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Nicolas Champagnat, Pierre-Emmanuel Jabin

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Abstract

We study strong existence and pathwise uniqueness for stochastic differential equations in $\mathbb{R}^d$ with rough coefficients, and without assuming uniform ellipticity for the diffusion matrix. Our approach relies on direct quantitative estimates on solutions to the SDE, assuming Sobolev bounds on the drift and diffusion coefficients, and L^p bounds for the solution of the corresponding Fokker-Planck PDE, which can be proved separately. This allows a great flexibility regarding the method employed to obtain these last bounds. Hence we are able to obtain general criteria in various cases, including the uniformly elliptic case in any dimension, the one-dimensional case and the Langevin (kinetic) case.

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1 Introduction

We investigate the well posedness of the Stochastic Differential Equation (SDE) in $\mathbb{R}^d$, $d \geq 1$,

$$dX_t = F(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad X_0 = \xi, \tag{1.1}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^r$ are Borel measurable function, $(W_t, t \geq 0)$ is a r-dimensional standard Brownian motion on some given complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and ξ is a $\mathcal{F}_0$-measurable random variable.

When σ and F are bounded, the law $u(t, dx)$ of X_t belongs to the set M_1 of functions from $\mathbb{R}_+$ with value in the set P_1 of probability measures on $\mathbb{R}^d$ such that, for all Borel subset Γ of $\mathbb{R}^d$, $t \mapsto u(t, \Gamma)$ is measurable. It is standard to deduce from Itô's formula that $u(t, dx)$ is a (weak, measure) solution to the Fokker-Planck PDE on $\mathbb{R}_+ \times \mathbb{R}^d$

$$\partial_t u + \nabla_x \cdot (Fu) = \nabla_x^2 : (au) = \sum_{1\leq i,j\leq d} \frac{\partial^2(a_{ij}u)}{\partial x_i \partial x_j}, \quad u(t = 0, dx) = u^0, \tag{1.2}$$

where $a = \frac{1}{2}\sigma\, \sigma^*$ and u^0 is the law of the initial r.v. ξ.

We first recall some classical terminology: weak existence holds for (1.1) if one can construct a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an adapted Brownian motion W and an adapted process X on this space solution to (1.1). Uniqueness in law holds if every solution X to (1.1), possibly on different probability space, has the same law. Strong existence means that one can find a solution to (1.1) on any given filtered probability space equipped with any given adapted Brownian motion. Finally, pathwise uniqueness means that, on any given filtered probability space equipped with any given Brownian motion, any two solutions to (1.1) with the same given $\mathcal{F}_0$-measurable initial condition ξ coincide. Our goal is to study *strong existence and pathwise uniqueness* for rough σ and F, through quantitative estimates on the difference between solutions and a priori bounds on the solutions to (1.2).

This question has been the object of many works aiming to improve the original result of Itô [10]. Krylov and Veretennikov [26, 27] studied the case of uniformly continuous a and bounded F, proving that only two cases are possible: either pathwise uniqueness holds, or strong existence does not hold. The question was studied again recently by Krylov and Röckner [16] and Zhang [29, 32]. All these works assume that the matrix a is uniformly elliptic, i.e. that $a(x) - c\,\mathrm{Id}$ is positive definite for all x for some constant

$c > 0$. The time-independent one-dimensional case was also deeply studied by Engelbert and Schmidt [7] (see also [28, 20]).

The main tools used in all the previous works are Krylov's inequality [12] and its extensions (see for example [6, 16, 17, 32]), Zvonkin's transformation [33] to remove the drift, and a priori estimates on solutions of the backward Kolmogorov equation or Fokker-Planck PDE (1.2) [26, 15, 16, 29]. Of great importance is also the result of Yamada and Watanabe [28], which proves that strong existence holds as soon as pathwise uniqueness and weak existence hold for all initial condition. Since general conditions for weak existence are well-known (see [13, 25, 23, 6, 17, 8]; see also [21] for a recent and deep study of the question), one only has to prove pathwise uniqueness to obtain strong existence. In dimension one, a key tool to prove pathwise uniqueness is the local time.

Another approach to strong existence and pathwise uniqueness was recently initiated by Le Bris and Lions in [18, 19], based on well-posedness results for the backward Kolmogorov equation. The authors define the notion of almost everywhere stochastic flows for (1.1), which combines existence and a flow property for almost all initial conditions, and give precise results in the case where $a = \mathrm{Id}$. The general case was recently studied deeply by P.-L. Lions in [21], who reduces the question to well-posedness, L^1 norms and stability properties for two backward Kolmogorov equations; the first one associated to the SDE (1.1) and the other one obtained by a doubling of variable technique. Note that this approach does not require assumptions of uniform ellipticity for a. In [19], the authors also define a stochastic transport equation whose solutions are in correspondence with the stochastic flow. This approach was also used in [30], where the existence of almost everywhere stochastic flows was obtained for $\mathrm{div} F$ and $\nabla\sigma$ bounded, for ∇F in $L \log L$ and with some bounds of $\nabla(\mathrm{div}\sigma)$, but without any assumption of uniform ellipticity for σ.

Most of the previous approaches also use estimates on the difference between two solutions of (possibly regularizations of) (1.1). The approach we present here is based on estimates on path functionals inspired by the method used by Crippa and De Lellis [4] to obtain an alternative proof of the results of Di Perna and Lions [5] on well-posedness for ODEs. The functional of [4] was used and adapted to obtain several extensions [11, 3] for deterministic systems. Note that other techniques exist to prove well posedness directly on characteristics of ODEs, see [9] for instance. Functionals inspired from [4] were already used in the context of SDEs in [30], in [22] to study weak uniqueness, and in [31] to study the case of SDEs in the sense of Stratonovich assuming $\sigma \in W^{2,2}$ and the case of standard SDEs assuming local exponential moments on $\nabla\sigma$.

The originality of the quantitative estimates we develop here is that they allow us to treat separately the strong existence and pathwise uniqueness for (1.1) from the question of bounds on solutions to (1.2). The typical result will hence assume that some estimate could be obtained on solutions to (1.2) (by whichever method) and conclude that strong existence and pathwise uniqueness hold provided that some bounds on σ and F in Sobolev spaces related to the bounds on u hold. The great advantage is the flexibility that one then enjoys as it is possible to choose the best method to deal with (1.2) according to any additional structure. For instance, ellipticity on σ is not required *a priori*. The second advantage of the method is its simplicity as it relies on some direct quantitative estimates on the solutions.

To give a better idea let us present a typical result that we obtain. For existence we consider sequence of approximations to (1.1)

$$dX_t^n = F_n(t, X_t^n)\, dt + \sigma_n(t, X_t^n)\, dW_t, \quad X_0^n = \xi, \tag{1.3}$$

with the same Brownian motion W_t for any n. And we introduce the corresponding approximation for (1.2)

$$\partial_t u_n + \nabla_x \cdot (F_n u_n) = \sum_{1\le i,j\le d} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}^n(t,x) u_n(t,x)), \quad u_n(t=0, dx) = u^0, \tag{1.4}$$

with $a^n = \sigma_n\, \sigma_n^*$ and $u_n \in M_1$.

The next result is not the most general we obtain, but it does not require any additional definition and illustrates the type of assumptions we need. We use the classical notations for L^p and Sobolev spaces with different exponents for space and time. For example, $L^q_{t,\mathrm{loc}}(W_x^{1,p})$ for $1 \le p, q \le \infty$ is the set of measurable functions f of the variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, such that, for almost all $t \ge 0$, $f(t, \cdot) \in W^{1,p}(\mathbb{R}^d)$ and $t \mapsto \|f(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d)} \in L^q([0, T])$ for all $T > 0$. We also call weak topology on the set M_1 of measurable functions of time with values in the set P_1 of probability measures on $\mathbb{R}^d$, the topology of weak-* convergence in time for the tight topology of probability measures on $\mathbb{R}^d$. In other words, $u_n \to u$ for the weak topology of M_1 iff $\langle u_n, f\rangle \to \langle u, f\rangle$ for all bounded continuous function f on $\mathbb{R}_+ \times \mathbb{R}^d$ with support included in $[0, T] \times \mathbb{R}^d$ for some $T > 0$.

Theorem 1.1 *Assume $d \ge 2$. One has*

(i) *Existence: Assume that there exists a sequence of smooth F_n, $\sigma_n \in L^\infty$ converging in the sense of distributions to F and σ respectively, such* *that the solution $u_n \in M_1$ to (1.4) satisfies for $1 \le p, q \le \infty$, with $1/p' + 1/p = 1$, $1/q + 1/q' = 1$*

$$\sigma_n - \sigma \longrightarrow 0 \text{ in } L^q_{t,loc}(L^p_x) \quad \text{and} \quad F_n - F \longrightarrow 0 \text{ in } L^q_{t,loc}(L^p_x),$$
$$\sup_n \left(\|\sigma_n\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} + \|F_n\|_{L^q_{t,loc}(W^{1,p}_x)} + \|F_n\|_{L^\infty} + \|\sigma_n\|_{L^\infty} \right) < \infty,$$
$$\sup_n \|u_n\|_{L^{q'}_{t,loc}(L^{p'}_x)} < \infty, \; u_n \longrightarrow u \text{ in the weak topology of } M_1.$$

Then there exists a strong solution X_t to (1.1) and $(X^n_t - \xi, t \in [0,T])_n$ converges in $L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$ and $T > 0$ to $(X_t - \xi, t \in [0,T])$, with X^n_t the solutions to (1.3). In addition, $u(t,dx)$ is the law of X_t for almost all $t \ge 0$.

(ii) *Uniqueness: Let X and Y be two solutions to (1.1) with one-dimensional time marginals $u_X(t,x)dx$ and $u_Y(t,x)dx$ both in $L^{q'}_{t,loc}(L^{p'}_x)$. Assume that $F, \sigma \in L^\infty$, $X_0 = Y_0$ a.s. and that*

$$\|F\|_{L^q_{t,loc}(W^{1,p}_x)} + \|\sigma\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < \infty$$

with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then one has pathwise uniqueness: $sup_{t \ge 0}|X_t - Y_t| = 0$ a.s.

We obtain better results in the one-dimensional case.

Theorem 1.2 *Assume $d = 1$.*

(i) *The existence result of Theorem 1.1 (i) holds under the same assumptions on F_n, σ_n, u_n, except that the assumption $\sup_n \|\sigma_n\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < \infty$ can be replaced by*

$$\sup_n \|\sigma_n\|_{L^{2q}_{t,loc}(W^{1/2,2p}_x)} < \infty$$

and in the case $p = 1$, the assumption $\sup_n \|F_n\|_{L^q_{t,loc}(W^{1,p}_x)} < \infty$ must be replaced by

$$\sup_n \|F_n\|_{L^q_{t,loc}(W^{1,1+\varepsilon}_x)} < \infty$$

for some $\varepsilon > 0$.

(ii) *The uniqueness result of Theorem 1.1 (ii) holds true under the same assumptions on F, σ, u_X, u_Y, except that $\|\sigma\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < \infty$ can be replaced by*

$$\|\sigma\|_{L^{2q}_{t,loc}(W^{1/2,2p}_x)} < \infty.$$

and in the case $p = 1$, the assumption $\|F\|_{L^q_{t,loc}(W^{1,p}_x)} < \infty$ must be replaced by

$$\|F\|_{L^q_{t,loc}(W^{1,1+\varepsilon}_x)} < \infty$$

for some $\varepsilon > 0$.

Note that no assumption of uniform ellipticity is needed in Theorems 1.1 and 1.2, provided one can prove a priori estimates on the various solutions u_n, u_X, u_Y to (1.4) and (1.2). Note also that pathwise uniqueness is proved only for particular solutions to (1.1), so we cannot use directly the result of Yamada and Watanabe to deduce strong existence. Hence our method proves separately strong existence and pathwise uniqueness; however they use very similar techniques.

Of course, as they are laws, u_n, u_X and u_Y all have bounded mass so Theorems 1.1 and 1.2 really depend on whether it is possible to obtain higher integrability for a solution of (1.2). From Theorem 1.1 we may for instance simply deduce

Corollary 1.3 *Assume that $d \geq 2$, $u^0 \in L^1 \cap L^\infty$, F, $\sigma \in L^\infty$, $F \in L^1_{t,loc}(W^{1,1}_x)$ and $\nabla\sigma \in L^q_{t,loc}(L^p_x)$, where $2/q + d/p = 1$ with $p > d$. Assume as well that σ is uniformly elliptic. Then one has existence of a strong solution to* (1.1) *with marginal distributions $u(t, dx)$ in $L^\infty_{t,loc}(L^\infty_x)$. In addition, pathwise uniqueness holds among all solutions with marginal distributions in $L^\infty_{t,loc}(L^\infty_x)$.*

However in many physical cases, uniform ellipticity is not necessary. For instance in the phase space problem

$$dX_t = V_t, \quad dV_t = F(t, X_t)dt + \sigma(t, X_t)\, dW_t, \quad X_0 = x,\ V_0 = v.$$

one obtains an even better result.

Corollary 1.4 *Assume that $\sigma \in L^\infty \cap L^2_{t,loc}(H^1_x)$ and $F \in L^1_{t,loc}(W^{1,1}_x)$. Assume also that the law $u_0 \in L^\infty$. Then one has both existence of a strong solution to* (2.22) *and pathwise uniqueness among all solutions with marginal distributions in $L^\infty_{t,loc}(L^\infty_x)$.*

The goal of Section 2 is to give the statement of all our results. We start in Subsection 2.1 by defining the norms and Banach spaces needed to state our most general results in Subsection 2.2. Theorems 1.1 and 1.2 are then obtained as corollaries of these general results. In Subsection 2.3, several corollaries of Theorems 1.1 and 1.2 are stated in various situations, including the uniformly elliptic case (Corollary 1.3), the non-degenerate one-dimensional case and the kinetic (Langevin) case (Corollary 1.4). The conditions for strong existence and pathwise uniqueness are then compared with the best conditions in the literature. The rest of the paper is devoted to the proofs of all the results stated in Section 2, and the organization of the rest of the paper is given in the end of Section 2.

2 Statement of the results

As usual one needs regularity assumptions on F and σ to ensure strong existence and pathwise uniqueness for (1.1). In our case, these are Sobolev norms with respect to some $u \in M_1$, defined in Subsection 2.1. Our general results are then stated in Subsection 2.2, and several consequences of these results are discussed in Subsection 2.3.

2.1 Norms and Banach spaces

The conditions we shall impose on F and σ can be roughly described as follows. We need σ to be L^2 in time and H^1 in space (in dimension $d \geq 2$) or $H^{1/2}$ in space (in dimension $d = 1$) w.r.t. the measure u solution to (1.2), and F to be L^1 in time and $W^{1,1}$ in space w.r.t. the measure u. Weighted Sobolev spaces have been extensively used and studied, but the key difference here is that no regularity is known on the weight u. It could very well be a sum of Dirac masses. This is why one must be careful and why maximal functions are required.

Remark that we are using here the maximal functions on the whole space $\mathbb{R}^d$. This is only by convenience. One would have exactly the same results if they were restricted to a smooth domain Ω s.t. $\operatorname{supp} u \subset \Omega$, i.e. by taking

$$M\,f(x) = \sup_r \frac{1}{|B(0,r)\cap\Omega|} \int_{B(0,r)\cap\Omega} f(x+z)\,dz.$$

The goal of the next Subsections is to give the precise definitions and basic properties of our spaces.

2.1.1 The space $H^1_T(u)$

Fix first $v \in P_1$. We start with the following definition.

Definition 2.1 *The space $H^1(v)$ is defined as the subspace of functions $f \in BV_{loc}(\mathbb{R}^d)$, the space of functions on $\mathbb{R}^d$ with locally bounded variations,* *such that*

$$\|f\|^2_{H^1(v)} := \int_{\mathbb{R}^d} \left((M\,|f|(x))^2 + (M\,|\nabla_x f|(x))^2\right)\, v(dx) < \infty,$$

where M is the usual maximal operator.

First of all, observe that the definition makes perfect sense. If $f \in BV_{\rm loc}(\mathbb{R}^d)$ then $|\nabla f|$ is a locally finite measure. This allows to define $M\,|\nabla f|$ per

$$M\,|\nabla f|(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |\nabla f|(x+dz), \quad \forall x \in \mathbb{R}^d.$$

In that case, it is well known (see [24]) that $M\,|\nabla f|$ is a Borel function with value in $\mathbb{R}_+ \cup \{+\infty\}$. It locally belongs in fact to the weak L^1 space, that is for any $R > 0$, there exists C_R s.t.

$$|\{x \in B(0,R),\ M\,|\nabla f|(t,x) > L\}| \leq \frac{C_R}{L}.$$

Therefore the integral of $(M\,|\nabla f|)^2$ against the Borelian measure v is well defined with value in $\mathbb{R}_+ \cup \{+\infty\}$, thus justifying the definition.

The main point of the definition is that we have a well behaved space independently of any regularity on v.

Theorem 2.2 *Assume that v belongs to P_1. Then $H^1(v)$ is a Banach space with norm* (2.3)*. Moreover the norm is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{H^1(v)} \leq \liminf_n \|f_n\|_{H^1(v)}. \tag{2.1}$$

And if for a given $f \in BV_{loc}(\mathbb{R}^d)$, v_n converges to v in the tight topology of probability measures then

$$\|f\|_{H^1(v)} \leq \liminf_n \|f\|_{H^1(v_n)}.$$

This result is proved in Section 3.

There are several technical reasons why we use $M|\nabla f|$ in the definition of the norm. Note however that the intuitive definition with just ∇f would most certainly be too weak as v could for instance vanish just at the points where ∇f is very large. In particular, without the maximal function in the definition of the norm (2.3), it would be very easy to find counterexamples to (2.1).

Now, given any $u \in M_1$, we give a second definition.

Definition 2.3 *For all $T > 0$, the space $H^1_T(u)$ is defined as the subspace of the set of measurable functions on $[0,T] \times \mathbb{R}^d$ such that, for almost all $t \in [0,T]$, $f(t,\cdot) \in H^1(u(t,\cdot))$ and*

$$\|f\|^2_{H^1_T(u)} = \int_0^T \|f(t,\cdot)\|^2_{H^1(u(t,\cdot))}\, dt < \infty. \tag{2.2}$$

In particular, if $u(t,\cdot)$ is the distribution of X_t solution to (1.1), then, for all $T > 0$ and $\sigma \in H^1_T(u)$,

$$\|\sigma\|^2_{H^1_T(u)} = \mathbb{E}\left(\int_0^T M\,|\sigma|^2(t,X_t)dt\right) + \mathbb{E}\left(\int_0^T (M|\nabla\sigma|(t,X_t))^2 dt\right). \tag{2.3}$$

We then have the following immediate consequence of Thm. 2.2.

Corollary 2.4 *Fix $T > 0$. Assume u belongs to M_1. Then $H^1_T(u)$ is a Banach space with norm (2.2). Moreover the norm is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{H^1_T(u)} \leq \liminf_n \|f_n\|_{H^1_T(u)}. \tag{2.4}$$

And if for a given f measurable on $\mathbb{R}_+ \times \mathbb{R}^d$ with $f(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$ for almost all $t \geq 0$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{H^1_T(u)} \leq \liminf_n \|f\|_{H^1_T(u_n)}.$$

2.1.2 The space $H^{1/2}_T(u)$

In the one dimensional case, we can prove strong existence and pathwise uniqueness using $H^{1/2}$ type of assumptions on σ. The definitions and properties of the spaces $H^{1/2}_T(u)$ follow exactly the same steps as before. We first fix $v \in P_1$.

Definition 2.5 *For any function $f \in L^1_{loc}(\mathbb{R}^d)$, one defines*

$$\partial_x^{1/2} f = \mathcal{F}^{-1}\,|\xi|^{1/2}\,\mathcal{F}\,f,$$

with $\mathcal{F}$ the Fourier transform in $\mathbb{R}^d$. The space $H^{1/2}(v)$ is defined as the subspace of functions $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_x^{1/2} f$ is a locally finite Radon measure and

$$\|f\|^2_{H^{1/2}(v)} = \int_{\mathbb{R}^d} \left((M\,|f|(x))^2 + (M\,|\partial_x^{1/2} f|(x))^2\right)\, v(dx) < \infty.$$

As for $H^1(v)$, the maximal function can be extended to measures by

$$M\,|\partial_x^{1/2} f|(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |\partial_x^{1/2} f|(x+dz), \quad \forall x \in \mathbb{R}^d.$$

One has again that $M\,|\partial_x^{1/2} f|$ is a Borel function with value in $\mathbb{R}_+ \cup \{+\infty\}$ belonging to the local weak L^1 space. The integral against the Borelian measure v is hence well defined in $\mathbb{R}_+ \cup \{+\infty\}$, independently of the regularity of v.

The next result is proved in Section 3.

Theorem 2.6 *Assume that v belongs to P_1. Then $H^{1/2}(v)$ is a Banach space with norm (2.7). Moreover the norm is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{H^{1/2}(v)} \leq \liminf_n \|f_n\|_{H^{1/2}(v)}. \tag{2.5}$$

And if, for a given $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_x^{1/2} f$ is a locally finite Radon measure, v_n converges to v in the tight topology of probability measures on $\mathbb{R}^d$, then

$$\|f\|_{H^{1/2}(v)} \leq \liminf_n \|f\|_{H^{1/2}(v_n)}.$$

Given any $u \in M_1$, we give a second definition.

Definition 2.7 *For all $T > 0$, the space $H_T^{1/2}(u)$ is defined as the subspace of the set of measurable functions on $[0,T] \times \mathbb{R}^d$ such that, for almost all $t \in [0,T]$, $f(t,\cdot) \in H^{1/2}(u(t,\cdot))$ and*

$$\|f\|^2_{H_T^{1/2}(u)} = \int_0^T \|f(t,\cdot)\|^2_{H^{1/2}(u(t,\cdot))}\, dt < \infty. \tag{2.6}$$

In particular, if $u(t,\cdot)$ is the distribution of X_t solution to (1.1), then, for all $T > 0$ and $\sigma \in H_T^{1/2}(u)$,

$$\|\sigma\|^2_{H_T^{1/2}(u)} = \mathbb{E}\left(\int_0^T M\,|\sigma|^2(t,X_t)dt\right) + \mathbb{E}\left(\int_0^T (M|\partial_x^{1/2}\sigma|(t,X_t))^2 dt\right). \tag{2.7}$$

Again, one has the following immediate consequence of Thm. 2.6.

Corollary 2.8 *Fix $T > 0$. Assume u belongs to M_1. Then $H_T^{1/2}(u)$ is a Banach space with norm (2.6). Moreover the norm is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{H_T^{1/2}(u)} \leq \liminf_n \|f_n\|_{H_T^{1/2}(u)}. \tag{2.8}$$

And if for a given $f \in L^1(\mathbb{R}_+ \times \mathbb{R}^d)$ s.t. $\partial_x^{1/2} f(t, \cdot)$ is a locally finite Radon measure for almost all $t \in [0, T]$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{H_T^{1/2}(u)} \leq \liminf_n \|f\|_{H_T^{1/2}(u_n)}.$$

2.1.3 The space $W_T^{\phi,\mathrm{weak}}(u)$

We also need some similar $W^{1,1}$ assumptions on F. Following the definition of $H^1(u)$, a first attempt would be

$$\|F\|^2_{W_T^{1,1}(u)} = \int_0^T \int_{\mathbb{R}^d} (M|F|(t,x) + M|\nabla F|(t,x))\, u(t,dx)\, dt. \tag{2.9}$$

Unfortunately, while this definition would work, it is slightly too strong in some cases. This is due to the fact that the maximal operator M is bounded on L^p, $p > 1$, but not on L^1. In particular if $u \in L^\infty$ then the norm defined in (2.3) would automatically be finite if σ is in the usual H^1 space but the norm defined in (2.9) would *not* be finite if $F \in W^{1,1}$ in general.

Therefore in order to obtain better assumptions we have to work with a more complicated space. We proceed as before and fix $v \in P_1$. We also introduce a super-linear function ϕ, i.e. a function ϕ on $[1, \infty)$ such that $\phi(\xi)/\xi$ is non-decreasing and converges to ∞ as $\xi \to \infty$.

Definition 2.9 *For any locally finite Radon measure μ, decomposing μ into a part absolutely continuous with respect to the Lebesgue measure μ_a and the singular part μ_s, one defines*

$$M_L \mu = \sqrt{\log L} + \int_{\mathbb{R}^d} \frac{|\mu_a|(z)\, \mathbb{1}_{|\mu_a(z)| \geq \sqrt{\log L}}\, dz + |\mu_s|(dz)}{(L^{-1} + |x - z|)\, |x - z|^{d-1}}.$$

For any function $f \in BV_{loc}(\mathbb{R}^d)$, the decomposition of ∇f, into a part absolutely continuous with respect to the Lebesgue measure $\nabla_a f$ and the singular part $\nabla_s f$, makes $M_L \nabla f$ well defined.

The space $W^{\phi,weak}(v)$ is hence defined as the subspace of functions $f \in BV_{loc}(\mathbb{R}^d)$ s.t.

$$\|f\|_{W^{\phi,weak}(v)} = \int_{\mathbb{R}^d} M\,|f|(x)\,v(dx) + \sup_{L\geq 1} \frac{\phi(L)}{L\,\log L} \int_{\mathbb{R}^d} M_L \nabla f\, v(dx) < \infty.$$

In this definition the maximal function is regularized so that $M_L\nabla f$ is locally integrable for any fixed L. The supremum is then taken outside.

Obviously the space heavily depends on the choice of ϕ. Note that $M_L\nabla f \geq \sqrt{\log L}$ so that

$$\|f\|_{W^{\phi,\mathrm{weak}}(v)} \geq \sup_{L\geq 1} \frac{\phi(L)}{L\,\sqrt{\log L}}.$$

In particular $\|f\|_{W^{\phi,\mathrm{weak}}(v)} = +\infty$ for all f if $\phi(L) \gg L\,\sqrt{\log L}$ asymptotically as $L \to +\infty$. On the other hand we want to choose ϕ superlinear as we need to control the integrability of $|\nabla f|$. This leads to the assumptions

$$\frac{\phi(L)}{L} \longrightarrow +\infty, \quad \frac{\phi(L)}{L\,\sqrt{\log L}} \longrightarrow 0, \quad \text{as } L \to +\infty. \tag{2.10}$$

Even with this assumption, $W^{\phi,\mathrm{weak}}(v)$ is not a Banach space and in particular $\|\cdot\|_{W^{\phi,\mathrm{weak}}(v)}$ is not a norm. Of course $\|0\|_{W^{\phi,\mathrm{weak}}(v)} \neq 0$ but this could easily be remedied by considering $\|\cdot\|_{W^{\phi,\mathrm{weak}}(v)} - \alpha_\phi$ instead, for the right constant α_ϕ.

The main problem is that $\|\lambda\, f\|_{W^{\phi,\mathrm{weak}}(v)} \neq |\lambda|\,\|f\|_{W^{\phi,\mathrm{weak}}(v)}$ and this cannot easily be corrected. It is in fact the same kind of issue that one has with the definition of so-called Orlicz spaces such as $L\log L$. The solution is similar and would consist in constructing the right norm by duality.

We did not feel that it was appropriate in this article however. Such a construction in the present case would be considerably more complex than for classical Orlicz space. It would also distract from our main goal while bringing very little to our results. It is worth recalling the main reason why we introduce the space $W^{\phi,\mathrm{weak}}$: it is a compromise between two requirements.

- The estimates that we perform later in the text would not work for instance with the simple requirement that

  $$\int (|f| + |\nabla f|)\, v(dx) < \infty,$$

  so the maximal operator is needed.

- We want to recover the classical assumption if v is bounded from below and above. That means that if $1/C \leq v \leq C$, then any $f \in W^{1,1}$ must be included in $W^{\phi,\text{weak}}(v)$ for some well chosen ϕ (depending on f). This is in particular why we do not use the direct extension $W^{1,1}(v)$ of the space $H^1(v)$ (where L^2 norms are replaced by L^1 norms).

The above definition of $W^{\phi,\text{weak}}(v)$ fulfills those two goals and therefore we do not study further this space.

Theorem 2.10 *Assume that v belongs to P^1, that ϕ is super-linear and continuous and that* (2.10) *holds. Then $W^{\phi,weak}(v)$ is well defined and $\|\cdot\|_{W^{\phi,weak}(v)}$ is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{W^{\phi,weak}(v)} \leq \liminf_n \|f_n\|_{W^{\phi,weak}(v)}. \tag{2.11}$$

And if for a given $f \in BV_{loc}(\mathbb{R}^d)$, v_n converges to v in the tight topology of probability measures then

$$\|f\|_{W^{\phi,weak}(v)} \leq \liminf_n \|f\|_{W^{\phi,weak}(v_n)}.$$

Moreover if $v \geq 1/C$ over a smooth open set Ω and $f \in W^{\phi,weak}(v)$ then $f \in W^{1,1}(K)$ for any compact set $K \subset \Omega$. Reciprocally if $v \leq C$ over Ω and $f \in W^{1,1}(\Omega)$ with compact support in Ω, then there exists a super-linear ϕ satisfying (2.10) *s.t. $f \in W^{\phi,weak}(v)$.*

Now, given $u \in M_1$ and a super-linear function ϕ, we define

Definition 2.11 *For all $T > 0$, the space $W_T^{\phi,weak}(u)$ is defined as the set of measurable f on $[0,T] \times \mathbb{R}^d$ such that $f(t,\cdot) \in W_T^{\phi,weak}(u(t,\cdot))$ for almost all $t \in [0,T]$ and*

$$\|f\|_{W_T^{\phi,weak}(u)} = \int_0^T \|f(t,\cdot)\|_{W^{\phi,weak}(u(t,\cdot))}\, dt < \infty.$$

In particular, if $u(t,\cdot)$ is the distribution of X_t solution to (1.1), then, for all $T > 0$ and $F \in W_T^{\phi,\text{weak}}(u)$,

$$\|F\|^2_{W_T^{\phi,\text{weak}}(u)} = \sup_{L\geq 1} \frac{\phi(L)}{L \log L} \mathbb{E}\left(\int_0^T (M\,|F|(t,X_t) + M_L |\nabla F|(t,X_t))\, dt \right). \tag{2.12}$$

Then we have

Corollary 2.12 *Fix $T > 0$, assume u belongs to M_1, and that ϕ is super-linear, continuous and satisfies* (2.10)*. Then $W_T^{\phi,weak}(u)$ is well-defined and $\|\cdot\|_{W_T^{\phi,weak}(u)}$ is lower semi-continuous with respect to convergence in the sense of distribution: If $f_n \longrightarrow f$ in the sense of distribution then*

$$\|f\|_{W_T^{\phi,weak}(u)} \leq \liminf_n \|f_n\|_{W_T^{\phi,weak}(u)}. \tag{2.13}$$

And if for a given f measurable on $\mathbb{R}_+ \times \mathbb{R}^d$ with $f(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$ for almost all $t \in [0,T]$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{W_T^{\phi,weak}(u)} \leq \liminf_n \|f\|_{W_T^{\phi,weak}(u_n)}.$$

Moreover if $u \geq 1/C$ over $[0,T] \times \Omega$ where $\Omega \subset \mathbb{R}^d$ is a smooth open set and $f \in W_T^{\phi,weak}(u)$ then $f \in L_t^1([0,T], W^{1,1}(K))$ for any compact set $K \subset \Omega$. Reciprocally if $u \leq C$ over $[0,T] \times \Omega$ and $f \in L_t^1([0,T], W^{1,1}(\Omega))$ with compact support in $[0,T] \times \Omega$, then there exists a super-linear ϕ satisfying (2.10) *s.t. $f \in W_T^{\phi,weak}(u)$.*

The first two points of Cor. 2.12 are direct consequences of Thm. 2.10, and the last statements about the cases where u is bounded from above or below can be proved exactly as the similar statement of Thm. 2.10 is proved in Section 3.

2.2 General results on strong solutions to (1.1)

In the multi-dimensional case, our most general result is the following one, proved in Section 4.

Theorem 2.13 *Assume that $d \geq 2$. One has*

(i) *Existence: Fix $T > 0$ and assume that there exists a sequence of smooth F_n, $\sigma_n \in L^\infty$ converging in the sense of distribution to F and σ respectively, such that the solution $u_n \in M_1$ to* (1.4) *satisfies for some super-linear ϕ*

$$\int_0^T \int_{\mathbb{R}^d} (|\sigma_n - \sigma| + |F_n - F|)\, du_n\, dt \longrightarrow 0, \tag{2.14}$$

$$\sup_n \left(\|F\|_{W_T^{\phi,weak}(u_n)} + \|\sigma\|_{H_T^1(u_n)} + \|F_n\|_{L^\infty} + \|\sigma_n\|_{L^\infty} \right) < \infty, \tag{2.15}$$

$$u_n \longrightarrow u \text{ for the weak topology of } M_1. \tag{2.16}$$

Then there exists a strong solution X_t to (1.1) *s.t.* $(X_t^n - \xi, t \in [0,T])_n$ *converges in* $L^p(\Omega, L^\infty([0,T]))$ *for all* $p > 1$ *to* $(X_t - \xi, t \in [0,T])$*, with* X_t^n *the solutions to* (1.3)*. In addition,* $u(t,dx)$ *is the law of* X_t *for almost all* $t \in [0,T]$*.*

(ii) *Uniqueness: Let X and Y be two solutions to* (1.1) *with one-dimensional time marginals* $u_X(t,\cdot)$ *and* $u_Y(t,\cdot)$ *on* $[0,T]$*. Assume that* $F, \sigma \in L^\infty$*,* $X_0 = Y_0$ *a.s. and that*

$$\|F\|_{W_T^{\phi,weak}(u_X)} + \|F\|_{W_T^{\phi,weak}(u_Y)} + \|\sigma\|_{H_T^1(u_X)} + \|\sigma\|_{H_T^1(u_Y)} < \infty \quad (2.17)$$

for some super-linear function ϕ*. Then one has pathwise uniqueness on* $[0,T]$*, i.e.* $sup_{t\in[0,T]}|X_t - Y_t| = 0$ *a.s.*

Note that we do not require any ellipticity on σ for this result. In that sense we cannot hope to have any smoothing effect from the Wiener process and the assumption on F must be enough to provide well posedness in the purely deterministic setting ($\sigma = 0$). In this case, taking any $u_0 \in L^\infty$, our result gives that there exists a unique solution of $\dot{X}_t = F(t, X_t)$ with $X_0 = \xi$ and with law $u \in L^\infty$ provided that there exists a sequence of regularized F_n s.t. $u_n \to u$ for the weak-$*$ topology with $u \in L^\infty$ and a super-linear ϕ s.t.

$$\sup_{L\geq 1} \frac{\phi(L)}{L \log L} \|F + M_L \nabla F\|_{L^1([0,T]\times\mathbb{R}^d)} < \infty.$$

The first point is for example implied by the assumption $\mathrm{div} F \in L^\infty$ and the second one can be proved to hold if $F \in L^1_{t,\mathrm{loc}}(W_x^{1,1})$ as in the proof of Corollary 1.1 in the Appendix. Hence, we recover the classical results of DiPerna and Lions [5] but not the optimal BV assumption from Ambrosio [2].

In dimension 1, the result is even better: we recover the $H^{1/2}$ type of assumption from [28, 20, 7], but we lose a little bit on F (we have to use (2.9) instead of (2.12)).

Theorem 2.14 *Assume that $d = 1$. One has*

(i) *Existence: Fix $T > 0$ and assume that there exists a sequence of smooth* $F_n, \sigma_n \in L^\infty$ *converging in the sense of distribution to F and σ respectively, such that the solution u_n to* (1.4) *satisfies*

$$\int_0^T \int_{\mathbb{R}} (|\sigma_n - \sigma| + |F_n - F|)\, du_n\, dt \longrightarrow 0,$$
$$\sup_n (\|F\|_{W_T^{1,1}(u_n)} + \|\sigma\|_{H_T^{1/2}(u_n)} + \|F_n\|_{L^\infty} + \|\sigma_n\|_{L^\infty}) < \infty,$$
$u_n \longrightarrow u$ *for the weak topology of* M_1*.*

Then there exists a strong solution X_t to (1.1) *s.t.* $(X_t^n - \xi, t \in [0,T])_n$ *converges in* $L^p(\Omega, L^\infty([0,T]))$ *for all* $p > 1$ *to* $(X_t - \xi, t \in [0,T])$*, with* X_t^n *the solutions to* (1.3)*. In addition,* $u(t,dx)$ *is the law of* X_t *for almost all* $t \in [0,T]$*.*

(ii) *Uniqueness: Let X and Y be two solutions to* (1.1) *with one-dimensional time marginals* $u_X(t,\cdot)$ *and* $u_Y(t,\cdot)$ *on* $[0,T]$*. Assume that* $F, \sigma \in L^\infty$*,* $X_0 = Y_0$ *a.s. and that*

$$\|F\|_{W_T^{1,1}(u_X)} + \|F\|_{W_T^{1,1}(u_Y)} + \|\sigma\|_{H_T^{1/2}(u_X)} + \|\sigma\|_{H_T^{1/2}(u_Y)} < \infty.$$

Then pathwise uniqueness holds on $[0,T]$*, i.e.* $sup_{t\in[0,T]}|X_t - Y_t| = 0$ *a.s.*

Of course, while precise, the norms given by (2.3)–(2.12) or (2.7)–(2.9) are not so simple to use. However it is quite easy to deduce more intuitive results with the more usual $W^{1,p}$ norms. We recall that M is continuous onto every L^p space for $1 < p \leq \infty$ and hence the norms $\|\cdot\|_{H_T^1(u)}$ and $\|\cdot\|_{W_T^{1,1}(u)}$ are controlled by appropriate Sobolev norms if some L^q estimate is available on the law u.

One complication occurs when $u_X \in L^\infty$ and one wants to obtain the close to optimal $W^{1,1}$ assumption on F (instead of $W^{1,p}$ for some $p > 1$) as the maximal function is not bounded onto L^1. This is the reason why we defined (2.12), which can be used following [11] (we recall the main steps in the appendix).

Therefore, Theorems 1.1 and 1.2 are simple corollaries of Theorems 2.13 and 2.14, respectively, except for the previous complication for Theorem 1.1.

In order to apply Theorems 1.1 and 1.2, we need to consider cases where it is possible to obtain better integrability than L^1 bounds for a solution to (1.2). This occurs in various situations, some of which will be studied in the next Subsection. One difficulty to apply Theorems 1.1 (ii) and 1.2 (ii) is to obtain pathwise uniqueness without restriction on the set of solutions considered. This will of course be ensured if uniqueness in law is known for (1.1). More precisely, if the conclusion of Theorem 1.1 (i) or Theorem 1.2 (i) holds, then either $\|F\|_{W_T^{\phi,\mathrm{weak}}(u)} + \|\sigma\|_{H_T^1(u)} < \infty$ (if $d \geq 2$) or $\|F\|_{W_T^{1,1}(u)} + \|\sigma\|_{H_T^{1/2}(u)} < \infty$ (if $d = 1$) by Cor. 2.4, 2.8 and 2.12. Since there is uniqueness in law for (1.1), then $u_X = u_Y = u$ for all solutions X and Y to (1.1) as in Theorem 1.1 (ii) or Theorem 1.2 (ii) and hence pathwise uniqueness holds. This argument will be used repeatedly in the next subsection. Note however that condition (2.17) may impose restrictions on the initial distribution. This issue will be studied in Prop. 2.23.

2.3 Consequences

Let us first consider the case where σ is uniformly elliptic: for all t, x,

$$\frac{1}{2}\sigma\,\sigma^*(t,x) = a(t,x) \geq c\,I \tag{2.18}$$

for some $c > 0$. For example if $F = 0$ and σ does not depend on time, then there exists a corresponding stationary measure $\bar{u} > 0$ in $L^{d/(d-1)}$ as per Aleksandrov [1]. In that case, when $u_0 \leq C\bar{u}$, then the unique solution u of (1.2) in $L^2_{t,\mathrm{loc}}(H^1_x)$ satisfies $u(t,dx) \leq C\bar{u}(x)dx$ for all $t \geq 0$ by the maximum principle.

Corollary 2.15 *Assume that $F = 0$ and $\sigma(x)$ satisfies* (2.18) *and belongs to $L^\infty \cap W^{1,2d}_x$ (or $L^\infty \cap H^{1/2}$ if $d = 1$). Assume also that $u_0 \leq C\bar{u}$ for some constant $C > 0$. Then one has both existence of a strong solution to* (1.1) *and pathwise uniqueness.*

Note that pathwise uniqueness holds without additional assumption since $\sigma \in W^{1,2d}$ implies that σ is continuous, and uniqueness in law holds in this case since σ is bounded and uniformly elliptic [25, Thm. 7.2.1].

Those results were later extended by Krylov in the parabolic, time dependent case [12, 14]. We may for example use the following version found in [32].

Theorem 2.16 *Assume that F and σ are bounded and σ satisfies* (2.18). *Then, for all solution X of* (1.1) *with any initial distribution, for all $T > 0$ and $p, q > 1$ such that*

$$\frac{d}{p} + \frac{2}{q} < 2,$$

there exists a constant C such that for all $f \in L^q_t(L^p_x)$

$$\mathbb{E}\left[\int_0^T f(t, X_t)dt\right] \leq C\|f\|_{L^q_t(L^p_x)}.$$

This result means that

$$u \in L^{q'}_t(L^{p'}_x),$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, and we obtain the following corollary.

Corollary 2.17 (i) *Assume that $d \geq 2$, F, $\sigma \in L^\infty$, σ satisfies* (2.18), *$F \in L^{q/2}_{t,loc}(W^{1,p/2}_x)$ and $\sigma \in L^q_{t,loc}(W^{1,p}_x)$ with $2/q + d/p < 1$. Then one has both existence of a strong solution to* (1.1) *and pathwise uniqueness for any initial condition ξ.*

(ii) *Assume that* $d = 1$, F, $\sigma \in L^\infty$, σ *satisfies* (2.18), $\sigma \in L^q_{t,loc}(W^{1/2,p}_x)$ *with* $2/q + 1/p < 1$ *and* $F \in L^{q/2}_{t,loc}(W^{1,p/2}_x)$ *if* $p > 2$, $F \in L^{q/2}_{t,loc}(W^{1,1+\varepsilon})$ *for some* $\varepsilon > 0$ *if* $p \leq 2$. *Then one has both existence of a strong solution to* (1.1) *and pathwise uniqueness for any initial condition* ξ.

Note that in this case, pathwise uniqueness holds without additional assumption since Krylov's inequality implies that $u \in L^{q'}_t(L^{p'}_x)$ for *all* solutions to (1.1).

In our setting, since we need additional regularity on σ, it is easy to obtain better a priori estimates for u than those given by Krylov's inequality. For instance:

Proposition 2.18 *For any* $d \geq 1$, *assume* $u^0 \in L^1 \cap L^\infty$, F, $\sigma \in L^\infty$, σ *satisfies* (2.18) *and* $\nabla\sigma \in L^q_{t,loc}(L^p_x)$ *satisfying* $2/q + d/p = 1$ *with* $p > d$. *Then any* u *solution to* (1.2), *limit for the weak topology in* M_1 *of smooth solutions, belongs to* $L^\infty_t(L^r_x)$ *for any* $1 \leq r \leq \infty$.

This proposition is based on classical energy estimates and hence we just give a very short proof of it in Section 6. Combined with Theorem 1.1 this gives slightly better conditions for σ and much better conditions for F, assuming additional conditions on the initial distribution. We obtain Corollary 1.3, restated here

Corollary 2.19 *Assume that* $d \geq 2$, $u^0 \in L^1 \cap L^\infty$, F, $\sigma \in L^\infty$, $F \in L^1_{t,loc}(W^{1,1}_x)$ *and* $\nabla\sigma \in L^q_{t,loc}(L^p_x)$, *where* $2/q + d/p = 1$ *with* $p > d$. *Assume as well that* σ *satisfies* (2.18). *Then one has existence of a strong solution to* (1.1) *with marginal distributions* $u(t, dx)$ *in* $L^\infty_{t,loc}(L^\infty_x)$. *In addition, pathwise uniqueness holds among all solutions with marginal distributions in* $L^\infty_{t,loc}(L^\infty_x)$.

As above, the pathwise uniqueness property could be improved if we could prove uniqueness in law. If $d = 2$, uniqueness in law holds when σ and F are bounded and σ is uniformly elliptic [13]. When $d \geq 3$, by Sobolev embedding, the assumption $\nabla\sigma \in L^q_{t,\mathrm{loc}}(L^p_x)$ implies that $x \mapsto \sigma(t, x)$ is continuous for almost all $t \geq 0$. This condition is not exactly sufficient to use the result of Stroock and Varadhan [25, Thm. 7.2.1], which assumes that $\sup_{t\in[0,T]} |\sigma(t, x) - \sigma(t, y)| \to 0$ when $y \to x$. This is true for example if $\nabla\sigma \in L^\infty_{t,\mathrm{loc}}(L^p_x)$ for $p > d$. Hence we obtain

Corollary 2.20 *Assume that* $d \geq 2$, $u^0 \in L^1 \cap L^\infty$, F, $\sigma \in L^\infty$, $F \in L^1_{t,loc}(W^{1,1}_x)$ *and* $\nabla\sigma \in L^q_{t,loc}(L^p_x)$ *where* $2/q + d/p = 1$ *with* $p > d$. *Assume* *as well that σ satisfies* (2.18)*, and if $d \geq 3$ that for all x,*

$$\sup_{t\in[0,T]} |\sigma(t,x) - \sigma(t,y)| \to 0 \quad \textit{when} \quad y \to x.$$

Then one has both existence of a strong solution to (1.1) *and pathwise uniqueness.*

This result can be compared with previous works dealing with the uniformly elliptic case. The best results in this case seem to be those of [32] and [21]. In the first work, strong existence and pathwise uniqueness are proved under the assumptions $\nabla\sigma \in L^q_{t,\mathrm{loc}}(L^p_x)$, $\sigma(t,x)$ uniformly continuous with respect to x and $F \in L^q_{t,\mathrm{loc}}(L^p_x)$ with $d/p + 2/q < 1$, so we obtain a slightly better condition on σ (we can handle the limit case $d/p + 2/q = 1$ and no uniform continuity is needed for strong existence), and a condition on F which is neither stronger nor weaker, since $L^1_{t,\mathrm{loc}}(W^{1,1}_x)$ neither contains nor is contained in $L^q_{t,\mathrm{loc}}(L^p_x)$ with $d/p + 2/q < 1$. In the second work, since the approach for pathwise uniqueness is very different, the conditions obtained are of a different nature as ours. In particular, this work requires additional boundedness assumptions on $\operatorname{div}\sigma$ and $(D\sigma)^2$.

In dimension 1 in the stationary case, even if (2.18) is not satisfied but instead only

$$\frac{1}{2}\sigma^2(x) = a(x) > 0, \tag{2.19}$$

then one has the a priori bound

$$u(t,x) \leq \frac{C}{a(x)}\, e^{\int_0^x \frac{F(y)}{a(y)}\,dy}, \quad \forall x \in \mathbb{R},$$

for solutions to (1.2) again provided that u^0 satisfies the same bound. Therefore, we obtain

Corollary 2.21 *Assume $d = 1$, $\sigma, F \in L^\infty$, σ satisfies* (2.19)*, $F/a \in L^1$,*

$$u_0(x) \leq \frac{C}{a(x)}\, e^{\int_0^x \frac{F(y)}{a(y)}\,dy}, \quad \forall x \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} \frac{(M|\partial^{1/2}\sigma|(x))^2}{a(x)}\,dx < \infty \quad \textit{and} \quad \int_{\mathbb{R}} \frac{M|\nabla F|}{a(x)}\,dx < \infty. \tag{2.20}$$

Then one has both existence of a strong solution to (1.1) *and pathwise uniqueness.*

Note that the assumptions (2.20) imply that $a^{-1} \in L^1_{\text{loc}}$, which is a necessary and sufficient condition for uniqueness in law when F is bounded [7].

We will prove in Lemma 3.3 of Section 3 that for all x, y

$$|\sigma(x) - \sigma(y)| \le \Big(M|\partial_x^{1/2}\sigma|(x) + M|\partial_x^{1/2}\sigma|(y)\Big)\ |x-y|^{1/2}. \tag{2.21}$$

This inequality allows us to compare our result with similar results of the literature [28, 33, 20, 7]. The best conditions in the time homogeneous case seem to be those of [7, Thm. 4.41], where pathwise uniqueness is proved to hold if $F/a \in L^1_{\text{loc}}$, $|\sigma(x)-\sigma(y)|^2 \le f(x)h(|y-x|)$ for all x, y with $f/a \in L^1_{\text{loc}}$ and $\int_{0+} h^{-1}(u)du = +\infty$. Our result gives worse conditions on F, and our condition on σ is slightly worse, since we need to take $h(u) = u$ in (2.21). However, we improve the conditions on σ of all the other references.

We point out that, in higher dimension as well, ellipticity is not always required for bounds on the law. We give the classical example of SDE's in the phase space $\mathbb{R}^{2d}$

$$dX_t = V_t, \quad dV_t = F(t, X_t)dt + \sigma(t, X_t)\, dW_t, \quad X_0 = x,\ V_0 = v. \tag{2.22}$$

The joint law $u(t, x, v)$ of the process $(X_t, V_t)_{t\ge 0}$ solves the kinetic equation

$$\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) + F(t,x)\cdot\nabla_v u(t,x,v) = \sum_{1\le i,j\le d} a_{ij}(t,x)\frac{\partial^2 u(t,x,v)}{\partial v_i \partial v_j}. \tag{2.23}$$

Eq. (2.23) is in fact better behaved than (1.2) for rough coefficients as its symplectic structure for instance guarantees that it satisfies a maximum principle for all measure-valued solutions that are limit of smooth solutions. In particular for any initial data $u^0 \in L^\infty(\mathbb{R}^{2d})$, there exists a measure-valued solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^{2d})$. This is true even though the diffusion in (2.22) is degenerate (there is no diffusion in the x direction, and σ can also be degenerate).

Hence in this situation, one may deduce as claimed Corollary 1.4 or

Corollary 2.22 *Assume that $\sigma \in L^\infty \cap L^2_{t,loc}(H^1_x)$ and $F \in L^1_{t,loc}(W^{1,1}_x)$. Assume also that $u_0 \in L^\infty$. Then one has both existence of a strong solution to* (2.22) *and pathwise uniqueness among all solutions with marginal distributions in $L^\infty_{t,loc}(L^\infty_x)$.*

To conclude, let us observe that most of the previous results give strong existence for non-deterministic initial distributions. However, one can use the next result to obtain strong existence and pathwise uniqueness for almost all deterministic initial conditions.

Proposition 2.23 *Under the assumptions of either Cor. 2.15, Cor. 2.20 or Cor. 2.21, for any complete filtered probability space* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *equipped with a* r*-dimensional standard Brownian motion* W*, there is strong existence and pathwise uniqueness for* (1.1) *on* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W)$ *for almost all deterministic initial condition* $\xi = x \in \mathbb{R}^d$*.*

The proofs of the previous results are organized as follows. We start in Section 3 with some simple technical proofs, including those of Theorems 2.2, 2.6 and 2.10, Section 4 is then devoted to the proof of Theorem 2.13, Section 5 to the proof of Theorem 2.14, Section 6 to the proof of Proposition 2.18, and Section 7 to the proof of Proposition 2.23. The proof of Theorem 1.1 is given in the Appendix.

3 Useful technical results

The results and proofs presented in this section are mostly easy extensions of well-known techniques, which we need in the following sections and hence include here for the sake of completeness.

3.1 Pointwise difference estimates

We often need to estimate the difference of the coefficients σ of F at two different points x and y during the proofs. We collect here all the results which allow us to do so and that we later use. In all those estimates, time is only a parameter and we accordingly omit the time variable in most formulas.

We start by recalling the classical inequality (it is for instance a direct consequence of [24, Thm. VII.1] and of basic properties of the Poisson Kernel): there exists a constant C_d depending only on the dimension d such that for all $\sigma(t,\cdot) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and for all $x, y \in \mathbb{R}^d$,

$$|\sigma(t,x) - \sigma(t,y)| \leq C_d(M|\nabla_x \sigma|(t,x) + M|\nabla_x \sigma|(t,y))\,|x-y|. \tag{3.1}$$

This inequality extends easily by approximation arguments to all $\sigma(t,\cdot) \in BV_{\mathrm{loc}}(\mathbb{R}^d)$. We next turn to an extension with the operator M_L used in the definition (2.12).

Lemma 3.1 *Fix* $t \geq 0$ *and assume that* $F(t,\cdot) \in BV(\mathbb{R}^d)$*. Then for any* $x,\ y \in \mathbb{R}^d$

$$|F(t,x) - F(t,y)| \leq C_d(h(t,x) + h(t,y)) \left(|x-y| + \frac{1}{L}\right),$$

with $h(t,x) = |F(t,x)| + M_L \nabla F(t,x)$*, for some constant* C_d *that depends only on* d*.*

Proof First observe that by the definition of h, the result is obvious if $|x-y| \geq 1$. Assume now that $|x-y| \leq 1$. We recall the Lemma from [11].

Lemma 3.2 *Assume* $F \in BV(\mathbb{R}^d)$*. There exists a constant* C *depending only on* d *s.t. for any* $x, y \in \mathbb{R}^d$*,*

$$|F(x) - F(y)| \leq C \int_{\tilde{B}(x,y)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}} \right) |\nabla F|(dz), \qquad (3.2)$$

where $\tilde{B}(x,y)$ *denotes the ball of center* $(x+y)/2$ *and diameter* $|x-y|$*.*

Now $|\nabla F| \leq |\nabla F|_s + \sqrt{\log L}\,\lambda + |\nabla F|_a\, \mathbb{1}_{|\nabla F|_a \geq \sqrt{\log L}}$ where λ is Lebesgue's measure on $\mathbb{R}^d$ and where we identified $|\nabla F|_a$ with its density w.r.t. λ. Thus, if $1/L \leq |x-y| \leq 1$,

$$\frac{1}{|x-y|} \int_{\tilde{B}(x,y)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}} \leq C \left(\sqrt{\log L} + \int_{B(x,1)} \frac{|\nabla F|_a(z)\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\, dz + |\nabla F|_s(dz)}{(1/L + |x-z|)\, |x-z|^{d-1}} \right),$$

where we used that if $z \in B(x,y)$ then $|x-z| + 1/L \leq 2\,|x-y|$. Similarly, if $|x-y| \leq 1/L$,

$$\int_{\tilde{B}(x,y)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}}\, dz \leq \frac{C}{L} \left(\sqrt{\log L} + \int_{B(x,1)} \frac{|\nabla F|_a(z)\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\, dz + |\nabla F|_s(dz)}{(1/L + |x-z|)\, |x-z|^{d-1}} \right),$$

where we used that if $z \in B(x,y)$, then $|x-z| + 1/L \leq 2/L$. By the definition of M_L, this concludes the proof. □

Let us turn now to our last bound which uses $\partial_x^{1/2} \sigma$

Lemma 3.3 *Fix* $t \geq 0$ *and assume that* $\sigma(t,\cdot) \in L^1_{loc}$ *and* $\partial_x^{1/2} \sigma(t,\cdot)$ *is a locally finite Radon measure. Then for any* $x, y \in \mathbb{R}^d$

$$|\sigma(t,x) - \sigma(t,y)| \leq \left(M|\partial_x^{1/2}\sigma|(t,x) + M|\partial_x^{1/2}\sigma|(t,y) \right) |x-y|^{1/2}.$$

Proof By the definition of $\partial_x^{1/2}\sigma$

$$\sigma(x) = K \star \partial_x^{1/2}\sigma,$$

for the convolution kernel K with $\mathcal{F}K = |\xi|^{-1/2}$, which implies that

$$|K(x)| \leq C\,|x|^{d-1/2}, \qquad |\nabla K(x)| \leq C\,|x|^{d+1/2}. \tag{3.3}$$

Now simply compute

$$\begin{aligned}|\sigma(x) - \sigma(y)| \leq &\int_{|z-x|\geq 2\,|x-y|} |K(x-z) - K(y-z)|\,|\partial_x^{1/2}\sigma|(dz)\\ &+ \int_{|z-x|\leq 2\,|x-y|} (|K(x-z)| + |K(y-z)|)\,|\partial_x^{1/2}\sigma|(dz).\end{aligned}$$

Denote $|x-y| = r$. One has by (3.3)

$$\begin{aligned}\int_{|z-x|\leq 2\,r} |K(x-z)|\,|\partial_x^{1/2}\sigma|(dz) &\leq C \sum_{n\geq -1} \int_{|z-x|\leq 2^{-n}\,r} \frac{2^{n(d-1/2)}}{r^{d-1/2}}\,|\partial_x^{1/2}\sigma|(dz)\\ &\leq C \sum_{n\geq -1} 2^{-n/2} r^{1/2}\, M|\partial_x^{1/2}\sigma|(x) = C\,r^{1/2}\,M|\partial_x^{1/2}\sigma|(x).\end{aligned}$$

Since $|z-x| \leq 2r$ implies that $|z-y| \leq 3r$, one has the same inequality

$$\int_{|z-x|\leq 2\,r} |K(y-z)|\,|\partial_x^{1/2}\sigma|(dz) \leq C\,r^{1/2}\,M|\partial_x^{1/2}\sigma|(y).$$

As for the last term, first note that if $|x-z| \geq 2\,|x-y|$ then $|y-z| \geq |x-z|/2$. Hence by (3.3) if $|x-z| \geq 2\,|x-y|$

$$|K(x-z) - K(y-z)| \leq C\,\frac{|x-y|}{|x-z|^{d+1/2}}.$$

Therefore

$$\begin{aligned}&\int_{|z-x|\geq 2\,|x-y|} |K(x-z) - K(y-z)|\,|\partial_x^{1/2}\sigma|(dz)\\ &\qquad \leq \sum_{n\geq 1} \int_{|z-x|\geq 2^n\,r} C\,\frac{r}{(2^n r)^{d+1/2}}\,|\partial_x^{1/2}\sigma|(dz)\\ &\qquad \leq C\,r^{1/2} \sum_{n\geq 1} 2^{-n/2}\,M|\partial_x^{1/2}\sigma|(x) \leq C\,r^{1/2}\,M|\partial_x^{1/2}\sigma|(x).\end{aligned}$$

Summing up the three estimates concludes the proof. □

3.2 Proof of Theorems 2.2, 2.6 and 2.10

Proof of Theorem 2.2 First of all, $\|\cdot\|_{H^1(v)}$ is indeed a norm on $H^1(v)$. By definition it is non negative and finite on $H^1(v)$. Next if $\lambda > 0$ then $M(|\lambda f|) = \lambda M |f|$ and thus $\|\lambda f\|_{H^1(v)} = |\lambda| \, \|f\|_{H^1(v)}$. The triangle inequality is also trivially satisfied as $M(f+g) \le M f + M g$.

Finally if $\|f\|_{H^1(v)} = 0$ then $M|f| = 0$ on the support of v which contains (at least) one point x_0 since v is a probability measure. But now $M|f|(x_0) = 0$ implies that $f = 0$ by the definition of the maximal function.

We now prove (2.1). Consider a sequence f_n in $H^1(v)$ s.t. f_n converges to some f in the sense of distributions and assume (possibly restricting to a subsequence)

$$\sup_n \|f_n\|_{H^1(v)} < \infty.$$

(Otherwise, there is nothing to prove.)

We notice that f_n is hence uniformly bounded in BV_{loc}. Indeed for any $R > 0$, and any $x \in B(0,R)$

$$|\nabla f_n|(B(0,R)) \le (2\,R)^d \, M\,|\nabla f_n|(x),$$

so that by Cauchy-Schwartz

$$|\nabla f_n|(B(0,R)) \le \frac{2^d\,R^d}{\left(\int_{B(0,R)} v(dx)\right)^{1/2}} \|f_n\|_{H^1(v)}. \tag{3.4}$$

As $f_n \longrightarrow f$ in $\mathcal{D}'$ then f belongs to BV_{loc} as well. Therefore $M\,|\nabla f|$ is well defined.

On the other hand ∇f_n converges to ∇f in $\mathcal{D}'$. Note that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \ge 0$, the map $\mu \mapsto \int \varphi|\mu|$ is convex and continuous on the set of locally finite Radon measures on $\mathbb{R}^d$ for the strong topology of total variation. Hence it is lower semi-continuous for the weak-* topology, and so

$$\int \varphi\,|\nabla f|(dx) \le \liminf \int \varphi\,|\nabla f_n|(dx).$$

Now fix any $c > 1$ and any $r > 0$ and note that the previous inequality implies that

$$\begin{aligned}\frac{1}{|B(0,r)|}\int_{B(0,r)} |\nabla f|(x+dz) &\le \frac{1}{|B(0,r)|}\liminf \int_{B(0,cr)} |\nabla f_n|(x+dz)\\ &\le c^d \liminf M|\nabla f_n|(x).\end{aligned}$$

Taking now the supremum in r, we deduce that for any $c > 1$

$$M|\nabla f|(x) \le c^d \liminf M|\nabla f_n|(x).$$

Apply now Fatou's lemma and let c go to 1 to deduce

$$\int (M|\nabla f|(x))^2 u(dx) \le \liminf \int (M|\nabla f_n|(x))^2 u(dx).$$

The same steps can be performed with $M\,|f_n|$ and $M\,|f|$ thus proving that $f \in H^1(v)$ and that (2.1) holds.

Let us now prove that $H^1(v)$ is complete which concludes the proof that $H^1(v)$ is a Banach space. Accordingly consider any Cauchy sequence f_n in $H^1(v)$.

The sequence f_n is then also Cauchy in BV_{loc}. Indeed using (3.4) for $f_n - f_m$, we obtain that for any $R > 0$

$$|\nabla\,(f_n - f_m)|(B(0,R)) \le \frac{2^d\,R^d}{\left(\int_{B(0,R)} v(dx)\right)^{1/2}}\,\|f_n - f_m\|_{H^1(v)}.$$

Therefore there exists $f \in BV_{\text{loc}}$ s.t. f_n converges toward f in BV_{loc}. In particular f_n converges to f in $\mathcal{D}'$ and we may use (2.1) a first time to deduce that $f \in H^1(v)$.

It remains to show that $\|f_n - f\|_{H^1(v)} \longrightarrow 0$. For that fix n and consider the sequence $f_n - f_m$ in m. This sequence converges in the sense of distribution to $f_n - f$. We conclude using again (2.1) that

$$\|f_n - f\|_{H^1(v)} \le \liminf_{m\to\infty} \|f_n - f_m\|_{H^1(v)}.$$

Let us now turn to the last part of Thm. 2.2. We first recall that if μ is a finite, non-negative Radon measure then $M\,\mu$ is lower semicontinuous. This follows from similar arguments to the ones above: Consider any $x_n \to x$, then for $c > 1$

$$\begin{aligned}\frac{1}{|B(0,r)|}\int_{B(0,r)} \mu(x+dz) &\le \frac{1}{|B(0,r)|}\liminf \int_{B(0,cr)} \mu(x_n+dz)\\ &\le c^d \liminf M\,\mu(x_n).\end{aligned}$$

The lower semicontinuity of $M\,\mu$ then follows taking the supremum in r and then the infimum in c.

Denote now $g = (M\,|\nabla f|)^2 + (M\,|f|)^2$, g is a non negative, Borel function with values in $\mathbb{R}_+ \cup \{+\infty\}$. By the previous remark it is also lower semi-continuous. Note that for any positive measure μ

$$\int g\,d\mu = \int_0^\infty \int \mathbb{1}_{g(x)>\xi}\mu(dx)\,d\xi.$$

Now assume $v_n \to v$ in the tight topology of P^1. Note that for any open set O

$$\int_O dv \leq \liminf \int_O dv_n.$$

Take $O = \{g(x) > \xi\}$ which is open by the lower semi-continuity of g. Therefore, Fatou's lemma entails

$$\int g\,dv \leq \liminf \int g\,dv_n,$$

which finishes the proof of Thm. 2.2. $\square$

Proof of Theorem 2.6 The proof is nearly identical to that of Thm. 2.2 and for this reason we omit it here. The only difference is that the space BV is replaced by the space of L^1_{loc} functions f s.t. $\partial_x^{1/2} f$ is a locally finite measure. $\square$

Proof of Theorem 2.10 The first part of the proof concerning the lower semi-continuity follows exactly the same steps as the proof of Thm. 2.2. One uses the same intermediate control through the BV norm as, for all $R, L \geq 1$,

$$\begin{aligned} M_L\,|\nabla f|(x_0) &\geq \sqrt{\log L} + \frac{1}{R^{d-1}(R+L^{-1})}\int_{B(0,R)} \Big(|\nabla_a f|(z)\,\mathbb{1}_{|\nabla_a f|(z)\geq\sqrt{\log L}}\,dz + |\nabla_s f|(dz)\Big) \\ &\geq \frac{1}{C\,\sqrt{L}\,(1+R^d)}\int_{B(0,R)} |\nabla f|(dz). \end{aligned}$$

One also has the same type of lower semi-continuity properties as for instance if $f_n \to f$ in the sense of distribution for f_n a sequence uniformly bounded in BV_{loc} then for any $L' < L$

$$M_{L'}\nabla f(x) \leq \liminf_n M_L\,\nabla f_n(x).$$

Taking the supremum over L leads to (2.11) as ϕ is continuous.

We skip the rest of the details for this first part and instead focus on the connection with $W^{1,1}$ which is the main novel feature of $W^{\phi,\text{weak}}$.

By contradiction assume that $f \in W^{\phi,\text{weak}}(v)$ and $v \geq 1/C$ over Ω but that $f \notin W^{1,1}(B(x_0,r))$ for some ball s.t. $B(x_0,2r) \subset \Omega$. Since $f \in BV_{\text{loc}}$, it implies that the singular part $|\nabla_s f|$ does not vanish on $B(x_0,r)$. On the other hand

$$\int_{\mathbb{R}^d} M_L \nabla f \, v(dx) \geq \frac{1}{C} \iint_{B(x_0,2r)^2} \frac{|\nabla_s f|(dz)}{(L^{-1}+|z-x|)\,|z-x|^{d-1}}\,dx.$$

Define the kernel

$$K_L = C_L \, \frac{\mathbb{1}_{|x|\leq 2r}}{(L^{-1}+|x|)\,|x|^{d-1}},$$

with C_L s.t. $\|K_L\|_{L^1}=1$. Observe that K_L is a standard approximation of the identity by convolution so in particular

$$\liminf_{L\to\infty} \int_{B(x_0,2r)} K_L \star (|\nabla_s f|)\,dx \geq \int_{B(x_0,r)} |\nabla_s f|(B(x_0,r)) > 0.$$

As $C_L \sim \log L$, this has for consequence that there exists $C>0$ s.t. for L large enough

$$\int_{B(x_0,2r)^2} \frac{|\nabla_s f|(dz)}{(L^{-1}+|z-x|)\,|z-x|^{d-1}}\,dx \geq \frac{\log L}{C}.$$

Therefore

$$\|f\|_{W^{\phi,\text{weak}}(v)} \geq \frac{1}{C}\sup_L \frac{\phi(L)}{L} = +\infty,$$

giving the desired contradiction.

Reciprocally, assume that $v \leq C$ on Ω and that $f \in W^{1,1}(K)$ compactly supported in $K \subset \Omega$. First, by Sobolev embedding, f and hence $M\,f$ belong to L^p for some $p>1$ and $M\,f \in L^\infty(\Omega^c)$. Therefore

$$\int M\,|f|(x)\,v(dx) < \infty.$$

Then for $x \notin \Omega$

$$M_L \nabla f(x) \leq \sqrt{\log L} + \frac{1}{d(x,K)^d} \int_K |\nabla f(z)|\,dz.$$

As a consequence for any ϕ satisfying (2.10), there exists some finite constant C_ϕ s.t.

$$\|f\|_{W^{\phi,\text{weak}}} \leq C_\phi + C \sup_L \frac{\phi(L)}{L\,\log L} \int_\Omega M_L \nabla f(x)\,dx.$$

Now decompose ∇f in level sets by defining for all $n \in \mathbb{Z}$

$$\omega_n = \{z \in K,\ 2^n \leq |\nabla f(z)| < 2^{n+1}\}.$$

Then

$$\begin{aligned}\int_\Omega M_L \nabla f(x)\,dx &\leq |\Omega|\,\sqrt{\log L} + \sum_{n \geq \log_2 L - 1} \iint_{\Omega\times K} \frac{2^{n+1}\,\mathbb{1}_{z\in\omega_n}\,dz\,dx}{(L^{-1}+|z-x|)\,|z-x|^{d-1}} \\ &\leq |\Omega|\,\sqrt{\log L} + \sum_{n\geq \log_2 L-1} 2^{n+1}\,|\omega_n|\,\log L.\end{aligned}$$

Since $\nabla f \in L^1$, one has $\sum 2^n\,|\omega_n| < \infty$ and thus

$$s_N = \sum_{n\geq N} 2^n\,|\omega_n| \longrightarrow 0, \quad \text{as } N\to\infty.$$

We can now define an appropriate ϕ: Choose any smooth function s.t. $\phi(x)/x$ is non-decreasing and

$$\phi(2^{N+1}) = 2^{N+1}\,\min\left(N^{1/4},\ s_N^{-1}\right).$$

Then ϕ satisfies (2.10) while

$$\sup_L \frac{\phi(L)}{L\,\log L} \int_\Omega M_L \nabla f(x)\,dx \leq 2\,\sup_N \frac{\phi(2^{N+1})}{2^N}\,s_N \leq 4,$$

therefore concluding that $f \in W^{\phi,\text{weak}}(v)$. □

4 Proof of Theorem 2.13

We use two types of estimates; one is based on an explicit quantitative estimate which generalizes the one in [4] for Ordinary Differential Equations and one which generalizes the local time which is used in dimension 1 in the classical approach [28, 20, 7]. We use the first quantitative estimate to prove existence and the second one to prove uniqueness (though with suitable modifications any one could be used for both existence and uniqueness).

The first method is more precise but slightly more complicated than the second.

4.1 Existence

We consider the sequence of solutions to the regularized problem (1.3), and assume it satisfies the assumptions of Th. 2.13. The proof is based on estimates on the expectation of the family of quantities

$$Q^{(\varepsilon)}_{nm}(t) = \log\left(1 + \frac{|X^n_t - X^m_t|^2}{\varepsilon^2}\right), \quad \varepsilon \in (0,1], \quad n,m \geq 1, \tag{4.1}$$

given in the next lemma.

Lemma 4.1 *There exists a constant C such that, for all $0 < \varepsilon \leq 1$ and $n, m \geq 1$,*

$$\sup_{t\in[0,T]} \mathbb{E}(Q^{(\varepsilon)}_{nm}(t)) \leq C\,(1 + |\log\varepsilon|\,\tilde\eta(\varepsilon)) + C\,\frac{\eta(n,m)}{\varepsilon^2}, \tag{4.2}$$

where $\eta(n,m) \to 0$ when $n,m \to +\infty$ and $\tilde\eta(\varepsilon) := (\varepsilon\phi(\varepsilon^{-1}))^{-1} \to 0$ when $\varepsilon \to 0$.

Proof Note that

$$|\nabla(\log(1 + |x|^2/\varepsilon))| = \left|\frac{2x}{\varepsilon^2 + |x|^2}\right| \leq \frac{C}{\varepsilon + |x|}$$

and

$$|\nabla^2(\log(1 + |x|^2/\varepsilon))| = \left|\nabla\left(\frac{2x}{\varepsilon^2 + |x|^2}\right)\right| \leq \frac{C}{\varepsilon^2 + |x|^2}.$$

By Itô's formula, for any C^2_b function f,

$$\begin{aligned}\mathbb{E}(f(X^n_t - X^m_t)) =& f(0) + \frac{1}{2}\int_0^t \mathbb{E}\Big[\nabla^2 f(X^n_s - X^m_s)\,(\sigma_n\sigma_n^*(X^n_s) \\ &+ \sigma_m\sigma^*_m(X^m_s) - \sigma_n(X^n_s)\sigma_m(X^m_s) - \sigma_m(X^m_s)\sigma^*_n(X^n_s))\Big]\,ds \\ &+ \int_0^t \mathbb{E}(\nabla f(X^n_s - X^m_s)\cdot(F_n(s,X^n_s) - F_m(s,X^m_s)))\,ds.\end{aligned}$$

Since $\sup_n(\|\sigma_n\|_\infty + \|F_n\|_\infty) < +\infty$, we deduce

$$\begin{aligned}\mathbb{E}(f(X^n_t - X^m_t)) \leq& f(0) + \frac{1}{2}\int_0^t \mathbb{E}\Big[|\nabla^2 f(X^n_s - X^m_s)\,\Big(|\sigma(X^n_s) - \sigma(X^m_s)|^2 \\ &+ \sup_k \|\sigma_k\|_{L^\infty}\,(|\sigma_n(X^n_s) - \sigma(X^n_s)| + |\sigma_m(X^m_s) - \sigma(X^m_s)|)\Big)\Big]\,ds \\ &+ \int_0^t \mathbb{E}(|\nabla f(X^n_s - X^m_s)|\,|F_n(s,X^n_s) - F_m(s,X^m_s)|)\,ds.\end{aligned} \tag{4.3}$$

Hence

$$\mathbb{E}(Q_{nm}^{(\varepsilon)}(t)) \leq C \int_0^t \mathbb{E}\left(\frac{|\sigma(s, X_s^n) - \sigma(s, X_s^m)|^2}{\varepsilon^2 + |X_s^n - X_s^m|^2}\right) ds + C\, \frac{\eta(n,m)}{\varepsilon^2}$$
$$+ C \int_0^t \mathbb{E}\left(\frac{|F(s, X_s^n) - F(s, X_s^m)|}{\varepsilon + |X_t^n - X_t^m|}\right) ds, \tag{4.4}$$

with C a constant independent of n and ε and $\eta(n,m) \to 0$ as n, $m \to \infty$ by Assumption (2.14).

Since $\|\sigma\|_{H_T^1(u_n)} + \|\sigma\|_{H_T^1(u_m)} < \infty$, denoting $h = M|\nabla \sigma|$,

$$\int_0^T \int h^2(t,x)\,(u_n(t,dx) + u_m(t,dx))\,dt \leq \|\sigma\|_{H_T^1(u_n)} + \|\sigma\|_{H_T^1(u_m)} \leq C,$$

with C independent of n, m and ε. Now, it follows from (3.1) that

$$\int_0^t \mathbb{E}\left(\frac{|\sigma(s, X_s^n) - \sigma(s, X_s^m)|^2}{\varepsilon^2 + |X_s^n - X_s^m|^2}\right) ds \leq C \int_0^t \mathbb{E}(h^2(s, X_s^n) + h^2(s, X_s^m))\,ds,$$

and so

$$\int_0^t \mathbb{E}\left(\frac{|\sigma(s, X_s^n) - \sigma(s, X_s^m)|^2}{\varepsilon^2 + |X_s^n - X_s^m|^2}\right) ds \leq C.$$

We now turn to the term involving F and introduce the corresponding $h = |F| + M_{1/\varepsilon}\nabla F$.

By Lemma 3.1

$$\int_0^t \mathbb{E}\left(\frac{|F(s, X_n^s) - F(s, X_m^s)|}{\varepsilon + |X_n^s - X_m^s|}\right) ds$$
$$\leq C \int_0^t \int h(s,x)\,(u_n(s,x) + u_m(s,x))\,dx\,ds,$$

and by (2.12),

$$\int_0^t \int h(s,x)\,(u_n(s,x) + u_m(s,x))\,dx\,ds$$
$$\leq \frac{1/\varepsilon \log(1/\varepsilon)}{\phi(1/\varepsilon)} \left(\|F\|_{W_T^{\phi,\mathrm{weak}}(u_n)} + \|F\|_{W_T^{\phi,\mathrm{weak}}(u_m)}\right) \leq C\, \frac{|\log \varepsilon|}{\varepsilon\, \phi(\varepsilon^{-1})}.$$

Note that we used the inequality $|F| \leq M|F|$ a.e., which follows from Lebesgue's points theorem since $BV_{\mathrm{loc}}(\mathbb{R}^d) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The function $\tilde{\eta}(\varepsilon) = (\varepsilon\, \phi(\varepsilon^{-1}))^{-1} \to 0$ as $\varepsilon \to 0$ since ϕ is super-linear.

Combining the previous inequalities, we obtain (4.2). $\square$

Fix $p > 1$. The next step consists in deducing from Lemma 4.1 that $(X_t^n - \xi)$ is a Cauchy sequence in $L^p(\Omega, L^\infty([0,T]))$. Since F_n and σ_n are uniformly bounded, it is standard to deduce from the Burkholder-Davis-Gundy inequality that $X_t^n - \xi$ are uniformly bounded in $L^p(\Omega,\ L^\infty([0,\ T]))$ for all $p > 1$, so we only need to prove the next lemma.

Lemma 4.2 *For all $p > 1$,*

$$\mathbb{E}\left(\sup_{t\in[0,\ T]} |X_t^n - X_t^m|^p\right) \longrightarrow 0 \qquad \text{as } n,\ m \to +\infty. \tag{4.5}$$

Proof For fixed t, for any ε and L s.t. $0 < \varepsilon < L$,

$$\begin{aligned}\mathbb{E}(|X_t^n - X_t^m|^p) \leq & \mathbb{E}(|X_t^n - X_t^m|^p; |X_t^n - X_t^m| \geq L) + \varepsilon^{p/2} \\ & + L^p \mathbb{P}(|X_t^n - X_t^m| \geq \sqrt{\varepsilon}).\end{aligned}$$

Note that

$$\mathbb{E}(|X_t^n - X_t^m|^p; |X_t^n - X_t^m| \geq L) \leq \frac{1}{L}(\mathbb{E}(|X_t^n - \xi|^{p+1}) + \mathbb{E}(|X_t^m - \xi|^{p+1})). \tag{4.6}$$

Now, the inequalities

$$\sup_{n\geq 1, t\in[0,T]} \mathbb{E}(|X_t^n - \xi|^{p+1}) < +\infty.$$

and

$$\mathbb{P}(|X_t^n - X_t^m| \geq \sqrt{\varepsilon}) \leq \frac{\mathbb{E}Q_{nm}^{(\varepsilon)}(t)}{|\log \varepsilon|}.$$

imply that

$$\mathbb{E}(|X_t^n - X_t^m|^p) \leq C\left[\frac{1}{L} + \varepsilon^{p/2} + \frac{L^p}{|\log \varepsilon|}\left(1 + |\log \varepsilon|\,\tilde{\eta}(\varepsilon) + \frac{\eta(n,m)}{\varepsilon^2}\right)\right].$$

Taking for example $\varepsilon^2 = \eta(n,m)$ and $L = \left(\frac{1}{|\log \varepsilon|} + \tilde{\eta}(\varepsilon)\right)^{-1/2p}$, one concludes that

$$\sup_{t\in[0,T]} \mathbb{E}(|X_t^n - X_t^m|^p) \to 0 \qquad \text{as } n, m \to +\infty.$$

In order to pass the supremum inside the expectation, it suffices to observe that the computation of (4.3–4.4) in the proof of Lemma 4.1 can be applied to $|A^n_{t\wedge\tau}-A^m_{t\wedge\tau}|^2\vee|M^n_{t\wedge\tau}-M^m_{t\wedge\tau}|^2$, where τ is any stopping time and $X^n_t=\xi+A^n_t+M^n_t$ is Doob's decomposition of the semi martingale X^n_t, i.e.

$$A^n_t=\int_0^t F(s,X^n_s)ds \qquad \text{and} \qquad M^n_t=\int_0^t \sigma(s,X^n_s)dW_s.$$

Note that to be fully rigorous, one first needs to regularize the supremum $\vee$.

Instead of (4.4), we obtain

$$\begin{aligned}\mathbb{E}\log&\left(1+\frac{|A^n_{t\wedge\tau}-A^m_{t\wedge\tau}|^2\vee|M^n_{t\wedge\tau}-M^m_{t\wedge\tau}|^2}{\varepsilon^2}\right)\\ &\le C\int_0^t\mathbb{E}\left(\frac{|\sigma(s,X^n_s)-\sigma(s,X^m_s)|^2}{\varepsilon^2+|A^n_t-A^m_t|^2\vee|M^n_t-M^m_t|^2}\right)ds+C\,\frac{\eta(n,m)}{\varepsilon^2}\\ &\quad+C\int_0^t\mathbb{E}\left(\frac{|F(s,X^n_s)-F(s,X^m_s)|}{\varepsilon+|A^n_t-A^m_t|\vee|M^n_t-M^m_t|}\right)ds,\end{aligned}$$

or

$$\begin{aligned}\mathbb{E}\log&\left(1+\frac{|A^n_{t\wedge\tau}-A^m_{t\wedge\tau}|^2\vee|M^n_{t\wedge\tau}-M^m_{t\wedge\tau}|^2}{\varepsilon^2}\right)\\ &\le C\int_0^t\mathbb{E}\left(\frac{|\sigma(s,X^n_s)-\sigma(s,X^m_s)|^2}{\varepsilon^2+\frac{1}{4}|X^n_s-X^m_s|^2}\right)ds+C\,\frac{\eta(n,m)}{\varepsilon^2}\\ &\quad+C\int_0^t\mathbb{E}\left(\frac{|F(s,X^n_s)-F(s,X^m_s)|}{\varepsilon+\frac{1}{2}|X^n_t-X^m_t|}\right)ds.\end{aligned}$$

Therefore, the same computation as in Lemma 4.1 gives

$$\sup_{t\in[0,T],\ \tau\text{ stopping time}}\mathbb{E}(|A^n_{t\wedge\tau}-A^m_{t\wedge\tau}|^p\vee|M^n_{t\wedge\tau}-M^m_{t\wedge\tau}|^p)\to 0 \quad \text{as } n,m\to+\infty.$$

Since $p>1$, Doob's inequality entails

$$\mathbb{E}(\sup_{t\in[0,T]}|M^n_t-M^m_t|^p)\to 0 \quad \text{as } n,m\to+\infty.$$

Fix $\eta>0$, and fix n_0 such that

$$\sup_{t\in[0,T],\ \tau\text{ stopping time}}\mathbb{E}(|A^n_{t\wedge\tau}-A^m_{t\wedge\tau}|^p)\le\eta$$

for all $n, m \geq n_0$. For all $M > 0$, let $\tau = \inf\{t \geq 0 : |A_t^n - A_t^m| \geq M\}$. Then

$$\mathbb{P}(\sup_{t\in[0,T]} |A_t^n - A_t^m| \geq M) = \mathbb{P}(\tau \leq T) \leq \frac{\eta}{M^p}.$$

Now, for all $1 < q < p$,

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,T]} |A_t^n - A_t^m|^q) &= q \int_0^{+\infty} x^{q-1} \mathbb{P}(\sup_{t\in[0,T]} |A_t^n - A_t^m| \geq x) dx \\ &\leq q \int_0^{+\infty} x^{q-1} \left(\frac{\eta}{x^p} \wedge 1\right) dx = \frac{p\ \eta^{q/p}}{p-q}.\end{aligned}$$

Therefore

$$\mathbb{E}(\sup_{t\in[0,T]} |A_t^n - A_t^m|^q) \to 0 \quad \text{ as } n, m \to +\infty,$$

which concludes the proof of (4.5). □

From the fact that $(X^n - \xi)$ is a Cauchy sequence in $L^p(\Omega, L^\infty([0,T]))$, it is standard to deduce the almost sure convergence for the L^∞ norm of a subsequence of $(X_t^n, t \in [0,T])_n$ to a process $(X_t, t \in [0,T])$ such that $(X_t - \xi, t \in [0,T]) \in L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$. Since the convergence holds for the L^∞ norm, the process X is a.s. continuous and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Since u_n converges to u in the weak topology of M_1, we have for all bounded continuous function f on $[0,T] \times \mathbb{R}^d$

$$\mathbb{E} \int_0^T f(t, X_t) dt = \int_{\mathbb{R}^d} \int_0^T f(t, x) u(dt, dx),$$

so $u(t, dx)$ is the law of X_t for almost all t.

Defining for all $t \in [0,T]$

$$Y_t := \int_0^t F(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

it only remains to check that $Y_t = X_t - \xi$ for all $t \in [0,T]$ a.s. As

$$X_t^n - \xi = \int_0^t F_n(s, X_s^n) ds + \int_0^t \sigma_n(s, X_s^n)\, dW_s,$$

one has $Y_t = X_t$ provided that

$$\int_0^t \mathbb{E}(|F_n(s, X_s^n) - F(s, X_s)| + |\sigma_n(s, X_s^n) - \sigma(x, X_s)|^2)\, ds \longrightarrow 0.$$

From the assumption (2.14) and the L^∞ bounds on F, σ and σ_n, this is implied by: For any fixed $\varepsilon > 0$,

$$\int_0^T \Big[\mathbb{P}(|F(s,X^n_s) - F(s,X_s)| > \varepsilon) + \mathbb{P}(|\sigma(s,X^n_s) - \sigma(x,X_s)| > \varepsilon)\Big]\, ds \longrightarrow 0.$$

We prove it for σ, the argument for F being fully similar.

By Cor. 2.4

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (M|\nabla\sigma(t,x)|)^2\,(u(t,dx) + u_n(t,dx))\,dt \\ \leq \|\sigma\|_{H^1_T(u_n)} + \liminf \|\sigma\|_{H^1_T(u_n)} \leq C.\end{aligned} \tag{4.7}$$

Now by (3.1)

$$\begin{aligned}&\mathbb{P}(|\sigma(s,X^n_s) - \sigma(s,X_s)| > \varepsilon)\\ &\leq \mathbb{P}((M|\nabla\sigma|(s,X^n_s) + M|\nabla\sigma|(s,X_s)) > \varepsilon/|X^n_s - X_s|)\\ &\leq \mathbb{P}(|X^n_s - X_s| > \varepsilon^2) + \mathbb{P}(M|\nabla\sigma|(s,X^n_s) \geq \frac{1}{2\varepsilon}) + \mathbb{P}(M|\nabla\sigma|(s,X_s) \geq \frac{1}{2\varepsilon}),\end{aligned}$$

and one easily concludes from (4.7) and the fact that $|X^n_s - X_s| \longrightarrow 0$ almost surely.

4.2 Uniqueness

Consider two solutions X and Y satisfying the assumptions of point (ii) in Th. 2.13. Define a family of functions $(L_\varepsilon)_\varepsilon$ in $C^\infty(\mathbb{R}^d)$ satisfying

$$L_\varepsilon(x) = 1 \text{ if } |x| \geq \varepsilon, \quad L_\varepsilon(x) = 0 \text{ if } |x| \leq \varepsilon/2, \quad \varepsilon\,\|\nabla L_\varepsilon\|_{L^\infty} + \varepsilon^2\,\|\nabla^2 L_\varepsilon\| \leq C,$$

with C independent of ε, and $L_\varepsilon(x) \geq L_{\varepsilon'}(x)$ for all $\varepsilon \leq \varepsilon'$ and $x \in \mathbb{R}^d$. Use Itô's formula

$$\begin{aligned}\mathbb{E}(L_\varepsilon(X_t - Y_t)) =& L(0) + \int_0^t \mathbb{E}\left(\nabla L_\varepsilon(X_s - Y_s)\cdot(F(s,X_s) - F(s,Y_s)\right)\, ds\\ &+ \int_0^t \mathbb{E}\Big(\nabla^2 L_\varepsilon(X_s - Y_s) : (\sigma\sigma^*(X_s)\\ &+ \sigma\sigma^*(Y_s) - \sigma(X_s)\sigma^*(Y_s) - \sigma(Y_s)\sigma^*(X_s))\Big)\, ds.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}(L_\varepsilon(X_t - Y_t)) \leq C \int_0^t \mathbb{E}\Big(\mathbb{1}_{\varepsilon/2 \leq |X_t - Y_t| \leq \varepsilon}\Big(\frac{|\sigma(s,X_s) - \sigma(s,Y_s)|^2}{\varepsilon^2}\\ + \frac{|F(s,X_s) - F(s,Y_s)|}{\varepsilon}\Big)\Big)\, ds.\end{aligned}$$

Now denote $h = M|\nabla\sigma|$ so that

$$\int_0^T \int |h(t,x)|^2 \,(u_X(t,dx) + u_Y(t,dx))\,dt \le C < \infty.$$

Define as well $\widetilde{h}_\varepsilon = |F| + M_{1/\varepsilon}\nabla F$ s.t.

$$\int_0^T \int \widetilde{h}_\varepsilon\,(u_X + u_Y)\,dx\,ds \le \frac{C\,|\log\varepsilon|}{\varepsilon\,\phi(\varepsilon^{-1})}.$$

The corresponding computation involving $\widetilde{h}_\varepsilon$ is now tricky, precisely because of the dependence on ε in $\widetilde{h}_\varepsilon$. To simplify it, we will use a slightly different definition.

First note that, as observed in Section 2.1.3, one can always assume that ϕ satisfies (2.10), and so $\phi(\xi)/\xi$ is a non-decreasing function which grows note faster than $\log\xi$. In particular, there exists a constant $C > 0$ s.t.

$$\frac{1}{C}\varepsilon\,\phi(\varepsilon^{-1}) \le \frac{\phi(\xi)}{\xi} \le C\,\varepsilon\,\phi(\varepsilon^{-1}) \quad \forall\xi \in [\varepsilon^{-1/2},\ \varepsilon^{-1}].$$

Consider the partition of $(0,1) = \bigcup_i I_i$ where the $I_i = [a_i,\ b_i)$ are disjoint with $b_i = \sqrt{a_i}$ (except for $I_0 := [1/2, 1)$). In particular, $|I_i| := b_i - a_i$ satisfies

$$|I_i| \sim \sqrt{a_i} \quad \text{when } i \to +\infty.$$

Now for any $\varepsilon \in I_i$, choose $\bar{h}_\varepsilon = \widetilde{h}_{a_i}$. One has

$$\int_0^T \int \bar{h}_\varepsilon(t,x)\,(u_X(t,x) + u_Y(t,x))\,dx\,dt \le C\,\frac{|\log\varepsilon|}{\varepsilon\,\phi(\varepsilon)} \le C'\,\frac{|\log b_i|}{b_i\,\phi(b_i^{-1})}.$$

Now by (3.1) and Lemma 3.1

$$\begin{aligned}\mathbb{E}(L_\varepsilon(X_t - Y_t)) \le& C\int_0^t \mathbb{E}\left[(h^2(s,X_s) + h^2(s,Y_s))\mathbb{1}_{\varepsilon/2\le|X_t-Y_t|\le\varepsilon}\right]\,ds\\ &+ C\int_0^t \mathbb{E}\left[(\bar{h}_\varepsilon(s,X_s) + \bar{h}_\varepsilon(s,Y_s))\,\mathbb{1}_{\varepsilon/2\le|X_t-Y_t|\le\varepsilon}\right]\,ds.\end{aligned}$$

Denote

$$\alpha_k = \int_0^t \mathbb{E}\left[(h^2(s,X_s) + h^2(s,Y_s))\mathbb{1}_{2^{-k-1}\le|X_t-Y_t|\le 2^{-k}}\right]\,ds.$$

Note that

$$\sum_k \alpha_k \le \int_0^t \mathbb{E}\left((h^2(s,X_s)+h^2(s,Y_s)\right)\, ds$$
$$= \int_0^t \int h^2(s,x)\,(u_X(dx,s)+u_Y(dx,s))\,ds \le C.$$

Therefore $\alpha_k \longrightarrow 0$ as $k \to +\infty$.

Denote similarly

$$\beta_k = \int_0^t \mathbb{E}\left((\bar{h}_{2^{-k}}(s,X_s)+\bar{h}_{2^{-k}}(s,Y_s))\,\mathbb{1}_{2^{-k-1}\le|X_t-Y_t|\le 2^{-k}}\right)\, ds.$$

Denote $J_i = \{k,\ [2^{-k-1},\ 2^{-k}) \subset I_i\}$. Note that $|J_i| \ge \frac{1}{C}\,|\log b_i|$ (in fact, $|J_i| = \frac{|\log b_i|}{2\log 2}$) and since $\bar{h}_\varepsilon$ is fixed on $\varepsilon \in I_i$

$$\frac{1}{|J_i|}\sum_{k\in J_i}\beta_k \le \frac{1}{|J_i|}\int_0^t \int \bar{h}_{b_i}(s,x)\,(u_X(dx,s)+u_Y(dx,s))\,ds$$
$$\le \frac{C}{b_i\,\phi(b_i^{-1})} \longrightarrow 0 \quad \text{as } i\to\infty.$$

Therefore $\beta_{n_k} \longrightarrow 0$ as $k \to +\infty$ for some subsequence $n_k \to +\infty$. Consequently, since the sequence of functions L_ε is non increasing,

$$\sup_{t\in[0,\ T]} \mathbb{E}(L_\varepsilon(X_t-Y_t)) \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

On the other hand

$$\mathbb{E}(L_\varepsilon(X_t-Y_t)) \ge \mathbb{P}(|X_t-Y_t|>\varepsilon),$$

and by taking the limit $\varepsilon \to 0$, we deduce that for any $t\in[0,T]$

$$\mathbb{P}(|X_t-Y_t|>0)=0.$$

Since X_t and Y_t have a.s. continuous paths, we finally deduce that

$$\mathbb{P}(\sup_{t\in[0,T]}|X_t-Y_t|=0)=1.$$

5 Proof of Theorem 2.14

This proof follows exactly the same steps as the general multi-dimensional case given in Section 4. The only differences are the functionals used and accordingly we skip the other parts of the proof which are identical.

Technically the reason why the one dimensional case is so special is that $|x|$ is linear except at $x=0$ (see Section 5.2).

5.1 Existence

For $d = 1$, we replace the functional $Q_{nm}^{(\varepsilon)}$ by

$$\tilde{Q}_{nm}^{(\varepsilon)}(t) = e^{-U_t^{n,m}}\, |X_t^n - X_t^m| \,\log\left(1 + \frac{|X_t^n - X_t^m|^2}{\varepsilon^2}\right),$$

for $U_t^{n,m}$ a nonnegative stochastic process with bounded variation satisfying $dU_t^{n,m} = \lambda_t^{n,m}\, dt$ with $\lambda_t^{n,m}$ an adapted process (measurable function of a continuous, adapted process) to be chosen later.

Note that $f(x) = |x| \,\log(1 + |x|^2/\varepsilon^2)$ satisfies

$$|f'(x)| \leq 4 \log\left(1 + \frac{|x|^2}{\varepsilon^2}\right) \qquad \text{and} \qquad |f''(x)| \leq \frac{C}{\varepsilon + |x|}.$$

Therefore by Itô's formula

$$\begin{aligned}\mathbb{E}(\tilde{Q}_{nm}^{(\varepsilon)}(t)) \leq & C + C \int_0^t \mathbb{E}\left(\frac{|\sigma(X_s^n) - \sigma(X_s^m)|^2}{\varepsilon + |X_s^n - X_s^m|}\right) ds + \frac{\eta(n,m)}{\varepsilon} \\ & + \int_0^t \mathbb{E}\Bigg(|X_s^n - X_s^m| \,\log(1 + |X_s^n - X_s^m|^2/\varepsilon^2) \\ & \qquad \left(4\, \frac{|F(s, X_s^n) - F(s, X_s^m)|}{|X_s^n - X_s^m|} - \lambda_t^{n,m}\right)\Bigg)\, ds.\end{aligned}$$

The first term is treated identically as for the multi-dimensional case. The only difference here is that the careful choice of $\tilde{Q}_{nm}^{(\varepsilon)}$ improved the exponent of $|X_s^n - X_s^m|$ to 1 instead of 2 in the denominator. Therefore this term can be controlled with the $H_T^{1/2}(u_{n,m})$ norm of σ by using Lemma 3.3 instead of estimate (3.1).

The drawback is that the term with F must be dealt with differently. We introduce $\tilde{h} = M|\nabla F|$ s.t.

$$\int_0^T \int_{\mathbb{R}^d} \tilde{h}(t,x)\, (u_m(t,dx) + u_n(t,dx))\, dt \leq C.$$

We then choose

$$\lambda_t^{n,m} = 4\, \left(\tilde{h}(t, X_s^m) + \tilde{h}(t, X_t^m)\right).$$

Therefore we deduce that

$$\sup_{t \leq T} \mathbb{E}(\tilde{Q}_{nm}^{(\varepsilon)}(t)) \leq C + \frac{\eta(n,m)}{\varepsilon}.$$

Using a similar method as in Thm. 2.13, we write for constants L and K to be chosen later

$$\mathbb{E}(|X_t^n - X_t^m|^p) \le \mathbb{E}(|X_t^n - X_t^m|^p;\ |X_t^n - X_t^m| \ge L) + \frac{1}{|\log\varepsilon|^{p/2}} + \mathbb{P}(U_t^{n,m} \ge \log K) + L^p \mathbb{P}\left(|X_t^n - X_t^m| \ge \frac{1}{\sqrt{|\log\varepsilon|}};\ U_t^{n,m} \le \log K\right)$$

Note that

$$\mathbb{E}(U_t^{n,m}) = \mathbb{E}\left(\int_0^t \lambda_s^{n,m}\,ds\right) \le 4\int_0^t \tilde{h}(s,x)\,(u_n(s,dx) + u_m(s,dx))\,ds \le C.$$

Consequently

$$\mathbb{P}(U_t^{n,m} \ge \log(K)) \le \frac{C}{\log K}.$$

In addition, for ε small enough,

$$\mathbb{P}\left(|X_t^n - X_t^m| \ge \frac{1}{\sqrt{|\log\varepsilon|}};\ U_t^{n,m} \le \log K\right) \le \frac{K\,\mathbb{E}\tilde{Q}_{nm}^{(\varepsilon)}(t)}{2\sqrt{|\log\varepsilon|}}.$$

Therefore, using (4.6) as in the proof of Lemma 4.2,

$$\mathbb{E}(|X_t^n - X_t^m|^p) \le C\left(\frac{1}{L} + \frac{1}{|\log\varepsilon|^{p/2}} + \frac{1}{\log K} + \frac{L^p\,K\left(1 + \frac{\eta(n,m)}{\varepsilon}\right)}{\sqrt{|\log\varepsilon|}}\right).$$

Taking for example $\varepsilon = \eta(n,m)$, $K = |\log\varepsilon|^{1/8}$ and $L = |\log\varepsilon|^{1/8p}$, we deduce that

$$\sup_{t\in[0,T]} \mathbb{E}(|X_t^n - X_t^m|^p) \to 0, \qquad \text{as } n,m \to +\infty.$$

The rest of the proof is similar.

5.2 Uniqueness

For simplicity, we assume here that $F = 0$. Otherwise it is necessary to introduce U_t as in the previous subsection but it is handled in exactly the same way.

We similarly change the definition of L_ε in

$$\tilde{L}_\varepsilon(x) = |x| \text{ if } |x| \ge \varepsilon, \quad \tilde{L}_\varepsilon(x) = 0 \text{ if } |x| \le \varepsilon/2, \quad \|\nabla \tilde{L}_\varepsilon\|_{L^\infty} + \varepsilon\,\|\nabla^2 \tilde{L}_\varepsilon\| \le C,$$

with C independent of ε.

Applying Itô's formula

$$\mathbb{E}(\tilde{L}_\varepsilon(X_t - Y_t)) \leq C \int_0^t \mathbb{E}\left(\mathbb{1}_{\varepsilon/2 \leq |X_t - Y_t| \leq \varepsilon} \frac{|\sigma(X_s) - \sigma(Y_s)|^2}{\varepsilon}\right) ds.$$

By using as before the assumptions, Lemma 3.3 and the corresponding definition of $H_T^{1/2}(u_X)$ and $H_T^{1/2}(u_Y)$, one deduces that

$$\mathbb{E}(\tilde{L}_\varepsilon(X_t - Y_t)) \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

This is slightly less strong than before ($L_\varepsilon(\varepsilon) \gg \tilde{L}_\varepsilon(\varepsilon)$ when $\varepsilon \to 0$) but still enough. In particular one has if $\alpha \geq \varepsilon$

$$\mathbb{P}(|X_t - Y_t| \geq \alpha) \leq \frac{1}{\alpha}\, \mathbb{E}(\tilde{L}_\varepsilon(X_t - Y_t)).$$

Therefore by taking $\varepsilon \to 0$, one still obtains that for any $t \in [0, T]$,

$$\mathbb{P}(|X_t - Y_t| > 0) = 0,$$

which allows to conclude as before.

6 Proof of Prop. 2.18

We simply use the energy estimates. The computations below are formal but could easily be made rigorous by taking a regularization of σ, F and hence a and then pass to the limit.

$$\begin{aligned}\frac{d}{dt}\int u^\alpha(t,x)\,dx = &-\alpha\,(\alpha-1)\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot F(t,x)\,dx\\ &-\alpha\,(\alpha-1)\int u^{\alpha-2}(t,x)\nabla u(t,x)^T\,a(t,x)\,\nabla u(t,x)\,dx\\ &-\alpha\,(\alpha-1)\int u^{\alpha-1}(t,x)\sum_{1\leq i,j\leq d}\frac{\partial u(t,x)}{\partial x_i}\,\frac{\partial a_{ij}(t,x)}{\partial x_j}\,dx.\end{aligned}$$

Note that by (2.18)

$$\int u^{\alpha-2}(t,x)\nabla u(t,x)^T\,a(t,x)\,\nabla u(t,x)\,dx \geq C\,\|\nabla u^{\alpha/2}\|_{L^2}^2.$$

On the other hand

$$\begin{aligned}\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot F(t,x)\,dx &\leq \|\nabla u^{\alpha/2}\|_{L^2}\,\|u^{\alpha/2}\|_{L^2}\,\|F\|_{L^\infty}\\ &\leq \frac{C}{4}\,\|\nabla u^{\alpha/2}\|_{L^2}^2 + C'\int u^\alpha(t,x)\,dx.\end{aligned}$$

And

$$\int u^{\alpha-1}(t,x) \sum_{1\leq i,j\leq d} \frac{\partial u(t,x)}{\partial x_i} \frac{\partial a_{ij}(t,x)}{\partial x_j}\, dx \leq \|\nabla u^{\alpha/2}\|_{L^2}\, \|u^{\alpha/2}\,\nabla a\|_{L^2}$$

$$\leq \|\nabla u^{\alpha/2}\|_{L^2}\, \|\nabla a\|_{L^p}\, \|u^{\alpha/2}\|_{L^r},$$

with $1/2 = 1/p + 1/r$, which can be done since $p > d \geq 2$. Now by Sobolev embedding

$$\|u^{\alpha/2}\|_{L^r} \leq \left(\int u^\alpha\, dx\right)^{\theta/2} \|\nabla u^{\alpha/2}\|_{L^2}^{1-\theta},$$

for some $\theta \in (0,\ 1]$, precisely $1/r = 1/2 - (1-\theta)/d$ or $(1-\theta)/d = 1/p$, provided that $p > d$. In that case we immediately deduce that

$$\frac{d}{dt}\int u^\alpha(t,x)\, dx + \frac{C}{2}\int |\nabla u^{\alpha/2}|^2\, dx \leq C'' \left(1 + \|\nabla a\|_{L^p}^{2/\theta}\right) \int u^\alpha\, dx.$$

This concludes the bound provided that

$$\int_0^T \|\nabla a\|_{L^p}^{2/\theta} < \infty,$$

which means that $\nabla a \in L^q_{t,\mathrm{loc}}(L^p_x)$ with $1/q = \theta/2 = 1/2 - d/2p$. This exactly corresponds to the condition $2/q + d/p = 1$ with $p > d$.

Note that $p = d$ is critical here in the sense that the result could still hold in that case provided that the norm of ∇a is small enough with respect to the constant of ellipticity.

Finally we hence deduce that for any t and any $\alpha < \infty$

$$\|u(t,.)\|_{L^\alpha} \leq \|u(t=0,.)\|_{L^\alpha} \leq C,$$

with C independent of α since $u_0 \in L^1 \cap L^\infty$. This implies that $\|u(t,.)\|_{L^\infty} \leq C$ and finishes the proof.

7 Proof of Prop. 2.23

We are going to prove this result under the assumptions of Corollary 2.20. The other cases are similar.

Fix a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ equipped with a r-dimensional standard Brownian motion W. Fix also $u_0 > 0$ in $L^1 \cap L^\infty$ such that $\int_{\mathbb{R}^d} u_0(x)\, dx = 1$. Then, by Corollary 2.20, on the probability space $(\mathbb{R}^d \times \Omega, (\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t)_{t\geq 0}, u_0(x)dx \times \mathbb{P}(d\omega))$, there is strong existence of a process $(X_t(x,\omega), t \geq 0)$ solution of (1.1) with $\xi(x,\omega) = x$ and pathwise uniqueness holds. We deduce that strong existence for almost every deterministic initial condition x holds for (1.1) on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, \mathbb{P})$. In addition, the family of laws $\mathbb{P}_x$ of the process $\omega \mapsto X(x,\omega)$ for $x \in \mathbb{R}^d$ forms a regular conditional probability of the law of X given ξ.

For uniqueness, the two key points are

- first, that we are always in cases where uniqueness in law is known for *all* initial conditions in (1.1), and in particular for all deterministic initial conditions;
- second, that $u \in L^\infty$ by Cor. 2.19 (or is bounded by an explicit function in the case of Cor. 2.15 and 2.21), $\sigma \in H^1(u)$ and $F \in W_T^{\phi,\text{weak}}(u)$ (this is implied by Cor. 2.4 and 2.12).

For all x such that strong existence holds for (1.1) with $\xi = x$, let X_t^x and $\hat{X}_t^x$ be two strong solutions of (1.1) such that $X_0^x = \hat{X}_0^x = x$ a.s. Repeating the proof of Lemma 4.1, we have

$$\mathbb{E}\log\left(1 + \frac{|X_t^x - \hat{X}_t^x|^2}{\varepsilon^2}\right) \leq C\int_0^t \mathbb{E}\left[M|\nabla\sigma|(s, X_s^x)^2 + M|\nabla\sigma|(s, \hat{X}_s^x)^2\right] ds$$
$$+ C\int_0^t \mathbb{E}\left[(|F| + M_{1/\varepsilon}\nabla F)(s, X_s^x) + (|F| + M_{1/\varepsilon}\nabla F)(s, \hat{X}_s^x)\right] ds$$

By uniqueness in law, the two processes X^x and $\hat{X}^x$ have the same distribution $\mathbb{P}_x$, and so

$$\mathbb{E}\log\left(1 + \frac{|X_t^x - \hat{X}_t^x|^2}{\varepsilon^2}\right)$$
$$\leq C\int_0^t \mathbb{E}_x\left[\left((M|\nabla\sigma|)^2 + |F| + M_{1/\varepsilon}\nabla F\right)(s, X_s)\right] ds.$$

Let us denote by $M_t^\varepsilon(x)$ the integral in the r.h.s. Note that the l.h.s. may not be a measurable function of x, but $M_t^\varepsilon(x)$ is, because $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ is a regular conditional probability of the law of X given ξ. Choosing ϕ as in the proof of Theorem 1.1 (see the Appendix)

$$\int_{\mathbb{R}^d} M_t^\varepsilon(x)\, u_0(x)\, ds = \int_0^t \int_{\mathbb{R}^d} \left((M|\nabla\sigma|)^2 + |F| + M_{1/\varepsilon}\nabla F\right)(s,x) u(s,dx)\, ds$$
$$\leq C\left(1 + \frac{|\log\varepsilon|}{\varepsilon\phi(\varepsilon^{-1})}\right).$$

Now, copying the proof of Lemma 4.2,

$$\mathbb{E}(|X_t^x - \hat{X}_t^x|) \leq C\left[\sqrt{\varepsilon} + \frac{1}{L} + \frac{LM_t^\varepsilon(x)}{|\log \varepsilon|}\right]$$

Let us denote by $N_t^\varepsilon(x)$ the r.h.s. Choosing $L = \left(\frac{1}{|\log\varepsilon|} + \tilde{\eta}(\varepsilon)\right)^{-1}$ with $\eta(\varepsilon) = (\varepsilon\phi(\varepsilon^{-1}))^{-1}$, we obtain

$$\int_{\mathbb{R}^d} N_t^\varepsilon(x)\, u_0(x)\, ds \leq C\left(\sqrt{\varepsilon} + \sqrt{\frac{1}{|\log\varepsilon|} + \tilde{\eta}(\varepsilon)}\right).$$

Since the r.h.s. converges to 0 when $\varepsilon \to 0$, there exists a sequence $\varepsilon_k \to 0$ such that $N_t^{\varepsilon_k}(x) \to 0$ for almost all x. The diagonal procedure then shows the existence of a subsequence $\varepsilon'_k \to 0$ such that $N_t^{\varepsilon'_k}(x) \to 0$ for almost all x and for all t in a dense denumerable subset of $[0, T]$. Since the paths of X^x and $\hat{X}^x$ are continuous, we deduce that pathwise uniqueness holds for almost all $x \in \mathbb{R}^d$.

Appendix: Sketch of the proof of Theorem 1.1

The only thing left to prove after Thm. 2.13 is: Assume $u \in L^{q'}_{t,\mathrm{loc}}(L^{p'}_x(\mathbb{R}^d))$ then show that, for some super-linear ϕ,

$$\|\sigma\|_{H^1_T(u)} \leq C\,\|\sigma\|_{L^{2q}_t([0,T],W^{1,2p}_x)}, \qquad \|F\|_{W^{\phi,\mathrm{weak}}_T(u)} \leq C\,\|F\|_{L^q_t([0,T],W^{1,p}_x)}.$$

From the fact that the maximal operator M is bounded on L^p, $p > 1$, this is straightforward for σ (as $2p \geq 2 > 1$).

Therefore the key point is how to prove that for F when $p \geq 1$. We give the proof for $p = 1$, the case $p > 1$ can be treated following the same lines.

Now fix $L \geq 1$ and denote

$$h(t,x) = M_L \nabla F = \sqrt{\log L} + \int_{\mathbb{R}^d} \frac{|\nabla F(t,z)|\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\, dz}{(L^{-1} + |x-z|)\, |x-z|^{d-1}}.$$

As $p' = \infty$, for almost any fixed t, $u(t,\cdot) \in L^{q'} \cap L^\infty$ and hence

$$\begin{aligned}\int h(t,x)\, u(t,x)\, dx \leq{}& \sqrt{\log L} + \max(1,\ \|u(t,\cdot)\|_{L^\infty}) \\ & \iint \min(1,\ u(t,x))\, \frac{|\nabla F(t,z)|\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\, dz}{(L^{-1} + |x-z|)\, |x-z|^{d-1}}\, dx \\ \leq{}& \sqrt{\log L} + C\, \log L\, (\|u(t,\cdot)\|_{L^\infty} + \|u(t,\cdot)\|_{L^1})\, \|\nabla F(t,\cdot)\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\|_{L^1},\end{aligned}$$

by Fubini's theorem. Note that the term $\min(1, u(t,x))$ was kept in the integral because the function $x \mapsto (L^{-1} + |x-z|)^{-1}|x-z|^{-(d-1)}$ is not integrable on $\{|x| > 1\}$.

Therefore integrating now in time, by Hölder's estimates

$$\int_0^T \int h(t,x)\, u(t,x)\, dx\, dt \leq \sqrt{\log L}\, T + C\, \log L\, \|\nabla F\, \mathbb{1}_{|\nabla F| \geq \sqrt{\log L}}\|_{L^q_t(L^1_x)}.$$

Now, if $\nabla F \in L^q_t([0,T], L^1_x)$, then de la Vallée Poussin classical integrability result means that there exists a super-linear ψ s.t.

$$\|\psi(\nabla F)\|_{L^q_t([0,T],L^1_x)} < \infty.$$

Consequently

$$\int_0^T \int h(t,x)\, u(t,x)\, dx\, dt \leq T\, \sqrt{\log L} + C\, \frac{(\log L)^{3/2}}{\psi(\sqrt{\log L})}.$$

We conclude that $\|\nabla F\|_{W_T^{\phi,\mathrm{weak}}(u)}$ is bounded for ϕ defined by

$$\frac{L}{\phi(L)} = \frac{C\, \sqrt{\log L}}{\log L} + \frac{C\, \sqrt{\log L}}{\psi(\sqrt{\log L})},$$

which is hence also super-linear.

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