Geometrical optics formulaes for Helmholtz equation

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GEOMETRICAL OPTICS FORMULAES FOR HELMHOLTZ EQUATION

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Abstract. The present work deals with high frequency Helmholtz equation resolution using geometrical optics. We give formulaes in dimension 2 and 3 for mixed Dirichlet, Neumann and Robin boundaries conditions.

1. Introduction

Let $(K_j)_{j=1,\cdots,N}$ be a set of regular, disjoint and strictly convex compacts in $\mathbb{R}^d$ $d = 2$ or 3. We suppose that for $j \neq l$, $K_j \bigcap K_l = \emptyset$. Let $\Omega_j = K_j^c$ and $\Omega = \bigcap_{j=1}^{N} \Omega_j$. We note Γ_j the boundary of K_j and $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$.

Let u^{inc} be the incident wave given by

$$u^{inc}(x) = e^{-ik\varphi^{inc}(x)} a^{inc}(x) \tag{1.1}$$

which satisfy Helmholtz equation in $\mathbb{R}^d$. We want to solve, at *high frequency*, Helmholtz equation

$$\Delta u + k^2 u = 0, \text{ in } \Omega \tag{1.2}$$

with boundaries conditions

$$u(\gamma) = -u^{inc}(\gamma), \quad \forall \gamma \in \Gamma_D, \tag{1.3}$$

$$\frac{\partial u}{\partial n}(\gamma) = -\frac{\partial u^{inc}}{\partial n}(\gamma), \quad \forall \gamma \in \Gamma_N, \tag{1.4}$$

$$\alpha(k)\frac{\partial u}{\partial n}(\gamma) + \beta(k) u(\gamma) = -\left(\alpha(k)\frac{\partial u^{inc}}{\partial n}(\gamma) + \beta(k) u^{inc}(\gamma)\right), \quad \forall \gamma \in \Gamma_R, \tag{1.5}$$

and outgoing Sommerfeld radiation condition

$$r^2 \left(\frac{\partial u}{\partial r} - iku\right) \text{ bound for } r = |x| \to \infty. \tag{1.6}$$

We assume that

$$\alpha(k) = \sum_{j=0}^{\infty} \frac{\alpha_j}{k^j}, \quad \alpha_j \in \mathbb{C}, \tag{1.7}$$

$$\beta(k) = \sum_{j=-1}^{\infty} \frac{\beta_j}{k^j}, \quad \beta_j \in \mathbb{C}, \tag{1.8}$$

and

$$\beta_{-1} + \imath\alpha_0\mu \neq 0 \ \ \forall \mu \in [0;1] \tag{1.9}$$

Here we take Γ_D, Γ_N and Γ_R such that $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_R$, $\overset{o}{\Gamma}_D \cap \overset{o}{\Gamma}_N = \emptyset$, $\overset{o}{\Gamma}_D \cap \overset{o}{\Gamma}_R = \emptyset$ and $\overset{o}{\Gamma}_N \cap \overset{o}{\Gamma}_R = \emptyset$.

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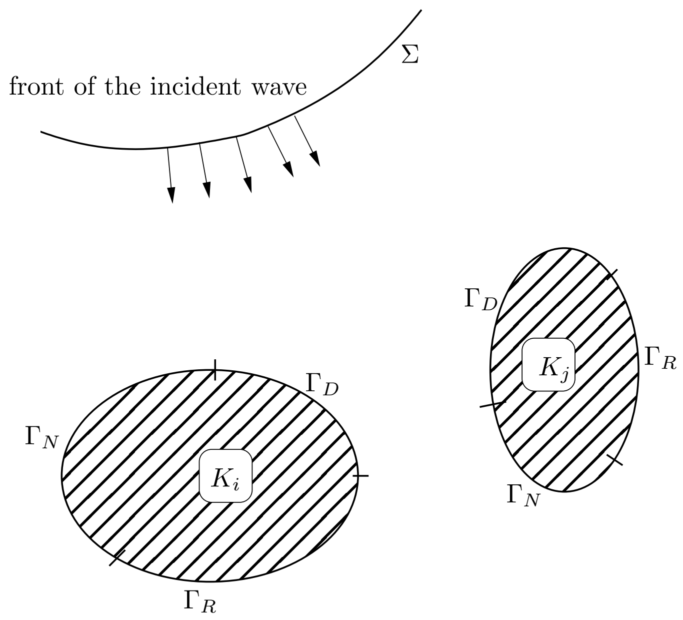

FIGURE 1. Exterior Helmholtz problem

We assume that, Σ, the front of the incident wave is given by

$$\Sigma = \left\{x \in \mathbb{R}^d \mid \varphi^{inc}(x) = \varphi_\Sigma\right\}$$

is a regular and orientable surface ($d = 3$) or curve ($d = 2$). We chose the orientation $N(\sigma) = \frac{\boldsymbol{\nabla}\,\varphi^{inc}(\sigma)}{\|\boldsymbol{\nabla}\,\varphi^{inc}(\sigma)\|}$.

Let $\Sigma_+ = \left\{x \in \mathbb{R}^d \mid \exists\sigma \in \Sigma, \exists t > 0, x = \sigma + t\,\boldsymbol{\nabla}\,\varphi^{inc}(\sigma)\right\}$, we also made following assumption :

(i) $\bigcup_{j=1}^{N} K_j \subset \Sigma_+$

(ii) curvature radius of Σ are negatives

At *high frequency* (means that wave length is small with respect to the obstacle boundary curvature radius), geometrical optics approximation for this problem is based on k^{-1} asymptotic expansion of $v(x) = u(x) + u^{inc}(x)$, where u is solution of problem (1.2,1.3,1.4,1.5,1.6). The first term of the k^{-1} expansion is the geometrical optics ansatz.

2. Notations and results

Definition 1 (Geometrical optic ray)**.** *Let $\rho = (\sigma, \gamma_1, \ldots, \gamma_l) = (\gamma_0, \gamma_1, \ldots, \gamma_l) \in \Sigma \times \Gamma^l$, we say that ρ is a geometrical optic ray coming from Σ, going through $x = \gamma_{l+1}$ and reflected l times if*

(1) $\gamma_j \neq \gamma_{j-1},\ \ \forall j \in \{1, \ldots, l+1\}$,

(2) $\left(\bigcup_{j=1}^{l+1}]\gamma_{j-1}; \gamma_j[\right) \cap \Gamma = \emptyset$,

(3) $\dfrac{\gamma_1 - \gamma_0}{|\gamma_1 - \gamma_0|} = \boldsymbol{\nabla}\,\varphi^{inc}(\gamma_0)$,

(4) ***reflections conditions*** *:* $\forall j \in \{1, \ldots, l\}$

$$\frac{\gamma_{j+1} - \gamma_j}{|\gamma_{j+1} - \gamma_j|} = \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|} - 2\left\langle \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|}, \boldsymbol{n}\,(\gamma_j)\right\rangle \boldsymbol{n}\,(\gamma_j),$$

(5) ***non grazing conditions*** *:* $\forall j \in \{1, \ldots, l\}$

$$\left\langle \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|}, \boldsymbol{n}\,(\gamma_j)\right\rangle \neq 0,$$

where $\boldsymbol{n}\,(\gamma)$ note the exterior normal at Γ in γ.

We note $\mathcal{R}_l(x)$ a such set of ρ and $\mathcal{R}(x) = \bigcup_{l \in \mathbb{N}^*} \mathcal{R}_l(x)$.

Definition 2 (Grazing ray)**.** *Let $\rho = (\sigma, \gamma_1, \dots, \gamma_l) = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \Sigma \times \Gamma^l$, we say that ρ is a grazing ray coming from Σ, going through $x = \gamma_{l+1}$ and reflected l times if conditions (1) to (4) of definition 1 are satisfied and not condition (5). We note $\mathcal{G}_l(x)$ a such set of ρ and $\mathcal{G}(x) = \bigcup_{l \in \mathbb{N}^*} \mathcal{G}_l(x)$.*

Definition 3 (Propagation operator)**.** *Let $t > 0$, we note $S_t^d \subset \mathbb{R}^{d\times d}$ the set of matrix $\mathbb{A}$ such that $\mathbb{I}+t\mathbb{A}$ is regular. We note $\mathcal{S}_t$ the following application :*

$$\begin{array}{rcccl} \mathcal{S}_t & : & S_t^d & \longrightarrow & \mathbb{R}^{d\times d} \\ & & \mathbb{A} & \longmapsto & \mathbb{A}(\mathbb{I}+t\mathbb{A})^{-1} \end{array}$$

Definition 4 (Reflection operator)**.** *Let $\mathbb{B} \in \mathbb{R}^{d\times d}$ a symmetric matrix. Let $\eta \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^d$ such that $\langle \zeta, \eta \rangle \neq 0$. We note $\mathcal{T}^d_{\mathbb{B},\eta,\zeta} : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{d\times d}$ given by :*
$\forall \mathbb{A} \in \mathbb{R}^{d\times d},\ \forall x \in \mathbb{R}^d$

$$\begin{aligned} (\mathcal{T}^d_{\mathbb{B},\eta,\zeta}(\mathbb{A}))x &= (\mathbb{A} - 2\langle \zeta, \eta\rangle \mathbb{B})x - 2\langle \eta, x\rangle (\mathbb{A}\eta + \mathbb{B}\zeta) \\ &\quad -2\langle \mathbb{A}\eta + \mathbb{B}\zeta, x\rangle \eta + 2\left[2\langle \mathbb{A}\eta, \eta\rangle - \tfrac{\langle \mathbb{B}\zeta, \zeta\rangle}{\langle \zeta, \eta\rangle}\right]\langle \eta, x\rangle \eta \end{aligned}$$

Definition 5 (Reflection coefficient)**.** *Let $\eta \in \mathbb{R}^d$, we note b_η the function defined on Γ by*

$$b_\eta(\gamma) = \begin{cases} -1, & \text{if } \gamma \in \Gamma_D, \\ 1, & \text{if } \gamma \in \Gamma_N, \\ \dfrac{\imath\alpha_0 \langle \eta, \boldsymbol{n}(\gamma)\rangle + \beta_{-1}}{\imath\alpha_0 \langle \eta, \boldsymbol{n}(\gamma)\rangle - \beta_{-1}}, & \text{if } \gamma \in \Gamma_R. \end{cases}$$

Theorem 1. *Let $x \in \Sigma_+$, such that $\mathcal{G}(x) = \emptyset$, and $\rho = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \mathcal{R}_l(x)$, $l \in \mathbb{N}^*$. We note $\mathbb{B}(\gamma)$ the curvature matrix of Γ at γ and $\boldsymbol{n}(\gamma)$ the exterior normal to Γ at γ. Let $u_\rho^{G.O.}(x)$ be the ray contribution :*

$$u_\rho^{G.O.}(x) = a_\rho(x) e^{-ik\varphi(x;\rho)}$$

with

$$\varphi(x;\rho) = \varphi_\rho^{(l)}(\gamma_l) + \|x - \gamma_l\|$$

$$a_\rho(x) = \frac{a_\rho^{(l)}(\gamma_l)}{\sqrt{\det(\mathbb{I} + \|x-\gamma_l\| \mathbb{A}_\rho^{(l)}(\gamma_l))}}$$

and for $j = l$ to 1,

$$\begin{cases} \varphi_\rho^{(j)}(\gamma_j) &= \varphi_\rho^{(j-1)}(\gamma_{j-1}) + \|\gamma_j - \gamma_{j-1}\| \\ a_\rho^{(j)}(\gamma_j) &= b_{\frac{\gamma_j - \gamma_{j-1}}{\|\gamma_j - \gamma_{j-1}\|}}(\gamma_j) \dfrac{a_\rho^{j-1}(\gamma_{j-1})}{\sqrt{\det\left(\mathbb{I} + \|\gamma_j - \gamma_{j-1}\| \mathbb{A}_\rho^{(j-1)}(\gamma_{j-1})\right)}} \\ \mathbb{A}_\rho^{(j)}(\gamma_j) &= \mathcal{T}^d_{\mathbb{B}(\gamma_j), \boldsymbol{n}(\gamma_j), \frac{\gamma_j - \gamma_{j-1}}{\|\gamma_j - \gamma_{j-1}\|}} \circ \mathcal{S}_{\|\gamma_j - \gamma_{j-1}\|}\left(\mathbb{A}_\rho^{(j-1)}(\gamma_{j-1})\right) \end{cases}$$

where $\varphi_\rho^{(0)}(\gamma_0) = \varphi^{inc}(\gamma_0)$, $\mathbb{A}_\rho^{(0)}(\gamma_0) = \operatorname{Hess}\varphi^{inc}(\gamma_0)$ and $a_\rho^{(0)}(\gamma_0) = a_0^{inc}(\gamma_0)$.

Then, we have

$$u^{inc}(x) = \sum_{\rho \in \mathcal{R}_0(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

and

$$u(x) = \sum_{\rho \in \mathcal{R}(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

3. Proof

To obtain theorem 1, we first give geometrical optics formulaes without obstacle, then with a stricly convox compact and finally with an union of stricly convex compacts.

3.1. **Without obstacles.** We want to solve, at high frequency, Helmholtz equation

$$\Delta u + k^2 u = 0 \ \text{ in } \mathbb{R}^d \tag{3.1}$$

using an asymptotic expansion in k^{-1} of the solution

$$u(x;k) = \sum_{j\in\mathbb{N}} \frac{a_j(x)}{k^j} e^{-\imath k\varphi(x)} \tag{3.2}$$

where we suppose that u is known on Σ.

By substituting expansion (3.2) in Helmholtz equation, we obtain

$$\begin{array}{rcl} k^2(1-\|\boldsymbol{\nabla}\,\varphi(x)\|^2)a_0(x) & & \\ +k\left[-\imath\left\{2(\boldsymbol{\nabla}\,\varphi(x),\boldsymbol{\nabla}\,a_0(x)) + a_0(x)\Delta\varphi(x)\right\} + (1-\|\boldsymbol{\nabla}\,\varphi(x)\|^2)a_1(x)\right) & & \\ +\sum_{j\in\mathbb{N}} k^{-j}\left(-\imath\left\{2(\boldsymbol{\nabla}\,\varphi(x),\boldsymbol{\nabla}\,a_{k+1}(x)) + a_{k+1}(x)\Delta\varphi(x)\right\} + \Delta a_k(x)\right) & = & 0 \end{array}$$

Equating the coefficients of different powers of k, we have :

- **Eikonal equation**

$$\|\boldsymbol{\nabla}\,\varphi(x)\|^2 = 1 \tag{3.3}$$

- **Transport equation for a_0**

$$\langle\boldsymbol{\nabla}\,\varphi(x),\boldsymbol{\nabla}\,a_0(x)\rangle + \frac{1}{2}a_0(x)\Delta\varphi(x) = 0 \tag{3.4}$$

- **Transport equation for a_j in function of a_{j-1}**

$$\langle\boldsymbol{\nabla}\,\varphi(x),\boldsymbol{\nabla}\,a_j(x)\rangle + \frac{1}{2}a_j(x)\Delta\varphi(x) = \frac{\imath}{2}\Delta a_{j-1}(x) \tag{3.5}$$

3.1.1. *Solution of Eikonal equation (3.3).* Derivating eikonal equation (3.3) give

$$\operatorname{Hess}\varphi(x)\,\boldsymbol{\nabla}\,\varphi(x) = 0 \tag{3.6}$$

This equation can be solved by the caracteristics method :
Let $X'(t) = \boldsymbol{\nabla}\,\varphi(X(t))$ with $X(0) = \sigma \in \Sigma$ then we have :

$$\begin{array}{rcl} \frac{d}{dt}\left(\boldsymbol{\nabla}\,\varphi(X(t))\right) & = & \operatorname{Hess}\varphi(X(t))X'(t) \\ & = & \operatorname{Hess}\varphi(X(t))\,\boldsymbol{\nabla}\,\varphi(X(t)) \\ & = & 0 \end{array}$$

That is to say

$$\boldsymbol{\nabla}\,\varphi(X(t)) = \boldsymbol{\nabla}\,\varphi(X(0))$$

hence

$$X(t) = X(0) + t\,\boldsymbol{\nabla}\,\varphi(X(0)).$$

and $\boldsymbol{\nabla}\,\varphi(X(t))$ is constant along the line $X(t) = X(0) + t\,\boldsymbol{\nabla}\,\varphi(X(0))$.
Owing to the construction of Σ, we have

$$X(t) = \sigma + t\boldsymbol{N}(\sigma)$$

with $\sigma \in \Sigma$ and $\boldsymbol{N}(\sigma) = \boldsymbol{\nabla}\,\varphi(\sigma)$ unitary normal vector to Σ at σ.

We also have :

$$\begin{array}{rcl} \frac{d}{dt}\left(\varphi(X(t))\right) & = & \langle\boldsymbol{\nabla}\,\varphi(X(t)), X'(t)\rangle \\ & = & \langle\boldsymbol{\nabla}\,\varphi(X(0)),\boldsymbol{\nabla}\,\varphi(X(0))\rangle \\ & = & 1 \end{array}$$

Thus

$$\varphi(X(t)) = \varphi(X(0)) + t.$$

We have proved following lemma :

Lemma 1. (1) *Caracteristic curves of Eikonal equation are geometrical optic rays,*

(2) *Phase is linear along geometrical optic rays :*

$$\forall\sigma\in\Sigma,\ \forall t\geq 0\ \ \varphi(\sigma + t\boldsymbol{N}(\sigma)) = \varphi(\sigma) + t. \tag{3.7}$$

(3) *We have* $\forall\sigma\in\Sigma$

$$\boldsymbol{\nabla}\,\varphi(\sigma) = \boldsymbol{N}(\sigma) \tag{3.8}$$

and

$$\operatorname{Hess}\varphi(\sigma)\boldsymbol{N}(\sigma) = 0. \tag{3.9}$$

3.1.2. *Solution of transport equation (3.4)* . We want to solve

$$\begin{cases} \langle \nabla \varphi(x), \nabla a_0(x)\rangle + \frac{1}{2}a_0(x)\Delta\varphi(x) = 0 \\ a_0 \text{ given on } \Sigma \end{cases} \tag{3.10}$$

In fact, we only have to solve this equation along the ray $\sigma + t\boldsymbol{N}(\sigma)$ where $\sigma \in \Sigma$ and $\boldsymbol{N}(\sigma) = \nabla \varphi(\sigma)$.

Thus, transport equation for a_0 becomes

$$\begin{cases} \frac{d}{dt}(a_0(\sigma + t\boldsymbol{N}(\sigma)) + \frac{1}{2}\Delta\varphi(\sigma + t\boldsymbol{N}(\sigma))a_0(\sigma + t\boldsymbol{N}(\sigma)) = 0, \\ a_0(\sigma) \text{ given on } \Sigma. \end{cases} \tag{3.11}$$

We now have to compute $\Delta\varphi(\sigma + t\boldsymbol{N}(\sigma)) = \operatorname{Tr}\operatorname{Hess}\varphi(\sigma(t))$.

Let $\sigma \in \Sigma$, we note for $d = 3$ (resp. $d = 2$) $\mathcal{B}_\Sigma(\sigma) = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{N}\}(\sigma)$ (resp. $\mathcal{B}_\Sigma(\sigma) = \{\boldsymbol{u}, \boldsymbol{N}\}(\sigma)$) the **direct orthonormal curvature basis** of Σ at σ. Here $\boldsymbol{u}$ and $\boldsymbol{v}$ are the direction of maximum and minimal principal curvature $k_1^{(0)}(\sigma)$ and $k_2^{(0)}(\sigma)$ (resp. $\boldsymbol{u}$ is the tangent vector and $k^{(0)}(\sigma)$ the curvature). By hypothesis on Σ, we have

$$k_2^{(0)}(\sigma) \le k_1^{(0)}(\sigma) \le 0 \text{ (resp. } k^{(0)}(\sigma) \le 0).$$

With these notations, we have in basis $\mathcal{B}_\Sigma(\sigma)$

$$\operatorname{Hess}\varphi(\sigma) = \begin{pmatrix} -k_1^{(0)}(\sigma) & 0 & 0 \\ 0 & -k_2^{(0)}(\sigma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (resp. } \operatorname{Hess}\varphi(\sigma) = \begin{pmatrix} -k^{(0)}(\sigma) & 0 \\ 0 & 0 \end{pmatrix})$$

Lemma 2. *Let $\sigma \in \Sigma$. We note $\sigma(t) = \sigma + t\boldsymbol{N}(\sigma)$ then*

$$\forall t \ge 0, \ (\operatorname{Hess}\varphi)(\sigma(t)) = \mathcal{S}_t(\operatorname{Hess}\varphi(\sigma)) \tag{3.12}$$

In basis $\mathcal{B}_\Sigma(\sigma)$ for $d = 3$ (resp. $d = 2$)

$$\operatorname{Hess}\varphi(\sigma(t)) = \begin{pmatrix} -k_1^{(0)}(\sigma(t)) & 0 & 0 \\ 0 & -k_2^{(0)}(\sigma(t)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (\textit{resp.} } \operatorname{Hess}\varphi(\sigma(t)) = \begin{pmatrix} -k^{(0)}(\sigma(t)) & 0 \\ 0 & 0 \end{pmatrix})$$

with

$$0 \ge k_j^{(0)}(\sigma(t)) = \frac{k_j^{(0)}(\sigma)}{1 - tk_j^{(0)}(\sigma)} \ge k_j^{(0)}(\sigma) \ \ j = 1, 2$$

$$(\textit{resp. } 0 \ge k^{(0)}(\sigma(t)) = \frac{k^{(0)}(\sigma)}{1 - tk^{(0)}(\sigma)} \ge k^{(0)}(\sigma))$$

Proof. To simplify notations, we note $\forall t \ge 0$

$$\mathbb{A}_t(\sigma) = (\operatorname{Hess}\varphi)(\sigma + t\boldsymbol{N}(\sigma)).$$

Let $\mathcal{V}(\sigma) \subset \mathbb{R}^d$ be a neighborhood of $\sigma \in \Sigma$. We have,$\forall x \in \mathcal{V}(\sigma)$

$$\nabla \varphi(x) = \nabla \varphi(\sigma) + \mathbb{A}_0(\sigma)(x - \sigma) + O(\|x - \sigma\|) \tag{3.13}$$

Substituing $\nabla \varphi(x)$ by (3.13) in Eikonal equation (3.3) give, $\forall x \in \mathcal{V}(\sigma)$:

$$\begin{aligned} 1 &= \left\| \nabla \varphi(\sigma) + \mathbb{A}_0(\sigma)(x - \sigma) + O\left(\|x - \sigma\|^2\right) \right\|^2 \\ &= \|\nabla \varphi(\sigma)\|^2 + 2\langle \mathbb{A}_0(\sigma)(x - \sigma), \nabla \varphi(\sigma)\rangle + O(\|x - \sigma\|^2) \\ &= 1 + 2\langle \mathbb{A}_0(\sigma)(x - \sigma), \boldsymbol{N}(\sigma)\rangle + O(\|x - \sigma\|^2). \end{aligned}$$

Using $\mathbb{A}_0(\sigma)$ symmetry, we obtain

$$\forall x \in \mathcal{V}(\sigma) \ , \ \langle \mathbb{A}_0(\sigma)\boldsymbol{N}(\sigma), x - \sigma\rangle = 0$$

Thus

$$\mathbb{A}_0(\sigma)\boldsymbol{N}(\sigma) = 0. \tag{3.14}$$

We now want to obtain a similar formula for $\mathbb{A}_t(\sigma)$.

We know that $\mathbb{A}_0(\sigma)$ is the curvature matrix of Σ at σ and in basis $\mathcal{B}_\Sigma(\sigma)$ we have for $d = 3$ (resp. $d = 2$)

$$\mathbb{A}_0(\sigma) = \begin{pmatrix} -k_1^{\Sigma}(\sigma) & 0 & 0 \\ 0 & -k_2^{\Sigma}(\sigma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (resp. } \mathbb{A}_0(\sigma) = \begin{pmatrix} -k^{\Sigma}(\sigma) & 0 \\ 0 & 0 \end{pmatrix}).$$

By hypothesis, curvature radius of Σ are negative and

$$k_2^\Sigma(\sigma) \le k_1^\Sigma(\sigma) \le 0 \ \ (\text{resp. } k^\Sigma(\sigma) \le 0)$$

Thus, $\forall t \ge 0$, $\mathbb{I} + t\mathbb{A}_0(\sigma)$ is regular. Let $x = \sigma + t\boldsymbol{N}(\sigma)$ with $t > 0$, then, there exists a neighborhood $\mathcal{V}(x) \subset \mathbb{R}^d$ of x such that

$$\forall x' \in \mathcal{V}(x),\ \exists \sigma' \in \mathcal{V}(\sigma) \text{ and } t' \text{ in a neighborhood of } t \text{ verifying } x' = \sigma' + t'\boldsymbol{N}(\sigma')$$

So we obtain by Taylor's developpement

$$\boldsymbol{\nabla}\varphi(x') = \boldsymbol{\nabla}\varphi(x) + \mathbb{A}_t(\sigma)(x' - x) + O(\|x' - x\|) \tag{3.15}$$

Substituing $\boldsymbol{\nabla}\varphi(x')$ for (3.15) in eikonal equation (3.3) give

$$\mathbb{A}_t(\sigma)\boldsymbol{N}(\sigma) = 0 \tag{3.16}$$

To prove formula (3.12) we write

$$\begin{aligned} x' - x &= \sigma' + t'\boldsymbol{N}(\sigma') - \sigma - t\boldsymbol{N}(\sigma) \\ &= \sigma' - \sigma + (t' - t)\boldsymbol{N}(\sigma) + t'(\boldsymbol{N}(\sigma') - \boldsymbol{N}(\sigma)). \end{aligned}$$

But, due to (3.13)

$$\boldsymbol{N}(\sigma') - \boldsymbol{N}(\sigma) = \mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|) \tag{3.17}$$

and, thus

$$x' - x = \sigma' - \sigma + (t' - t)\boldsymbol{N}(\sigma) + t'\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|). \tag{3.18}$$

Substituing $x' - x$ for (3.18) in (3.15) and using (3.16) give

$$\boldsymbol{N}(\sigma') = \boldsymbol{N}(\sigma) + \mathbb{A}_t(\sigma)(\sigma' - \sigma) + t'\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\| + \|t' - t\|).$$

Now, with formula (3.17) we found

$$\begin{gathered} \mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|) \\ = \\ \mathbb{A}_t(\sigma)(\sigma' - \sigma) + t'\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\| + \|t' - t\|) \end{gathered}$$

That is to say

$$\begin{aligned} \mathbb{A}_0(\sigma) &= \mathbb{A}_t(\sigma) + t\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma) \\ &= \mathbb{A}_t(\sigma)(\mathbb{I} + t\mathbb{A}_0(\sigma)). \end{aligned}$$

But, $\forall t \ge 0$, the matrix $(\mathbb{I} + t\mathbb{A}_0(\sigma))$ is regular and formula (3.12) is proved. □

Lemma 3. *Let $\sigma \in \Sigma$. Along the ray $\sigma(t) = \sigma + t\boldsymbol{N}(\sigma)$, $\forall t \ge 0$, we have*

$$a_0(\sigma(t)) = \frac{a_0(\sigma)}{\sqrt{\det(\mathbb{I} + t\,\mathrm{Hess}\,\varphi(\sigma))}} \tag{3.19}$$

and $\forall t > 0$, $\forall t' \ge 0$ such that $t > t'$

$$|a_0(\sigma(t))| \le |a_0(\sigma(t'))|. \tag{3.20}$$

The previous inegality becomes strict if, at least, one of the curvature radius of Σ is strictly negative.

Proof. By definition of $\mathbb{A}_t(\sigma)$ (see proof of lemma 2)

$$\Delta\varphi(\sigma(t)) = \mathrm{Tr}(\mathbb{A}_t(\sigma))$$

Then we solve equation (3.11):

$$\begin{aligned} a_0(\sigma(t)) &= a_0(\sigma)e^{-\frac{1}{2}\int_0^t \mathrm{Tr}\,\mathbb{A}_\tau(\sigma)d\tau} \\ &= a_0(\sigma)e^{-\frac{1}{2}\int_0^t \mathrm{Tr}\left[\mathbb{A}_0(\sigma)(I + \tau\mathbb{A}_0(\sigma))^{-1}\right]d\tau} \\ &= a_0(\sigma)e^{-\frac{1}{2}[\log\det(\mathbb{I} + \tau\mathbb{A}_0(\sigma))]_0^t} \\ &= \frac{a_0(\sigma)}{\sqrt{\det(\mathbb{I} + t\mathbb{A}_0(\sigma))}}. \end{aligned}$$

We have seen, in proof of lemma 2, that eigenvalues of $\mathbb{A}_0(\sigma)$ are positive, thus

$$\det(\mathbb{I} + t'\mathbb{A}_0(\sigma)) \ge \det(\mathbb{I} + t\mathbb{A}_0(\sigma)) \ge 1.$$

and inequality (3.20) is immediatly prooved.

We can remark that if, at least, one of the curvature of Σ at σ is strictly negative then, for $d = 3$ (resp. $d = 2$)

$$k_2^\Sigma(\sigma) < k_1^\Sigma(\sigma) \le 0 \ \ (\text{resp. } \ k^\Sigma(\sigma) < 0)$$

and thus inequality (3.20) become strict.

□

3.1.3. *Conclusion.* The knowledge of a_0, φ, $\boldsymbol{\nabla}\varphi$ and $\operatorname{Hess}\varphi$ on Σ is sufficient to compute them for all x in Σ_+ using formulaes along geometrical optic rays comming through x and we have

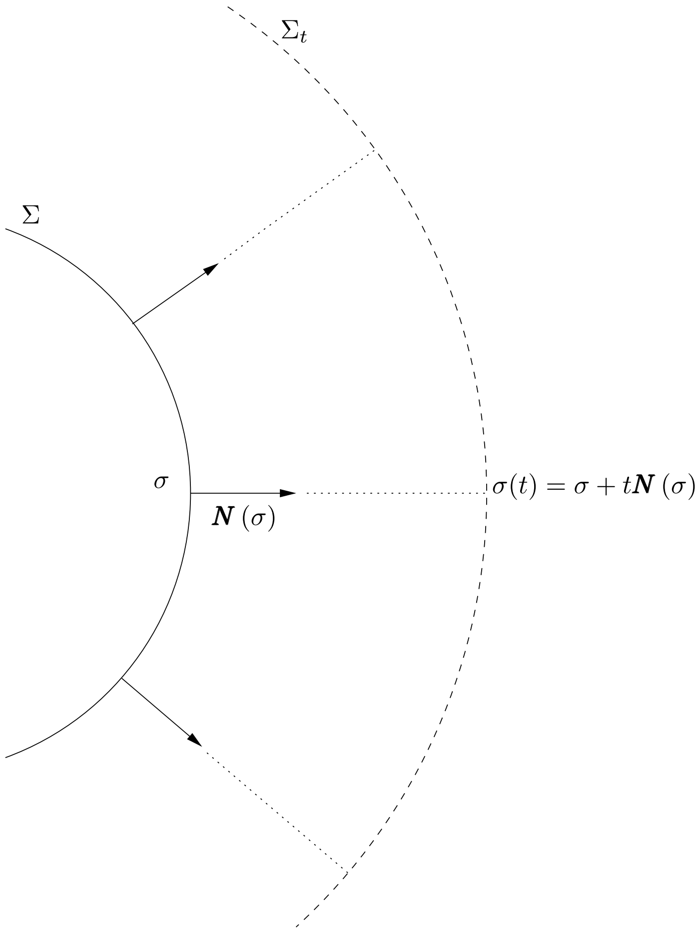

FIGURE 2. Wave Propagation

Theorem 2. *Let $x \in \Sigma_+$, then there exist an unique $\sigma \in \Sigma$ such that*

$$x = \sigma + \|x - \sigma\| \boldsymbol{N}(\sigma) \tag{3.21}$$

and solution of (3.1-3.2) is given by

$$u(x;k) = \frac{a_0(\sigma)}{\sqrt{\det\left(\mathbb{I} + \|x-\sigma\| \operatorname{Hess}\varphi(\sigma)\right)}} e^{-ik(\varphi(\sigma)+\|x-\sigma\|)} + O\left(\frac{1}{k}\right) \tag{3.22}$$

Proof. By definition of Σ_+, we have existence of $\sigma \in \Sigma$ satisfying (3.21).
For unicity, we suppose there exists $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$ satisfying (3.21). By convexity hypothesis on Σ, we have

$$\langle \sigma_2 - \sigma_1, \boldsymbol{N}(\sigma_1) \rangle \leq 0$$

and then

$$\langle \sigma_2 - \sigma_1, x - \sigma_1 \rangle \leq 0.$$

But

$$\langle \sigma_2 - \sigma_1, x - \sigma_1 \rangle = \langle \sigma_2 - \sigma_1, x - \sigma_2 \rangle + \|\sigma_2 - \sigma_1\|^2$$

so we obtain

$$\langle \sigma_2 - \sigma_1, x - \sigma_2 \rangle \leq 0. \tag{3.23}$$

By convexity hypothesis on Σ, we have

$$\langle \sigma_1 - \sigma_2, \boldsymbol{N}(\sigma_2) \rangle \leq 0$$

and then

$$\langle \sigma_1 - \sigma_2, x - \sigma_2 \rangle \leq 0. \tag{3.24}$$

Thus, inequalities (3.23) and (3.24) give us $\sigma_1 \equiv \sigma_2$.

Formula (3.22) is just an application of propagation Lemmas (lemma 1 and lemma 3) along the ray $\sigma \in \mathcal{R}_0(x)$. □

3.2. **Outside a strictly convex compact.** Let $K \subset \mathbb{R}^d$ be a regular and stricly convex compact, $\Omega = K^c$ and $\Gamma = \partial K$ be its boundary.

Remark 1. *Owing to the strict convexity of K and the hypotheses on Σ, we have*

$$\forall x \in \Sigma_+ \cap \Omega,\ \forall l > 1,\ \mathcal{R}_l(x) = \emptyset.$$

There is only one reflexion on K.

We denote by

$$\Gamma^s = \{\gamma \in \Gamma \mid \mathcal{R}_0(\gamma) = \emptyset\} \text{ and } \Gamma^e = (\Gamma^s)^c$$

Let $\gamma \in \Gamma^e$ and $\sigma \in \mathcal{R}_0(\gamma)$. Using previous formulas give us $a_0^{(0)}$, $\varphi^{(0)}$, $\boldsymbol{\nabla}\varphi^{(0)}$ and $\operatorname{Hess}\varphi^{(0)}$ on γ.

If we can compute reflected wave on γ (i.e. $a_0^{(1)}$, $\varphi^{(1)}$, $\boldsymbol{\nabla}\varphi^{(1)}$ and $\operatorname{Hess}\varphi^{(1)}$ on γ) then we can use propagation formulaes along the reflected ray given by $\gamma + t\,\boldsymbol{\nabla}\varphi^{(1)}(\gamma),\ t > 0$

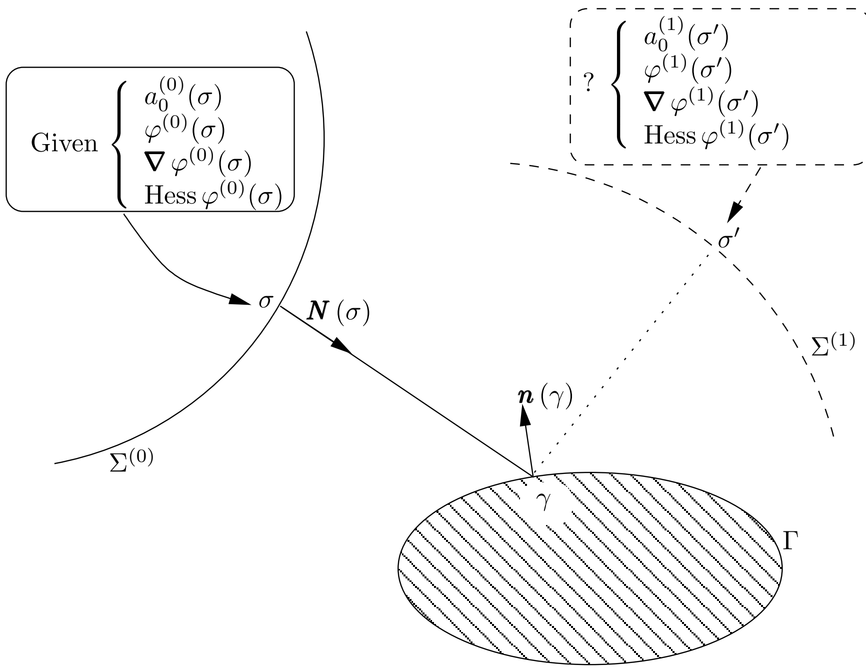

FIGURE 3. Wave reflection

So, the local problem is only to find how to compute the reflected wave in γ. That is to say :

$$\left\{\begin{array}{l} a_0^{(0)}(\gamma) \\ \varphi^{(0)}(\gamma) \\ \boldsymbol{\nabla}\varphi^{(0)}(\gamma) \\ \operatorname{Hess}\varphi^{(0)}(\gamma) \end{array}\right. \text{ given, how to compute } \left\{\begin{array}{l} a_0^{(1)}(\gamma) \\ \varphi^{(1)}(\gamma) \\ \boldsymbol{\nabla}\varphi^{(1)}(\gamma) \\ \operatorname{Hess}\varphi^{(1)}(\gamma) \end{array}\right. \quad ?$$

3.2.1. *Computation of $\varphi^{(1)}$ and $\boldsymbol{\nabla}\varphi^{(1)}$ in γ.*

Lemma 4. *Let $\gamma \in \Gamma$, then*

$$\varphi^{(1)}(\gamma) = \varphi^{(0)}(\gamma) \tag{3.25}$$

Proof. Due to boundary condition :

- If $\gamma \in \Gamma_D$, we have

$$a^{(1)}(\gamma)e^{-ik\varphi^{(1)}(\gamma)} = -a^{(0)}(\gamma)e^{-ik\varphi^{(0)}(\gamma)} \tag{3.26}$$

- If $\gamma \in \Gamma_N$, we have

$$\left(\frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma)\right) e^{-\imath k \varphi^{(1)}(\gamma)} = -\left(\frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right) e^{-\imath k \varphi^{(0)}(\gamma)} \tag{3.27}$$

- If $\gamma \in \Gamma_R$, we have

$$\left[\alpha(k)\left(\frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma)\right) + \beta(k) a^{(1)}(\gamma)\right] e^{-ik\varphi^{(1)}(\gamma)} = -\left[\alpha(k)\left(\frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right) + \beta(k) a^{(0)}(\gamma)\right] e^{-ik\varphi^{(0)}(\gamma)} \tag{3.28}$$

By identification, we immediately obtain (3.25). □

Lemma 5. *Let $\gamma \in \Gamma$, and $\boldsymbol{n}(\gamma)$ the exterior normal to Γ at γ. We have*

$$\boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) = \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma) - 2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{n}(\gamma)\right\rangle \boldsymbol{n}(\gamma) \tag{3.29}$$

Proof. Let $\mathcal{B}_\Gamma(\gamma) = \{\boldsymbol{u}(\gamma), \boldsymbol{v}(\gamma), \boldsymbol{n}(\gamma)\}$ be the **direct orthonormal curvature basis** of Γ at γ. Then we have the local parametrization of Γ at γ :

$$\gamma(u, v) = \gamma + (u, v, g(u, v))$$

with $g(u,v) = \frac{1}{2}(k_1^\Gamma(\gamma)u^2 + k_2^\Gamma(\gamma)v^2) + o(u^2+v^2)$

Taylor's expansion at order 1 of $\varphi^{(0)}$ and $\varphi^{(1)}$ on Γ at point γ are

$$\varphi^{(0)}(\gamma(u,v)) = \varphi^{(0)}(\gamma) + \left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), u\boldsymbol{u}(\gamma) + v\boldsymbol{v}(\gamma)\right\rangle + O(u^2+v^2)$$

$$\varphi^{(1)}(\gamma(u,v)) = \varphi^{(1)}(\gamma) + \left\langle \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma), u\boldsymbol{u}(\gamma) + v\boldsymbol{v}(\gamma)\right\rangle + O(u^2+v^2)$$

Due to relation (3.25) we obtain

$$\left\langle \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) - \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), u\boldsymbol{u}(\gamma) + v\boldsymbol{v}(\gamma)\right\rangle + O(u^2+v^2)$$

that is to say

$$\left\langle \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) - \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{u}(\gamma)\right\rangle = \left\langle \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) - \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{v}(\gamma)\right\rangle = 0$$

So, exists $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) = \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma) + \lambda\boldsymbol{n}(\gamma)$$

Taking the norm of previous relation and using eikonal equation (3.6) give

$$\lambda = -2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{n}(\gamma)\right\rangle.$$

A similar proof will act in dimension 2. □

3.2.2. *Computation of $a_0^{(1)}$ in γ.*

Lemma 6. *Let $\gamma \in \Gamma^e$. If we know $a_0^{(0)}(\gamma)$ and $\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma)$ then*

$$a_0^{(1)}(\gamma) = b_{\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma)}(\gamma) a_0^{(0)}(\gamma). \tag{3.30}$$

where the function $b_{\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma)}$ is given in definition 5.

Proof.

- If $\gamma \in \Gamma_D \cap \Gamma^e$, formula (3.26) give

$$a^{(1)}(\gamma) = -a^{(0)}(\gamma)$$

Using asymptotic expansions of $a^{(0)}$ and $a^{(1)}$

$$\sum_{j\in\mathbb{N}} k^{-j} a_j^{(1)}(\gamma) = -\sum_{j\in\mathbb{N}} k^{-j} a_j^{(0)}(\gamma)$$

With high frequency hypothesis, we obtain the leading term in power of k:

$$a_0^{(1)}(\gamma) = -a_0^{(0)}(\gamma)$$

- If $\gamma \in \Gamma_N \cap \Gamma^e$, formula (3.27) give

$$\frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -\left(\frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right)$$

  Using asymptotic expansions of $a^{(0)}$ and $a^{(1)}$

$$\begin{array}{c}\sum_{j\in\mathbb{N}}\left(k^{-j}\frac{\partial a_j^{(1)}}{\partial n}(\gamma) - \imath k^{-j+1}a_j^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma)\right)\\ = \\ -\sum_{j\in\mathbb{N}}\left(k^{-j}\frac{\partial a_j^{(0)}}{\partial n}(\gamma) - \imath k^{-j+1}a_j^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right)\end{array}$$

  The leading term in power of k is

$$a_0^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -a_0^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$$

  But lemma 5 give $\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$. Like $\gamma \in \Gamma^e$, we have $\left\langle \boldsymbol{\nabla}\, \varphi^{(0)}(\gamma), \boldsymbol{n}\,(\gamma)\right\rangle \neq 0$. So we obtain

$$a_0^{(1)}(\gamma) = a_0^{(0)}(\gamma)$$

- If $\gamma \in \Gamma_R \cap \Gamma^e$, formula (3.28) give

$$\begin{aligned}&\alpha(k)\left(\frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma)\right) + \beta(k)a^{(1)}(\gamma)\\ &= \quad -\left[\alpha(k)\left(\frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right) + \beta(k)a^{(0)}(\gamma)\right]\end{aligned}$$

  Using asymptotic expansions of $a^{(0)}$, $a^{(1)}$, $\alpha(k)$ and $\beta(k)$

$$\begin{aligned}&\left(\sum_{j\in\mathbb{N}} k^{-j}\alpha_j\right) \times \sum_{j\in\mathbb{N}}\left(k^{j}\frac{\partial a_j^{(1)}}{\partial n}(\gamma) - \imath k^{-j+1}a_j^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma)\right)\\ &+ \quad \left(\sum_{j\in\mathbb{N}} k^{-j+1}\beta_{j-1}\right) \times \sum_{j\in\mathbb{N}} k^{-j}a_j^{(1)}(\gamma)\\ &= \quad -\left(\sum_{j\in\mathbb{N}} k^{-j}\alpha_j\right) \times \sum_{j\in\mathbb{N}}\left(k^{j}\frac{\partial a_j^{(0)}}{\partial n}(\gamma) - \imath k^{-j+1}a_j^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)\right)\\ &- \quad \left(\sum_{j\in\mathbb{N}} k^{-j+1}\beta_{j-1}\right) \times \sum_{j\in\mathbb{N}} k^{-j}a_j^{(0)}(\gamma)\end{aligned}$$

  The leading term in power of k is

$$-\imath\alpha_0 a_0^{(1)}(\gamma)\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) + \beta_{-1}a_0^{(1)}(\gamma) = -\left(-\imath\alpha_0 a_0^{(0)}(\gamma)\frac{\partial \varphi^{(0)}}{\partial n}(\gamma) + \beta_{-1}a_0^{(0)}(\gamma)\right)$$

  But $\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$, so

$$a_0^{(1)}(\gamma)\left(\beta_{-1} + \imath\alpha_0\frac{\partial \varphi^{(0)}}{\partial n}\right) = -a_0^{(0)}(\gamma)\left(\beta_{-1} - \imath\alpha_0\frac{\partial \varphi^{(0)}}{\partial n}\right)$$

  with hypothesis $|\beta_{-1}| > |\alpha_0|$ we have

$$\beta_{-1} + \imath\alpha_0\frac{\partial \varphi^{(0)}}{\partial n} \neq 0$$

  So we obtain

$$a_0^{(1)}(\gamma) = -\frac{\beta_{-1} - \imath\alpha_0\frac{\partial \varphi^{(0)}}{\partial n}}{\beta_{-1} + \imath\alpha_0\frac{\partial \varphi^{(0)}}{\partial n}}a_0^{(0)}(\gamma)$$

$\square$

We notice that

$$\forall \gamma \in \Gamma^e \quad \left| b_{\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma)}(\gamma) \right| = 1. \tag{3.31}$$

This the case when there is no absorption in the boundary condition.

3.2.3. *Calculus of* $\operatorname{Hess}\varphi^{(1)}$ *at* $\gamma \in \Gamma^e$.

Lemma 7. *Let $\gamma \in \Gamma^e$. We note $\mathbb{B}(\gamma)$ the curvature matrix of Γ at γ. Then, in dimension $d = 2$ or 3, we have*

$$\operatorname{Hess}\varphi^{(1)}(\gamma) = \mathcal{T}^d_{\mathbb{B}(\gamma),\boldsymbol{n}(\gamma),\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma)}(\operatorname{Hess}\varphi^{(0)}(\gamma)). \tag{3.32}$$

Proof. We have seen (lemma 4) that

$$\varphi^{(1)}(\gamma) = \varphi^{(0)}(\gamma).$$

Let $\mathcal{V}_\Gamma(\gamma)$ be a neighboorhood of γ in Γ. Taylor's expansion at order 1 of $\boldsymbol{\nabla}\,\varphi^{(0)}$ and $\boldsymbol{\nabla}\,\varphi^{(1)}$ are, for all $\gamma' \in \mathcal{V}_\Gamma(\gamma)$,

$$\boldsymbol{\nabla}\,\varphi^{(0)}(\gamma') = \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma) + \operatorname{Hess}\varphi^{(0)}(\gamma)(\gamma' - \gamma) + o(|\ \gamma' - \gamma\ |) \tag{3.33}$$

and

$$\boldsymbol{\nabla}\,\varphi^{(1)}(\gamma') = \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) + \operatorname{Hess}\varphi^{(1)}(\gamma)(\gamma' - \gamma) + o(|\ \gamma' - \gamma\ |) \tag{3.34}$$

Let $\gamma(u, v)$ be the local parametrization of Γ at γ define in section **??** We compute now, Taylor's expansion at order 1 of $\boldsymbol{n}\,(\gamma(u, v))$ in dimension 2 and 3. For that, we first evaluate

$$\begin{aligned}
\left(\frac{\partial\gamma}{\partial u} \wedge \frac{\partial\gamma}{\partial v}\right)(u, v) &= \begin{pmatrix} -k_1^\Gamma(\gamma)u \\ -k_2^\Gamma(\gamma)v \\ 1 \end{pmatrix}_{\mathcal{B}_\Gamma(\gamma)} + O(u^2 + v^2), \\
&= \boldsymbol{n}\,(\gamma) + \mathbb{B}(\gamma)(\gamma(u, v) - \gamma) + O(u^2 + v^2).
\end{aligned}$$

We obtain

$$\boldsymbol{n}\,(\gamma(u, v)) = \frac{\boldsymbol{n}\,(\gamma) + \mathbb{B}(\gamma)(\gamma(u, v) - \gamma)}{(1 + (k_1^\Gamma(\gamma)u)^2 + (k_2^\Gamma(\gamma)v)^2)^{\frac{1}{2}}} + O(u^2 + v^2).$$

But

$$(1 + (k_1^\Gamma(\gamma)u)^2 + (k_2^\Gamma(\gamma)v)^2)^{-\frac{1}{2}} = 1 + O(u^2 + v^2)$$

so, we have

$$\boldsymbol{n}\,(\gamma') = \boldsymbol{n}\,(\gamma) + \mathbb{B}(\gamma)(\gamma' - \gamma) + o(|\ \gamma' - \gamma\ |) \tag{3.35}$$

In similar way, we obtain the same formula in dimension $d = 2$.

Using formulas (3.33) and (3.35) in equation (3.29) gives

$$\begin{aligned}
\boldsymbol{\nabla}\,\varphi^{(1)}(\gamma') = \ & \boldsymbol{\nabla}\,\varphi^{(1)}(\gamma) + (\operatorname{Hess}\varphi^{(0)}(\gamma) - 2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{n}\,(\gamma)\right\rangle \mathbb{B}(\gamma))(\gamma' - \gamma) \\
& -2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(\gamma' - \gamma)\right\rangle \boldsymbol{n}\,(\gamma) \\
& -2\left\langle \operatorname{Hess}\varphi^{(0)}(\gamma)(\gamma' - \gamma), \boldsymbol{n}\,(\gamma)\right\rangle \boldsymbol{n}\,(\gamma) + o(|\ \gamma' - \gamma\ |)
\end{aligned}$$

So we obtain with formula (3.34), for all $\gamma' \in \mathcal{V}_\Gamma(\gamma)$,

$$\begin{aligned}
\operatorname{Hess}\varphi^{(1)}(\gamma)(\gamma' - \gamma) = \ & (\operatorname{Hess}\varphi^{(0)}(\gamma) - 2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{n}\,(\gamma)\right\rangle \mathbb{B}(\gamma))(\gamma' - \gamma) \\
& -2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(\gamma' - \gamma)\right\rangle \boldsymbol{n}\,(\gamma) \\
& -2\left\langle \operatorname{Hess}\varphi^{(0)}(\gamma)(\gamma' - \gamma), \boldsymbol{n}\,(\gamma)\right\rangle \boldsymbol{n}\,(\gamma) + o(|\ \gamma' - \gamma\ |)
\end{aligned}$$

Now, we want to extend the previous formula for all x in $\mathcal{V}(\gamma) \subset \mathbb{R}^d$, a neighborhood of γ. So we must add to previous formula a function from $\mathbb{R}^d$ to $\mathbb{R}^d$ which vanishes on Γ. That is to say at order 2, there exists a constant vector $C(\gamma) \in \mathbb{R}^d$ such that :
for all $x \in \mathcal{V}(\gamma)$

$$\begin{aligned}
\operatorname{Hess}\varphi^{(1)}(\gamma)(x - \gamma) = \ & (\operatorname{Hess}\varphi^{(0)}(\gamma) - 2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \boldsymbol{n}\,(\gamma)\right\rangle \mathbb{B}(\gamma))(x - \gamma) \\
& -2\left\langle \boldsymbol{\nabla}\,\varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(x - \gamma)\right\rangle \boldsymbol{n}\,(\gamma) \\
& -2\left\langle \operatorname{Hess}\varphi^{(0)}(\gamma)(x - \gamma), \boldsymbol{n}\,(\gamma)\right\rangle \boldsymbol{n}\,(\gamma) \\
& + \langle x - \gamma, \boldsymbol{n}\,(\gamma)\rangle\, C(\gamma) + o(|\ x - \gamma\ |)
\end{aligned} \tag{3.36}$$

To compute $C(\gamma)$, we take $x \in \mathcal{V}(\gamma)$ such that $x - \gamma = \varepsilon\,\boldsymbol{\nabla}\,\varphi^{(1)}(\gamma)$ with $\varepsilon > 0$ and use (3.6) for $\varphi^{(0)}$ and $\varphi^{(1)}$

$$\operatorname{Hess}\varphi^{(0)}(x)\,\boldsymbol{\nabla}\,\varphi^{(0)}(x) = 0$$

and

$$\operatorname{Hess}\varphi^{(1)}(x)\,\boldsymbol{\nabla}\,\varphi^{(1)}(x) = 0$$

So, formula (3.36) become

$$\begin{array}{ll}(\operatorname{Hess}\varphi^{(0)}(\gamma)-2\left\langle\boldsymbol{\nabla}\varphi^{(0)}(\gamma),\boldsymbol{n}(\gamma)\right\rangle\mathbb{B}(\gamma))\boldsymbol{\nabla}\varphi^{(1)}(\gamma) & \\ -2\left\langle\boldsymbol{\nabla}\varphi^{(0)}(\gamma),\mathbb{B}(\gamma)(\boldsymbol{\nabla}\varphi^{(1)}(\gamma))\right\rangle\boldsymbol{n}(\gamma) & \\ -2\left\langle\operatorname{Hess}\varphi^{(0)}(\gamma)(\boldsymbol{\nabla}\varphi^{(1)}(\gamma)),\boldsymbol{n}(\gamma)\right\rangle\boldsymbol{n}(\gamma) & \\ +\left\langle\boldsymbol{\nabla}\varphi^{(1)}(\gamma),\boldsymbol{n}(\gamma)\right\rangle C(\gamma) & =0\end{array}$$

Using (3.29) and $\mathbb{B}(\gamma)\boldsymbol{n}(\gamma)=0$ we obtain

$$\mathbb{B}(\gamma)\boldsymbol{\nabla}\varphi^{(1)}(\gamma)=\mathbb{B}(\gamma)\boldsymbol{\nabla}\varphi^{(0)}(\gamma)$$

and so

$$\begin{aligned}C(\gamma) &= 2\left[2\left\langle\operatorname{Hess}\varphi^{(0)}(\gamma)\boldsymbol{n}(\gamma),\boldsymbol{n}(\gamma)\right\rangle-\frac{\left\langle\mathbb{B}(\gamma)\boldsymbol{\nabla}\varphi^{(0)}(\gamma),\boldsymbol{\nabla}\varphi^{(0)}(\gamma)\right\rangle}{\left\langle\boldsymbol{\nabla}\varphi^{(0)}(\gamma),\boldsymbol{n}(\gamma)\right\rangle}\right]\boldsymbol{n}(\gamma)\\ &\quad -2\left[\operatorname{Hess}\varphi^{(0)}(\gamma)\boldsymbol{n}(\gamma)+\mathbb{B}(\gamma)\boldsymbol{\nabla}\varphi^{(0)}(\gamma)\right]\end{aligned}$$

Replacing $C(\gamma)$ by previous formula in (3.36) immediately give (3.32).

□

These results are given in [eC89]

3.2.4. *Properties of* $\operatorname{Hess}\varphi^{(1)}(\gamma)$.

Lemma 8. *The matrix* $\operatorname{Hess}\varphi^{(1)}(\gamma)$ *is symmetric and*

$$\operatorname{Hess}\varphi^{(1)}(\gamma)\boldsymbol{\nabla}\varphi^{(1)}(\gamma)=0. \tag{3.37}$$

In dimension $d=3$, *eigenvalues of* $\operatorname{Hess}\varphi^{(1)}(\gamma)$ *are* $(-k_1^{(1)}(\gamma),-k_2^{(1)}(\gamma),0)$ *and*

$$k_2^{(1)}(\gamma)\le k_1^{(1)}(\gamma)<0,\;\; k_1^{(1)}(\gamma)+k_2^{(1)}(\gamma)<k_1^{(0)}(\gamma)+k_2^{(0)}(\gamma)\quad\textit{and}\quad k_1^{(1)}(\gamma)k_2^{(1)}(\gamma)<k_1^{(0)}(\gamma)k_2^{(0)}(\gamma).$$

In dimension $d=2$, *eigenvalues of* $\operatorname{Hess}\varphi^{(1)}(\gamma)$ *are* $(-k^{(1)}(\gamma),0)$ *and*

$$k^{(1)}(\gamma)<k^{(0)}(\gamma)\le 0.$$

Proof. As we have seen in section 3.1.2, in dimension $d=3$ (resp. $d=2$), eigenvalues of $\operatorname{Hess}\varphi^{(0)}(\gamma)$ are $(-k_1^{(0)}(\gamma),-k_2^{(0)}(\gamma),0)$ (resp. $-k^{(0)}(\gamma),0)$) where $k_2^{(0)}(\gamma)\le k_1^{(0)}(\gamma)\le 0$ (resp. $k^{(0)}(\gamma)\le 0$). So we only have to apply lemma 12 in dimension 2 or lemma 13 in dimension 3 to end the proof. □

3.2.5. *Conclusion.* Outside a stricly convex compact, we proved following result

Theorem 3. *Let* $x\in\Sigma_+\cap\Omega$, *such that* $\mathcal{G}(x)=\emptyset$. *Then* $\#\mathcal{R}(x)=\#\mathcal{R}_1(x)\le 1$, $\#\mathcal{R}_0(x)\le 1$, *and*

$$u^{inc}(x)=\sum_{\rho\in\mathcal{R}_0(x)}u_\rho^{G.O.}(x)+O\left(\frac{1}{k}\right)$$

and

$$u(x)=\sum_{\rho\in\mathcal{R}_1(x)}u_\rho^{G.O.}(x)+O\left(\frac{1}{k}\right)$$

3.3. **Outside an union of strictly convex compacts.** To prove theorem 1, we just have to verify that we can use propagation and reflection lemmas along each ray coming through x. For that, we have

Lemma 9. *Let* $x\in\Sigma_+\cap\Omega$, *such that* $\mathcal{G}(x)=\emptyset$. *Let* $l\in\mathbb{N}^*$ *such that* $\mathcal{R}_l(x)\neq\emptyset$ *and* $\rho=(\gamma_0,\gamma_1,\ldots,\gamma_l)\in\mathcal{R}_l(x)$. *In dimension* $d=3$ *(resp.* $d=2$*), if eigenvalues of the symmetric matrix* $\mathbb{A}_\rho^{(0)}(\gamma_0)$ *are* ($\lambda_1^{(0)}(\gamma_0)$, $\lambda_2^{(0)}(\gamma_0)$, 0) *(resp.* $(\lambda^{(0)}(\gamma_0),0)$*) where*

$$0\le\lambda_1^{(0)}(\gamma_0)\le\lambda_2^{(0)}(\gamma_0)\;\;\textit{(resp. }0\le\lambda^{(0)}(\gamma_0)\textit{)}$$

then $\forall j\in\{1,\cdots,l\}$ *eigenvalues of* $\mathbb{A}_\rho^{(j)}(\gamma_j)$ *are* $(\lambda_1^{(j)}(\gamma_j),\lambda_2^{(j)}(\gamma_j),0)$ *(resp.* $(\lambda^{(j)}(\gamma_j),0)$*) where*

$$0<\lambda_1^{(j)}(\gamma_j)\le\lambda_2^{(j)}(\gamma_j)\;\;\textit{(resp. }0<\lambda^{(j)}(\gamma_j)\textit{)}$$

Proof. Due to hypothesis we can apply propagation lemma 2 to obtain that

$$\mathbb{A}_\rho^{(0)}(\gamma_1) = \mathcal{S}_{\|\gamma_1-\gamma_0\|}(\mathbb{A}_\rho^{(0)}(\gamma_0))$$

is well defined. and its eigenvalues are $(\lambda_1^{(0)}(\gamma_1), \lambda_2^{(0)}(\gamma_1), 0)$ where

$$0 \leq \lambda_1^{(0)}(\gamma_1) \leq \lambda_2^{(0)}(\gamma_1).$$

By hypothesis $\rho \in \mathcal{R}_l(x)$ so $\langle \gamma_1 - \gamma_0, \boldsymbol{n}(\gamma_1)\rangle < 0$ and we can apply reflection lemma 7 to obtain that

$$\mathbb{A}_\rho^{(1)}(\gamma_1) = \mathcal{T}_{\mathbb{B}(\gamma_1), \boldsymbol{n}(\gamma_1), \frac{\gamma_1-\gamma_0}{\|\gamma_1-\gamma_0\|}}\left(\mathbb{A}_\rho^{(0)}(\gamma_1)\right)$$

is well defined.
Furthermore, lemma 8 give us that matrix $\mathbb{A}_\rho^{(1)}(\gamma_1)$ is symmetric and its eigenvalues are $(\lambda_1^{(1)}(\gamma_1), \lambda_2^{(1)}(\gamma_1), 0)$ where

$$0 < \lambda_1^{(1)}(\gamma_1) \leq \lambda_2^{(1)}(\gamma_1).$$

Then a simply recurrence proof give us lemma :
Let $j \in \{1, \cdots, l-1\}$ and suppose that $\mathbb{A}_\rho^{(j)}(\gamma_j)$ is symmetric and its eigenvalues are $(\lambda_1^{(j)}(\gamma_j), \lambda_2^{(j)}(\gamma_j), 0)$ where

$$0 < \lambda_1^{(j)}(\gamma_j) \leq \lambda_2^{(j)}(\gamma_j)$$

we can apply propagation lemma 2 to obtain that

$$\mathbb{A}_\rho^{(j)}(\gamma_{j+1}) = \mathcal{S}_{\|\gamma_{j+1}-\gamma_j\|}(\mathbb{A}_\rho^{(j)}(\gamma_j))$$

is well defined. and its eigenvalues are $(\lambda_1^{(j)}(\gamma_{j+1}), \lambda_2^{(j)}(\gamma_{j+1}), 0)$ where

$$0 < \lambda_1^{(j)}(\gamma_{j+1}) \leq \lambda_2^{(j)}(\gamma_{j+1}).$$

By hypothesis $\rho \in \mathcal{R}_l(x)$ so $\langle \gamma_{j+1} - \gamma_j, \boldsymbol{n}(\gamma_{j+1})\rangle < 0$ and we can apply reflection lemma 7 to obtain that

$$\mathbb{A}_\rho^{(j+1)}(\gamma_{j+1}) = \mathcal{T}_{\mathbb{B}(\gamma_{j+1}), \boldsymbol{n}(\gamma_{j+1}), \frac{\gamma_{j+1}-\gamma_j}{\|\gamma_{j+1}-\gamma_j\|}}\left(\mathbb{A}_\rho^{(j)}(\gamma_{j+1})\right)$$

is well defined.
Furthermore, lemma 8 give us that matrix $\mathbb{A}_\rho^{(j+1)}(\gamma_{j+1})$ is symmetric and its eigenvalues are $(\lambda_1^{(j+1)}(\gamma_{j+1}), \lambda_2^{(j+1)}(\gamma_{j+1}), 0)$ where

$$0 < \lambda_1^{(j+1)}(\gamma_{j+1}) \leq \lambda_2^{(j+1)}(\gamma_{j+1}).$$

A similar proof will act in dimension $d = 2$. $\square$

With theorem 1 hypothesis and this lemma, we immediatly have

$$\forall j \in \{1, \ldots, l\}, \ \det\left(\mathbb{I} + \|\gamma_j - \gamma_{j-1}\| \, \mathbb{A}_\rho^{(j-1)}(\gamma_{j-1})\right) > 0$$

and

$$\det\left(\mathbb{I} + \|x - \gamma_l\| \, \mathbb{A}_\rho^{(l)}(\gamma_l)\right) > 0.$$

So along each ray comming throug x we can apply propagation and reflection lemmas. Then, by adding the contribution of each ray we obtain theorem 1 results.

Lemma 10. *Let $x \in \Sigma_+ \cap \Omega$, such that $\mathcal{G}(x) = \emptyset$. Let $l > 1$ such that $\mathcal{R}_l(x) \neq \emptyset$ and $\rho = (\gamma_0, \gamma_1, \ldots, \gamma_l) \in \mathcal{R}_l(x)$. We have*

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{\left[\left(1 + 2d(x)\lambda_{\min}^\Gamma\right)\left(1 + 2d_{\min}\lambda_{\min}^\Gamma\right)^{l-1}\right]^{\frac{d-1}{2}}} \tag{3.38}$$

where

$$0 < d(x) = \min_{\gamma \in \Gamma} \|x - \gamma\|,$$

$$0 < \lambda_{\min}^\Gamma = \begin{cases} \min\limits_{\gamma \in \Gamma} \lambda_1(\gamma) & \textit{if } d = 3, \\ \min\limits_{\gamma \in \Gamma} \lambda(\gamma) & \textit{if } d = 2, \end{cases},$$

and

$$0 < d_{\min} = \min_{i \neq j} \min_{\gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j} \|\gamma_i - \gamma_j\|.$$

Proof. Using (3.31) and definition of $a_\rho^{(j)}(\gamma_j)$, we obtain

$$\forall j \in \{1,\cdots,l\},\ \ |a_\rho^{(j+1)}(\gamma_{j+1})| = \frac{|a_\rho^j(\gamma_j)|}{\sqrt{\det\left(\mathbb{I} + \|\gamma_{j+1}-\gamma_j\|\,\mathbb{A}_\rho^{(j)}(\gamma_j)\right)}} \tag{3.39}$$

where $t_j = \|\gamma_{j+1}-\gamma_j\|$ and $\gamma_{j+1} = x$.

In dimension $d = 2$, we can use equation (4.1) of lemma 12 to obtain

$$\lambda^{(j)}(\gamma_j) = \lambda^{(j-1)}(\gamma_j) - 2t_{j-1}\frac{\lambda^\Gamma(\gamma_j)}{\langle \boldsymbol{n}(\gamma_j), \gamma_j - \gamma_{j-1}\rangle}$$

But we have $\frac{1}{t_{j-1}}\langle \boldsymbol{n}(\gamma_j), \gamma_j - \gamma_{j-1}\rangle \in [-1;0[$ and then

$$\lambda^{(j)}(\gamma_j) \geq \lambda^{(j-1)}(\gamma_j) + 2\lambda^\Gamma_{\min} > 0.$$

From proposition (1) we get

$$\lambda^{(j-1)}(\gamma_j) = \frac{\lambda^{(j-1)}(\gamma_{j-1})}{1 + t_{j-1}\lambda^{(j-1)}(\gamma_{j-1})}$$

and so

$$\begin{array}{lcll} \lambda^{(j-1)}(\gamma_j) & > & 0 & \forall j \in \{2,\cdots,l\} \\ \lambda^{(0)}(\gamma_1) & \geq & 0 & \end{array}.$$

From equation (3.39), we immediately obtain

$$\begin{array}{lcll} |a_\rho^{(1)}(\gamma_1)| & \leq & |a_\rho^{(0)}(\gamma_0)| & \\ |a_\rho^{(j+1)}(\gamma_{j+1})| & < & |a_\rho^{(j)}(\gamma_j)|\left(1 + 2d_{\min}\lambda^\Gamma_{\min}\right)^{-1/2} & \forall j \in \{1,\cdots,l-1\} \\ |a_\rho^{(l)}(x)| = |a_\rho(x)| & < & |a_\rho^{(l)}(\gamma_l)|\left(1 + 2d(x)\lambda^\Gamma_{\min}\right)^{-1/2} & \end{array}$$

and thus

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{\left[\left(1 + 2d(x)\lambda^\Gamma_{\min}\right)\left(1 + 2d_{\min}\lambda^\Gamma_{\min}\right)^{l-1}\right]^{\frac{1}{2}}}.$$

In dimension $d = 3$, We have

$$\det\left(\mathbb{I} + t_j\mathbb{A}_\rho^{(j)}(\gamma_j)\right) \ = \ 1 + (\lambda_1^{(j)}(\gamma_j) + \lambda_2^{(j)}(\gamma_j))t_j + \lambda_1^{(j)}(\gamma_j)\lambda_2^{(j)}(\gamma_j)t_j^2$$

and we can use equations (4.5) and (4.6) of lemma 13 to obtain

$$\lambda_1^{(j)}(\gamma_j) + \lambda_2^{(j)}(\gamma_j) \geq \lambda_1^{(j-1)}(\gamma_j) + \lambda_2^{(j-1)}(\gamma_j) + 4\lambda^\Gamma_{\min} > 0,$$
$$\lambda_1^{(j)}(\gamma_j)\lambda_2^{(j)}(\gamma_j) \geq \lambda_1^{(j-1)}(\gamma_j)\lambda_2^{(j-1)}(\gamma_j) + 4(\lambda^\Gamma_{\min})^2 > 0.$$

So, with these inequalities, we have

$$\begin{array}{rcl} \det\left(\mathbb{I} + t_j\mathbb{A}_\rho^{(j)}(\gamma_j)\right) & \geq & 1 + 4\lambda^\Gamma_{\min}t_j + 4(\lambda^\Gamma_{\min}t_j)^2 + \left(\lambda_1^{(j-1)}(\gamma_j) + \lambda_2^{(j-1)}(\gamma_j)\right)t_j \\ & & + \left(\lambda_1^{(j-1)}(\gamma_j)\lambda_2^{(j-1)}(\gamma_j)\right)t_j^2 \end{array}$$

From proposition (1) we get

$$\lambda_\alpha^{(j-1)}(\gamma_j) = \frac{\lambda_\alpha^{(j-1)}(\gamma_{j-1})}{1 + t_{j-1}\lambda_\alpha^{(j-1)}(\gamma_{j-1})},\ \ \alpha \in \{1,2\}$$

and, as $\lambda_2^{(j-1)}(\gamma_{j-1}) \geq \lambda_1^{(j-1)}(\gamma_{j-1}) > 0$ for $j \in \{2,\cdots,l+1\}$ and $\lambda_2^{(0)}(\gamma_0) \geq \lambda_1^{(0)}(\gamma_0) \geq 0$ (see lemma 13) we have

$$\begin{array}{lcllll} \lambda_2^{(j-1)}(\gamma_j) & \geq & \lambda_1^{(j-1)}(\gamma_j) & > & 0 & \forall j \in \{2,\cdots,l\} \\ \lambda_2^{(0)}(\gamma_1) & \geq & \lambda_1^{(0)}(\gamma_1) & \geq & 0 & \end{array}.$$

With these inequalities, we have

$$\begin{array}{lcll} \det\left(\mathbb{I} + t_j\mathbb{A}_\rho^{(j)}(\gamma_j)\right) & > & \left(1 + 2\lambda^\Gamma_{\min}t_j\right)^2 & \text{if } j > 1 \\ \det\left(\mathbb{I} + t_1\mathbb{A}_\rho^{(1)}(\gamma_1)\right) & \geq & \left(1 + 2\lambda^\Gamma_{\min}t_1\right)^2 & \end{array}$$

From equation (3.39), we immediately obtain

$$\begin{array}{lcl} |a_\rho^{(1)}(\gamma_1)| & \leq & |a_\rho^{(0)}(\gamma_0)| \\ |a_\rho^{(j+1)}(\gamma_{j+1})| & < & |a_\rho^{(j)}(\gamma_j)| \left(1+2d_{\min}\lambda_{\min}^{\Gamma}\right)^{-1} \quad \forall j \in \{1, \cdots, l-1\} \\ |a_\rho^{(l)}(x)| = |a_\rho(x)| & < & |a_\rho^{(l)}(\gamma_l)| \left(1+2d(x)\lambda_{\min}^{\Gamma}\right)^{-1} \end{array}$$

and thus

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{\left(1+2d(x)\lambda_{\min}^{\Gamma}\right)\left(1+2d_{\min}\lambda_{\min}^{\Gamma}\right)^{l-1}}.$$

□

4. Technical results

Proposition 1. *Let $t > 0$ and $\mathbb{A} \in S_t^d$. Assume that λ is an eigenvalue of $\mathbb{A}$ with corresponding eingenvector ζ then $\frac{\lambda}{1+t\lambda}$ is an eigenvalue of $\mathcal{S}_t(\mathbb{A})$ with corresponding eingenvector ζ.*

Proof. By definition $\mathcal{S}_t(\mathbb{A}) = \mathbb{A}(\mathbb{I}+t\mathbb{A})^{-1}$ and by hypothesis

$$(\mathbb{I}+t\mathbb{A})\zeta = (1+t\lambda)\zeta$$

with $1+t\lambda \neq 0$. So $(1+t\lambda)(\mathbb{I}+t\mathbb{A})^{-1}\zeta = \zeta$ and we obtain

$$(1+t\lambda)\mathbb{A}(\mathbb{I}+t\mathbb{A})^{-1}\zeta = \mathbb{A}\zeta.$$

Thus

$$\mathbb{A}(\mathbb{I}+t\mathbb{A})^{-1}\zeta = \frac{\lambda}{1+t\lambda}\zeta.$$

□

Lemma 11. *Let $\eta \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^d$ such that $\|\eta\| = 1$ and $\langle \eta, \zeta \rangle \neq 0$ Let $\mathbb{A}$ and $\mathbb{B}$ symmetric matrices in $\mathbb{R}^{d\times d}$. Assume $\mathbb{A}\zeta = 0$ and $\mathbb{B}\eta = 0$ then*

$$(\mathcal{T}_{\mathbb{B},\eta,\zeta}^d(\mathbb{A}))(\zeta - 2\langle \zeta, \eta \rangle \eta) = 0$$

Proof. Note $\xi = \zeta - 2\langle \zeta, \eta \rangle \eta)$ then, by definition,

$$\begin{array}{rcl} (\mathcal{T}_{\mathbb{B},\eta,\zeta}^d(\mathbb{A}))\xi & = & \mathbb{A}\xi - 2\langle \zeta, \eta \rangle \mathbb{B}\xi - 2\langle \eta, \xi \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) \\ & & -2\langle \mathbb{A}\eta + \mathbb{B}\zeta, \xi \rangle \eta + 2\left[2\langle \mathbb{A}\eta, \eta \rangle - \frac{\langle \mathbb{B}\zeta, \zeta \rangle}{\langle \zeta, \eta \rangle}\right]\langle \eta, \xi \rangle \eta \end{array}$$

Due to hypothesis, we have

$$\mathbb{A}\xi = -2\langle \zeta, \eta \rangle \mathbb{A}\eta, \ \mathbb{B}\xi = \mathbb{B}\zeta, \ \langle \eta, \xi \rangle = -\langle \zeta, \eta \rangle$$

and

$$\langle \mathbb{A}\eta + \mathbb{B}\zeta, \xi \rangle = -2\langle \zeta, \eta \rangle \langle \mathbb{A}\eta, \eta \rangle + \langle \mathbb{B}\zeta, \zeta \rangle.$$

So, we obtain

$$\begin{array}{rcl} (\mathcal{T}_{\mathbb{B},\eta,\zeta}^d(\mathbb{A}))\xi & = & -2\langle \zeta, \eta \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) + 2\langle \zeta, \eta \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) + 4\langle \zeta, \eta \rangle \langle \mathbb{A}\eta, \eta \rangle \eta \\ & & -2\langle \mathbb{B}\zeta, \zeta \rangle \eta - 2\left[2\langle \mathbb{A}\eta, \eta \rangle - \frac{\langle \mathbb{B}\zeta, \zeta \rangle}{\langle \zeta, \eta \rangle}\right]\langle \eta, \zeta \rangle \eta \\ & = & 0. \end{array}$$

□

Lemma 12. *Let $\mathcal{B} = (\boldsymbol{u}, \boldsymbol{w})$ and $\mathcal{B}^{(I)} = (\boldsymbol{u}^{(I)}, \boldsymbol{w}^{(I)})$ two direct orthonormal basis of $\mathbb{R}^2$ such that $\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle < 0$. Let $\mathbb{B} = \operatorname{diag}(\lambda, 0)$ in $\mathcal{B}$ basis and $\mathbb{A}^{(I)} = \operatorname{diag}(\lambda^{(I)}, 0)$ in $\mathcal{B}^{(I)}$ basis. Then $\mathbb{A}^{(R)} = \mathcal{T}^2_{\mathbb{B},\boldsymbol{w},\boldsymbol{w}^{(I)}}(\mathbb{A}^{(I)})$ is a symmetric matrix having for eingenvalues $(\lambda^{(R)}, 0)$ where*

$$\lambda^{(R)} = \lambda^{(I)} - 2\frac{\lambda}{\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle} \tag{4.1}$$

and

$$\mathbb{A}^{(R)}\left(\boldsymbol{w}^{(I)} - 2\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle \boldsymbol{w}\right) = 0 \tag{4.2}$$

If $\lambda > 0$ and $\lambda^{(I)} \geq 0$ then

$$\lambda^{(R)} > \lambda^{(I)}. \tag{4.3}$$

Proof. In basis $\mathcal{B}$, we note $\boldsymbol{u}^{(I)} = (u_j)_{j=1}^2$, $\boldsymbol{w}^{(I)} = (w_j)_{j=1}^2$ and $\mathbb{A}^{(I)} = (a_{ij})_{i,j=1}^2$. By hypothesis, $\mathbb{A}^{(I)}$ is symmetric and $\mathbb{A}^{(I)}\boldsymbol{w}^{(I)} = 0$.

As $\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle \neq 0$, we can compute $\mathbb{A}^{(R)}$: $\forall x \in \mathbb{R}^2$

$$\begin{aligned}\mathbb{A}^{(R)}x \quad = \quad & \left(\mathbb{A}^{(I)} - 2\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle \mathbb{B}\right) x \\ & -2\left\langle \boldsymbol{w}, x\right\rangle \left(\mathbb{A}^{(I)}\boldsymbol{w} + \mathbb{B}\boldsymbol{w}^{(I)}\right) \\ & -2\left\langle \mathbb{A}^{(I)}\boldsymbol{w} + \mathbb{B}\boldsymbol{w}^{(I)}, x\right\rangle \boldsymbol{w} \\ & +2\left[2\left\langle \mathbb{A}^{(I)}\boldsymbol{w}, \boldsymbol{w}\right\rangle - \frac{\left\langle \mathbb{B}\boldsymbol{w}^{(I)}, \boldsymbol{w}^{(I)}\right\rangle}{\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)}\right\rangle}\right]\left\langle \boldsymbol{w}, x\right\rangle \boldsymbol{w}\end{aligned}$$

in basis $\mathcal{B}$, $x = (x_j)_{j=1}^2$ and

$$\begin{aligned}\mathbb{A}^{(R)}x \quad = \quad & \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix} - 2w_2\begin{pmatrix}\lambda x_1 \\ 0\end{pmatrix} - 2x_2\begin{pmatrix} a_{12} + \lambda w_1 \\ a_{22}\end{pmatrix} \\ & -2\left\langle \begin{pmatrix} a_{12} + \lambda w_1 \\ a_{22}\end{pmatrix}, \begin{pmatrix} x_1 \\ x_2\end{pmatrix}\right\rangle \begin{pmatrix} 0 \\ 1\end{pmatrix} + 2\left[2a_{22} - \frac{\lambda w_1^2}{w_2}\right] x_2 \begin{pmatrix} 0 \\ 1\end{pmatrix}\end{aligned}$$

That is to say,

$$\mathbb{A}^{(R)} = \begin{pmatrix} a_{11} - 2\lambda w_2 & -a_{12} - 2\lambda w_1 \\ -a_{12} - 2\lambda w_1 & a_{22} - 2\lambda \frac{w_1^2}{w_2}\end{pmatrix}$$

So we have $\mathbb{A}^{(R)}$ symmetry. We can remark that formula (4.2) is a direct application of Lemma 11.

Now, we study eingenvalues of $\mathbb{A}^{(R)}$. Let $\mathbb{H}$ be the normal matrix given by

$$\mathbb{H} = \begin{pmatrix} u_1 & w_1 \\ -u_2 & -w_2\end{pmatrix}$$

Then $\mathbb{H}\mathbb{A}^{(R)}\mathbb{H}^t$ is a similar to $\mathbb{A}^{(R)}$.

Using $\mathbb{A}^{(I)}\boldsymbol{u}^{(I)} = \lambda^{(I)}\boldsymbol{u}^{(I)}$ and $\mathbb{A}^{(I)}\boldsymbol{w}^{(I)} = 0$ give

$$\begin{aligned}\mathbb{A}^{(R)}\mathbb{H} \quad & = \begin{pmatrix} (a_{11} - 2\lambda w_2)u_1 + (a_{12} + 2\lambda w_1)u_2 & (a_{11} - 2\lambda w_2)w_1 + (a_{12} + 2\lambda w_1)w_2 \\ (a_{12} + 2\lambda w_1)u_1 - (a_{22} - 2\lambda \frac{w_1^2}{w_2})u_2 & (a_{12} + 2\lambda w_1)w_1 - (a_{22} - 2\lambda\frac{w_1^2}{w_2})w_2\end{pmatrix} \\ & = \begin{pmatrix} \lambda^{(I)}u_1 + 2\lambda(w_1u_2 - w_2u_1) & 0 \\ -\lambda^{(I)}u_2 + 2\lambda(\frac{w_1^2}{w_2}u_2 - w_1u_1) & 0\end{pmatrix}\end{aligned}$$

But $(\boldsymbol{u}^{(I)}, \boldsymbol{w}^{(I)})$ is a direct and orthonormal basis, so

$$w_2u_1 - w_1u_2 = 1,\ w_1u_1 + w_2u_2 = 0 \text{ and } u_1^2 + u_2^2 = 1.$$

Then

$$\begin{aligned}\mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H} \quad & = \begin{pmatrix} \lambda^{(I)}(u_1^2 + u_2^2) + 2\lambda\left[(w_1u_2 - w_2u_1)u_1 - \frac{w_1^2}{w_2}u_2^2 + w_1u_1u_2\right] & 0 \\ \lambda^{(I)}(u_1w_1 + u_2w_2) + 2\lambda(w_1^2u_2 - w_2u_1w_1 - w_1^2u_2 + w_2u_1w_1) & \end{pmatrix} \\ & = \begin{pmatrix} \lambda^{(I)} - 2\frac{\lambda}{w_2}\left[w_2u_1 - w_1u_2\right]^2 & 0 \\ 0 & 0\end{pmatrix} \\ & = \begin{pmatrix} \lambda^{(I)} - 2\frac{\lambda}{w_2} & 0 \\ 0 & 0\end{pmatrix}\end{aligned}$$

□

Lemma 13. *Let $\mathcal{B} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ and $\mathcal{B}^{(I)} = (\boldsymbol{u}^{(I)}, \boldsymbol{v}^{(I)}, \boldsymbol{w}^{(I)})$ two direct orthonormal basis of $\mathbb{R}^3$ such that $\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)}\right\rangle < 0$. Let $\mathbb{B} = \mathrm{diag}(\lambda_1, \lambda_2, 0)$ in $\mathcal{B}$ basis and the symmetric matrix $\mathbb{A}^{(I)} = \mathrm{diag}(\lambda_1^{(I)}, \lambda_2^{(I)}, 0)$ in $\mathcal{B}^{(I)}$ basis. Then $\mathbb{A}^{(R)} = \mathcal{T}^3_{\mathbb{B}, \boldsymbol{w}, \boldsymbol{w}^{(I)}}(\mathbb{A}^{(I)})$ is a symmetric matrix having for eingenvalues $(\lambda_1^{(R)}, \lambda_2^{(R)}, 0)$ and*

$$\mathbb{A}^{(R)}\left(\boldsymbol{w}^{(I)} - 2\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)}\right\rangle \boldsymbol{w}\right) = 0. \tag{4.4}$$

Under the hypothesis

$$0 < \lambda_1 \leq \lambda_2 \text{ and } 0 \leq \lambda_1^{(I)} \leq \lambda_2^{(I)} \tag{H}$$

we have

$$\lambda_1^{(R)} + \lambda_2^{(R)} \geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1 > 0, \tag{4.5}$$

$$\lambda_1^{(R)}\lambda_2^{(R)} \geq \lambda_1^{(I)}\lambda_2^{(I)} + 4(\lambda_1)^2 > 0 \tag{4.6}$$

and thus

$$\lambda_1^{(R)} > 0,\ \lambda_2^{(R)} > 0 \tag{4.7}$$

Proof. In basis $\mathcal{B}$, $\boldsymbol{u}^{(I)} = (u_j)_{j=1}^3$, $\boldsymbol{v}^{(I)} = (v_j)_{j=1}^3$, $\boldsymbol{w}^{(I)} = (w_j)_{j=1}^3$ and $\mathbb{A}^{(I)} = (a_{ij})_{i,j=1}^3$. By hypothesis, $\mathbb{A}^{(I)}$ is symmetric and $\mathbb{A}^{(I)}\boldsymbol{w}^{(I)} = 0$.

As $\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle \neq 0$, we can compute $\mathbb{A}^{(R)}$: $\forall x \in \mathbb{R}^3$

$$\begin{array}{rcl} \mathbb{A}^{(R)}x & = & \left(\mathbb{A}^{(I)} - 2\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)} \right\rangle \mathbb{B}\right) x \\ & & -2\left\langle \boldsymbol{w}, x\right\rangle \left(\mathbb{A}^{(I)}\boldsymbol{w} + \mathbb{B}\boldsymbol{w}^{(I)}\right) \\ & & -2\left\langle \mathbb{A}^{(I)}\boldsymbol{w} + \mathbb{B}\boldsymbol{w}^{(I)}, x\right\rangle \boldsymbol{w} \\ & & +2\left[2\left\langle \mathbb{A}^{(I)}\boldsymbol{w}, \boldsymbol{w}\right\rangle - \frac{\left\langle \mathbb{B}\boldsymbol{w}^{(I)}, \boldsymbol{w}^{(I)}\right\rangle}{\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)}\right\rangle}\right] \left\langle \boldsymbol{w}, x\right\rangle \boldsymbol{w} \end{array}$$

in basis $\mathcal{B}$, $x = (x_j)_{j=1}^3$ and

$$\begin{array}{rcl} \mathbb{A}^{(R)}x & = & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - 2w_3 \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ 0 \end{pmatrix} \\ & & -2x_3 \begin{pmatrix} a_{13} + \lambda_1 w_1 \\ a_{23} + \lambda_2 w_2 \\ a_{33} \end{pmatrix} \\ & & -2\left\langle \begin{pmatrix} a_{13} + \lambda_1 w_1 \\ a_{23} + \lambda_2 w_2 \\ a_{33} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ & & +2\left[2a_{33} - \dfrac{\lambda_1 w_1^2 + \lambda_2 w_2^2}{w_3}\right] x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

That is to say,

$$\mathbb{A}^{(R)} = \begin{pmatrix} a_{11} - 2\lambda_1 w_3 & a_{12} & -a_{13} - 2\lambda_1 w_1 \\ a_{12} & a_{22} - 2\lambda_2 w_3 & -a_{23} - 2\lambda_2 w_2 \\ -a_{13} - 2\lambda_1 w_1 & -a_{23} - 2\lambda_2 w_2 & a_{33} - \frac{2}{w_3}(\lambda_1 w_1^2 + \lambda_2 w_2^2) \end{pmatrix}$$

So we have $\mathbb{A}^{(R)}$ symmetry.

To proove formula (4.4), we make computation in basis $\mathcal{B}$ where

$$\boldsymbol{w}^{(I)} - 2\left\langle \boldsymbol{w}, \boldsymbol{w}^{(I)}\right\rangle \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ -w_3 \end{pmatrix}$$

So

$$\mathbb{A}^{(R)}\boldsymbol{w}^{(R)} = \begin{pmatrix} a_{11}w_1 - 2\lambda_1 w_3 w_1 + a_{12}w_2 + a_{13}w_3 + 2\lambda_1 w_1 w_3 \\ a_{12}w_1 + a_{22}w_2 - 2\lambda_2 w_3 w_2 + a_{23}w_3 + 2\lambda_2 w_2 w_3 \\ -a_{13}w_1 - 2\lambda_1 w_1^2 - a_{23}w_2 - 2\lambda_2 w_2^2 - a_{33}w_3 + 2(\lambda_1 w_1^2 + \lambda_2 w_2^2) \end{pmatrix}$$

Then, using $\mathbb{A}^{(I)}\boldsymbol{w}^{(I)} = 0$ give immediatly formula (4.4).

To establish the formulas (4.5)-(4.6)-(4.7) under the assumptions (H), we will study the eigenvalues of matrix $\mathbb{A}^{(R)}$. For that, let $\mathbb{H}$ be the normal matrix given by

$$\mathbb{H} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ -u_3 & -v_3 & -w_3 \end{pmatrix}$$

Then $\mathbb{H}\mathbb{A}^{(R)}\mathbb{H}^t$ is similar to $\mathbb{A}^{(R)}$. Using that $(\boldsymbol{u}^{(I)}, \boldsymbol{v}^{(I)}, \boldsymbol{w}^{(I)})$ is a direct and orthonormal basis, we have

$$\boldsymbol{u}^{(I)} \wedge \boldsymbol{v}^{(I)} = \boldsymbol{w}^{(I)} \Leftrightarrow \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{4.8}$$

$$\boldsymbol{v}^{(I)} \wedge \boldsymbol{w}^{(I)} = \boldsymbol{u}^{(I)} \Leftrightarrow \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -v_1 w_3 + v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{4.9}$$

$$\boldsymbol{w}^{(I)} \wedge \boldsymbol{u}^{(I)} = \boldsymbol{v}^{(I)} \Leftrightarrow \begin{pmatrix} w_2u_3 - w_3u_2 \\ -w_1u_3 + w_3u_1 \\ w_1u_2 - w_2u_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{4.10}$$

$$\left\|\boldsymbol{u}^{(I)}\right\| = 1 \Leftrightarrow u_1^2 + u_2^2 + u_3^2 = 1 \tag{4.11}$$

$$\left\|\boldsymbol{v}^{(I)}\right\| = 1 \Leftrightarrow v_1^2 + v_2^2 + v_3^2 = 1 \tag{4.12}$$

$$\left\|\boldsymbol{w}^{(I)}\right\| = 1 \Leftrightarrow w_1^2 + w_2^2 + w_3^2 = 1 \tag{4.13}$$

Using $\mathbb{A}^{(I)}\boldsymbol{u}^{(I)} = \lambda_1^{(I)}\boldsymbol{u}^{(I)}$, $\mathbb{A}^{(I)}\boldsymbol{v}^{(I)} = \lambda_1^{(I)}\boldsymbol{v}^{(I)}$ and $\mathbb{A}^{(I)}\boldsymbol{w}^{(I)} = 0$ give

$$\mathbb{A}^{(R)}\mathbb{H} = \begin{pmatrix} \lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1) & \lambda_2^{(I)}v_1 + 2\lambda_1(w_1v_3 - w_3v_1) & 0 \\ \lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2) & \lambda_2^{(I)}v_2 + 2\lambda_2(w_2v_3 - w_3v_2) & 0 \\ \left\{-\lambda_1^{(I)}u_3 \quad -2\lambda_1(w_1u_1 - \frac{u_3w_1^2}{w_3}) \right. & \left\{-\lambda_2^{(I)}v_3 \quad -2\lambda_1(w_1v_1 - \frac{v_3w_1^2}{w_3}) \right. & \\ \left. \qquad -2\lambda_2(w_2u_2 - \frac{u_3w_2^2}{w_3})\right\} & \left. \qquad -2\lambda_2(w_2v_2 - \frac{v_3w_2^2}{w_3})\right\} & 0 \end{pmatrix}$$

To obtain $\mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}$, we compute $\left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{x}, \boldsymbol{y}\right\rangle \quad \forall x, y \in \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ We can remark that

$$\left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{x}, \boldsymbol{w}\right\rangle = 0 \;\; \forall x \in \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$$

Now, we compute the six leading terms of matrix $\mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}$

- Calculus of $\left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{u}\right\rangle$

$$\begin{aligned} \left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{u}\right\rangle &= \left(\lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1)\right)u_1 \\ &+ \left(\lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2)\right)u_2 \\ &- \left(-\lambda_1^{(I)}u_3 - 2\lambda_1(w_1u_1 - \frac{u_3w_1^2}{w_3}) - 2\lambda_2(w_2u_2 - \frac{u_3w_2^2}{w_3})\right)u_3 \end{aligned}$$

By hypothesis $\left\|\boldsymbol{u}^{(I)}\right\| = 1$ so

$$\begin{aligned} \left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{u}\right\rangle &= \lambda_1^{(I)} + 2\frac{\lambda_1}{w_3}\left(2w_3w_1u_3u_1 - w_3^2u_1^2 - w_1^2u_3^2\right) \\ &\quad + 2\frac{\lambda_2}{w_3}\left(2w_3w_2u_2u_3 - w_3^2u_2^2 - w_2^2u_3^2\right) \\ &= \lambda_1^{(I)} - 2\frac{\lambda_1}{w_3}(w_3u_1 - w_1u_3)^2 - 2\frac{\lambda_2}{w_3}(w_3u_2 - w_2u_3)^2 \end{aligned}$$

Using formula (4.10) give

$$\left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{u}\right\rangle = \lambda_1^{(I)} - \frac{2}{w_3}\left(\lambda_1v_2^2 + \lambda_2v_1^2\right)$$

- Calculus of $\left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{v}\right\rangle$

$$\begin{aligned} \left\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\boldsymbol{u}, \boldsymbol{v}\right\rangle &= \left(\lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1)\right)v_1 \\ &+ \left(\lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2)\right)v_2 \\ &- \left(-\lambda_1^{(I)}u_3 - 2\lambda_1(w_1u_1 - \frac{u_3w_1^2}{w_3}) - 2\lambda_2(w_2u_2 - \frac{u_3w_2^2}{w_3})\right)v_3 \end{aligned}$$

By hypothesis $\left\langle \boldsymbol{u}^{(I)}, \boldsymbol{v}^{(I)} \right\rangle = 0$ so

$$\begin{aligned}
\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{v} \right\rangle &= 2\frac{\lambda_1}{w_3}\left(w_3 w_1 u_3 v_1 - w_3^2 u_1 v_1 + w_3 w_1 u_1 v_3 - w_1^2 u_3 v_3\right) \\
&\quad +2\frac{\lambda_2}{w_3}\left(w_3 w_2 u_3 v_2 - w_3^2 u_2 v_2 + w_3 w_2 u_2 v_3 - w_2^2 u_3 v_3\right) \\
&= 2\frac{\lambda_1}{w_3}(w_3 u_1 - w_1 u_3)(w_1 v_3 - w_3 v_1) + 2\frac{\lambda_2}{w_3}(w_3 u_2 - w_2 u_3)(w_2 v_3 - w_3 v_2)
\end{aligned}$$

Using formulas (4.9) and (4.10) give

$$\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{v} \right\rangle = \frac{2}{w_3}(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1)$$

- Calculus of $\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{w} \right\rangle$

$$\begin{aligned}
\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{w} \right\rangle &= \left(\lambda_1^{(I)} u_1 + 2\lambda_1(w_1 u_3 - w_3 u_1)\right) w_1 \\
&\quad + \left(\lambda_1^{(I)} u_2 + 2\lambda_2(w_2 u_3 - w_3 u_2)\right) w_2 \\
&\quad - \left(-\lambda_1^{(I)} u_3 - 2\lambda_1(w_1 u_1 - \frac{u_3 w_1^2}{w_3}) - 2\lambda_2(w_2 u_2 - \frac{u_3 w_2^2}{w_3})\right) w_3
\end{aligned}$$

By hypothesis $\left\langle \boldsymbol{u}^{(I)}, \boldsymbol{w}^{(I)} \right\rangle = 0$ so

$$\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{w} \right\rangle = 0$$

- Calculus of $\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{u} \right\rangle$

$$\begin{aligned}
\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{u} \right\rangle &= \left(\lambda_2^{(I)} v_1 + 2\lambda_1(w_1 v_3 - w_3 v_1)\right) u_1 \\
&\quad + \left(\lambda_2^{(I)} v_2 + 2\lambda_2(w_2 v_3 - w_3 v_2)\right) u_2 \\
&\quad - \left(-\lambda_2^{(I)} v_3 - 2\lambda_1(w_1 v_1 - \frac{v_3 w_1^2}{w_3}) - 2\lambda_2(w_2 v_2 - \frac{v_3 w_2^2}{w_3})\right) u_3
\end{aligned}$$

By hypothesis $\left\langle \boldsymbol{u}^{(I)}, \boldsymbol{v}^{(I)} \right\rangle = 0$ so

$$\begin{aligned}
\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{u} \right\rangle &= 2\frac{\lambda_1}{w_3}\left(w_1 w_3 u_1 v_3 - w_3^2 u_1 v_1 + w_1 w_3 u_3 v_1 - w_1^2 u_3 v_3\right) \\
&\quad +2\frac{\lambda_2}{w_3}\left(w_2 w_3 u_2 v_3 - w_3^2 u_2 v_2 + w_2 w_3 u_3 v_2 - w_2^2 u_3 v_3\right) \\
&= 2\frac{\lambda_1}{w_3}(w_3 u_1 - w_1 u_3)(w_1 v_3 - w_3 v_1) + 2\frac{\lambda_2}{w_3}(w_3 u_2 - w_2 u_3)(w_2 v_3 - w_3 v_2)
\end{aligned}$$

Using formulas (4.9) and (4.10) give

$$\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{u} \right\rangle = \frac{2}{w_3}\left(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1\right)$$

- Calculus of $\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{v} \right\rangle$

$$\begin{aligned}
\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{v} \right\rangle &= \left(\lambda_2^{(I)} v_1 + 2\lambda_1(w_1 v_3 - w_3 v_1)\right) v_1 \\
&\quad + \left(\lambda_2^{(I)} v_2 + 2\lambda_2(w_2 v_3 - w_3 v_2)\right) v_2 \\
&\quad - \left(-\lambda_2^{(I)} v_3 - 2\lambda_1(w_1 v_1 - \frac{v_3 w_1^2}{w_3}) - 2\lambda_2(w_2 v_2 - \frac{v_3 w_2^2}{w_3})\right) v_3
\end{aligned}$$

By hypothesis $\left\|\boldsymbol{v}^{(I)}\right\| = 1$ so

$$\begin{aligned}\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{v} \right\rangle &= \lambda_2^{(I)} + 2\frac{\lambda_1}{w_3}\left(2w_1w_3v_1v_3 - w_3^2v_1^2 - w_1^2v_3^2\right) \\ &\quad +2\frac{\lambda_2}{w_3}\left(2w_2w_3v_2v_3 - w_3^2v_2^2 - w_2^2v_3^2\right) \\ &= \lambda_2^{(I)} - 2\frac{\lambda_1}{w_3}\left(w_3v_1 - w_1v_3\right)^2 - 2\frac{\lambda_2}{w_3}\left(w_3v_2 - w_2v_3\right)^2\end{aligned}$$

Using formula (4.9) give

$$\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{v} \right\rangle = \lambda_2^{(I)} - \frac{2}{w_3}\left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)$$

- Calculus of $\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{w} \right\rangle$

$$\begin{aligned}\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{v}, \boldsymbol{w} \right\rangle &= \left(\lambda_2^{(I)} v_1 + 2\lambda_1(w_1v_3 - w_3v_1)\right) w_1 \\ &\quad + \left(\lambda_2^{(I)} v_2 + 2\lambda_2(w_2v_3 - w_3v_2)\right) w_2 \\ &\quad - \left(-\lambda_2^{(I)} v_3 - 2\lambda_1(w_1v_1 - \frac{v_3w_1^2}{w_3}) - 2\lambda_2(w_2v_2 - \frac{v_3w_2^2}{w_3})\right) w_3\end{aligned}$$

By hypothesis $\left\langle \boldsymbol{v}^{(I)}, \boldsymbol{w}^{(I)} \right\rangle = 0$ so

$$\left\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \boldsymbol{u}, \boldsymbol{w} \right\rangle = 0$$

With all these formulas we obtain :

$$\mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} = \begin{pmatrix} \lambda_1^{(I)} - \frac{2}{w_3}\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right) & \frac{2}{w_3}(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) & 0 \\ \frac{2}{w_3}\left(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1\right) & \lambda_2^{(I)} - \frac{2}{w_3}\left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.14}$$

As $\mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H}$ is similar to $\mathbb{A}^{(R)}$, we refind that 0 is an eigenvalue of $\mathbb{A}^{(R)}$ and the two others eigenvalues $\lambda_1^{(R)}$ and $\lambda_2^{(R)}$ are also eingenvalues of

$$\mathbb{A} = \begin{pmatrix} \lambda_1^{(I)} - \frac{2}{w_3}\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right) & \frac{2}{w_3}(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) \\ \frac{2}{w_3}\left(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1\right) & \lambda_2^{(I)} - \frac{2}{w_3}\left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right) \end{pmatrix}$$

Furthermore, we have that

$$\mathrm{Tr}(\mathbb{A}) = \lambda_1^{(R)} + \lambda_2^{(R)} \tag{4.15}$$

$$\det(\mathbb{A}) = \lambda_1^{(R)} \lambda_2^{(R)} \tag{4.16}$$

To establish inequality 4.5, we use equation (4.15) under hypothesis (H) and $w_3 < 0$ to get

$$\lambda_1^{(R)} + \lambda_2^{(R)} \geq \lambda_1^{(I)} + \lambda_2^{(I)} - \frac{2}{w_3}\lambda_1((u_1^2 + u_2^2) + (v_1^2 + v_2^2)).$$

We remark that

$$\begin{aligned}((u_1^2 + u_2^2) + (v_1^2 + v_2^2)) &= (u_1 + v_2)^2 + (u_2 - v_1)^2 - 2(u_1v_2 - u_2v_1) \\ &= (u_1 + v_2)^2 + (u_2 - v_1)^2 - 2w_3\end{aligned}$$

and so, we obtain

$$\begin{aligned}\lambda_1^{(R)} + \lambda_2^{(R)} &\geq \lambda_1^{(I)} + \lambda_2^{(I)} - \frac{2}{w_3}\lambda_1\left((u_1 + v_2)^2 + (u_2 - v_1)^2 - 2w_3\right) \\ &\geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1 + 4\lambda_1\left((u_1 + v_2)^2 + (u_2 - v_1)^2\right) \\ &\geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1.\end{aligned}$$

To establish inequality 4.6, we use equation (4.16) to get

$$\begin{aligned}
\lambda_1^{(R)}\lambda_2^{(R)} &= \left(\lambda_1^{(I)} - \frac{2}{w_3}\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\right)\left(\lambda_2^{(I)} - \frac{2}{w_3}\left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\right) \\
&\quad -\frac{4}{w_3^2}\left(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1\right)^2 \\
&= \lambda_1^{(I)}\lambda_2^{(I)} + \frac{4}{w_3^2}\left[\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right) - \left(\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1\right)^2\right] \\
&\quad -\frac{2}{w_3}\left(\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\lambda_2^{(I)} + \left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\lambda_1^{(I)}\right) \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3}\left(\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\lambda_2^{(I)} + \left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\lambda_1^{(I)}\right) \\
&\quad +\frac{4}{w_3^2}\left[\lambda_1^2 u_2^2 v_2^2 + \lambda_1\lambda_2 u_1^2 v_2^2 + \lambda_1\lambda_2 v_1^2 u_2^2 - \lambda_1^2 u_2^2 v_2^2 - 2\lambda_1\lambda_2 u_1 v_1 u_2 v_2 - \lambda_1\lambda_2 v_1^2 u_2^2\right] \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3}\left(\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\lambda_2^{(I)} + \left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\lambda_1^{(I)}\right) \\
&\quad +\frac{4}{w_3^2}\lambda_1\lambda_2\left[u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + v_1^2 u_2^2\right] \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3}\left(\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\lambda_2^{(I)} + \left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\lambda_1^{(I)}\right) \\
&\quad +\frac{4}{w_3^2}\lambda_1\lambda_2\left(u_1 v_2 - v_1 u_2\right)^2
\end{aligned}$$

From formula (4.8), we have $u_1 v_2 - v_1 u_2 = w_3$ and so we obtain

$$\begin{aligned}
\lambda_1^{(R)}\lambda_2^{(R)} &= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3}\left(\left(\lambda_1 v_2^2 + \lambda_2 v_1^2\right)\lambda_2^{(I)} + \left(\lambda_1 u_2^2 + \lambda_2 u_1^2\right)\lambda_1^{(I)}\right) \\
&\quad +4\lambda_1\lambda_2
\end{aligned}$$

Under hypothesis (H) and $w_3 < 0$, we clearly have

$$\lambda_1^{(R)}\lambda_2^{(R)} \geq \lambda_1^{(I)}\lambda_2^{(I)} + 4\lambda_1^2 > 0$$

and we obtain formula (4.6).

□

References

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