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ABSTRACT

The computational complexity of a problem arising in the context of sparse optimization is considered, namely, the projection onto the set of k-cosparse vectors w.r.t. some given matrix $\boldsymbol{\Omega}$. It is shown that this projection problem is (strongly) NP-hard, even in the special cases in which the matrix $\boldsymbol{\Omega}$ contains only ternary or bipolar coefficients. Interestingly, this is in contrast to the projection onto the set of k-sparse vectors, which is trivially solved by keeping only the k largest coefficients.

***Index Terms*—** Compressed Sensing, Computational Complexity, Cosparsity, Cosparse Analysis, Projection

1. INTRODUCTION

A central problem in compressed sensing (CS), see, e.g., [1, 2, 3], is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{P$_0$}$$

for a given matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m < n$ and right hand side vector $\boldsymbol{b} \in \mathbb{R}^m$, where $\|\boldsymbol{x}\|_0$ denotes the ℓ_0-quasi-norm, i.e., the number of nonzero entries in $\boldsymbol{x}$. This problem is known to be strongly NP-hard, cf. [MP5] in [4]; the same is true for the variant in which $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is replaced by $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2 \leq \varepsilon$, see [5, 6].

Two related problems arise in signal and image processing, where the unknown signal $\boldsymbol{x}$ to be estimated from a low-dimensional observation $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ cannot directly be modeled as being sparse. In the most standard approach, $\boldsymbol{x}$ is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary $\boldsymbol{D}$, i.e., $\boldsymbol{x} = \boldsymbol{D}\boldsymbol{z}$, where the representation vector $\boldsymbol{z}$ is sparse. The problem of minimizing $\|\boldsymbol{z}\|_0$ such that $\boldsymbol{A}\boldsymbol{D}\boldsymbol{z} = \boldsymbol{b}$ is obviously also NP-hard.

The alternative *cosparse analysis model* [7] assumes that $\boldsymbol{\Omega}\boldsymbol{x}$ has many zeros, where $\boldsymbol{\Omega}$ is an analysis operator. Typical examples include finite difference operators: They are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\boldsymbol{\Omega}\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{C$_0$}$$

and was also shown to be NP-hard [7, Section 4.1].

Due to the hardness of problem (P$_0$), various heuristics have been developed. A popular approach is well-illustrated by the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error $\|\boldsymbol{A}\tilde{\boldsymbol{x}} - \boldsymbol{b}\|_2$ and a hard-thresholding step, i.e., the (Euclidean) projection onto the set of k-sparse vectors. Under restricted isometry properties (RIP) on $\boldsymbol{A}$, the IHT algorithm can be proven to converge to the solution of (P$_0$) [8, 9]. A desirable RIP often holds with high probability when $\boldsymbol{A}$ has i.i.d. sub-Gaussian entries, see, e.g., [10], but is (strongly) NP-hard to verify in general [11].

Adaptations of IHT and related algorithms to the cosparse analysis setting have been proposed [12, 13], based on a general scheme for unions of subspaces [14]. A key step is the projection onto the set of k-cosparse vectors, as an analogous replacement for hard-thresholding.

The main contribution of this note is to show that this projection is in fact strongly NP-hard in general, which contrasts with the extremely simple and fast hard-thresholding operation.

2. COMPLEXITY OF COSPARSE PROJECTION PROBLEMS

This section is mainly devoted to proving the following central result.

Theorem 1. *Consider any $p \in \mathbb{N} \cup \{\infty\}$, $p > 1$, and let $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. Given $\boldsymbol{\Omega} \in \mathbb{Q}^{r\times n}$ ($r > n$), $\boldsymbol{\omega} \in \mathbb{Q}^n$, and a positive integer $k \in \mathbb{N}$, it is NP-hard*

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in the strong sense to solve the k-cosparse ℓ_p-norm projection problem

$$\min_{\boldsymbol{z}\in\mathbb{R}^n} \{ \|\boldsymbol{\omega}-\boldsymbol{z}\|_p^q \,:\, \|\boldsymbol{\Omega z}\|_0 \le k \}, \qquad (k\text{-CoSP}_p)$$

even when $\boldsymbol{\omega} \in \{0,1\}^n$ (with exactly one nonzero entry) and $\boldsymbol{\Omega}$ contains only entries in $\{-1,0,+1\}$ or $\{-1,1\}$, respectively.

Let us first provide some comments on the statement of the theorem.

Remark 1. *Theorem 1 shows the NP-hardness of (k-CoSP$_p$) in the strong sense (i.e., (k-CoSP$_p$) is strongly NP-hard). Thus, since the objective is always polynomially bounded (never exceeding $\|\boldsymbol{\omega}\|_p^p$ or $\|\boldsymbol{\omega}\|_\infty$), not even a* fully polynomial-time approximation scheme (FPTAS) *can exist, unless P=NP. An FPTAS is an algorithm that, given an approximation factor $\epsilon > 0$, computes a solution that has an objective function value at most $(1+\epsilon)$ times the minimum value, with running time polynomial in the encoding length of the instance and $1/\epsilon$. Moreover, there also cannot exist a* pseudo-polynomial (exact) algorithm, *i.e., one that runs in time polynomial in the* numeric *value of the input, but not its* encoding length, *unless P=NP; for details, see [4], and also [15].*

Remark 2. *It is perhaps not immediately clear why the "min" in (k-CoSP$_p$) is attained, since the constraint set is not bounded, in general. Nevertheless, the infimum is finite since the objective is lower-bounded by zero. Moreover, since $\boldsymbol{z} = 0$ is always feasible, the optimal value is upper-bounded by $\|\boldsymbol{\omega}\|_p^q$. Since the level set $\{ \boldsymbol{z} : \|\boldsymbol{\Omega z}\|_0 \le k,\ 0 \le \|\boldsymbol{\omega}-\boldsymbol{z}\|_p^q \le \|\boldsymbol{\omega}\|_p^q \}$ is compact, the infimum is attained, justifying the use of "min" instead of "inf" in (k-CoSP$_p$). (See also Remark 6 below.)*

Before stating the proof of Theorem 1, we need some preliminaries. Let MINULR$_0^=$($\boldsymbol{A}$,K) be the problem to decide, for a given matrix $\boldsymbol{A} \in \mathbb{Q}^{m\times n}$ and a positive integer $K \in \mathbb{N}$, whether there exists a vector $\boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{z} \neq 0$ and at most K of the m equalities in the system $\boldsymbol{Az} = 0$ are violated (MINULR stands for "minimum number of unsatisfied linear relations"). This homogeneous equality version, in which the trivial all-zero solution is excluded, was proven to be NP-complete [16], and even NP-hard to approximate within any constant factor [17]; more results about the problem's approximation complexity can be found in [18]. For the proof of Theorem 1, we utilize the following result:

Theorem 2 (Theorem 1 and Corollary 2 in [16])**.** *The problem MINULR$_0^=$($\boldsymbol{A}$,K) is strongly NP-hard, even when $\boldsymbol{A}$ contains only ternary entries, i.e., $\boldsymbol{A} \in \{-1,0,+1\}^{m\times n}$, or for bipolar $\boldsymbol{A} \in \{-1,+1\}^{m\times n}$.*

The cited results from [16] are in fact more general than the statements we use here in that they also hold for inhomogeneous systems, where the right hand side vector is part of the input. Moreover, [16] deals with the complementary problem of MINULR$_0^=$, namely the *Maximum Feasible Subsystem* (MAXFS) problem (w.r.t. homogeneous linear equality systems) in which one asks for the largest possible number of simultaneously satisfied equalities from a given linear system. It is easy to see that MINULR and MAXFS are equivalent when solved to optimality (see also [18]).

Note also that the above two results state NP-hardness in the strong sense, although this is not made explicit in the original version [16]. The NP-hardness proofs, however, are by reduction from the *Exact Cover by 3-Sets* (X3C) problem, which is well-known to be strongly NP-hard (see, e.g., [4]), and preserve polynomial boundedness of the constructed numbers as well as of their encoding lengths.

Proof of Theorem 1. Let $p \in \mathbb{N}\cup\{\infty\}$, $p > 1$, and $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. Given an instance $(\boldsymbol{A}, K)$ of MINULR$_0^=$ (w.l.o.g., $\boldsymbol{A} \in \mathbb{Q}^{r\times n}$ with $r > n$), we will reduce it to n instances of (k-CoSP$_p$).

For all $i = 1,\dots,n$, we define a k-cosparse projection instance given by $\boldsymbol{\Omega} = \boldsymbol{A}$, $\boldsymbol{\omega} = \boldsymbol{e}_i$ and $k = K$ (where $\boldsymbol{e}_i$ denotes the i-th unit vector in $\mathbb{Q}^n$). Note that each such instance obviously has encoding length polynomially bounded by that of $\boldsymbol{A}$ and K. Writing

$$f_{i,k}^{(p)} := \min_{\boldsymbol{z}\in\mathbb{R}^n} \{ \|\boldsymbol{e}_i-\boldsymbol{z}\|_p^q \,:\, \|\boldsymbol{\Omega z}\|_0 \le k \}$$
$$\text{and} \quad f_k^{(p)} := \min_{1\le i\le n} f_{i,k}^{(p)},$$

we observe that since $\boldsymbol{z} = 0$ is always a feasible point, it holds that $f_{i,k}^{(p)} \le 1$ for all i, and hence $f_k^{(p)} \le 1$. We claim that $f_k^{(p)} < 1$ if and only if there exists a nonzero vector $\boldsymbol{z}$ that violates at most $k = K$ equations in $\boldsymbol{Az} = 0$.

1. If $f_k^{(p)} < 1$ (i.e., there exists some $i \in \{1,\dots,n\}$ such that $f_{i,k}^{(p)} < 1$) then there exists a vector $\boldsymbol{z}$ such that $\|\boldsymbol{\Omega z}\|_0 \le k$ and $\|\boldsymbol{e}_i - \boldsymbol{z}\|_p^q < 1$ (in particular, $\boldsymbol{z}_i \neq 0$). But this of course means that at most K equalities in the system $\boldsymbol{Az} = 0$ are violated, i.e., MINULR$_0^=$$(\boldsymbol{A}, K)$ has a positive answer.

2. Conversely, assume that there exists a nonzero vector $\boldsymbol{z}$ that violates at most K equations in $\boldsymbol{Az} = 0$, i.e., $\|\boldsymbol{\Omega z}\|_0 \le k$ and MINULR$_0^=$$(\boldsymbol{A}, K)$ has a positive answer. We will prove that

$$f_k^{(p)} \le \min_{i,\lambda} \|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^q < 1. \qquad (1)$$

In fact, for an arbitrary scalar $\lambda \in \mathbb{R}$, $\boldsymbol{z}' := \lambda\boldsymbol{z}$ obeys $\|\boldsymbol{\Omega z}'\|_0 \le k$ as well. If $\boldsymbol{z}$ contains only one nonzero component z_i, λ can be chosen sucht that $\boldsymbol{z}' = \boldsymbol{e}_i$ and consequently $f_k^{(p)} = f_{i,k}^{(p)} = 0$, which shows the claim. Thus, it remains to deal with the case in which $\boldsymbol{z}$ contains at least two nonzero components. Consider some i

such that $z_i \neq 0$. Without loss of generality, we can also assume that $z_i > 0$ (otherwise replace $\boldsymbol{z}$ with $-\boldsymbol{z}$). We consider the case $p < \infty$ first. It holds that

$$\begin{aligned}\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^p &= \|\lambda\boldsymbol{z}\|_p^p - |\lambda z_i|^p + |1-\lambda z_i|^p \\ &= |\lambda|^p\big(\|\boldsymbol{z}\|_p^p - |z_i|^p\big) + |1-\lambda z_i|^p\end{aligned}$$

for any λ. The function $|1-\lambda z_i|^p$ is convex (in λ), and it is easy to see that we have $|1-\lambda z_i|^p \leq 1 - \lambda z_i$ for $0 \leq \lambda \leq 1/z_i$. Consequently, for $0 \leq \lambda \leq 1/z_i$,

$$\begin{aligned}&|\lambda|^p\big(\|\boldsymbol{z}\|_p^p - |z_i|^p\big) + |1-\lambda z_i|^p \\ &\leq 1 + |\lambda|^p\big(\|\boldsymbol{z}\|_p^p - |z_i|^p\big) - \lambda z_i. \qquad (2)\end{aligned}$$

Since $\boldsymbol{z}$ contains at least two nonzero components, it follows that $\alpha := \|\boldsymbol{z}\|_p^p - |z_i|^p > 0$. Since $p > 1$, it is easy to see that $|\lambda|^p\alpha - \lambda z_i < 0$ for sufficiently small positive λ. In conclusion, there exists a solution $\lambda\boldsymbol{z}$ with $\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^p < 1$ (and thus $f_k^{(p)} < 1$), which proves the reverse direction for $p < \infty$.

Finally, for the case $p = \infty$, note that

$$\min_{i,\lambda}\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_\infty \leq \min_{i,\lambda}\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_{\tilde{p}},$$

for all $1 < \tilde{p} < \infty$, whence we can employ the above reasoning to reach the same conclusion.

To summarize, we could reduce the NP-hard problem MinULR$_0^{=}(\boldsymbol{A}, K)$ to n instances of the k-cosparse projection problem (k-CoSP$_p$), which therefore is NP-hard as well. Moreover, by Theorem 2 (and since all numerical values in the constructed instances obviously remain polynomially bounded by the input size), the NP-hardness of (k-CoSP$_p$) holds in the *strong* sense, even for $\boldsymbol{\Omega}$ $(= \boldsymbol{A})$ with ternary or bipolar coefficients and, by the construction above, for $\boldsymbol{\omega}$ binary (and indeed, with only one nonzero entry). ☐

Remark 3. *The reduction in the proof of Theorem 1 does not work for* $0 < p \leq 1$. *Indeed, consider the vector with all entries equal to* 1, $\boldsymbol{z} = \mathbb{1}$. *It is not hard to verify that* $\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^p = |1-\lambda|^p + (n-1)|\lambda|^p \geq 1$ *for all* i *and* λ. *Thus,* $\min_{i,\lambda}\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^p \geq 1$ *does not imply* $\boldsymbol{z} = 0$ *when* $p \in (0,1]$*; vice versa,* $\boldsymbol{z} \neq 0$ *does not imply that* $\min_{i,\lambda}\|\boldsymbol{e}_i - \lambda\boldsymbol{z}\|_p^p < 1$.

Remark 4. *It is noteworthy that the claim in the proof of Theorem 1 is in fact true for every real-valued* $p > 1$. *However, one may encounter irrational (i.e., not finitely representable) numbers when working with* p*-th powers for arbitrary real, or even rational,* p. *Since retaining finite rational representations is crucial for NP-hardness proofs, we restricted ourselves to integer* p *(or* $p = \infty$*).*

The following result is an immediate consequence of Theorem 1.

Corollary 1. *It is strongly NP-hard to compute a minimizer for the problem* (k-CoSP$_p$)*, even under the input data restrictions specified in Theorem 1. In particular, it is strongly NP-hard to compute the Euclidean projection*

$$\Pi_{\boldsymbol{\Omega},k}(\boldsymbol{\omega}) := \arg\min_{\boldsymbol{z}\in\mathbb{R}^n}\{\,\|\boldsymbol{\omega} - \boldsymbol{z}\|_2 \,:\, \|\boldsymbol{\Omega}\boldsymbol{z}\|_0 \leq k\,\}. \qquad (k\text{-CoSP})$$

Proof. Clearly, if a minimizer was known, we would also know the optimal value of (k-CoSP$_p$). Hence, computing a minimizer is at least as hard as solving (k-CoSP$_p$), and the complexity results of Theorem 1 carry over directly. In particular, computing $\Pi_{\boldsymbol{\Omega},k}(\boldsymbol{\omega})$ is also strongly NP-hard, since Theorem 1 applies to the ℓ_2-norm ($p = 2$) and the minimizers of $\|\boldsymbol{\omega} - \boldsymbol{z}\|_2$ (the objective in (k-CoSP)) are of course the same as those of $\|\boldsymbol{\omega} - \boldsymbol{z}\|_2^2$. ☐

Let us comment on a few more subtle aspects concerning (k-CoSP$_p$) and the proof of Theorem 1.

Remark 5. *The above reduction from* MinULR$_0^{=}$ *is an example of what is called a* Turing reduction*; more commonly used are* Karp reductions*, cf. [4, 15]. In the latter, the known NP-hard problem is reduced to a* single instance *of the problem under consideration. For* (k-CoSP$_p$) *with* $p \in \mathbb{N}, p > 1$ *(excluding the case* $p = \infty$*), we could also obtain a Karp reduction by constructing the instance*

$$\begin{aligned}&\tilde{\boldsymbol{\Omega}}^\top := (\boldsymbol{A}^\top, \dots, \boldsymbol{A}^\top) \in \mathbb{Q}^{n\times rn}, \\ &\tilde{\boldsymbol{\omega}}^\top := (\boldsymbol{e}_1^\top, \dots, \boldsymbol{e}_m^\top) \in \{0,1\}^{n^2}, \quad \tilde{k} := nK,\end{aligned}$$

which is obviously still polynomially related to the input $(\boldsymbol{A}, K)$ *of the given* MinULR$_0^{=}$ *instance. Then, defining*

$$\tilde{f}_{\tilde{k}}^{(p)} := \min_{\boldsymbol{z}\in\mathbb{R}^{rn}}\{\,\|\tilde{\boldsymbol{\omega}} - \boldsymbol{z}\|_p^p \,:\, \|\tilde{\boldsymbol{\Omega}}\boldsymbol{z}\|_0 \leq \tilde{k}\,\},$$

it is easy to see that $\tilde{f}_{\tilde{k}}^{(p)} < n\,(= \|\tilde{\boldsymbol{\omega}}\|_p^p)$ *if and only if* MinULR$_0^{=}(A, K)$ *has a positive answer. In particular, a solution can be seen to be comprised of the solutions to the* n *separate* (k-CoSP$_p$)*-problems considered in the Turing reduction, stacked on top of each other. We chose the Turing reduction for our proof of Theorem 1 because it allows us to conclude NP-hardness even if* $\boldsymbol{\omega}$ *is a unit vector, i.e., binary with exactly one nonzero entry, and also for* $p = \infty$.

Remark 6. *The optimization problem* (k-CoSP$_p$) *could be replaced by its corresponding decision version:*

$$\text{Does there exist some } \boldsymbol{z} \in \mathbb{R}^n \text{ such that } \|\boldsymbol{\omega} - \boldsymbol{z}\|_p^q < \gamma \text{ and } \|\boldsymbol{\Omega}\boldsymbol{z}\|_0 \leq k\,? \qquad (k\text{-CoSP}_p\text{-Dec})$$

Indeed, the proof of Theorem 1 corresponds to showing hardness of this decision problem for $\gamma = 1$*; the Karp reduction sketched in the previous remark would use* $\gamma = n$.

Note that (k-CoSP$_p$-Dec) *is in fact contained in NP (at least for* $p =$ 1, 2, *or* ∞*): The set* $\{\,\boldsymbol{z} \,:\, \|\boldsymbol{\Omega}\boldsymbol{z}\|_0 \leq k\,\}$

defined by the constraints yields (exponentially many) affine subspaces of $\mathbb{R}^n$ defined by $n-k$ or more homogeneous equalities, and the projection of some $\boldsymbol{\omega}$ onto it (w.r.t. $\|\cdot\|_p^q$) is clearly equivalent to that onto one (unknown) of these affine subspaces. For $p=2$, the (Euclidean) projection onto such spaces has a known explicit formula which keeps all entries in the solution rational if the input ($\boldsymbol{\Omega}$) is rational. Similarly, for $p=1$ or $p=\infty$, (k-CoSP$_p$) can be seen as a linear program over the unknown correct affine subspace; hence, here, the solution is also rational.

Thus, for $p=1$, 2, or ∞, a certificate of a positive answer exists that has encoding length polynomially bounded by that of the input. Hence, by Theorem 1 and the above discussion, we have that (k-CoSP$_p$-Dec) *is in fact* strongly NP-complete *for $p=2$ or $p=\infty$. (The case $p=1$ is not covered by Theorem 1, and for the remaining values of p it is not immediately clear how one could guarantee the existence of a certificate with polynomially bounded encoding length.)*

3. CONCLUSIONS

Theorem 1 and Corollary 1 show that no polynomial algorithm to compute the projection onto the set of k-cosparse vectors can exist unless P=NP.

In theoretical algorithmic applications of the Euclidean k-cosparse projection operation (k-CoSP) in [12, 13], it had so far been assumed that the projection problem (k-CoSP$_2$) can be approximated efficiently. While our result refutes this assumption to a certain degree (cf. Remark 1), it is not clear whether other (general polynomial-time) approximation algorithms exist that may still be useful in practice despite exhibiting theoretical running time bounds that depend exponentially on (at least) the approximation quality $1/\epsilon$.

Moreover, as for most NP-hard problems, there are specific instances which are known to be much easier to handle than the general case. For instance, when $\boldsymbol{\Omega}$ is the identity (or, more generally, a unitary) matrix, hard-tresholding—i.e., zeroing all but the k entries with largest absolute values—achieves the projection onto the k-cosparse set w.r.t. any ℓ_p-norm. Other examples include the Euclidean case ($p=2$) when $\boldsymbol{\Omega}$ is the 1D finite difference operator or the 1D fused Lasso operator, respectively: The projection is then achieved using dynamic programming [13].

For the problem MINULR$_0^=$ (or its minimization variant, respectively), strong non-approximability results were derived in [17, 16, 18]; for instance, it cannot be approximated within any constant unless P=NP. However, these results do not carry over to the k-cosparse projection problem, since the objectives differ: In the optimization version of MINULR$_0^=$, we wish to minimize the number K of violated equalities, while in (k-CoSP$_p$), the goal is minimizing the distance of $\boldsymbol{z}$ to a given point (under the *constraint* that the number of nonzeros in $\boldsymbol{\Omega z}$ does not exceed K). Thus, despite the link between the two problems exploited in the proof of Theorem 1, (hypothetical) approximation guarantees for this distance unfortunately do not yield any (non-)approximability statements for (k-CoSP$_p$) by means of those for MINULR$_0^=$.

Thus, it remains a challenge to find (practically) efficient approximation schemes for the k-cosparse projection problem (k-CoSP$_p$), or to establish further (perhaps negative) results concerning its approximability. Other open questions are the complexity of (k-CoSP$_p$) for $0<p\leq 1$, or containment in NP of (k-CoSP$_p$-Dec) for $p\in\mathbb{N}\setminus\{1,2\}$, cf. Remarks 3 and 6.

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