Adaptive Spatial Aloha, Fairness and Stochastic Geometry

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***Abstract*—This work aims at combining adaptive protocol design, utility maximization and stochastic geometry. We focus on a spatial adaptation of Aloha within the framework of ad hoc networks. We consider quasi-static networks in which mobiles learn the local topology and incorporate this information to adapt their medium access probability (MAP) selection to their local environment. We consider the cases where nodes cooperate in a distributed way to maximize the global throughput or to achieve either proportional fair or max-min fair medium access. In the proportional fair case, we show that nodes can compute their optimal MAPs as solutions to certain fixed point equations. In the maximum throughput case, the optimal MAPs are obtained through a Gibbs Sampling based algorithm. In the max min case, these are obtained as the solution of a convex optimization problem. The main performance analysis result of the paper is that this type of distributed adaptation can be analyzed using stochastic geometry in the proportional fair case. In this case, we show that, when the nodes form a homogeneous Poisson point process in the Euclidean plane, the distribution of the optimal MAP can be obtained from that of a certain shot noise process w.r.t. the node Poisson point process and that the mean utility can also be derived from this distribution. We discuss the difficulties to be faced for analyzing the performance of the other cases (maximal throughput and max-min fairness). Numerical results illustrate our findings and quantify the gains brought by spatial adaptation in such networks.**

I. Introduction

Stochastic geometry has recently been used for the analysis and performance evaluation of wireless (ad hoc as well as cellular) networks; in this approach, one models node locations as a spatial point process, e.g., homogeneous Poisson point processes, and one computes various network statistics, e.g., interference, successful transmission probability, coverage (or, outage) probability etc. as spatial averages. This often leads to tractable performance metrics that are amenable to parametric optimization with respect to network parameters (node density, protocol parameters, etc.). More precisely, this approach yields spatial averages of the performance metrics for given network parameters; then the parameters can be chosen to optimize performance. This approach takes a macroscopic view of the network with the underlying assumption that all nodes in the network have identical statistical characteristics.

In practice, due to randomness and heterogeneity in networks, nodes need to adapt to local spatial and temporal conditions (e.g., channel conditions and topology) to reach optimum network wide performance. For example, nodes in wireless LANs adjust their window sizes based on acknowledgment feedback; in cellular networks nodes are scheduled based on channel conditions and adapt their transmit powers based on the measured SINRs, which in turn depend on the transmit powers set by other nodes. In all such scenarios, distributed adaptive algorithms are used to reach a desired network wide operating point e.g. that maximizing some utility. While the behavior of such distributed optimization protocols is often well understood on a given topology, there are usually no analytical characterizations of the statistical properties of the optimal state in large random and heterogeneous networks.

The main aim of this work is to use stochastic geometry to study spatial adaptations of medium access control in Aloha that aim at optimizing certain utilities. While we identify a utility for which stochastic geometry can be used to compute the spatial distribution of MAP and the expected utility, we are far from being able to do so for all types of utilities within the α-fair class and we discuss the difficulties to be faced.

Let us start with a review of the state of the art on Aloha. Wireless spectrum is well known to be a precious and scarce shared resource. Medium Access Control (MAC) algorithms are employed to coordinate access to the shared wireless medium. An efficient MAC protocol should ensure high system throughput, and should also distribute the available bandwidth fairly among the competing nodes. The simplest of the MAC protocols, Aloha and slotted Aloha, with a "random access" spirit, were introduced by Abramson [1] and Roberts [17] respectively. In these protocols, only one node could successfully transmit at a time. Reference [4] modeled node locations as spatial point processes, and also modeled channel fadings, interferences and SINR based reception. This allowed for spatial reuse and multiple simultaneous successful transmissions depending on SINR levels at the corresponding receivers. All the above protocols prescribe identical attempt probabilities for all the nodes. Reference [5] further proposed opportunistic Aloha in which nodes' transmission attempts are modulated by their channel conditions.

Among the initial attempts of MAP adaptation in Aloha, reference [11] analyzed protocol model and proposed stochastic approximation based strategies that were based on receiver feedback and were aimed at stabilizing the network. References [4], [5] also optimized nodes' attempt probabilities (or thresholds) in order to maximize the spatial density of successful transmissions. Reference [13] analyzed both plain and opportunistic Aloha in a network where all the nodes communicate to one access point. They assumed statistically identical Rayleigh faded channels with no dependence on geometry (i.e., no path loss components). They demonstrated a paradoxical behavior where plain Aloha yields better aggregate throughput than the opportunistic one. Reference [14] also studied optimal random access with SINR based reception. However, they considered constant channel gains. They developed a centralized algorithm that maximizes the network throughput, and also an algorithm that leads to max-min fair operation. Reference [18] modeled network as an undirected graph and studied Aloha under the protocol model. They designed distributed algorithms that are either proportional fair or max-min fair. Reference [12] built upon the model of [4], and formulated the channel access problem as a noncooperative game among users. They considered throughput and delay as performance metrics and proposed pricing schemes that induce socially optimum behavior at equilibrium. However, they set time average quantities (e.g., throughput, delay) as utilities (or costs), and concentrated on symmetric Nash equilibria. Consequently, in their analysis, dependence on local conditions vanishes.

In none of the above Aloha models, nodes account for both wireless channel randomness and local topology for making their random access decisions, as we do in the present paper.[1]

There is a vast literature on the modeling of CSMA by stochastic geometry which will not be reviewed in detail here. The very nature of this MAC protocol is adaptive as each node senses the network and acts in order to ensure that certain exclusion rules are satisfied, namely that neighboring nodes do not access the channel simultaneously. However, CSMA as such is designed to guarantee a reasonable scheduling, not to optimize any utility of the throughput. The closest reference to our work is probably [6] where the authors study an adaptation of the exclusion range and of the transmit power of a CSMA node to the location of the closest interferer. This adaptation aims at maximizing the mean number of nodes transmitting per unit time and space (while respecting the above exclusion rules). This mean number is however only a surrogate of the rate. In addition, the adaptation is only w.r.t. to the location of the nearest interferer.

We study spatial adaptation of Aloha in ad hoc networks. The network setting is described in Section II. We consider quasi-static networks in which mobiles learn the topology, and incorporate this information in their medium access probability (MAP) selection. We consider the cases where nodes are benevolent and cooperate in a distributed way to maximize the global network throughput or to reach either a proportional fair or max-min fair sharing of the network resources. We analyze the case where nodes account only for their closest interferers, for all nodes in a given ball around them, or even all nodes in the network.

Section III is focused on the distributed algorithms that maximize the aggregate throughput or lead to max-min fairness in such networks. In the proportional fair case, we show that nodes can compute the optimal MAPs as solutions to certain fixed point equations. In the maximum throughput case, the optimal MAPs are obtained through a Gibbs Sampling based algorithm. In the max min case, the optimal MAPs are obtained as the solution of a convex optimization problem.

Section IV contains the stochastic geometry results. The model features nodes forming a realization of a homogeneous Poisson point process in the Euclidean plane. We compute the MAP distribution in such a network in the proportional fair case using shot noise field theory. To the best of our knowledge, this distribution is the first example of successful combination of stochastic geometry and adaptive protocol design aimed at optimizing ceratin utility function within this Aloha setting. We also show that the mean value of the logarithm of the throughput obtained by a typical node can be derived from this distribution. Finally, we discuss the difficulties to be faced in order to extend the result to other types of utilities.

The numerical results are gathered in Section V. The aim of this section is two-fold: 1) check the analytical results against simulation and 2) quantify the gains brought by adaption within this setting.

II. Network Model

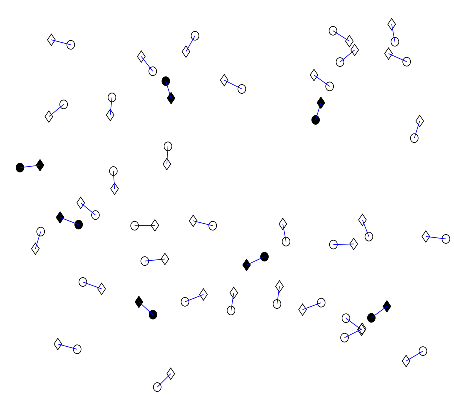

Fig. 1. A snapshot of bipolar MANET with Aloha as the medium access protocol. The *diamonds* represent transmitters, and the connected *circles* the corresponding receivers. The solid diamonds represent the nodes that are transmitting in a slot.

We model the ad-hoc wireless network as a set of transmitters and their corresponding receivers, all located in the Euclidean plane. This is often referred to as "bipole model" [3,

[1] In view of this distinction, we refer to the spatial Aloha protocol of [4] as plain Aloha.

Chapter 16]. There are N transmitter-receiver pairs communicating over a shared channel. The transmitters follow the slotted version of the Aloha medium access control (MAC) protocol (see Figure 1). A transmitter, in each transmission attempt, sends one packet which occupies one slot. Each transmitter uses unit transmission power. We assume that each node has an infinite backlog of packets to transmit to its receiver. The Euclidean distance between transmitter j and receiver i is r_{ji}, and the path-loss exponent is α ($\alpha > 2$). We also assume Rayleigh faded channels with h_{ji} being the random fading between transmitter j and receiver i. Moreover, we assume that the random variables $h_{ij}, 1 \le i \le N, 1 \le j \le N$ are independent and identically distributed with mean $1/\mu$.[2] Thus all h_{ji}s have cumulative distribution function (cdf) $F(x) = 1 - e^{-\mu x}$ with $x \ge 0$. All the receivers are also subjected to white Gaussian thermal noise with variance w, which is also constant across slots. We assume that a receiver successfully receives the packet of the corresponding transmitter if the received SINR exceeds a threshold T.

Let e_i be the indicator variable indicating whether transmitter i transmits in a slot, and p_i be i's medium access probability. Thus $\mathbb{P}(e_i = 1) = p_i$. When node i transmits, the received SINR at the corresponding receiver is

$$\gamma_i = \frac{h_{ii} r_{ii}^{-\alpha}}{\sum_{j \neq i} e_j h_{ji} r_{ji}^{-\alpha} + w}.$$

Then the probability of successful reception q_i can be calculated as follows.

$$\begin{aligned}
&\mathbb{P}\left(\gamma_i \ge T \middle| \{(h_{ji}, e_j) : j \neq i\}\right) \\
&= \mathbb{P}\left(h_{ii} \ge \sum_{j \neq i} e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T + \frac{wT}{r_{ii}^{-\alpha}} \middle| \{(h_{ji}, e_j) : j \neq i\}\right) \\
&= \exp\left(-\mu T\left(\sum_{j \neq i} e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} + \frac{w}{r_{ii}^{-\alpha}}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
q_i &= \mathbb{E}_{\{(h_{ji}, e_j): j \neq i\}} \exp\left(-\mu T\left(\sum_{j \neq i} e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} + \frac{w}{r_{ii}^{-\alpha}}\right)\right) \\
&= \mathrm{e}^{\frac{-\mu w T}{r_{ii}^{-\alpha}}} \prod_{j \neq i} \mathbb{E}_{(h_{ji}, e_j)} \exp\left(-\mu e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T\right) \\
&= \mathrm{e}^{\frac{-\mu w T}{r_{ii}^{-\alpha}}} \prod_{j \neq i} \mathbb{E}_{h_{ji}} \left((1 - p_j) + p_j \exp\left(-\mu h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T\right)\right) \\
&= \mathrm{e}^{-\mu w T r_{ii}^{\alpha}} \prod_{j \neq i} \left((1 - p_j) + \frac{p_j}{1 + 1/b_{ji}}\right),
\end{aligned}$$

where $b_{ji} = \frac{1}{T}\left(\frac{r_{ji}}{r_{ii}}\right)^{\alpha}$. Further simplifying,

$$q_i = \mathrm{e}^{-\mu w T r_{ii}^{\alpha}} \prod_{j \neq i} \left(1 - \frac{p_j}{1 + b_{ji}}\right). \tag{1}$$

[2] The independence assumption is justified if the distance between two receivers is larger than the coherence distance of the wireless channel [3]. We assume this to be the case.

Then, the rate or throughput of transmitter i is given by $p_i q_i$.

The thermal noise appears merely as a constant multiplicative factor in the expression for the successful transmission probability (see (1)). Moreover, in interference limited networks, the impact of thermal noise is negligible as compared to interference. We focus on such networks, and thus we ignore the thermal noise factor throughout.

III. Adaptive Spatial Aloha and Fairness

In this section, we analyze adaptations of spatial Aloha that maximize aggregate throughput or achieve proportional fairness or max-min fairness.

A. Maximum Throughput Medium Access

The throughput maximizing medium access probabilities solve the following optimization problem.

$$\begin{aligned}
\text{maximize} \quad & \Theta := \sum_i p_i \prod_{j \neq i} \left(1 - \frac{p_j}{1 + b_{ji}}\right), \\
\text{subject to} \quad & 0 \le p_i \le 1, \ i \in \mathcal{N}.
\end{aligned}$$

We first argue that the optimum in the above optimization problem is attained at one of the vertices of the hypercube formed by the constraint set. To see this, suppose $\mathbf{p}^* \in [0,1]^{\mathcal{N}}$ is an optimal solution, and $p_i^* \in (0, 1)$ for some $i \in \mathcal{N}$. Clearly,

$$\begin{aligned}
&\frac{\partial \Theta}{\partial p_i}\Big|_{\mathbf{p} = \mathbf{p}^*} \\
&= \prod_{j \neq i} \left(1 - \frac{p_j^*}{1 + b_{ji}}\right) - \sum_{j \neq i} \frac{p_j^*}{1 + b_{ij}} \prod_{k \neq i,j} \left(1 - \frac{p_k^*}{1 + b_{kj}}\right) \\
&= 0.
\end{aligned}$$

Since the partial derivative is independent of p_i, p_i can be set to either 0 or 1 without reducing the value of the objective function. This proves our claim. In the following we focus only on such extreme solutions. Then the above problem is equivalent to finding an $\mathcal{M} \subset N$ such that $p_i = 1$ if and only if $i \in \mathcal{M}$ is an optimal solution. Thus we are interested in

$$\underset{\mathcal{M} \subset \mathcal{N}}{\text{maximize}} \quad \sum_{i \in \mathcal{M}} \prod_{j \in \mathcal{M} \setminus \{i\}} \left(1 - \frac{1}{1 + b_{ji}}\right).$$

An iterative solution: We can pose this problem as a strategic form game with the users as players [16]. For each player its action a_i lies in $\{0, 1\}$, and the utility function $u_i : \mathbf{a} \mapsto \mathbb{R}$ is given by

$$\begin{aligned}
u_i(0, \mathbf{a}_{-i}) &= 0, \\
u_i(1, \mathbf{a}_{-i}) &= 1 - \prod_{j \in \mathcal{M} \setminus \{i\}} \left(1 - \frac{1}{1 + b_{ji}}\right) \\
&\quad - \sum_{j \in \mathcal{M} \setminus \{i\}} \frac{1}{1 + b_{ij}} \prod_{k \in \mathcal{M} \setminus \{i,j\}} \left(1 - \frac{1}{1 + b_{kj}}\right).
\end{aligned}$$

This is a potential game with the above objective function as the potential function [15]. Thus the best response dynamics converges to a Nash equilibrium. This algorithm can be implemented in a distributed fashion if each node i knows b_{ij}, b_{ji} for all j, and also $\mathcal{M}$ and $\prod_{k \in \mathcal{M} \setminus \{j\}} (1 - (1 + b_{kj})^{-1})$ for all

$j \in \mathcal{M}$ after each iteration. However, a Nash equilibrium can be a suboptimal solution to the above optimization problem. To alleviate this problem, we propose a Gibbs sampler based distributed algorithm, wherein each node i chooses action 1 with probability

$$p_i = \frac{\mathrm{e}^{u_i(1,\mathbf{a}_{-i})/\tau}}{1+\mathrm{e}^{u_i(1,\mathbf{a}_{-i})/\tau}}.$$

The parameter τ is called the temperature. The Gibbs sampler dynamics converges to a steady state which is the Gibbs distribution associated with the aggregate throughput and the temperature τ [9]. In other words, we are led to the following distribution on the action profiles:

$$\pi_\tau(\mathbf{a}) = u\mathrm{e}^{\sum_{i\in\mathcal{N}} u_i(\mathbf{a})},$$

where u is a normalizing constant. When τ goes to 0 in an appropriate way (i.e., as $1/\log(1+t)$, where t is the time), the distribution $\pi_\tau(\cdot)$ converges to a dirac mass at the action profile $\mathbf{a}^*$ with maximum aggregate utility if it is unique. Notice that the aggregate utility $\sum_{i\in\mathcal{N}} u_i(\mathbf{a})$ equals the aggregate throughput. Thus the action profile $\mathbf{a}^*$ is a solution to the original throughput optimization problem.

Remark 3.1: The first two terms in the utility function $u_i(1,\mathbf{a}_{-i})$ can be seen as "selfish" part of user i, whereas the last summation term is "altruistic" part. The user makes a decision based on whether the "selfish" part dominates or viceversa.

Remark 3.2: In a quasi-static network where topology continuously changes, although at a slower time scale, different sets of nodes are likely to be scheduled to transmit under different topologies. Thus, in terms of long term performance, maximum throughput medium access is not grossly unfair.

B. Proportional Fair Medium Access

The proportional fair medium access problem can be formulated as follows.

$$\begin{aligned} &\text{maximize} \quad && \sum_i \log(p_i q_i), \\ &\text{subject to} \quad && 0 \le p_i \le 1,\ i \in \mathcal{N}. \end{aligned}$$

The objective function can be rewritten as

$$\sum_i \left(\log p_i + \sum_{j\neq i} \log\left(1 - \frac{p_j}{1+b_{ji}}\right)\right).$$

We thus have a convex separable optimization problem. The partial derivative of the objective function with respect to p_i is

$$\frac{1}{p_i} - \sum_{j\neq i} \frac{1}{1+b_{ij}-p_i}, \tag{2}$$

which is continuous and decreasing in p_i over $[0,1]$. We conclude that at optimality, for each $1 \le i \le N$,

$$p_i = \begin{cases} f_i(p_i) := \left(\sum_{j\neq i} \frac{1}{1+b_{ij}-p_i}\right)^{-1} & \text{if } f_i(1) < 1, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that user i's optimal attempt probability is independent of the attempt probabilities of other users. In particular, if $f_i(1) < 1$, user i can perform iterations $p_i^{k+1} = f_i(p_i^k)$ autonomously. Furthermore,

$$f_i'(p_i) = -\sum_{j\neq i} \frac{1}{(1+b_{ij}-p_i)^2} \left(\sum_{j\neq i} \frac{1}{1+b_{ij}-p_i}\right)^{-2}.$$

Clearly, $|f_i'(p_i)| < 1$, i.e., $f_i(\cdot)$ is a contraction. Thus the fixed point iterations converge to the optimal p_i starting from any $p_i \in [0,1]$.

Remark 3.3: The characterization of the optimal attempt probabilities reflects the altruistic behavior of users. More precisely, user i's attempt probability is a function of $\{b_{ij}, j\neq i\}$ which are measures of i's interference to all other users. In particular, if $\sum_{j\neq i}\frac{1}{b_{ij}} < 1$, i.e., if i's transmission does not cause significant interference to the other users, then i transmits in all the slots. Unlike the throughput maximization problem, there is no "selfish" component in the decision making rule.

Remark 3.4: As the target SINR $T \to \infty$, $b_{ij} \to 0$ for all i, j, and the proportional fair attempt probabilities satisfy

$$\frac{1}{p_i} = \sum_{j\neq i} \frac{1}{1-p_i} = \frac{N-1}{1-p_i}$$

for all $i \in \mathcal{N}$. This yields $p_i = \frac{1}{N}$ for all $i \in \mathcal{N}$. This is expected, because in the limiting case a transmission can succeed if and only if there is no other concurrent transmission. This is hence Aloha without spatial reuse, and it is well known that in this case, the optimal access probability is $1/N$ asymptotically [7].

C. Max-min Fair Medium Access

Our analysis in this section follows [18], [19]. The max-min fair medium access problem can be formulated as

$$\begin{aligned} &\text{maximize} \quad && \theta, \\ &\text{subject to} \quad && \theta \le p_i \prod_{j\neq i}\left(1 - \frac{p_j}{1+b_{ji}}\right),\ i \in \mathcal{N}, \end{aligned}$$

where constraint functions are defined for all $\mathbf{p} \in [0,1]^{\mathcal{N}}$. The following is an equivalent convex optimization problem (see [18] for details):

$$\begin{aligned} &\text{minimize} \quad && \frac{1}{2}\theta^2, \\ &\text{subject to} \quad && \theta \le \log p_i + \sum_{j\neq i}\log\left(1 - \frac{p_j}{1+b_{ji}}\right),\ i \in \mathcal{N}. \end{aligned}$$

The Lagrange function of this problem is given by [8]

$$\frac{1}{2}\theta^2 + \sum_{i\in\mathcal{N}} \lambda_i \left(\theta - \log p_i - \sum_{j\neq i}\log\left(1 - \frac{p_j}{1+b_{ji}}\right)\right),$$

with $\lambda_i \ge 0, i \in \mathcal{N}$ being the Lagrange multipliers.

Minimization of the Lagrange function (which is concave in $\mathbf{p}$ and θ) gives

$$p_i = \begin{cases} \lambda_i \left(\sum_{j \neq i} \frac{\lambda_j}{1+b_{ij}-p_i} \right)^{-1} & \text{if } \frac{1}{\lambda_i} \sum_{j \neq i} \frac{\lambda_j}{b_{ij}} > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

$$\theta = -\sum_{i \in N} \lambda_i. \tag{4}$$

Wang and Kar [18] suggest that the Lagrange multipliers be updated using the gradient projection method. More precisely, for all $i \in \mathcal{N}$,

$$\lambda_i(n+1) = \Bigg[\lambda_i(n) + \beta(n) \tag{5}$$

$$\left(\theta - \log p_i - \sum_{j \neq i} \log \left(1 - \frac{p_j}{1+b_{ji}} \right) \right) \Bigg]^+ , \tag{6}$$

where $\beta(n)$ is the step size at the nth iteration. Furthermore, [18, Theorem 2] implies that a solution arbitrary close to an optimal solution can be reached via appropriate choice of step sizes. However, all the users need to exchange variables in order to perform updates.

Finally, the *directed link graph* corresponding to our network is a directed graph in which each vertex stands for a user (i.e., a transmitter-receiver pair) in the network. There is an edge from vertex i to vertex j in the directed link graph if transmission of user i affects the success of transmission of user j. Two vertices i and j are said to be connected if either of the following two conditions hold:

1) there is an edge from i to j or viceversa,
2) there are vertices $v_0 = i, v_1, \ldots, v_{n-1}, v_n = j$ such that v_m and v_{m+1} are connected for $m = 0, 1, \ldots, n-1$.

Clearly, the directed link graph for our network model is a complete graph; for any pair of vertices i and j there is an edge from i to j and also from j to i. In particular, the directed link graph is a single *strongly connected component* [19]. Thus [19, Corollary 1] implies that the above optimization also obtains the lexicographic max-min fair medium access probabilities that yield identical rates for all the users.

D. Closest Interferer Case

Note that a user needs to know the entire topology, and in a few cases, also needs to communicate with all the nodes to implement the adaptation rules developed in Sections III-A-III-C. In this section, we carry out analysis under the simplifying assumption that the aggregate interference at a receiver is dominated by the transmission from the closest interferer. This is a reasonable approximation in a moderately dense network, specifically when the path loss attenuations are high. Throughout this section, we use the notation

$$c(i) := \operatorname*{argmin}_{j \neq i} r_{ji},$$
$$C(i) := \{j : c(j) = i\},$$

for all $1 \leq i \leq N$. In words, $c(i)$ is the strongest interferer of node i, and $C(i)$ is the set of nodes to which node i is the strongest interferer. We assume that there is always a unique $c(i)$ for each i. Then, accounting only for the nearest interferer, the approximate probability of successful transmission for node i is

$$\tilde{q}_i = 1 - \frac{p_{c(i)}}{1+b_{c(i)i}}.$$

The analysis of Sections III can be adapted to this simplified scenario.

1) Maximum Throughput Medium Access: The throughput maximization problem can now be posed as follows.

$$\begin{aligned} \text{maximize} \quad & \tilde{\Theta} := \sum_i p_i \tilde{q}_i, \\ \text{subject to} \quad & 0 \leq p_i \leq N, \ i \in \mathcal{N}. \end{aligned}$$

As in Section III-A, we can argue that some $\mathbf{p}^* \in \{0,1\}^{\mathcal{N}}$ attains the optimal throughput. Again, an equivalent optimization problem is

$$\operatorname*{maximize}_{\mathcal{M} \subset \mathcal{N}} \quad \sum_{i \in \mathcal{M}} \left(1 - \frac{\mathbb{1}\{c(i) \in \mathcal{M}\}}{1+b_{c(i)i}} \right),$$

or alternatively,

$$\operatorname*{maximize}_{\mathcal{M} \subset \mathcal{N}} \quad \sum_{i \in \mathcal{M}} \left(1 - \sum_{j \in C(i)} \frac{\mathbb{1}\{j \in \mathcal{M}\}}{1+b_{ij}} \right).$$

We now formulate a strategic form game among users, with action sets $\{0,1\}$ and utility functions given by

$$\begin{aligned} u_i(0, \mathbf{a}_{-i}) =& 0, \\ u_i(1, \mathbf{a}_{-i}) =& 1 - \frac{a_{c(i)}}{1+b_{c(i)i}} - \sum_{j \in C(i)} \frac{a_j}{1+b_{ij}}. \end{aligned}$$

Again a Gibbs sampler based algorithm yields the optimal set of the transmitting users. Also, user i only needs to know the distances of user $c(i)$ and all the receivers in $C(i)$ and their actions to make its decision.

Remark 3.5: Notice that user i must choose $a_i = 1$ if

$$1 - \frac{1}{1+b_{c(i)i}} - \sum_{j \in C(i)} \frac{1}{1+b_{ij}} > 0.$$

Such users can set their actions to 1, and need not undergo Gibbs sampler updates.

Discussion: Consider a scenario where a node's closest interferer does not transmit, i.e., has zero attempt probability. Nonetheless, this node always has an active closest interferer (unless there are no other nodes in the network). A better approximation of the success probabilities, and hence of the throughput, is obtained by always accounting for the closest active interferer. Towards this, let us define

$$c(i, \mathcal{M}) := \operatorname*{argmin}_{j \in \mathcal{M}, j \neq i} r_{ji},$$
$$C(i, \mathcal{M}) := \{j \in \mathcal{M} : c(j, \mathcal{M}) = i\},$$

for all $i \in \mathcal{M}$. We are now faced with the following optimization problem.

$$\underset{\mathcal{M}\subset\mathcal{N}}{\text{maximize}} \quad \sum_{i\in\mathcal{M}} \left(1 - \sum_{j\in C(i,\mathcal{M})} \frac{1}{1+b_{ij}}\right).$$

We can now define users' utility functions as follows.

$$\begin{aligned} u_i(0,\mathbf{a}_{-i}) =&0,\\ u_i(1,\mathbf{a}_{-i}) =&1 - \frac{1}{1+b_{c(i,\mathcal{M})i}}\\ &- \sum_{j\in C(i,\mathcal{M}\cup\{i\})} \left(\frac{1}{1+b_{ij}} - \frac{1}{1+b_{c(j,\mathcal{M})j}}\right), \end{aligned}$$

where $\mathcal{M} = \{j \in \mathcal{N} : j \neq i, a_j = 1\}$. The analogous distributed algorithm (Gibbs sampler based) can again be shown to lead to the optimal solution.

2) Proportional Fair Medium Access: We now aim to solve the following optimization problem.

$$\begin{aligned} &\text{maximize} \quad \sum_i \log{(p_i\tilde{q}_i)},\\ &\text{subject to} \quad 0 \le p_i \le 1,\ i \in \mathcal{N}. \end{aligned}$$

Following the discussion in Section III-B, we obtain

$$p_i = \begin{cases} \left(\sum_{j\in C(i)} \frac{1}{1+b_{ij}-p_i}\right)^{-1} & \text{if } \sum_{j\in C(i)} \frac{1}{b_{ij}} > 1,\\ 1 & \text{otherwise.} \end{cases}$$

Again, if $\sum_{j\in C(i)} \frac{1}{b_{ij}} > 1$, iterations $p_i^{k+1} = f_i(p_i^k)$ converge to the optimal p_i starting from any $p_i \in [0,1]$.

Remark 3.6: If $\sum_{j\in C(i)} \frac{1}{b_{ij}} < 1$, i.e., if i's transmission does not cause significant interference to the users for whom i is closest interferer, then i transmits in all the slots. The same user may not transmit (in any slot) under the maximum throughput objective if $1 - \frac{1}{1+b_{c(i)i}} \approx 0$ and $c(i)$ and users in $C(i)$ transmit.

We can have explicit formulae for the attempt probabilities in a few special cases.

1) Suppose $C(i)$ is singleton for each i. If $C(i) = \{j\}$, then

$$p_i = \begin{cases} \frac{1+b_{ij}}{2} & \text{if } b_{ij} < 1,\\ 1 & \text{otherwise.} \end{cases}$$

2) *Linear Network Topology:* We now consider a scenario where nodes are placed along a line, say $\mathbb{R}$, and are indexed sequentially. We also assume that for any node i the potential interferers are the two immediate neighbors $i-1$ and $i+1$. This also amounts to assuming $C(i) = \{i-1, i+1\}$ for all i. Then, assuming $\frac{1}{b_{i,i-1}} + \frac{1}{b_{i,i+1}} > 1$, the proportional fair attempt probability of node i satisfies

$$\frac{1}{p_i} = \frac{1}{1+b_{i,i-1}-p_i} + \frac{1}{1+b_{i,i+1}-p_i}.$$

This is quadratic equation in p_i, which on solving gives[3]

$$p_i = \frac{2 + b_{i,i-1} + b_{i,i+1} - \sqrt{\begin{matrix}(b_{i,i-1}-b_{i,i+1})^2\\ +(1+b_{i,i-1})(1+b_{i,i+1})\end{matrix}}}{3}.$$

3) Max-min Fair Medium Access: Similar to Section III-C, the max-min fair medium access problem is

$$\begin{aligned} &\text{maximize} \quad \theta,\\ &\text{subject to} \quad \theta \le p_i \left(1 - \frac{p_{c(i)}}{1+b_{c(i)i}}\right),\ i \in \mathcal{N}. \end{aligned}$$

Again, the constraint functions are defined for all $\mathbf{p} \in [0,1]^{\mathcal{N}}$.

Let us recall the definition of the directed link graph associated with the network. In this section, we only account for the interference due the closest interferer. Thus there is an edge from vertex i to vertex j if and only if $j \in C(i)$. We assume that the directed link graph is connected (i.e., all the vertices in the graph are connected with each other). If it is not connected, the max-min fair medium access problem on the entire graph decomposes into separate max-min fair medium access problems on each of the connected subgraphs, which can be solved independently.

We now pursue the following convex optimization problem which is equivalent to the above max-min fair medium access optimization problem.

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\sum_{i\in\mathcal{N}} \theta_i^2,\\ &\text{subject to} \quad \theta_i \le \log p_i + \log\left(1 - \frac{p_{c(i)}}{1+b_{c(i)i}}\right),\ i \in \mathcal{N},\\ &\qquad\qquad \theta_i \le \theta_j, j \in C(i), j = c(i). \end{aligned}$$

The last set of constraints along with the connected assumption (of the directed link graph) forces θ_i to be equal for all $i \in \mathcal{N}$. This confirms equivalence to the initial optimization problem. Now the Lagrange function is

$$\begin{aligned} \frac{1}{2}\sum_{i\in\mathcal{N}} \theta_i^2 + \sum_{i\in\mathcal{N}} \lambda_i \left(\theta_i - \log p_i - \log\left(1 - \frac{p_{c(i)}}{1+b_{c(i)i}}\right)\right)\\ + \sum_{i\in\mathcal{N}} \sum_{j\in C(i)\cup\{c(i)\}} \mu_{ij}(\theta_i - \theta_j), \end{aligned}$$

where $\lambda_i \ge 0, \mu_{ij} \ge 0, j \in C(i)\cup\{c(i)\}, i \in \mathcal{N}$, are the Lagrange multipliers. An approach similar to Section III-C prescribes the following update rules. For all $i \in \mathcal{N}, j \in$

[3]The other root is greater than 1, and thus is not a valid probability.

$C(i) \cup \{c(i)\}$,

$$p_i = \begin{cases} \lambda_i \left(\sum_{j \in C(i)} \frac{\lambda_j}{1+b_{ij}-p_i} \right)^{-1} & \text{if } \frac{1}{\lambda_i} \sum_{j\in C(i)} \frac{\lambda_j}{b_{ij}} > 1, \\ 1 & \text{otherwise,} \end{cases} \tag{7}$$

$$\theta_i = \begin{cases} -\lambda_i - & \sum_{j \in C(i)\cup\{c(i)\}} (\mu_{ij} - \mu_{ji}) \\ & \text{if } \lambda_i + \sum_{j\in C(i)\cup\{c(i)\}}(\mu_{ij}-\mu_{ji}) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

$$\begin{aligned} \lambda_i(n+1) =& \Bigg[\lambda_i(n) + \beta(n)\Bigg(\theta_i \\ & - \log p_i - \log\left(1 - \frac{p_c(i)}{1+b_{c(i)i}}\right)\Bigg)\Bigg]^+, \end{aligned} \tag{9}$$

$$\mu_{ij}(n+1) = [\mu_{ij}(n) + \beta(n)(\theta_i - \theta_j)]^+, \tag{10}$$

where $\beta(n)$ is the step size at the nth iteration as before. Observe that any user i can perform updates (7)-(10) via local information exchange. More precisely, it needs to communicate only with user $c(i)$ and the users in $C(i)$.

E. A Note on Distributed Implementation

The aim of the present paper is primarily of theoretical nature and it is beyond our scope to discuss implementation issues. Let us however stress that the discussed adaptations of the MAP are implementable. We will focus on the proportional fair case in view the main focus of the paper.

Assume each receiver has a distinctive pilot signal with fixed power. Since we assumed a quasi-static network in which the nodes move at a slower time scale, each node can then learn the distance that separates it from a given receiver by listening to its pilot signal and by performing a time average over (so as to smooth out fading). Once this data is available for all receivers, a given transmitter can then solve the key fixed point equation that characterizes its optimal MAP. Distinctive pilot signals can be obtained by a collection of orthogonal codes chosen at random by the receivers. In practice, it is enough for a transmitter to detect the "dominant" receivers (i.e., those within a certain distance to it), so that the scheme will work in an infinite network with a finite (properly tuned) number of such codes.

IV. Stochastic Geometry Analysis

A. Network and Communication Model

We now assume that the transmitting nodes are scattered on the Euclidian plane according to a homogeneous Poisson point process of intensity λ. For each transmitter, its corresponding receiver is at distance r_0 in a random direction. The traffic and channel models are the same as in Section II. As before the transmitters use slotted Aloha to access the channel, and a receiver successfully receives the packet from its transmitter if the received SINR exceeds a threshold T. Finally, transmitters adapt their attempt probabilities as described in Section III.

Each transmitter is associated with a multi dimensional mark that carries information about the adaptive transmission probability and the transmission status. Let $\tilde{\Phi} = \{X_n, Z_n\}$ denote a marked Poisson point, where

- $\Phi = \{X_n\}$ denotes the Poisson point process of intensity λ, representing the location of transmitters in the Euclidean plane.
- $\{Z_n = (\phi_n, p_n, e_n)\}$ denote the marks of the Poisson point process $\tilde{\Phi}$, which consist of three components:
  - $\{\phi_n\}$ denote the angles from transmitters to receivers. These angles are i.i.d. and uniform on $[0, 2\pi]$ and independent of Φ. We will call them the primary marks.
  - $\{p_n\}$ denote the MAPs of the nodes; p_n is a secondary mark (i.e. a functionals of Φ and its primary marks, see below).
  - $\{e_n\}$ are indicator functions that take value one if a given node decides to transmit in a given time slot, and zero otherwise. Clearly, $\mathbb{P}(e_n = 1) = p_n = 1 - \mathbb{P}(e_n = 0)$. In particular, given p_n, e_n is independent of everything else including $\{e_m\}_{m\neq n}$.

The locations of the receivers will be denoted by $\Phi^r = \{Y_n = X_n + (r_0, \phi_n)\}$ with $(r_0, \phi) := (r_0 \cos\phi, r_0 \sin\phi)$. It follows from the displacement theorem [2] that Φ^r is also a homogeneous Poisson point process of intensity λ.

The above assumptions will be referred to as the Poisson model. We will also consider below a more general case where the above marked point process is just stationary.

B. Proportional Fair Spatial Aloha

1) MAP distribution: Let us consider response functions $L : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ defined for each $0 \le p \le 1$ as follows

$$L_\rho(x, y) = \frac{\rho}{\frac{\|x-y\|^\alpha}{Tr_0^\alpha} + 1 - \rho}.$$

For all $0 \le \rho \le 1, x \in \mathbb{R}^2$, the shot noise field $J_{\Phi^r}(\rho, x)$ associated with the above response function and the marked point process $\tilde{\Phi}$ is

$$J_{\Phi^r}(\rho, x) = \int_{\mathbb{R}^2} L_\rho(x, y)\Phi^r(\mathrm{d}y) = \sum_{Y_n \in \Phi^r} L_\rho(x, Y_n).$$

Notice that this shot noise is not that representing the interference at x. It rather measures the effect of the presence of a transmitter at x on the whole set of receivers.

Consider a typical node at the origin, $X_0 = 0$, with marks p_0, ϕ_0. Let $\mathbb{P}^0$ denote the Palm distribution of the stationary marked point process $\tilde{\Phi}$ [2, Chapter 1]. The fixed point equation determining the MAP of node $X_0 = 0$ reads (see (2))

$$\frac{1}{p_0} = \sum_{n\neq 0} \frac{1}{\frac{\|Y_n\|^\alpha}{Tr_0^\alpha} + 1 - p_0}.$$

We have a similar equation for each node and the sequence $\{p_n\}$ is readily seen to be a sequence of marks of Φ and its primary marks.

It follows directly from monotonicity arguments that

$$\left\{\frac{1}{p_0} < \frac{1}{\rho}\right\} \quad \text{iff} \quad \left\{\sum_{n\neq 0} \frac{\rho}{\frac{\|Y_n\|^\alpha}{Tr_0^\alpha} + 1 - \rho} < 1\right\}.$$

Notice that we have not used the specific assumptions on the point process so far. Hence we have the following general connection between the optimal MAP distribution and the shot noise J_{Φ^r}:

Theorem 4.1: For all stationary marked point processes $\widetilde{\Phi}$ (not necessarily Poisson), for all $0 < \rho < 1$,

$$\mathbb{P}^0(p_0 > \rho) = \mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(\rho, 0) < 1\right),$$

and

$$\mathbb{P}^0(p_0 = 1) = \mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(1, 0) < 1\right),$$

with $\mathbb{P}^0$ the Palm distribution of $\tilde{\Phi}$.

We now use the fact that $\tilde{\Phi}$ is an independently marked Poisson point process [2, Definition 2.1]. From Slivnyak's theorem [2, Theorem 1.13],

$$\mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(\rho, 0) < 1\right) = \mathbb{P}\left(J_{\Phi^r}(\rho, 0) < 1\right)$$

for all $0 \leq \rho \leq 1$. Consequently,

$$\mathbb{P}^0(p_0 > \rho) = \mathbb{P}\left(J_{\Phi^r}(\rho, 0) < 1\right),$$

and

$$\mathbb{P}^0(p_0 = 1) = \mathbb{P}\left(J_{\Phi^r}(1, 0) < 1\right).$$

It follows from [2, Proposition 2.6] and from the fact that Φ^r is a homogeneous Poisson point process that one can write the Laplace transform $\mathcal{L}_{J(\rho,0)}(s)$ of the shot noise $J_{\Phi^r}(\rho, 0)$ as

$$\mathcal{L}_{J(\rho,0)}(s) = \exp\left\{-2\pi\lambda \int_0^\infty \left(1 - \mathrm{e}^{-\frac{s\rho\bar{r}_0}{r^\alpha + (1-\rho)\bar{r}_0}}\right) r\mathrm{d}r\right\}, \tag{11}$$

where $\bar{r}_0 := Tr_0^\alpha$.

Theorem 4.2: Under the above Poisson assumptions, the attempt probability of the typical node has the distribution

$$\mathbb{P}^0(p_0 > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_{J(\rho,0)}(iw) \frac{\mathrm{e}^{iw} - 1}{iw} \mathrm{d}w, \tag{12}$$

with $\mathcal{L}_{J(\rho,0)}(\cdot)$ given by (11).

Proof: Let $g_\rho(\cdot)$ denote the density of the shot noise field $J_\Phi(\rho, 0)$. Then

$$\mathbb{P}^0(p_0 > \rho) = \int_0^1 g_\rho(t)\mathrm{d}t = \int_{-\infty}^{\infty} g_\rho(t)u(t)\mathrm{d}t,$$

where $u(t) = 1$ if $0 \leq t \leq 1$ and 0 otherwise. Now using Parseval's theorem

$$\mathbb{P}^0(p_0 > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_{J(\rho,0)}(w)\mathcal{F}_u^*(w)\mathrm{d}w,$$

with $\mathcal{F}_A(w) = \mathbb{E}\exp(-iwA)$ the Fourier transform of the real valued random variable A and B^* the complex conjugate of B. The claim follows after substituting $\mathcal{F}_u(w) = \frac{1-\mathrm{e}^{-iw}}{iw}$ and $\mathcal{F}_{J(\rho,0)}(w) = \mathcal{L}_{J(\rho,0)}(iw)$. ∎

Remark 4.1: For $\alpha = 4$, the Laplace transform $\mathcal{L}_{J(p,0)}(s)$ can be simplified as

$$\mathcal{L}_{J(p,0)}(s) = \exp\left\{-2\pi\lambda\sqrt{(1-p)}Tr_0^2 \int_0^1 \frac{1 - \mathrm{e}^{-spv^2/(1-p)}}{v^2\sqrt{1-v^2}}\mathrm{d}v\right\}.$$

2) Mean Utility: This subsection is devoted to the analysis of the mean value of the logarithm of the throughput of the typical node:

$$\theta = \mathbb{E}^0 \log((p_0)) + \mathbb{E}^0 \log((q_0)),$$

with q_0 defined in (1). Since we know the cdf f of p_0, the first term poses no problem. The second term can be rewritten as (see (1))

$$\mathbb{E}^0 \log((q_0)) = \mathbb{E}^0\left[\sum_{n\neq 0} \log\left(1 - \frac{p_n}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1}\right)\right].$$

Under the law $\mathbb{P}^0$, the points $\{X_n\}_{n\neq 0}$ of Φ form a homogeneous Poisson point process of intensity λ. However, the marks $\{p_n\}_{n\neq 0}$ do not have the law identified in the last section. In fact, the mark p_n of a point X_n $(n \neq 0)$ satisfies the following modified fixed point equation:

$$\frac{1}{p_n} = \frac{1}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1 - p_n} + \sum_{m\neq 0,n} \frac{1}{\frac{\|X_n - Y_m\|^\alpha}{Tr_0^\alpha} + 1 - p_n},$$

with the convention that $p_n = 1$ if there is no solution in $[0, 1]$. We can use the same argument as above to conclude that $\frac{1}{p_n} < \frac{1}{\rho}$ iff

$$\frac{\rho}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1 - \rho} + \sum_{m\neq 0,n} \frac{\rho}{\frac{\|X_n - Y_m\|^\alpha}{Tr_0^\alpha} + 1 - \rho} < 1.$$

Conditioned on there being two nodes at 0 and x, the other points form a homogeneous Poisson point process of intensity λ. This allows one to prove the following.

Theorem 4.3: Under the above Poisson assumptions, given that there is a node at 0 and a node at $x \in \mathbb{R}^2$, the attempt probability of the node at x has the distribution

$$\mathbb{P}^{0,x}(p_x > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_{J_x(\rho,0)}(iw) \frac{\mathrm{e}^{iw} - 1}{iw} \mathrm{d}w,$$

with

$$\mathcal{L}_{J_x(\rho,0)}(s) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(-\frac{s\rho\bar{r}_0}{||x - (r_0, \phi)||^\alpha + (1-\rho)\bar{r}_0}\right) \mathrm{d}\phi \; \exp\left\{-2\pi\lambda \int_0^\infty \left(1 - \mathrm{e}^{-\frac{s\rho\bar{r}_0}{r^\alpha + (1-\rho)\bar{r}_0}}\right) r\mathrm{d}r\right\},$$

and $(r_0, \phi) := (r_0 \cos\phi, r_0 \sin\phi)$.

Due to the circular symmetry, the first integral in the expression of $\mathcal{L}_{J_x(\rho,0)}(s)$ depends on x only through $\|x\|$. Thus the density of p_x also depends on $\|x\|$ only, and it will be denoted by f_r when $\|x\| = r$. The density of p_0 identified in the last subsection will be denoted by f. The main result of this section is:

Theorem 4.4: Under the above Poisson assumptions, in the proportional fair case, the mean utility of a typical node is

$$\theta = \int_0^1 \log(u) f(\mathrm{d}u) + \frac{1}{2\pi} \int_{\phi\in(0,2\pi)} \int_{x\in\mathbb{R}^2} \int_v \log\left(1 - \frac{v\overline{r}_0}{||x-(r_0,\phi)||^\alpha + \overline{r}_0}\right) \mathrm{d}\phi f_{||x-(r_0,\phi)||}(\mathrm{d}v)\mathrm{d}x. \quad (13)$$

Proof: See Appendix A. ■

C. Discussion of the Other Cases

A preliminary concern when trying to use Euclidean stochastic geometry of the infinite plane in the maximum throughput and max-min fairness cases is that it is not clear whether the associated infinite dimensional optimization problems make sense in the first place. In the proportional fair case, each node computes its optimizing MAP in one step as the solution of a fixed point equation that is almost surely well defined (in terms of a shot noise) even in the infinite Poisson population case. Unfortunately, this does not extend to the other two cases.

This does not mean that there is no hope at all. Consider for instance the maximum throughput case accounting only for the closest interferer, and further simplify it by measuring interference at the transmitter rather than at the receiver. In other words, consider the same optimization problem as in Section III-D1 but with $c(i)$ being the closest transmitter to transmitter i (rather than to receiver i). Then, in the Poisson case, the infinite dimensional optimization problem can be shown to reduce to a *countable collection of finite optimization problems*. This follows from the fact that there are no infinite "descending chains" in a homogeneous Poisson point process [10, Chapter 2] (a *descending chain* is a sequence of nodes $i_1, i_2, \ldots$ such that $c(i_n) = i_{n+1}$ for all n). As a result, the Poisson point process can be decomposed into a countable collection of finite "descending trees", where each path from the leaves to the root is a descending chain. The associated optimization is hence well defined and the problem can be reduced to evaluating the solution of the optimization problem of Section III-D1 on the typical descending tree. Hence, there is hope to progress on this and on related cases. This will however not be pursued here and is left for future research.

V. Numerical Results

In this section, we study the proposed adaptive spatial Aloha schemes quantitatively. We compute various metrics formulated through the stochastic geometry based analysis, and we also perform simulation. The simulation not only validates the analytical model, but also illustrates the performance of schemes for which we do not have an analytical characterization.

A. Computation of the Integrals

We used Maple and Matlab to evaluate the integrals of Section IV. The infinite integral that shows up in the expression of the Laplace transform (11) is handled without truncation by Maple and Matlab. The singularity at $w = 0$ in the contour integrals (12) leveraging Parseval's theorem is a false singularity and it is also handled without further work by either Maple or Matlab. The Matlab code is particularly efficient and is used throughout the analytical evaluations described below.

B. Simulation Setting

We consider a two dimensional square plane with side length L, and N nodes placed independently over the plane according to the uniform distribution; this corresponds to $\lambda = N/L^2$ in the stochastic geometry model.[4] Each node has its receiver randomly located on the unit circle around it, again as per the uniform distribution. Thus $r_{ii} = 1$ for all i. We set $\alpha = 4$ and $T = 10$. To nullify the edge effect, we take into account only the nodes falling in the $L/2 \times L/2$ square around the center while computing various metrics. While all other parameters remain, we vary L and N for different simulations. For each parameter set we calculate the average of the performance metric of interest over 1000 independent network realizations.

C. Joint Validation of the Analysis and the Simulation

We validate the analytical expression against the simulation for the case of proportional fair medium access. For illustration, we plot the cumulative distribution function (c.d.f.) of the MAP in Figure 2. Here we set $L = 40$ and consider two values of N, $N = 400$ and $N = 800$, which correspond to $\lambda = 0.5$ and $\lambda = 0.25$ respectively. The plots show that the stochastic geometry based formula (see Theorem 4.2) quite accurately predicts the nodes' behavior in simulation.

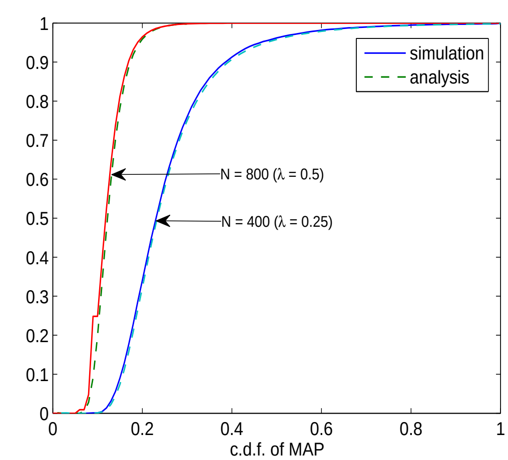

Fig. 2. Cumulative distribution function of the MAP for the proportional fair case.

We also study the distributions $\mathbb{P}^{0,x}(p_x > \rho)$ (also referred to as f_r for $\|x\| = r$) defined in Theorem 4.3. Figure 3 shows

[4] A finite snapshot of a Poisson random proces would contain a Poisson distributed number of nodes. However, for large λL^2, the Poisson random variable with mean λL^2 is highly concentrated around its mean. Thus we can use λL^2 nodes for all the realizations in our simulation.

their plots for $\lambda = 0.25$ and two values of $\|x\|$, $\|x\| = 1$ a $\|x\| = 10$. Again the plots based on the analytical expressi and those based on simulation closely match. Clearly, unc $\mathbb{P}^0$, nodes closer to the origin are more likely to be inactiv

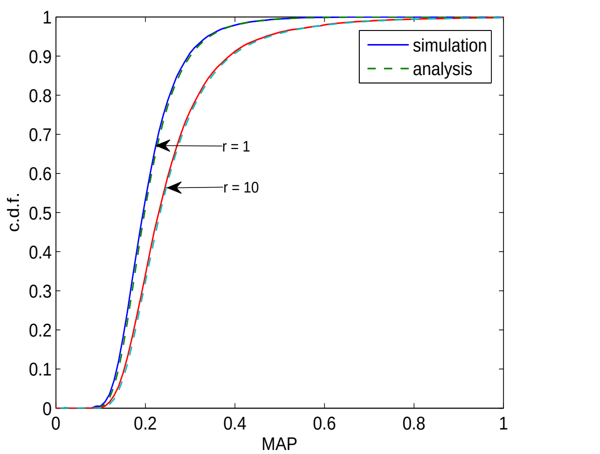

Fig. 3. Demonstration of f_r, the MAP distribution of a node at distance r from origin, under $\mathbb{P}^0$.

D. Performance of the Adaptive Protocols

In this section we illustrate the performance of various adaptive schemes and their benefit over plain Aloha. We compute the performance metrics via simulation and also through analytical expressions whenever the latter are available. In such cases the analytical results and the simulation validate each other.

First we set $L = 20$ and $N = 50$. We consider the maximum throughput medium access, however, only accounting for the closest interferers. In figure 4, we show steady state behavior of our Gibbs sampling based algorithms; we have set the temperature $\tau(t) = 1/\log(1 + t)$. As expected, the improved maximum throughput medium access (see the discussion at the end of Section III-D1) insures a better exclusion behavior. Under this scheme, a lesser number of nodes transmit, and neighboring nodes are unlikely to transmit simultaneously. So, this is expected to deliver better aggregate throughput.

Now we keep L fixed at 20, but vary N from 10 to 100; this corresponds to varying λ from 0.025 to 0.25 in the analytical expressions. We evaluate the aggregate throughputs of various Aloha schemes including plain Aloha. The average throughputs are plotted in Figure 5. *Although in some of the schemes we derive the attempt probabilities only considering the closest interferers, we always take into account the aggregate interference while calculating the throughput*. When the number of nodes is small, both the throughput maximizing medium access and plain aloha have identical performance; both prescribe attempt probabilities close to one for all the nodes. When the number of nodes increases beyond 45, the throughput maximizing medium access significantly underestimates the interference, and thus its performance deteriorates. On the other hand, the aggregate interference based

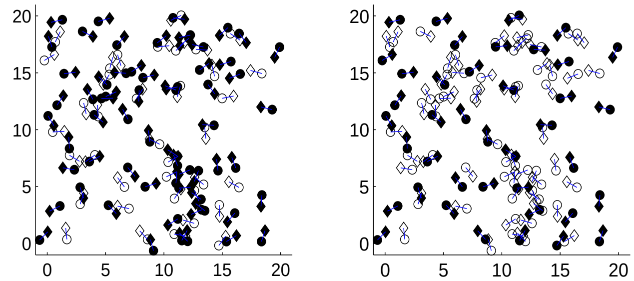

Fig. 4. Throughput maximizing medium access: There are 100 transmitter receiver pairs. The *diamonds* represent candidate transmitters, and the connected *circles* the corresponding receivers. The solid diamonds represent the nodes that transmit in all the slots; others never transmit. The left plot corresponds to the maximum throughput medium access and the right one to its improved version.

proportional fair scheme significantly outperforms plain Aloha in terms of aggregate throughput also. This benefit is sustained even as the number of nodes increases. We also notice that the improved version of maximum throughput medium access (see the discussion at the end of Section III-D1) yields best performance among all the schemes, and its performance does not deteriorate until a much higher number of nodes.

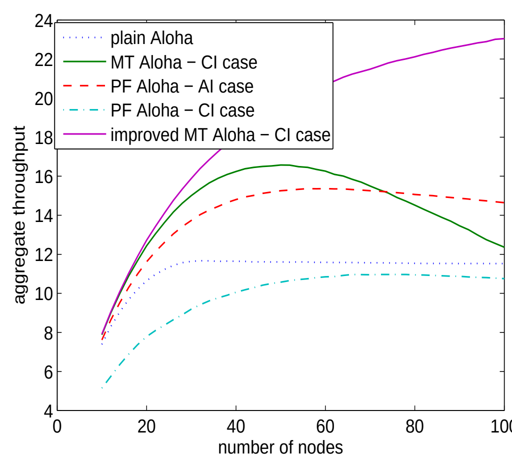

Fig. 5. Throughputs of various medium access schemes as a function of the number of nodes. MT, PF, CI and AI stand for *maximum throughput, proportional fair, closest interferer* and *aggregate interference* respectively.

In Figure 6, we plot the logarithms of the aggregate throughputs corresponding to the two proportional fair medium access schemes. The figure illustrates that as the number of nodes increases, the performance of the closest interferer based medium access worsens in comparison to the performance of the aggregate interference based scheme - this is not visible merely looking at the corresponding aggregate throughputs (see Figure 5)).

In Figure 7, we plot the c.d.f. of the MAP for the proportional fair case. Our objective is to compare the case when nodes account for the aggregate interference with when they account for the closest interferer only. We set $L = 40$ and

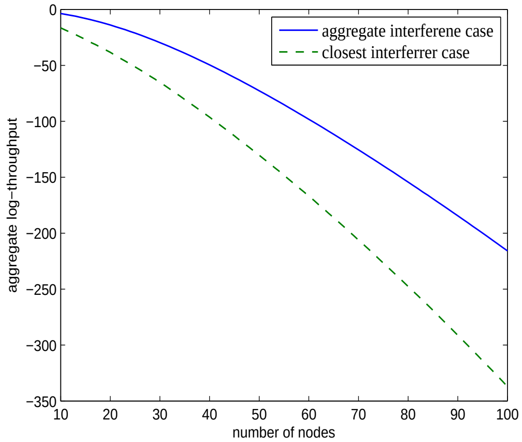

Fig. 6. Performance of the two proportional fair medium access schemes as a function of the number of nodes.

plot MAP distributions corresponding to two values of N, 400 and 800 (corresponding to $\lambda = 0.25$ and $\lambda = 0.5$ respectively). As expected, the nodes attempt more aggressively when they account for the closest interferer only. While there is a favored probability interval in the aggregate interference case, there are more than one such intervals in the closest interferer case. Also, in the latter case, about 34% of nodes attempt in almost all slots, irrespective of N, the total number of nodes. This can be understood by noticing that the probability that a node is not the closest interferer to any other node is not sensitive to N.

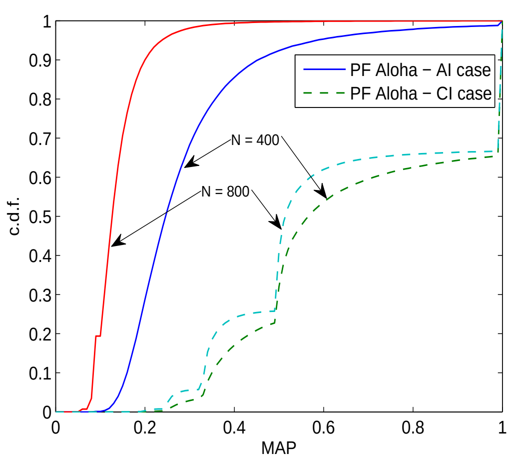

Fig. 7. Cumulative distribution function of the MAP for the proportional fair case.

VI. Conclusion

We have shown the feasibility of the performance analysis of distributed adaptive protocols that aim at maximizing some global utility in a large random network using stochastic geometry.

More precisely, the most natural distributed adaptation of the medium access probability of Aloha that aims at proportional fairness optimization was shown to have a tractable optimal MAP distribution. This distribution is obtained from the law of a certain shot noise field that describes the interference created by a typical node to all receivers but his. In the Poisson case, the distribution of the optimal MAP is obtained as a non-singular contour integral which is amenable to an efficient evaluation using classical numerical tools. The network performance at optimum can in turn be deduced from the latter using Campbell's formula.

This approach is shown to provide an analytic way of quantifying the gains brought by this proportionally fair adaptive version of Aloha compared to plain Aloha.

This line of thoughts opens several research directions. The first one is the extension to other types of fairness, still in the framework of Aloha. We indicate that this is possible at least under certain simplifications of the interference model. The second and broader question is whether this approach can be extended to MAC protocols other than Aloha. An example would be an adaptation of the exclusion radius of CSMA/CA to the full environment of a node aiming at maximizing some utility of the throughput. A third general question concerns the evaluation of the "price of decentralization". When the discussed protocols are suboptimal because of their greedy/distributed nature, is it possible to use stochastic geometry to evaluate the typical discrepancy between the performance of the distributed scheme and the optimal centralized one?

VII. Acknowledgments

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APPENDIX

Let $\Psi = \{Y'_m\}$ be a point process with marks and $\psi \circ \theta_z$ be the point process ψ shifted by $-z$. Given that $Y_0 = (r_0, \phi)$,

$$p_n = h(||X_n - (r_0, \phi)||, \Phi^r \setminus \{(r_0, \phi), Y_n\} \circ \theta_{X_n}),$$

where the mapping $h(u, \Psi)$ associates with Ψ and the real number u the solution of

$$\frac{1}{p} = \frac{1}{\frac{u^\alpha}{Tr_0^\alpha} + 1 - p} + \sum_m \frac{1}{\frac{\|Y'_m\|^\alpha}{Tr_0^\alpha} + 1 - p},$$

with the usual convention if there is no solution in $[0, 1]$. It follows from Slivnyak's theorem that

$$\mathbb{E}^0 \left[\sum_{n \neq 0} \log \left(1 - \frac{p_n}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1} \right) \right] =$$

$$\mathbb{E}^0 \left[\sum_{n \neq 0} \log \left(1 - \frac{h(||X_n - Y_0||, \Phi^r \setminus \{Y_0, Y_n\} \circ \theta_{X_n})}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1} \right) \right] =$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathbb{E} \sum_n \log \left(1 - \frac{h(||X_n - (r_0, \phi)||, \Phi^r \setminus \{Y_n\} \circ \theta_{X_n})}{\frac{\|X_n - (r_0, \phi)\|^\alpha}{Tr_0^\alpha} + 1} \right) \mathrm{d}\phi.$$

It now follows from Campbell's formula that

$$\mathbb{E} \left[\sum_{n \neq 0} \log \left(1 - \frac{h(||X_n - (r_0, \phi)||, \Phi^r \setminus \{Y_n\} \circ \theta_{X_n})}{\frac{\|X_n - (r_0, \phi)\|^\alpha}{Tr_0^\alpha} + 1} \right) \right]$$

$$= \int_{\mathbb{R}^2} \int_v \log \left(1 - \frac{v \overline{r}_0}{||x - (r_0, \phi)||^\alpha + \overline{r}_0} \right) \lambda \mathrm{d}x$$
$$\mathbb{P}^0(h(||x - (r_0, \phi)||, \Phi^r \setminus \{Y_0\}) = \mathrm{d}v)$$
$$= \int_{\mathbb{R}^2} \int_v \log \left(1 - \frac{v \overline{r}_0}{||x - (r_0, \phi)||^\alpha + \overline{r}_0} \right)) \lambda \mathrm{d}x$$
$$\mathbb{P}(h(||x - (r_0, \phi)||, \Phi^r) = \mathrm{d}v),$$

where the last relation follows from Slivnyak's theorem.