1) = a^{\text{eff}}\,\partial_{x_1}^2 u_1(t,x_1)+f(t,x_1), & (t,x_1)\in(t_0,T)\times(-1,1),\\ u_1(t,\pm1)=\hat w^\pm\partial_{x_1}u(t,\pm1), & t\in(t_0,T),\\ u_1(t_0,x_1)=0, & x_1\in(-1,1), \end{cases} \tag{2.70}$$

where, N_2 being a solution of (2.35), $f(t,x_1)$ is given by

$$f(t,x_1)=\partial_{\xi_1}^3u(t,x_1)\int\limits_Y\big[a_{1j}^\Theta\partial_{y_j}N_2+a_{11}^\Theta N-b_1^\Theta N_2-a^{\text{eff}}\,\rho_\Theta\,N\big]\,dy.$$

As in the proof of Theorem 2.2, one can prove that the following estimate holds

$$|\tilde{v}_1^\varepsilon - \tilde{U}^\varepsilon| \le C\,\varepsilon^2,\ t \ge t_0,\ x \in G_\varepsilon,$$

with the constant C independent of ε. On the other hand, because of the exponential stabilization of the boundary layer v_*^-, we have

$$|\tilde{U}^\varepsilon - U^\varepsilon| \le C\,\varepsilon\,(1-x_1), \quad t \ge t_0,\ x \in G_\varepsilon,$$

where U^ε is given by (2.65). This yields

$$|\tilde{v}_1^\varepsilon - U^\varepsilon| \le C\,\varepsilon\,(1-x_1),\ t \ge t_0,\ x \in G_\varepsilon. \tag{2.71}$$

We proceed by estimating the solution $\tilde{v}_2^\varepsilon$ of (2.67). Let $\phi^\varepsilon(t,x)$ be a solution of the following problem

$$\begin{cases} \rho_\Theta^\varepsilon(x)\,\partial_t\phi^\varepsilon + A_\Theta^\varepsilon\phi^\varepsilon = 0, & \text{in } (t_0,T)\times G_\varepsilon, \\ B_\Theta^\varepsilon\phi^\varepsilon = 0, & \text{on } (t_0,T)\times\Sigma_\varepsilon, \\ \phi^\varepsilon(t,\pm 1,x') = 0, & x' \in (t_0,T)\times\varepsilon Q, \\ \phi^\varepsilon(t_0,x) = 1, & x \in G_\varepsilon. \end{cases} \tag{2.72}$$

Then, by the maximum principle,

$$|\tilde{v}_2^\varepsilon(t,x)| \le \phi^\varepsilon(t,x)\,\max_{x\in G_\varepsilon}|\varepsilon^{-2}v^\varepsilon(t_0,x) - u(t_0,x_1)|, \quad (t,x)\in(t_0,T)\times G_\varepsilon.$$

In view of Theorem 2.2,

$$\max_{x\in G_\varepsilon}|\varepsilon^{-2}v^\varepsilon(t_0,x) - u(t_0,x_1)| \le C\,\varepsilon,$$

thus,

$$|\tilde{v}_2^\varepsilon(t,x)| \le C\,\varepsilon\,\phi^\varepsilon(t,x), \quad (t,x)\in(t_0,T)\times G_\varepsilon.$$

By standard homogenization it easy to prove that

$$|\phi^\varepsilon(t,x)| \le C\,(1-x_1), \quad (t,x)\in(2\,t_0,T)\times G_\varepsilon.$$

Combining the last two estimates yields

$$|\tilde{v}_2^\varepsilon(t,x)| \le C\,\varepsilon\,(1-x_1), \quad (t,x)\in(2\,t_0,T)\times G_\varepsilon. \tag{2.73}$$

Estimates (2.71), (2.73) imply the statement of Theorem 2.3. The proof is complete. □

3 The case of a layer

We now consider the case of a layer in $\mathbb{R}^d$. More precisely, the domain Ω is defined as the layer $\{x\in\mathbb{R}^d : x' = (x_1,\cdots,x_{d-1})\in\mathbb{R}^{d-1},\ -1\le x_d\le 1\}$ (see Figure 2).

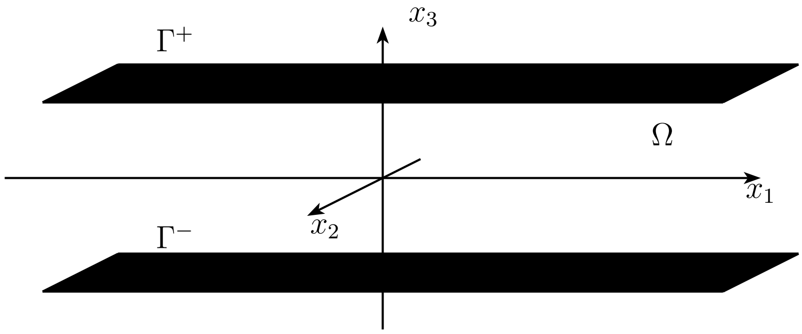

Figure 2: The layer Ω

Note that we change the notations from the previous section since a point $x \in \mathbb{R}^d$ is now denoted $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$. The boundary of Ω consists of two hyperplanes $\Gamma^\pm = \{x \in \mathbb{R}^d : x_d = \pm 1\}$. We study the homogenization of the non-stationary convection-diffusion problem (1.1) which, in the case of a layer, reads

$$\begin{cases} \partial_t u^\varepsilon + A_\varepsilon u^\varepsilon = 0, \quad \text{in } (0,T)\times\Omega, \\ u^\varepsilon = 0, \qquad \text{on } (0,T)\times(\Gamma^+\cup\Gamma^-), \\ u^\varepsilon(0,x) = u_0(x), \qquad\qquad \text{in } \Omega, \end{cases} \tag{3.1}$$

where, as before,

$$A_\varepsilon u^\varepsilon = -\mathrm{div}\big(a^\varepsilon \nabla u^\varepsilon\big) + \frac{1}{\varepsilon} b^\varepsilon \cdot \nabla u^\varepsilon,$$

and the coefficients of the equation are still given by (2.2), namely $a^\varepsilon_{ij}(x) = a_{ij}(x/\varepsilon)$ and $b^\varepsilon_i(x) = b_i(x/\varepsilon)$. In the case of a layer our main assumptions are slightly different from those in the previous section. We assume that the following conditions are satisfied.

(A1) The coefficients of the equation $a_{ij}, b_j \in L^\infty(\Omega)$ are Y-periodic, $Y = (0,1]^d$ being the periodicity cell.

(A2) The $d \times d$ matrix $a(y)$ is symmetric and satisfies a uniform ellipticity condition with a coercivity constant $\Lambda > 0$.

(A3) The initial data u_0 has compact support with respect to $x' = (x_1, \cdots, x_{d-1})$, namely $u_0(x) \in C^1_0(\mathbb{R}^{d-1}; C^1[-1,1])$.

(A4) For simplicity we assume that $\varepsilon = 1/N$, $N \in \mathbb{Z}^+$, so that an entire number of periodicity cells fits in the thickness of the layer Ω.

As in the case of a thin rod, we study the asymptotic behaviour of solutions $u^\varepsilon(t,x)$ of problem (3.1), as $\varepsilon \to 0$.

3.1 Auxiliary spectral problems, factorization and main result.

In order to simplify the original problem, we make use of the factorization principle, as in Section 2 (with respect to x_d instead of x_1), and then construct the asymptotics of the new unknown function. However, the main difference with the previous case of a rod is that we must use moving coordinates (see [3], [12], [18]) in the directions parallel to the layer. This makes the equation homogenizable at the price that the initial condition becomes asymptotically singular. As before, we circumvent this difficulty of singular initial data by constructing the asymptotics of the Green function of the factorized problem.

We recall that the cell operator A is defined by (2.3) and its adjoint A^* by (2.4). For $\theta \in \mathbb{R}$, we introduce two families of spectral problems, similar to (2.5),

$$\begin{cases} e^{-\theta\, y_d}\, A\, e^{\theta\, y_d}\, p_\theta(y) = \lambda(\theta)\, p_\theta(y), \quad \text{in } Y, \\ y \to p_\theta(y) \quad \text{Y-periodic,} \end{cases} \tag{3.2}$$

$$\begin{cases} e^{\theta\, y_d}\, A^*\, e^{-\theta\, y_d}\, p^*_\theta(y) = \lambda(\theta)\, p^*_\theta(y), \quad \text{in } Y, \\ y \to p^*_\theta(y) \quad \text{Y-periodic.} \end{cases}$$

By the Krein-Rutman theorem, for each $\theta \in \mathbb{R}$, the first eigenvalue $\lambda_1(\theta)$ of problem (3.2) is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, the statement of Lemma 2.1 remains valid, and we call Θ the unique maximum point of $\lambda_1(\theta)$. The eigenfunctions p_θ and p^*_θ are normalized by (2.6) as above. Arguments similar to those in Section 2 yield

$$\frac{d\lambda_1}{d\theta}\Big|_{\theta=0} = \int\limits_Y \big(b_d\, p^*_\theta + a_{dj}\, \partial_{y_j} p^*_\theta\big)\, dy = \bar{b}_d, \tag{3.3}$$

where $\bar{b}_d$ is called the normal effective drift (normal to the layer). Hence, $\bar{b}_d = 0$ if and only if $\Theta = 0$. If the normal effective drift is zero, i.e., $\bar{b}_d = 0$, then the method of homogenization in moving coordinates can be applied directly (see [3], [12], [18]). Therefore, we assume that $\bar{b}_d \neq 0$ (or, equivalently, $\Theta \neq 0$).

In what follows we consider the case $\bar{b}_d > 0$, the other case $\bar{b}_d < 0$ being symmetric. If $\bar{b}_d > 0$, then we perform the change of unknown function as follows

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\,(x_d+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, v^\varepsilon(t,x). \tag{3.4}$$

Substituting (3.4) into (3.1), one obtains that the new unknown function v^ε solves the following problem

$$\begin{cases} \rho^\varepsilon_\Theta\, \partial_t v^\varepsilon + A^\varepsilon_\Theta\, v^\varepsilon = 0, \qquad (t,x) \in (0,T) \times \Omega, \\ v^\varepsilon = 0, \qquad (t,x) \in (0,T) \times (\Gamma^+ \cup \Gamma^-), \\ v^\varepsilon(0,x) = u_0(x)\, p^{-1}_\Theta\big(\frac{x}{\varepsilon}\big)\, e^{-\frac{\Theta\,(x_d+1)}{\varepsilon}}, \quad x \in \Omega, \end{cases} \tag{3.5}$$

where $\rho^\varepsilon_\Theta(x) = \varrho_\Theta(x/\varepsilon)$,

$$A^\varepsilon_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) + \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v,$$

and the coefficients of the operator are given by

$$\begin{aligned} &a^\Theta_{ij}(y) = \varrho_\Theta(y)\, a_{ij}(y), \quad \varrho_\Theta(y) = p_\Theta(y)\, p^*_\Theta(y),\\ &b^\Theta_i(y) = \varrho_\Theta(y)\, b_i(y) - 2\,\varrho_\Theta(y)\, a_{id}(y)\,\Theta \\ &+a_{ij}(y)\,\big[p_\Theta(y)\,\partial_{y_j} p^*_\Theta(y) - p^*_\Theta(y)\,\partial_{y_j} p_\Theta(y)\big]. \end{aligned} \tag{3.6}$$

The matrix a^Θ is positive definite since both p_Θ and p^*_Θ are positive functions. The vector-field b^Θ, for each $\theta\in\mathbb{R}$, is divergence-free and its last component b^Θ_d has zero mean, that is

$$\int\limits_Y b^\Theta_d(y)\,dy = 0; \quad \mathrm{div}\, b^\theta = 0,\ \forall\,\theta. \tag{3.7}$$

The averages of the other components are denoted by

$$\beta^\Theta_i = \int\limits_Y b^\Theta_i(y)\,dy, \quad i = 1,\cdots,d. \tag{3.8}$$

The vector β^Θ is called the effective convection (note that its formula is different from that of the normal effective drift $\bar b_d$ defined in (3.3)). Since $\beta^\Theta_d = 0$ because of (3.7), the convection is parallel to the layer. When the effective convection β^Θ is not equal to zero, contrary to the case of the rod, we cannot use classical homogenization methods for (3.5), and, rather, we rely on the method of moving coordinates (see [3], [12], [18]).

Theorem 3.1. *Suppose that conditions* **(A1)-(A4)** *are fulfilled, the normal effective drift (defined by* (3.3)*) satisfies* $\bar b_d > 0$ *and* $u_0(\cdot,-1)\neq 0$. *Then, for* $t\geq t_0>0$, *the asymptotics of the solution* u^ε *of problem* (3.1) *takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_d+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\,\big[u\big(t, x-\frac{\beta^\Theta}{\varepsilon}t\big) + r_\varepsilon(t,x)\big],$$

where $u(t,x)$ *is the solution of the homogenized problem*

$$\begin{cases} \partial_t u(t,x) = \mathrm{div}(a^{\mathrm{eff}}\nabla u(t,x)), & (t,x)\in(0,T)\times\Omega,\\ u(t,x) = 0, & (t,x)\in(0,T)\times(\Gamma^-\cup\Gamma^+).\\ u(0,x) = -M\,u_0(x',-1)\,\delta'(x_d+1), & x\in\Omega, \end{cases} \tag{3.9}$$

with a positive definite matrix a^{eff} *defined by* (3.14) *and the constant* M *defined by*

$$M = \int\limits_{(0,1]^{d-1}}\int\limits_0^{+\infty} \big[z_d + N^*_d(z) + v^-(z)\big]\, p^{-1}_\Theta(z)\, e^{-\Theta z_d} dz_d\, dz', \tag{3.10}$$

where N_d^* is a solution of the cell problem (3.16) *and the boundary layer v^- is defined by* (3.25)*. The remainder term satisfies, for $t \geq t_0 > 0$,*

$$|r_\varepsilon(t,x)| \leq C\,\varepsilon \quad \text{for any } x \in \Omega \text{ such that } x_d \in I^+ \Subset (-1,1],$$

and the constant C depends solely on I^+, Λ, d.

Remark 3.1. *In the case $u_0(x',-1) = \cdots = \partial_{\xi_d}^{k-1} u_0(x',-1) = 0$ and $\partial_{x_d}^k u_0(x',-1) \neq 0$ for some k, the asymptotics of u^ε takes the form*

$$u^\varepsilon(t,x) = \varepsilon^{2+k}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_d+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, \big[u(t, x - \frac{\beta^\Theta}{\varepsilon} t) + r_\varepsilon(t,x)\big],$$

where $|r_\varepsilon(t,x)| \leq C\,\varepsilon$, for $t \geq t_0 > 0$ and $x \in \Omega$ such that $x_d \in I^+ \Subset (-1,1]$ and $u(t,x)$ solves the problem

$$\begin{cases} \partial_t u(t,x) = \mathrm{div}(a^{\mathrm{eff}} \nabla u(t,x)), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma^- \cup \Gamma^+). \\ u(0,x) = -M_k\, \partial_{x_d}^k u_0(x',-1)\, \delta'(x_d - 1), & x \in \Omega, \end{cases}$$

with the constant M_k given by

$$M_k = \frac{1}{k!} \int\limits_{(0,1]^{d-1}} \int\limits_0^{+\infty} (z_d)^k\, \big[z_d + N_d^*(z) + v^-(z)\big]\, p_\Theta^{-1}(z)\, e^{-\Theta z_d} dz_d\, dz'.$$

Remark 3.2. *Similarly to the case of a rod (see Remarks 2.2 and 2.6), the error estimate for the remainder term r_ε is not precise enough in the region of interest where $u^\varepsilon(t,x)$ achieves its maximum. Indeed, the homogeneous Dirichlet boundary condition for $u(t,x)$, together with the exponential $e^{\frac{\Theta(x_d+1)}{\varepsilon}}$ shows that $u^\varepsilon(t,x)$ attains its maximum at a distance of order ε of the plane Γ^+: there, by a Taylor expansion, $u(t,x)$ is of the order of ε, like the remainder term $r_\varepsilon(t,x)$ which is thus not negligible. A better ansatz with a better error estimate will be given in Theorem 3.2 below.*

3.2 Proof of Theorem 3.1

The proof is partly similar to that of Theorem 2.1 and relies on the representation formula for v^ε

$$v^\varepsilon(t,x) = \int\limits_\Omega K_\varepsilon(t,x,\xi)\, u_0(\xi)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\, d\xi, \tag{3.11}$$

where $K_\varepsilon(t,x,\xi)$ is the Green function of problem (3.5). However, one major difference with the previous case of a rod is that, as was already pointed out, in the case $\beta^\Theta \neq 0$, the classical homogenization methods do not apply to problem (3.5). To overcome this difficulty, we shall use moving coordinates.

Recall that, for any x, K_ε solves the adjoint problem

$$\begin{cases} \varrho_\Theta\big(\frac{\xi}{\varepsilon}\big)\,\partial_t K_\varepsilon(t,x,\xi) + A_\Theta^{*,\varepsilon}\,K_\varepsilon(t,x,\xi) = 0, & (t,\xi)\in(0,T)\times\Omega,\\ K_\varepsilon(t,x,\xi) = 0, & (t,\xi)\in(0,T)\times(\Gamma^-\cup\Gamma^+),\\ K_\varepsilon(0,x,\xi) = \delta(x-\xi), & \xi\in\Omega, \end{cases} \tag{3.12}$$

$$A_\Theta^{*,\varepsilon} v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) - \frac{1}{\varepsilon}\,b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v.$$

Since b_Θ is divergence-free, $A_\Theta^{*,\varepsilon}$ differs from A_Θ^ε by the sign in front of the first-order term. For any $\xi\in\Omega$, K_ε solves the direct problem with respect to (t,x), but since we are interested in the asymptotics of K_ε w.r.t ξ, we prefer to interpret it from the very beginning as a solution of adjoint problem (3.12).

We study the asymptotic behaviour of K_ε, as $\varepsilon\to 0$, and then from (3.11) derive the asymptotics for v^ε.

3.2.1 Asymptotic behaviour of $K_\varepsilon(t,x,\xi)$

As in the proof of Theorem 2.1, instead of analyzing directly K_ε, we consider the difference

$$V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi) - K_\varepsilon(t,x,\xi),$$

where Φ_ε is the fundamental solution in $\mathbb{R}^d$, that is, for any $x\in\mathbb{R}^d$, Φ_ε solves the problem

$$\begin{cases} \varrho_\Theta\big(\frac{\xi}{\varepsilon}\big)\;\partial_t\Phi_\varepsilon + A_\Theta^{*,\varepsilon}\Phi_\varepsilon = 0, & (t,\xi)\in(0,T)\times\mathbb{R}^d,\\ \Phi_\varepsilon(0,x,\xi) = \delta(x-\xi), & \xi\in\mathbb{R}^d. \end{cases}$$

In this way, for all $x\in\Omega$, V_ε satisfies the problem

$$\begin{cases} \varrho_\Theta\big(\frac{\xi}{\varepsilon}\big)\,\partial_t V_\varepsilon(t,x,\xi) + A_\Theta^{*,\varepsilon}\,V_\varepsilon(t,x,\xi) = 0, & (t,\xi)\in(0,T)\times\Omega,\\ V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi), & (t,\xi)\in(0,T)\times(\Gamma^-\cup\Gamma^+),\\ V_\varepsilon(0,x,\xi) = 0, & \xi\in\Omega. \end{cases} \tag{3.13}$$

We emphasize that V_ε is a regular function of ξ, for x such that $x_d\neq\pm1$.

The asymptotics of Φ_ε is easier to establish. First, we introduce its zero-order approximation $\Phi_0(t,x,\xi)$, the fundamental solution of the effective problem

$$\begin{cases} \partial_t\Phi_0 = \mathrm{div}_\xi(a^{\mathrm{eff}}\nabla_\xi\Phi_0), & (t,\xi)\in(0,T)\times\mathbb{R}^d,\\ \Phi_0(0,x,\xi) = \delta(x-\xi), & \xi\in\mathbb{R}^d \end{cases}$$

with a^{eff} given by

$$\begin{aligned} a_{ij}^{\mathrm{eff}} &= \int_Y \big(a_{ij}^\Theta(y) + a_{ik}^\Theta(y)\partial_{y_k}N_j(y) - b_i^\Theta(y)\,N_j(y) + \beta_j^\Theta\,\rho_\Theta\,N_j(y)\big)\,dy\\ &= \int_Y \big(a_{ij}^\Theta(\eta) + a_{ik}^\Theta(\eta)\partial_{y_k}N_j^*(\eta) + b_i^\Theta(\eta)\,N_j^*(\eta) - \beta_j^\Theta\,\rho_\Theta\,N_j^*(\eta)\big)\,d\eta. \end{aligned} \tag{3.14}$$

The vector functions N and N^* solve the following cell problems (direct and adjoint, respectively)

$$\begin{cases} -\mathrm{div}(a^\Theta \nabla N_i) + b^\Theta \cdot \nabla N_i = \partial_{y_j} a^\Theta_{ij}(y) - b^\Theta_i(y) + \beta^\Theta_i, \quad \text{in } Y, \\ y \mapsto N_i \quad Y-\text{periodic}; \end{cases} \tag{3.15}$$

$$\begin{cases} -\mathrm{div}(a^\Theta \nabla N^*_i) - b^\Theta \cdot \nabla N^*_i = \partial_{y_j} a^\Theta_{ij}(y) + b^\Theta_i(y) - \beta^\Theta_i, \quad \text{in } Y, \\ y \mapsto N^*_i \quad Y-\text{periodic}. \end{cases} \tag{3.16}$$

Notice that, although the above cell problems (3.15) and (3.16) are of the same type as (2.23) and (2.24), they contain additional β^Θ_i term on the right-hand side. Observe that, by the very definition of β^Θ, the compatibility conditions for (3.15) and (3.16) are satisfied.

We further introduce the second-order corrector functions N^2_{ij}, N^{2*}_{ij}, solutions of

$$\begin{cases} A_\Theta N^2_{ij} = \partial_{y_k}(a^\Theta_{ki} N_j) + a^\Theta_{ik} \partial_{y_k} N_j + a^\Theta_{ij} \\ -b^\Theta_i N_j + \beta^\Theta_i \rho_\Theta N_j - a^{\mathrm{eff}}_{ij} \rho_\Theta, \quad \text{in } Y, \\ y \mapsto N^2_{ij} \text{ is periodic}; \end{cases} \tag{3.17}$$

$$\begin{cases} A^*_\Theta N^{2*}_{ij} = \partial_{y_k}(a^\Theta_{ki} N^*_j) + a^\Theta_{ik} \partial_{y_k} N^*_j + a^\Theta_{ij} \\ +b^\Theta_i N^*_j - \beta^\Theta_i \rho_\Theta N^*_j - a^{\mathrm{eff}}_{ij} \rho_\Theta, \quad \text{in } Y, \\ y \mapsto N^{2*}_{ij} \text{ is periodic}, \end{cases} \tag{3.18}$$

where A_Θ and A^*_Θ are defined by (2.12) and (2.13), respectively.

Then we define the first- and second-order approximations of Φ_ε

$$\Phi^\varepsilon_1\big(t,x,\widetilde{\xi}\big) = \Phi_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big) \cdot \nabla_x \Phi_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big) \cdot \nabla_{\widetilde{\xi}} \Phi_0\big(t,x,\widetilde{\xi}\big), \tag{3.19}$$

$$\begin{aligned} \Phi^\varepsilon_2\big(t,x,\widetilde{\xi}\big) &= \Phi^\varepsilon_1\big(t,x,\widetilde{\xi}\big) + \varepsilon^2 N^2_{ij}\big(\frac{x}{\varepsilon}\big)\, \partial_{x_i}\partial_{x_j} \Phi_0(t,x,\widetilde{\xi}) \\ &+ \varepsilon^2 N^{2*}_{ij}\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_i}\partial_{\xi_j} \Phi_0(t,x,\widetilde{\xi}) + \varepsilon^2 N_i\big(\frac{x}{\varepsilon}\big)\, N^*_j\big(\frac{\xi}{\varepsilon}\big)\, \partial_{x_i}\partial_{\xi_j} \Phi_0(t,x_1,\widetilde{\xi}), \end{aligned} \tag{3.20}$$

where $\widetilde{\xi}$ is the moving coordinate defined by

$$\widetilde{\xi} = \xi + \frac{\beta^\Theta}{\varepsilon}\, t. \tag{3.21}$$

Remark 3.3. *The variables x and ξ being dual, the moving coordinate for x is defined with the opposite velocity, namely*

$$\widetilde{x} = x - \frac{\beta^\Theta}{\varepsilon}\, t.$$

By the same techniques, as in [1], one can prove

Lemma 3.1. *Assume that conditions* **(A1)-(A2)** *are fulfilled. Then, for* $x, \xi \in \mathbb{R}^d$ *and* $t \geq \varepsilon^2$*, the estimate holds*

$$\Big|\Phi_\varepsilon(t,x,\xi) - \Phi_k^\varepsilon\big(t,x,\xi + \frac{\beta^\Theta}{\varepsilon}\,t\big)\Big| \leq C\,\frac{\varepsilon^{k+1}}{t^{(d+k+1)/2}}, \quad k=0,1,2,$$

where β^Θ *is defined by* (3.8).

Turning back to V_ε, its zero-order approximation is V_0, defined for any $x \in \Omega$, as a solution of the homogenized problem

$$\begin{cases} \partial_t V_0 = \mathrm{div}_\xi(a^{\mathrm{eff}} \nabla_\xi V_0), & (t,\xi) \in (0,T)\times\Omega, \\ V_0(t,x,\xi) = \Phi_0(t,x,\xi), & (t,\xi) \in (0,T)\times(\Gamma^-\cup\Gamma^+), \\ V_0(0,x,\xi) = 0, & \xi \in \Omega. \end{cases}$$

Note that $V_0(t,x,\xi) \in C^\infty([0,T]\times\Omega\times\overline{\Omega})$ and for $(t,\xi) \in [0,T]\times\Omega$ one has

$$|\partial_t^k\,\partial_x^l\,\partial_\xi^m\,V_0(t,x,\xi)| \leq \frac{C}{\mathrm{dist}(K,(\Gamma^-\cup\Gamma^+))^{2k+l+m+d}}, \quad x \in K \Subset \Omega.$$

The first-order approximation of V_ε is defined by

$$\begin{aligned} V_1^\varepsilon(t,x,\xi) &= V_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,N_j\big(\frac{x}{\varepsilon}\big)\,\partial_{x_j} V_0\big(t,x,\widetilde{\xi}\big) \\ &+ \varepsilon\,N_j^*\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_j} V_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,V_1\big(t,x,\widetilde{\xi}\big) + \varepsilon\,V_{bl}^\varepsilon(t,x,\xi), \end{aligned} \tag{3.22}$$

where $\widetilde{\xi}$ is the moving coordinate defined by (3.21), and V_1, V_{bl}^ε are defined below. A higher order asymptotic expansion for V_ε takes the form

$$W_\varepsilon\big(t,x,\xi\big) = V_1^\varepsilon(t,x,\xi) + \varepsilon^2\,V_2^\varepsilon(t,x,\xi) + \varepsilon^2\,\varphi_{\mathrm{bl}}^\varepsilon(t,x,\xi) + \varepsilon^3\,\psi_{\mathrm{bl}}^\varepsilon(t,x,\xi) \tag{3.23}$$

with

$$\begin{aligned} V_2^\varepsilon(t,x,\xi) &= N_{ij}^2(x/\varepsilon)\,\partial_{x_i}\partial_{x_j} V_0(t,x,\widetilde{\xi}) \\ &+ N_{ij}^{2*}(\xi/\varepsilon)\,\partial_{\xi_i}\partial_{\xi_j} V_0(t,x,\widetilde{\xi}) + N_i(x/\varepsilon)\,N_j^*(\xi/\varepsilon)\,\partial_{x_i}\partial_{\xi_j} V_0(t,x,\widetilde{\xi}) \\ &+ N_i(x/\varepsilon)\,\partial_{x_i} V_1(t,x,\widetilde{\xi}) + N_i^*(\xi/\varepsilon)\,\partial_{\xi_i} V_1(t,x,\widetilde{\xi}). \end{aligned} \tag{3.24}$$

In order to define V_1 and the first boundary layer corrector $V_{\mathrm{bl}}^\varepsilon$, we consider auxiliary problems in semi-infinite cylinders $\mathbb{G}^\mp = (0,1]^{d-1}\times(0,\mp\infty)$:

$$\begin{cases} A_\Theta^* v^\pm = 0, & \eta \in \mathbb{G}^\mp, \\ v^\pm(\eta',0) = -N_d^*(\eta',0), \\ \eta' \mapsto v^\pm(\eta',\eta_d) \text{ is } (0,1]^{d-1}-\text{periodic}. \end{cases} \tag{3.25}$$

Since $\beta_d = 0$, such functions $v^\pm$ exist, are uniquely defined and stabilize to some constants $\hat{v}^\pm$ at an exponential rate, as $\eta_d \to \mp\infty$ (see [22]):

$$\begin{aligned} &|v^\pm(\eta',\eta_d) - \hat{v}^\pm| \leq C_0\,e^{-\gamma\,|\eta_d|}, \quad C_0,\gamma > 0; \\ &\|\nabla v^+\|_{L^2((n-1,n)\times Q)} \leq C\,e^{-\gamma\,n}, \quad \forall n < 0, \\ &\|\nabla v^-\|_{L^2((n,n+1)\times Q)} \leq C\,e^{-\gamma\,n}, \quad \forall n > 0. \end{aligned} \tag{3.26}$$

The first boundary layer corrector is given by

$$
\begin{aligned}
V_{bl}^{\varepsilon}(t,x,\xi) \;&= \Big[v^{-}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big)-\hat v^{-}\Big]\,\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi-\frac{\beta^{\Theta}}{\varepsilon}t\big)\Big|_{\xi_d=-1} \\
&+\Big[v^{+}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d-1}{\varepsilon}\big)-\hat v^{+}\Big]\,\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi-\frac{\beta^{\Theta}}{\varepsilon}t\big)\Big|_{\xi_d=1}.
\end{aligned}
\tag{3.27}
$$

Then, V_1, for $x\in\Omega$, is defined as the solution of

$$
\begin{cases}
\partial_t V_1 = \operatorname{div}_{\xi}(a^{\text{eff}}\,\nabla_{\xi}V_1)+F(t,x,\xi), & (t,\xi)\in(0,T)\times\Omega,\\
V_1(t,x,\xi)=\hat v^{\pm}\,\partial_{\xi_d}\,(V_0-\Phi_0)(t,x,\xi), & (t,\xi)\in(0,T)\times\Gamma^{\pm},\\
V_1(0,x,\xi)=0, & \xi\in\Omega,
\end{cases}
\tag{3.28}
$$

where

$$
\begin{aligned}
F(t,x,\xi)=\partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j}V_0(t,x,\xi)\int_Y\big[a^{\Theta}_{kl}\partial_{\eta_l}N^{2*}_{ij}\\
+\,a^{\Theta}_{ij}N^{*}_{k}+b^{\Theta}_{k}N^{2*}_{ij}-\beta^{\Theta}_{k}\rho_{\Theta}N^{2*}_{ij}-a^{\text{eff}}_{ij}\,\rho_{\Theta}\,N^{*}_{k}\big]\,d\eta.
\end{aligned}
$$

The second boundary layer corrector $\varphi^{\varepsilon}_{\text{bl}}$ is defined as follows

$$
\begin{aligned}
&\varphi^{\varepsilon}_{bl}(t,x,\xi)\\
&=\Big[\varphi^{-}_{k}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big)-\hat\varphi^{-}_{k}\Big]\,\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=-1}\Big)\\
&+\Big[\varphi^{+}_{k}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d-1}{\varepsilon}\big)-\hat\varphi^{+}_{k}\Big]\,\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=1}\Big).
\end{aligned}
$$

Remark that, since $\beta^{\Theta}_d=0$, we have $\xi_d=\widetilde{\xi}_d$ and the above definition makes sense when we enforce $\xi_d=-1$. The functions $\varphi^{\pm}_k$ solve nonhomogeneous problems

$$
\begin{cases}
A^{*}_{\Theta}\varphi^{\pm}_{k}=\partial_{\eta_i}(a^{\Theta}_{ik}(v^{\pm}-\hat v^{\pm}))+a^{\Theta}_{ik}\,\partial_{\eta_i}v^{\pm}\\
+(b^{\Theta}_{k}-\beta^{\Theta}_{k}\rho_{\Theta})(v^{\pm}-\hat v^{\pm}), \qquad \eta\in\mathbb{G}^{\mp},\\
\varphi^{\pm}_{k}(\eta',0)=0,\\
\eta'\mapsto\varphi^{\pm}_{k}(\eta',\eta_d)\text{ is }(0,1]^{d-1}-\text{periodic}.
\end{cases}
$$

The right-hand side of the above equation, due to (3.26), is an exponentially decaying function. Since $\beta^{\Theta}_d=0$, the functions $\varphi^{\pm}_k$ exist, are uniquely defined and stabilize to some constants $\hat\varphi^{\pm}_k$ at an exponential rate, as $\eta_d\to\pm\infty$ (see [22]). The corrector $\varphi^{\varepsilon}_{\text{bl}}$ is introduced to compensate the terms of order ε^0 which will appear on the right-hand side after substituting $V^{\varepsilon}_{\text{bl}}$ into the original equation.

The last boundary layer corrector ψ^{ε}_{bl} is defined by

$$
\begin{aligned}
&\psi^{\varepsilon}_{bl}(t,x,\xi)\\
&=\Big[\psi^{-}_{ik}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big)-\hat\psi^{-}_{ik}\Big]\,\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=-1}\Big)\\
&+\Big[\psi^{+}_{ik}\big(\frac{\xi'}{\varepsilon},\frac{\xi_d-1}{\varepsilon}\big)-\hat\psi^{+}_{ik}\Big]\,\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=1}\Big).
\end{aligned}
$$

The functions $\psi_{ik}^{\pm}$ solve nonhomogeneous problems

$$\begin{cases} A_\Theta^* \psi_{ik}^{\pm} = (a_{ik}^\Theta - a_{ik}^{\rm eff}\rho_\Theta)(v^\pm - \hat{v}^\pm) + \partial_{\eta_i}(a_{ij}^\Theta(\varphi_k^\pm - \hat{\varphi}_k{}^\pm)) \\ +a_{ij}^\Theta \partial_{\eta_j}\varphi_k^\pm + (b_i^\Theta - \beta_i^\Theta)(\varphi_k^\pm - \hat{\varphi}_k{}^\pm), \qquad \eta \in \mathbb{G}^\mp, \\ \psi_{ik}^\pm(\eta', 0) = 0, \\ \eta' \mapsto \psi_{ik}^\pm(\eta', \eta_d) \text{ is } (0,1]^{d-1} - \text{periodic.} \end{cases}$$

The right-hand side of the above equation is again an exponentially decaying function. Thus, the functions $\psi_{ik}^{\pm}$ exist, are uniquely defined and stabilize to some constants $\hat{\psi}_j^\pm$ at an exponential rate, as $\eta_d \to \mp\infty$. The boundary layer corrector $\psi_{\rm bl}^\varepsilon$ is designed in order to compensate the terms of order ε on the right-hand side of equation (3.13) which comes from $V_{\rm bl}^\varepsilon$ and $\varphi_{\rm bl}^\varepsilon$ being substituted into this equation.

This completes the construction of the formal expansion. We proceed with its justification. Recall that the functions V_1 and $V_{\rm bl}^\varepsilon$ are introduced to satisfy the boundary conditions on $\Gamma^\pm$ up to second order in ε, while the purpose of $V_2^\varepsilon, \varphi_{\rm bl}^\varepsilon$ and $\psi_{\rm bl}^\varepsilon$ is to guarantee the required accuracy, and the latter terms will not show up in the final result.

Proposition 3.1. *Let V_1^ε be the first-order approximation of V_ε defined by* (3.22)*. Then, for x such that $x_d \in I \Subset (-1,1)$ and for $t \ge 0$, we have*

$$\int_\Omega |V_\varepsilon - V_1^\varepsilon|^2 \, dx \le C\, \varepsilon^4 \tag{3.29}$$

with the constant C depending only on $\mathrm{dist}(x, \Gamma^- \cup \Gamma^+), \Lambda$ *and d.*

Proof. Let us substitute ansatz (3.23) into (3.13) and compute the discrepancy

$$\begin{aligned} &\rho_\Theta^\varepsilon \partial_t (W_\varepsilon - V_\varepsilon) + A_\Theta^{*,\varepsilon}(W_\varepsilon - V_\varepsilon) \\ &= \varepsilon\, R_1\big(t,x,\widetilde{\xi}; y,\eta\big) + \varepsilon\, \mathrm{div}_\eta (a^\Theta(\eta) \nabla_{\widetilde{\xi}} V_2(t,x,\widetilde{\xi};y,\eta))\Big|_{y=\frac{x}{\varepsilon},\eta=\frac{\xi}{\varepsilon}} \\ &+\varepsilon^2 R_2\big(t,x,\widetilde{\xi};\eta\big) + \varepsilon^3 R_3\big(t,x,\widetilde{\xi};\eta\big)\Big|_{y=\frac{x}{\varepsilon},\eta=\frac{\xi}{\varepsilon}}, \end{aligned} \tag{3.30}$$

where $\widetilde{\xi}$ is the moving coordinate defined by (3.21) and

$$\begin{aligned} &R_1\big(t,x,\widetilde{\xi};y,\eta\big) = -\rho_\Theta(\eta)\partial_t V_1(t,x,\widetilde{\xi}) - \rho_\Theta(\eta) N_j^*(\eta)\partial_t \partial_{\xi_j} V_0(t,x,\widetilde{\xi}) \\ &-\rho_\Theta(\eta) N_j(y) \partial_t \partial_{x_j} V_0(t,x,\widetilde{\xi}) - \rho_\Theta(\eta)\beta_j^\Theta \partial_{\widetilde{\xi}_j} V_2(t,x,\widetilde{\xi};y,\eta) \\ &+\mathrm{div}_\xi (a^\Theta(\eta)\nabla_\eta V_2(t,x,\widetilde{\xi};y,\eta)) + \mathrm{div}_\xi (a^\Theta(\eta)\nabla_\xi (N^*(\eta)\cdot \nabla_\xi V_0(t,x,\widetilde{\xi})) \\ &+\mathrm{div}_\xi (a^\Theta(\eta)\nabla_\xi (N(y)\cdot \nabla_x V_0(t,x,\widetilde{\xi})) + \mathrm{div}_\xi (a^\Theta(\eta)\nabla_\xi V_1(t,x,\widetilde{\xi})) \\ &+b_j^\Theta(\eta)\partial_{\xi_j} V_2(t,x,\widetilde{\xi};y,\eta), \end{aligned}$$

and

$$R_2\big(t,x,\widetilde{\xi};\eta\big)=\Big\{(a^{\rm eff}_{ij}-a^{\Theta}_{ij}(\eta))(\varphi_k(\eta)-\hat{\varphi}_k)$$
$$-\partial_{\eta_j}(a^{\Theta}_{jl}(\eta)(\psi_{ik}(\eta)-\hat{\psi}_{ik}))-a^{\Theta}_{jl}(\eta)\partial_{\eta_l}\psi_{ik}(\eta)$$
$$+(\beta^{\Theta}_j-b^{\Theta}_j(\eta))(\psi_{ik}(\eta)-\hat{\psi}_{ik})\Big\}$$
$$\times\partial_{\xi_j}\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=1}\Big);$$
$$R_3\big(t,x,\widetilde{\xi};\eta\big)=(\rho_\Theta(\eta)a^{\rm eff}_{jl}-a^{\Theta}_{jl})(\psi_{ik}(\eta)-\hat{\psi}_{ik})$$
$$\times\partial_{\xi_l}\partial_{\xi_j}\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\widetilde{\xi}\big)\Big|_{\xi_d=1}\Big).$$

Notice that, in view of (3.24) and (3.28),

$$\int\limits_Y R_1\big(t,x,\widetilde{\xi};y,\eta\big)\,d\eta=0.$$

Thus, there exists $\chi\big(t,x,\widetilde{\xi};y,\eta\big)$, periodic in η, such that

$$-{\rm div}_\eta\chi=R_1\big(t,x,\widetilde{\xi};y,\eta\big).$$

Consequently,

$$R_1\big(t,x,\widetilde{\xi};y,\frac{\xi}{\varepsilon}\big)=-\varepsilon{\rm div}_\xi\chi\big(t,x,\widetilde{\xi};y,\frac{\xi}{\varepsilon}\big)+\varepsilon{\rm div}_\xi\chi\big(t,x,\widetilde{\xi};y,\eta\big)\Big|_{\eta=\frac{\xi}{\varepsilon}}.$$

It is easy to see that, for sufficiently small ε,

$$\int\limits_\Omega\big[\chi\big(t,x,\widetilde{\xi};y,\frac{\xi}{\varepsilon}\big)\big]^2\,d\xi\le C\int\limits_\Omega\int\limits_Y\big[R_1\big(t,x,\xi;y,\eta\big)\big]^2\,d\eta\,d\xi$$

with the constant C independent of ε. To estimate the norm on the right-hand side of the last inequality, we notice that each term in R_1 is a product of the form

$$F(y,\eta)\,\partial_t^r\partial_{\xi_j}^m V_0(t,x,\widetilde{\xi})$$

with a bounded periodic function $F(y,\eta)$. It is a classical matter to show that the derivatives V_0 are exponentially decreasing at infinity. Consequently,

$$\int\limits_\Omega\big[\chi\big(t,x,\widetilde{\xi};y,\frac{\xi}{\varepsilon}\big)\big]^2\,d\xi\le C$$

for $x_d\in I$. Then, multiplying equation (3.30) by $W_\varepsilon-V_\varepsilon$, integrating by parts taking into account (3.7), the exponential decay of boundary layers and of V_0, we obtain

$$\int\limits_\Omega|W_\varepsilon-V_\varepsilon|^2\,d\xi\le C\,\varepsilon^4,\quad t\ge0. \tag{3.31}$$

Note that due to the presence of the boundary layer correctors, the boundary conditions on $\Gamma^+\cap\Gamma^-$ in (3.13) are satisfied up to the second order in ε. It remains to notice that for $t\ge0$ and $x\in\Omega$ such that $x_d\in I\Subset(-1,1)$

$$\int\limits_\Omega|W_\varepsilon(t,x,\xi)-V_1^\varepsilon(t,x,\xi)|^2 d\xi\le C\,\varepsilon^4,$$

where V_1^ε is the first-order approximation of V_ε defined by (3.22). Combining the last two estimates finishes the proof of Proposition 3.1. □

Combining the previous estimates on the aproximations of Φ_ε (Lemma 3.1) and of V_ε (Proposition 3.1), we deduce similar result for the asymptotics of the Green function $K_\varepsilon(t,x,\xi)$. We do not give the proofs of the two lemmas below since they are very similar to their counterpart given in Section 2 in the case of a rod.

Lemma 3.2. *Assume that conditions* (**A1**) − (**A2**) *are satisfied. Let* K_ε *be the Green function solving* (3.12). *For* $t \geq t_0 > 0$ *and* $x \in \Omega$ *such that* $x_d \in I \Subset (-1,1)$, *we have*

$$\int\limits_\Omega |K_\varepsilon(t,x,\xi) - K_1^\varepsilon\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)|^2 d\xi \leq C\,\varepsilon^4,$$

where K_1^ε *is a first-order approximation of* K_ε *given by*

$$\begin{aligned} K_1^\varepsilon\big(t,x,\widetilde{\xi}\big) &= K_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\cdot\nabla_x K_0\big(t,x,\widetilde{\xi}\big) \\ &+\varepsilon\, N\big(\frac{\xi}{\varepsilon}\big)\cdot\nabla_\xi K_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\, K_1(t,x,\widetilde{\xi}) - \varepsilon\, V_{bl}^\varepsilon(t,x,\widetilde{\xi}), \end{aligned} \tag{3.32}$$

$\widetilde{\xi}$ *is the moving coordinate defined by* (3.21), $K_0 = \Phi_0 - V_0$ *is the Green function of the effective problem* (3.9), N, N^* *are the cell solutions of* (3.15), (3.16), *respectively,* $V_{\mathrm{bl}}^\varepsilon$ *is defined by* (3.27) *and* $K_1(t,x,\xi) = -V_1(t,x,\xi)$ *with* V_1 *the solution of* (3.28).

Lemma 3.3. *Denote by* I^+, I^- *compact subsets of* $(-1,1]$ *and* $[-1,1)$, *respectively. Let conditions* (**A1**) − (**A2**) *be fulfilled. Then, for* $x,\xi \in \Omega$ *such that* $x_d \in I^+$, $\xi_d \in I^-$, *and* $t \geq t_0 > 0$, *the following estimate holds true:*

$$|K_\varepsilon(t,x,\xi) - K_1^\varepsilon\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)| \leq C\,\varepsilon^2, \tag{3.33}$$

with the constant C *depending on* I^+, I^-, Λ, d *and independent of* ε.

3.2.2 Asymptotics of $u^\varepsilon(t,x)$

Recall that v^ε as a solution of (3.4), is represented in terms of the Green function K_ε by (3.11). Obviously,

$$\begin{aligned} v^\varepsilon(t,x) = &\int\limits_\Omega K_1^\varepsilon\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi \\ +&\int\limits_\Omega \Big(K_\varepsilon(t,x,\xi) - K_1^\varepsilon\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\Big)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi, \end{aligned} \tag{3.34}$$

where K_1^ε is the first order approximation of K_ε given by (3.32). Suppose that the initial function is such that $u_0(x',-1) \neq 0$. The case $u_0(x',-1) = \cdots =$

$\partial_{\xi_d}^{k-1} u_0(x', -1) = 0$, $\partial_{\xi_d}^{k} u_0(x', -1) \neq 0$ can be considered similarly. With the help of Lemma 3.3 we estimate the second integral in (3.34).

$$\Big|\int\limits_{\Omega} \Big(K_\varepsilon(t,x,\xi) - K_1^\varepsilon\big(t, x - \frac{\beta^\Theta}{\varepsilon}\, t, \xi\big)\Big)\, u_0(\xi)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\, d\xi\Big|$$
$$\le C\,\varepsilon^3 \int\limits_{\mathbb{R}^{d-1}} |u_0(\xi', -1)|\, d\xi' \int\limits_0^{+\infty} e^{-\Theta z_d}\, dz_d \le C\,\varepsilon^3.$$

To complete the proof it remains to compute the asymptotic behavior of the first integral in (3.34). Denote

$$v_0^\varepsilon(\xi) = u_0(\xi)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}.$$

Then, by definition (3.32) of K_1^ε,

$$\begin{aligned}
&\int\limits_{\Omega} K_1^\varepsilon(t,x,\xi)\, v_0^\varepsilon(\xi)\, d\xi = \int\limits_{\Omega} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_{\Omega} N_j^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_{\Omega} v^-\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d+1}{\varepsilon}\big)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_{\Omega} N_j\big(\frac{x}{\varepsilon}\big)\, \partial_{x_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_{\Omega} \Big(K_1\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big) - \hat v^-\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1}\Big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_{\Omega} \Big(v^+\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d-1}{\varepsilon}\big) - \hat v^+\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=1}\Big)\, v_0^\varepsilon(\xi)\, d\xi.
\end{aligned} \tag{3.35}$$

Notice that $K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big) = K_0\big(t, x, \xi + \frac{\beta^\Theta}{\varepsilon} t\big)$ since $\beta_d^\Theta = 0$ and Ω is bounded only in the x_d-direction. Expanding K_0 and u_0 into Taylor series with respect to ξ_d, for $t \ge t_0 > 0$, we obtain

$$\begin{aligned}
&\int\limits_{\Omega} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&= \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' \\
&\times \int\limits_{-1}^{1} (\xi_d+1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_d+1)}{\varepsilon}} d\xi_d + O(\varepsilon^3) \\
&= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' \\
&\times \int\limits_0^{+\infty} z_d\, p_\Theta^{-1}\big(\frac{\xi'}{\varepsilon}, z_d\big)\, e^{-\Theta z_d} dz_d + O(\varepsilon^3).
\end{aligned}$$

The function

$$\psi(\zeta') = \int\limits_0^{+\infty} z_d\, p_\Theta^{-1}\big(\zeta', z_d\big)\, e^{-\Theta z_d} dz_d,$$

is $(0,1]^{d-1}$-periodic and belongs to $H^1((0,1]^{d-1})$. By the classical mean-value theorem, we deduce the asymptotic behavior of the first term in (3.35)

$$\begin{aligned}
&\int\limits_\Omega K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' \\
&\times \int\limits_{(0,1]^{d-1}} \int\limits_0^{+\infty} z_d\, p_\Theta^{-1}(z', z_d)\, e^{-\Theta z_d} dz_d\, dz' + O(\varepsilon^3).
\end{aligned}$$

By similar arguments, the other terms in (3.35) admit the representations

$$\begin{aligned}
&\int\limits_\Omega N_j^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' \\
&\times \int\limits_{(0,1]^{d-1}} \int\limits_0^{+\infty} N_d^*(z)\, p_\Theta^{-1}(z)\, e^{-\Theta z_d} dz_d\, dz' + O(\varepsilon^3)
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_\Omega v^-\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d+1}{\varepsilon}\big)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' \\
&\times \int\limits_{(0,1]^{d-1}} \int\limits_0^{+\infty} v^-(z)\, p_\Theta^{-1}(z)\, e^{-\Theta z_d} dz_d\, dz' + O(\varepsilon^3).
\end{aligned}$$

Noticing that $K_1\big|_{\xi_d=-1} = \hat{v}^- \partial_{\xi_d} K_0\big|_{\xi_d=-1}$, and $\partial_{x_j} K_0\big|_{\xi_d=-1} = 0$, one can see that the last three integrals in (3.35) are of order ε^3. We emphasize that, in view of (3.26), the terms containing boundary layer correctors near Γ^+ are negligible. Finally,

$$v^\varepsilon(t,x) = \varepsilon^2 M \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi' + O(\varepsilon^3),$$

where the constant M is given by (3.10). This completes the proof of Theorem 3.1. □

As already said in Remark 3.2, Theorem 3.1 provides only the leading term of the asymptotics of u^ε. However, due to the presence of the exponentially large factor $e^{\Theta(x_d+1)/\varepsilon}$, we are mostly interested in the asymptotics of u^ε in a ε-neighbourhood of Γ^+, where u^ε is maximum and where both, the leading and the corrector terms (including the boundary layer corrector) are of the same order.

In Theorem 3.2 below we construct the corrector terms for u^ε, that improves significantly the asymptotics of u^ε in the vicinity of Γ^+ and, therefore, makes the result of Theorem 3.1 complete.

Let us define the first-order approximation for u^ε by

$$\begin{aligned} U^\varepsilon(t,x) &= u\big(t,x-\frac{\beta^\Theta}{\varepsilon}t\big) + \varepsilon\, N_k\big(\frac{x}{\varepsilon}\big)\, \partial_{x_k} u\big(t,x-\frac{\beta^\Theta}{\varepsilon}t\big) \\ &+\varepsilon u_1\big(t,x-\frac{\beta^\Theta}{\varepsilon}t\big) + \varepsilon\Big[v_*^+\big(\frac{x'}{\varepsilon},\frac{x_d-1}{\varepsilon}\big) - \hat v_*^+\Big]\,\partial_{x_1} u\big(t,x-\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{x_d=1}. \end{aligned} \tag{3.36}$$

Here $u(t,x)$ is the solution of the homogenized problem (3.9), N solves (3.15). The boundary layer corrector $v_*^+(y)$ are defined similarly to $v^+(y)$ (see (3.27)), except for the fact that the adjoint operator is replaced with the direct one. Namely, v_*^+ solves the following problem in $\mathbb{G}^- = (0,1]^{d-1}\times(-\infty,0)$:

$$\begin{cases} A_\Theta v_*^+ = 0, & y\in\mathbb{G}^-,\\ v_*^+(y',0) = -N_d(y',0), & \\ y'\mapsto v_*^+(y',y_d) \text{ is } (0,1]^{d-1}-\text{periodic}. & \end{cases}$$

Since $\beta_d = 0$, there exists a unique bounded solution v_*^+ and it stabilizes to some constant $\hat v_*^+$ at an exponential rate, as $y_d\to-\infty$.

The function $u_1(t,x)$ in (3.36) solves the following problem

$$\begin{cases} \partial_t u_1 = \mathrm{div}(a^{\mathrm{eff}}\,\nabla u_1) + F(t,x), & (t,x)\in(0,T)\times\Omega,\\ u_1(t,x) = \hat v_*^+\,\partial_{x_d} u(t,x), & (t,x)\in(0,T)\times(\Gamma^-\cup\Gamma^+),\\ u_1(0,x) = 0, & x\in\Omega, \end{cases}$$

where

$$\begin{aligned} F(t,x) = \partial_{x_k}\partial_{x_i}\partial_{x_j} u(t,x) \int\limits_Y \big[a^\Theta_{kl}\partial_{\eta_l} N^2_{ij} \\ + a^\Theta_{ij} N_k - b^\Theta_k N^2_{ij} + \beta^\Theta_k \rho_\Theta N^2_{ij} - a^{\mathrm{eff}}_{ij}\,\rho_\Theta\, N_k\big]\, d\eta. \end{aligned}$$

Theorem 3.2. *Let the assumptions of Theorem 3.1 be fulfilled. The refined asymptotics of the solution u^ε of problem* (3.1)*, for $t\ge t_0>0$ and $x\in\Omega$, takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_d+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\,\big[U^\varepsilon(t,x) + R_\varepsilon(t,x)\big],$$

where U^ε is given by(3.36)*, and, for some constant $C = C(\Lambda,d)$, the remainder term satisfies the estimate*

$$|R_\varepsilon(t,x)| \le C\,\varepsilon(1-x_d),$$

which is uniform for $t\ge t_0>0$, $x\in\Omega$.

The proof of Theorem 3.2 is similar to that of Theorem 2.3 in the case of a rod. We leave it to the reader.

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