Local rigidity for SL (3,C) representations of 3-manifolds groups

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LOCAL RIGIDITY FOR SL$(3,\mathbb{C})$ REPRESENTATIONS OF 3-MANIFOLDS GROUPS

NICOLAS BERGERON, ELISHA FALBEL, ANTONIN GUILLOUX, PIERRE-VINCENT KOSELEFF AND FABRICE ROUILLIER

Abstract. Let M be a non-compact hyperbolic 3-manifold that has a triangulation by positively oriented ideal tetraedra. We explain how to produce local coordinates for the variety defined by the gluing equations for $\mathrm{SL}(3,\mathbb{C})$-representations. In particular we prove local rigidity of the "geometric" representation in $\mathrm{SL}(3,\mathbb{C})$, recovering a recent result of Menal-Ferrer and Porti. More generally we give a criterion for local rigidty of $\mathrm{SL}(3,\mathbb{C})$-representations and provide detailed analysis of the figure eight knot sister manifold exhibiting the different possibilities that can occur.

Contents

1. Introduction

Let M be a compact orientable 3-manifold with boundary a union of ℓ tori. Assume that the interior of M carries a hyperbolic metric of finite volume and let $\rho : \pi_1(M) \to \mathrm{SL}(3,\mathbb{C})$ be the corresponding holonomy composed with the 3-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$.

Building on [1] we give a combinatorial proof of the following theorem first proved by Menal-Ferrer and Porti [6].

1.1. **Theorem.** *The class $[\rho]$ of ρ in the algebraic quotient of $\mathrm{Hom}(\pi_1(M),\mathrm{SL}(3,\mathbb{C}))$ by the action of $\mathrm{SL}(3,\mathbb{C})$ by conjugation is a smooth point with local dimension 2ℓ.*

We do not solely consider the *geometric* representation ρ and in fact our proof applies to an explicit open subset of the (decorated) representation variety. It also provides explicit coordinates and a description of the possible deformations. We analyse in the last section the figure eight knot sister manifold: we describe all the (decorated) representations whose restriction to the boundary torus is unipotent.

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It turns out that there exist rigid points (i.e. isolated points in the (decorated) unipotent representation variety) together with non-rigid components.

There is a natural map h from the (decorated) representation variety of M to the representation variety of its boundary. It is known that its image is a Lagrangian subvariety and the map is a local isomorphism on a Zariski-open set. Our remark 5.8 proves in a combinatorial way these facts. When M is a knot complement and one considers the group $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{SL}(3,\mathbb{C})$, this image is the algebraic variety defined by the famous A-polynomial of the knot. In this paper, we explore more precisely the map h and exhibit a complicated fiber.

2. Ideal triangulation

2.1. An *ordered simplex* is a simplex with a fixed vertex ordering. Recall that an orientation of a set of vertices is a numbering of the elements of this set up to even permutation. The face of an ordered simplex inherits an orientation. We call *abstract triangulation* a pair $\mathcal{T} = ((T_\mu)_{\mu=1,\dots,\nu}, \Phi)$ where $(T_\mu)_{\mu=1,\dots,\nu}$ is a finite family of abstract ordered simplicial tetrahedra and Φ is a matching of the faces of the T_μ's reversing the orientation.

For any simplicial tetrahedron T, we define $\mathrm{Trunc}(T)$ as the tetrahedron truncated at each vertex. We call *triangulation* — or rather *ideal triangulation* — of a compact 3-manifold M with boundary an abstract triangulation $\mathcal{T}$ and an oriented homeomorphism

$$h : \bigsqcup_{\mu=1}^{\nu} \mathrm{Trunc}(T_\mu)/\Phi \to M.$$

In the following we will always assume that the boundary of M is a disjoint union of a finite collection of 2-dimensional tori. The most important family of examples being the compact 3-manifolds whose interior carries a complete hyperbolic structure of finite volume. The existence of an ideal triangulation for M still appears to be an open question.[1] Luo, Schleimer and Tillmann [5] nevertheless prove that, passing to a finite regular cover, we may assume that M admits an ideal triangulation. In the following paragraphs we assume that M itself admits an ideal triangulation $\mathcal{T}$ and postpone to the proof of Theorem 1.1 the task of reducing to this case (see lemma 5.10). Recall that the number of edges of $\mathcal{T}$ is equal to the number ν of tetrahedra.

2.2. Parabolic decorations. We recall from [1] the notion of a *parabolic decoration* of the pair $(M, \mathcal{T})$: to each tetrahedron T_μ of $\mathcal{T}$ we associate non-zero complex coordinates $z_\alpha(T_\mu)$ $(\alpha \in I)$ where

$$I = \{\text{vertices of the (red) arrows in the triangulation given by figure 1}\}.$$

We let $J^2_{T_\mu} = \mathbb{Z}^I$ be the 16-dimensional abstract free $\mathbb{Z}$-module and denote the canonical basis $\{e_\alpha\}_{\alpha\in I}$ of $J^2_{T_\mu}$. It contains *oriented edges* e_{ij} (edges oriented from j to i) and *faces* e_{ijk}. Using these notations the 16-tuple of complex parameters $(z_\alpha(T_\mu))_{\alpha\in I}$ is better viewed as an element

$$z(T_\mu) \in \mathrm{Hom}(J^2_{T_\mu}, \mathbb{C}^\times) \cong \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^*.$$

[1] Note however that starting from the Epstein-Penner decomposition of M into ideal polyhedra, Petronio and Porti [7] produce a degenerate triangulation of M.

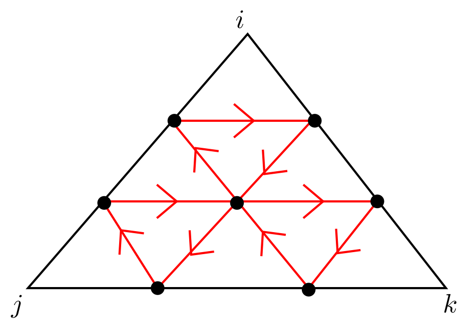

FIGURE 1. Combinatorics of W

We refer to [1] for details. Such an element uniquely determines a tetrahedron of flags if and only if the following relations are satisfied:

$$z_{ijk} = -z_{il}z_{jl}z_{kl}, \tag{2.2.1}$$

$$z_{ij}z_{ik}z_{il} = -1, \tag{2.2.2}$$

and

$$z_{ik} = \frac{1}{1 - z_{ij}}. \tag{2.2.3}$$

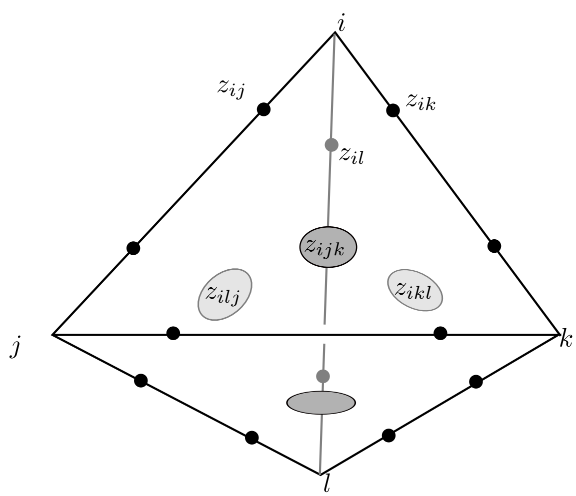

FIGURE 2. The z-coordinates for a tetrahedron

2.3. Let J^2 denote the direct sum of the $J^2_{T_\mu}$'s and consider an element $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*$ as as a set of parameters of the triangulation $\mathcal{T}$. As usual, these coordinates are subject to consistency relations after gluing by Φ: given two adjacent tetrahedra T_μ, $T_{\mu'}$ of T with a common face (ijk) then

$$z_{ijk}(T_\mu)z_{ikj}(T_{\mu'}) = 1. \tag{2.3.1}$$

And given a sequence $T_1, \ldots, T_\mu$ of tetrahedra sharing a common edge ij and such that ij is an inner edge of the sub complex composed by $T_1 \cup \cdots \cup T_\mu$ then

$$z_{ij}(T_1) \cdots z_{ij}(T_\mu) = z_{ji}(T_1) \cdots z_{ji}(T_\mu) = 1. \tag{2.3.2}$$

2.4. Consider a fundamental domain of the triangulation of the universal cover $\tilde{M}$ lifted from the one of M. A decoration of the complex is then equivalent to an assignment of a flag to each of its vertices; together with an additional transversality condition on the flags to ensure that the z_α's do not vanish.

3. The representation variety

We define the *representation variety* $\mathcal{R}(M, \mathcal{T})$ as:

$$\mathcal{R}(M, \mathcal{T}) = g^{-1}(1, \ldots, 1)$$

where $g = (h, a, f) : \mathbb{C}^\times \otimes (J^2)^* \to (\mathbb{C}^\times)^{8\nu} \times (\mathbb{C}^\times)^{4\nu} \times \mathrm{Hom}(C_1^{\mathrm{or}} + C_2, \mathbb{C}^\times) \cong \mathbb{C}^{16\nu}$ is the product of the three maps h, a, f, defined below.

3.1. First $h = (h_1, \ldots, h_\nu)$ is the product of the maps $h_\mu : \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^* \to \mathbb{C}^8$ $(\mu = 1, \ldots, \nu)$ associated to the T_μ's and which are defined by

$$h_\mu(z) = \Big(-\frac{z_{ijk}}{z_{il}z_{jl}z_{kl}}, -\frac{z_{ikl}}{z_{ij}z_{kj}z_{lj}}, -\frac{z_{ilj}}{z_{ik}z_{lk}z_{jk}}, -\frac{z_{kjl}}{z_{ki}z_{ji}z_{lj}}, \\ -z_{ij}z_{ik}z_{il}, -z_{ji}z_{jk}z_{jl}, -z_{ki}z_{kj}z_{kl}, -z_{li}z_{lj}z_{lk} \Big)$$

here $z = z(T_\mu) \in \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^*$, cf. (2.2.1) and (2.2.2).

3.2. Next we define the map a, cf. (2.2.3). Let $a_\mu : \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^* \to \mathbb{C}^4$ $(\mu = 1, \ldots \nu)$ associated to T_μ be the map defined by

$$a_\mu(z) = (z_{ik}(1 - z_{ij}), z_{jl}(1 - z_{ji}), z_{ki}(1 - z_{kl}), z_{lj}(1 - z_{lk})).$$

We define then $a = (a_1, \ldots, a_\nu)$.

3.3. Finally we let C_1^{or} be the free $\mathbb{Z}$-module generated by the oriented 1-simplices of $\mathcal{T}$ and C_2 the free $\mathbb{Z}$-module generated by the 2-faces of $\mathcal{T}$. As in [1], we define a map

$$F : C_1^{\mathrm{or}} + C_2 \to J^2$$

by, for $\bar{e}_{ij}$ an oriented edge of K,

$$F(\bar{e}_{ij}) = e^1_{ij} + \ldots + e^\mu_{ij}$$

where $T_1, \ldots, T_\mu$ is a sequence of tetrahedra sharing the edge $\bar{e}_{ij}$ such that $\bar{e}_{ij}$ is an inner edge of the subcomplex $T_1 \cup \cdots \cup T_\mu$ and each e^μ_{ij} gets identified with the *oriented* edge $\bar{e}_{ij}$ in $\mathcal{T}$. And for a 2-face $\bar{e}_{ijk}$,

$$F(\bar{e}_{ijk}) = e^\mu_{ijk} + e^{\mu'}_{ikj},$$

where μ and μ' index the two 3-simplices having the common face $\bar{e}_{ijk}$. We then define the map

$$f : \mathrm{Hom}(J^2, \mathbb{C}^\times) \to \mathrm{Hom}(C_1^{\mathrm{or}} + C_2, \mathbb{C}^\times)$$

by $f(z) = z \circ F$, compare (2.3.1) and (2.3.2).

3.4. From an element in $\mathcal{R}(M,\mathcal{T})$, one may reconstruct a representation (up to conjugacy) by computing the holonomy of the complex of flags (see [1, section 5]). Restating the remark 2.4, a decoration is equivalent to a map, equivariant under $\pi_1(M)$, from the space of cusps of $\tilde{M}$ to the space of flags with a transversality condition. Note that each flag is then invariant by the holonomy of the cusp.

Moreover, the map from $\mathcal{R}(M,\mathcal{T})$ to $\mathrm{Hom}(\pi_1(M),\mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ is open: if you have a representation ρ, its decoration equip each cusp p of M with a flag F_p invariant by the holonomy of the fixator Γ_p of p. Now, if you deform the representation ρ in ρ', for each cusp p, you can deform F_p into a flag F'_p invariant under $\rho'(\Gamma_p)$. The transversality condition being open, this gives a decoration for any decoration ρ' near ρ.

4. Infinitesimal deformations

4.1. Let $z=(z(T_\mu))_{\mu=1,\dots,\nu}\in\mathcal{R}(M,\mathcal{T})$. The exponential map identifies $T_z(\mathbb{C}^\times\otimes_{\mathbb{Z}}(J^2)^*)$ with $\mathbb{C}\otimes(J^2)^*=\mathrm{Hom}(J^2,\mathbb{C})$. Under this identification the differential d_zg defines a linear map which we write as a direct sum $d_zh\oplus d_za\oplus d_zf$.

In the following three lemmas we identify the kernel of each of these three linear maps. Before that we recall from [1] that each $J^2_{T_\mu}$ is equipped with a bilinear skew-symmetric form given by

$$\Omega^2(e_\alpha,e_\beta)=\varepsilon_{\alpha\beta}.$$

Here given α and β in I we set (recall figure 1):

$$\varepsilon_{\alpha\beta}=\#\{\text{oriented (red) arrows from }\alpha\text{ to }\beta\}-\#\{\text{oriented (red) arrows from }\beta\text{ to }\alpha\}.$$

We let (J^2,Ω^2) denote the orthogonal sum of the spaces $(J^2_{T_\mu},\Omega^2)$. We denote by e^μ_α the e_α-element in $J^2_{T_\mu}$. Let

$$p:J^2\to(J^2)^*$$

be the homomorphism $v\mapsto\Omega^2(v,\cdot)$. On the basis (e_α) and its dual (e^*_α), we can write

$$p(e_\alpha)=\sum_\beta\varepsilon_{\alpha\beta}e^*_\beta.$$

Let J be the quotient of J^2 by the kernel of Ω^2. The latter is the subspace generated on each tetrahedron by elements of the form

$$\sum_{\alpha\in I}b_\alpha e_\alpha$$

for all $\{b_\alpha\}\in\mathbb{Z}^I$ such that $\sum_{\alpha\in I}b_\alpha\varepsilon_{\alpha\beta}=0$ for every $\beta\in I$. Equivalently it is the subspace generated by $e_{ij}+e_{ik}+e_{il}$ and $e_{ijk}-(e_{il}+e_{jl}+e_{kl})$.

We let $J^*\subset(J^2)^*$ be the dual subspace which consists of the linear maps which vanish on the kernel of Ω^2. Note that we have $J^*=\mathrm{Im}(p)$ and that it is 8-dimensional. The form Ω^2 induces a — now non-degenerate — symplectic form Ω on J. This yields a canonical identification between J and J^*; we denote by Ω^* the corresponding symplectic form on J^*.

4.2. Lemma. *As a subspace of $\mathbb{C}\otimes(J^2)^*$ the kernel of d_zh is equal to $\mathbb{C}\otimes J^*$.*

Proof. It follows from the definitions that $\xi \in \mathbb{C} \otimes (J^2)^*$ belongs to the kernel of $d_z h$ if and only if it vanishes on the subspace $\mathrm{Ker}(\Omega^2)$ generated by $e^\nu_{ij} + e^\nu_{ik} + e^\nu_{il}$ and $e^\nu_{ijk} - (e^\nu_{il} + e^\nu_{jl} + e^\nu_{kl})$. On the other hand Ω^2 induces a (non-degenerate) symplectic form on $J^2_{T_\mu}/\mathrm{Ker}(\Omega^2)$ which yields a canonical identification between $J^2_{T_\mu}/\mathrm{Ker}(\Omega^2)$ and $\mathrm{Im}(p)$. We conclude that ξ belongs to $\mathrm{Im}(p)$ if and only if $\mathrm{Ker}(\Omega) \subset \mathrm{Ker}(\xi)$. This concludes the proof. $\square$

4.3. Lemma. *As a subspace of $\mathbb{C} \otimes (J^2)^*$ the kernel of $d_z a$ is equal to the subspace $\mathcal{A}(z)$ defined as:*

$$\left\{ \xi \in \mathrm{Hom}(J^2, \mathbb{C}) \ : \ \begin{array}{c} \xi(e^\mu_{ij}) + z_{il}(T_\mu)\xi(e^\mu_{ik}) = 0, \ \xi(e^\mu_{ji}) + z_{jk}(T_\mu)\xi(e^\mu_{jl}) = 0 \\ \xi(e^\mu_{ki}) + z_{kl}(T_\mu)\xi(e^\mu_{kj}) = 0, \ \xi(e^\mu_{lj}) + z_{lk}(T_\mu)\xi(e^\mu_{li}) = 0 \end{array} , \forall \mu \right\}.$$

Proof. Here again we only have to check this on each tetrahedra T_μ of $\mathcal{T}$. All four coordinates of a_μ can be dealt with in the same way, we only consider the first coordinate:

$$z \mapsto z_{ik}(1 - z_{ij}).$$

Taking the differential of the logarithm we get:

$$\frac{dz_{ik}}{z_{ik}} - \frac{dz_{ij}}{1 - z_{ij}} = 0.$$

Equivalently,

$$\frac{dz_{ij}}{z_{ij}} = \left(\frac{1 - z_{ij}}{z_{ij}} \right) \frac{dz_{ik}}{z_{ik}}.$$

Since $z \in \mathcal{R}(M, \mathcal{T})$, we have $h_\nu(z) = a_\mu(z) = 1$. In particular

$$(1 - z_{ij}) = \frac{1}{z_{ik}} \quad \text{and} \quad z_{ij} z_{ik} = -\frac{1}{z_{il}}.$$

We conclude that

$$\frac{dz_{ij}}{z_{ij}} + z_{il} \frac{dz_{ik}}{z_{ik}}.$$

Under the identification of $T_z(\mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*)$ with $\mathbb{C} \otimes (J^2)^* = \mathrm{Hom}(J^2, \mathbb{C})$ this proves the lemma. $\square$

We denote by $F^* : (J^2)^* \to C_1^{\mathrm{or}} + C_2$ the dual map to F (here we identify $C_1^{\mathrm{or}} + C_2$ with its dual by using the canonical basis). It is the "projection map":

$$(e^\mu_\alpha)^* \mapsto \bar{e}_\alpha$$

when $(e^\mu_\alpha)^* \in (J^2_{\mathrm{int}})^*$ and maps $(e^\mu_\alpha)^*$ to 0 if $(e^\mu_\alpha)^* \notin (J^2_{\mathrm{int}})^*$. By definition of f we have:

4.4. Lemma. *As a subspace of $\mathbb{C} \otimes (J^2)^*$ the kernel of $d_z f$ is equal to $\mathbb{C} \otimes \mathrm{Ker}(F^*)$.*

Lemma 4.2, 4.3 and 4.4 therefore imply that

$$\mathrm{Ker}\ d_z g = (\mathbb{C} \otimes (\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)) \cap \mathcal{A}(z). \tag{4.4.1}$$

Note that among these three spaces, two are defined over $\mathbb{Z}$ and do not depend one the point z, but the last one, $\mathcal{A}(z)$, is actually depending on z. We shall give examples where the dimension of the intersection vary and describe the corresponding deformations in $\mathcal{R}(M, \mathcal{T})$. But first we consider an open subset of $\mathcal{R}(M, \mathcal{T})$ which we prove to be a manifold.

5. The complex manifold $\mathcal{R}(M, \mathcal{T}^+)$

5.1. Let

$$\mathcal{R}(M, \mathcal{T}^+) = \{z = (z(T_\mu))_{\mu=1,\dots,\nu} \in \mathcal{R}(M, \mathcal{T}) \ : \ \mathrm{Im}\ z_{ij}(T_\mu) > 0, \ \forall \mu, i, j\}$$

be the subspace of $\mathcal{R}(M, \mathcal{T})$ whose *edge* coordinates have positive imaginary parts. Note that coordinates corresponding to the geometric representation belong to $\mathcal{R}(M, \mathcal{T}^+)$. The main theorem of this section is a generalization of a theorem of Choi [2]; it states that $\mathcal{R}(M, \mathcal{T}^+)$ is a smooth complex manifold and gives local coordinates.

Assume that ∂M is the disjoint union of ℓ tori. For each boundary torus T_s ($s = 1, \dots, \ell$) of M we fix a symplectic basis (a_s, b_s) of the first homology group $H_1(T_s)$. Given a point z in the representation variety $\mathcal{R}(M, \mathcal{T})$ we may consider the holonomy elements associated to a_s, resp. b_s. They preserve a flag associated to the torus by the decoration. In a basis adapted to this flag, those matrices are of the form (for notational simplicity, we write them in $\mathrm{PGL}(3, \mathbb{C})$ rather than $\mathrm{SL}(3, \mathbb{C})$):

$$\begin{pmatrix} \frac{1}{A_s^*} & * & * \\ 0 & 1 & * \\ 0 & 0 & A_s \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{B_s^*} & * & * \\ 0 & 1 & * \\ 0 & 0 & B_s \end{pmatrix}.$$

Now the diagonal entries of the first matrix A_s and A_s^* for each torus define a map

$$\mathcal{R}(M, \mathcal{T}) \to (\mathbb{C}^\times)^{2\ell}; \quad z \mapsto (A_s, A_s^*)_{s=1,\dots,\ell}. \tag{5.1.1}$$

5.2. Theorem. *Assume that ∂M is the disjoint union of ℓ tori. Then the complex variety $\mathcal{R}(M, \mathcal{T}^+)$ is a smooth complex manifold of dimension 2ℓ.*

Moreover: the map (5.1.1) *restricts to a local biholomorphism from $\mathcal{R}(M, \mathcal{T}^+)$ to $(\mathbb{C}^\times)^{2\ell}$.*

Proof. Consider

$$C_1^{\mathrm{or}} + C_2 \xrightarrow{F} J^2 \xrightarrow{p} (J^2)^* \xrightarrow{F^*} C_1^{\mathrm{or}} + C_2.$$

The skew-symmetric form Ω^* on J^* is non-degenerate but its restriction to $\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)$ has a kernel. In [1] we relate this form with "Goldman-Weil-Petersson" forms on the peripheral tori: there is a form wp_s on each $H^1(T_s, \mathbb{Z}^2)$, $s = 1, \dots, \ell$, defined as the coupling of the cup product on H^1 with the scalar product $\langle , \rangle$ on $\mathbb{Z}^2$ defined by: [2]

$$\langle \begin{pmatrix} n \\ m \end{pmatrix}, \begin{pmatrix} n' \\ m' \end{pmatrix} \rangle = \frac{1}{3}(2nn' + 2mm' + nm' + n'm),$$

see [1, section 7.2].

For our purpose we rephrase the content of [1, Corollary 7.10] in the following:

5.3. Lemma. *We have* $\mathrm{Ker}(\Omega^*_{|\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)}) = \mathrm{Im}(p \circ F)$. *The skew-symmetric form Ω^* therefore induces a* symplectic *form on the quotient*

$$(J^* \cap \mathrm{Ker}(F^*))/\mathrm{Im}(p \circ F).$$

Moreover: there is a symplectic isomorphism — defined over $\mathbb{Q}$ — between this quotient and the space $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{C}^2)$ equipped with the direct sum $\oplus_s \mathrm{wp}_s$, still denoted wp.

[2] This product should be interpreted as the Killing form on the space of roots of $\mathfrak{sl}(3, \mathbb{C})$ through a suitable choice of basis.

5.4. Let us briefly explain how Lemma 5.3 follows from [1]. First recall from [1, section 7.3] that given an element $z \in \mathcal{R}(M, \mathcal{T})$ we may compute the holonomy of a loop $c \in H_1(T_s)$ and get an upper triangular matrix; let $(\frac{1}{C^*}, 1, C)$ be its diagonal part. The application which maps $c \otimes \begin{pmatrix} n \\ m \end{pmatrix}$ to $C^m (C^*)^n$ yields the *holonomy map*

$$\mathrm{hol} : \mathcal{R}(M, \mathcal{T}) \to \oplus_{s=1}^{\ell} \mathrm{Hom}(H_1(T_s, \mathbb{Z}^2), \mathbb{C}^\times).$$

Whatever determination log of the logarithm we may choose, the differential of $\log \circ \mathrm{hol}$ is well defined. Given an element $z \in \mathcal{R}(M, \mathcal{T})$ and identifying $T_z \mathcal{R}(M, \mathcal{T})$ with a subspace of $\mathbb{C} \otimes (J^2)^*$, Lemma 7.5 of [1] then express twice the differential of $\log \circ \mathrm{hol}$ as the restriction of a linear map

$$\mathbb{C} \otimes (J^2)^* \to \mathbb{C}^{2\ell}. \tag{5.4.1}$$

This map is obtained — after tensorization with $\mathbb{C}$ — as the composition of the map h^*, dual to the map h defined in [1, section 7.4], with the projection to $\oplus_s H^1(T_s, \mathbb{C}^2) \cong \mathbb{C}^{4\ell}$ (using the bases (a_s, b_s)). The symplectic isomorphism of Lemma 5.3 is then given (up to a rationnal constant) by the differential of $\log \circ \mathrm{hol}$.

Now let $z \in \mathcal{R}(M, \mathcal{T}^+)$. The key point of the proof of Theorem 5.2 is the following:

5.5. Lemma.
- *For every $\xi \neq 0$ in $\mathcal{A}(z)$, we have $\Omega^*(\xi, \bar{\xi}) \neq 0$,*
- $(\mathbb{C} \otimes (\mathrm{Im}(p \circ F)) \cap \mathcal{A}(z) = \{0\}$.

Proof. The second point is a direct consequence of the first one. Indeed let

$$\xi \in (\mathbb{C} \otimes (\mathrm{Im}(p \circ F)) \cap \mathcal{A}(z).$$

It follows from the first point in Lemma 5.3 that $\Omega^*(\xi, \bar{\xi}) = 0$. If the first point holds, then it forces ξ to be null.

Now $\Omega^*(\xi, \bar{\xi})$ can be computed locally on each tetrahedron T_μ: Since ξ belongs to the subspace $\mathbb{C} \otimes J^* \subset \mathbb{C} \otimes (J^2)^*$, it is determined by the coordinates $\xi_{ij}^\mu = \xi(e_{ij}^\mu)$, $\xi_{ik}^\mu = \xi(e_{ik}^\mu)$ etc... Now, w.r.t. the symplectic form Ω, the basis vector e_{ij}^μ is orthogonal to all the basis vector except e_{ik}^μ and $\Omega(e_{ij}^\mu, e_{ik}^\mu) = -1$. By duality we therefore have

$$\begin{aligned} \Omega^*(\xi, \bar{\xi}) &= \sum_{\mu=1}^{\nu} \sum_{i=1}^{4} (\bar{\xi}_{ij}^\mu \xi_{ik}^\mu - \xi_{ij}^\mu \bar{\xi}_{ik}^\mu) \\ &= \sum_{\mu=1}^{\nu} \sum_{i=1}^{4} |\xi_{ij}^\mu|^2 \left(\frac{1}{\overline{z_{il}(T_\mu)}} - \frac{1}{z_{il}(T_\mu)} \right). \end{aligned}$$

Here the last equality follows from the fact that $\xi \in \mathcal{A}(z)$. We conclude because for each μ and i we have (up to a nonzero constant):

$$\mathrm{Im} \left(\frac{1}{\overline{z_{il}(T_\mu)}} - \frac{1}{z_{il}(T_\mu)} \right) > 0.$$

□

5.6. Let $\mathcal{L}(z)$ be the image of $\mathcal{A}(z)$ in $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{C}^2)$. It follows from the previous lemma and the fact that the map is defined over $\mathbb{Q}$ (see Lemma 5.3) that $\mathcal{L}(z)$ is a totally isotropic subspace isomorphic to $\mathcal{A}(z) \cap (\mathbb{C} \otimes (J^* \cap \mathrm{Ker}(F^*))$ and verifies that for any $\chi \neq 0$ in $\mathcal{L}(z)$, we have $\mathrm{wp}(\chi, \bar{\chi}) \neq 0$.

The space $\oplus_{s=1}^{\ell} H^1(T_s,\mathbb{C}^2)$ decomposes as the sum of two subspaces: $\sum_s [a_s]\otimes\mathbb{C}^2$ and $\sum_s [b_s]\otimes\mathbb{C}^2$ (where $[a_s]$, resp $[b_s]$, denotes the Poincaré dual to a_s, resp b_s). Both are Lagrangian subspaces and are invariant under complex conjugation. To prove theorem 5.2, it remains to prove that $\mathcal{L}(z)$ projects surjectivily onto $\sum_s [a_s]\otimes\mathbb{C}^2$. The dimension $\dim\mathcal{L}(z)$ may be computed. In fact, by duality, we have:

$$\dim(J^*\cap\operatorname{Ker}(F^*)) = \dim(\operatorname{Im}(p)\cap\operatorname{Ker}(F^*)) = \dim(J^2)^* - \dim(\operatorname{Im}(F)+\operatorname{Ker}(p)).$$

But we obviously have:

$$\dim(\operatorname{Im}(F)+\operatorname{Ker}(p)) = \dim\operatorname{Ker}(p) + \dim\operatorname{Im}(F) - \dim(\operatorname{Ker}(p)\cap\operatorname{Im}(F)).$$

On the other hand we have $\dim J^2 = 16\nu$, $\dim\operatorname{Ker}(p) = 8\nu$ and[3] $\dim\operatorname{Im}(F) = \dim C_1^{\mathrm{or}} + \dim C_2 = 4\nu$. It finally follows from the proof of [1, Lemma 7.13] that $\dim(\operatorname{Ker}(p)\cap\operatorname{Im}(F)) = 2\ell$. We conclude that

$$\dim(J^*\cap\operatorname{Ker}(F^*)) = 4\nu + 2\ell.$$

Now $\dim\mathcal{A}(z) = 4\nu$. The intersection $\mathcal{A}(z)\cap J^*\cap\operatorname{Ker}(F^*)$ is therefore of dimension at least 2ℓ and $\mathcal{L}(z)$ is a totally isotropic subspace of dimension at least 2ℓ in a symplectic space of dimension 4ℓ: it is a Lagrangian subspace. Theorem 5.2 now immediately follows from the following lemma. $\square$

Remark. The preceeding considerations give a combinatorial proof that the image of $\mathcal{R}(M,\mathcal{T})$ is a Lagragian subvariety of the space of representation of the fundamental group of the boundary of M.

5.7. Lemma. *We have:*

$$\mathcal{L}(z)\cap\sum_s [b_s]\otimes\mathbb{C}^2 = \{0\}.$$

Proof. Suppose that χ belongs to this intersection. Since $\sum_s [b_s]\otimes\mathbb{C}^2$ is a Lagrangian subspace invariant under complex conjugation, the complex conjugate $\bar\chi$ also belongs to $\sum_s [b_s]\otimes\mathbb{C}^2$ and we have

$$\mathrm{wp}(\chi,\bar\chi) = 0.$$

Since χ also belongs to $\mathcal{L}(z)$, Lemma 5.5 finally implies that $\chi = 0$. $\square$

5.8. Rigid points. In general if $z\in\mathcal{R}(M,\mathcal{T})$, the space $\mathcal{L}(z)$ is still a Lagrangian subspace. Replacing Lemma 5.5 by the *assumption* that

$$(\mathbb{C}\otimes(\operatorname{Im}(p\circ F))\cap\mathcal{A}(z) = \{0\}, \tag{5.8.1}$$

the proof of Theorem 5.2 still implies that $\mathcal{R}(M,\mathcal{T})$ is (locally around z) a smooth complex manifold of dimension 2ℓ and the choice of a 2ℓ-dimensional subspace of $\oplus_{s=1}^{\ell} H^1(T_s,\mathbb{C}^2)$ transverse to $\mathcal{L}(z)$ yields a choice of local coordinates. A point z verifying (5.8.1) is called a *rigid point* of $\mathcal{R}(M,\mathcal{T})$: indeed, at such a point, you cannot deform the representation without deforming its trace on the boundary tori. Note that if there exists a point $z\in\mathcal{R}(M,\mathcal{T})$ such that the condition (5.8.1) is satisfied, then (5.8.1) is satisfied for almost every point in the same connected component: this transversality condition may be expressed as the non-vanishing of a determinant of a matrix with entries in $\mathbb{C}(z)$. In the next section we provide explicit examples of all the situations that can occur.

[3] Note that the map F is injective.

5.9. Proof of Theorem 1.1. Theorem 1.1 does not immediately follow from Theorem 5.2 since M may not admit an ideal triangulation. Recall however that M has a finite regular cover M' that do admit an ideal triangulation. We may therefore apply Theorem 1.1 to M' and the proof follows from the general (certainly well known) lemma.

5.10. Lemma. *Let M' be a finite regular cover of M. Let ρ and ρ' be the geometric representations for M and M'.*

Then one cannot deform ρ without deforming ρ'.

Proof. Let γ_i be a finite set of loxodromic element generating $\pi_1(M)$. Let n be the index of $\pi_1(M')$ in $\pi_1(M)$. Then γ_i^n is a loxodromic element of $\pi_1(M')$.

Hence $\rho'(\gamma_i^n) = (\rho(\gamma_i))^n$ is a loxodromic elements in $\mathrm{PGL}(3,\mathbb{C})$. The crucial though elementary remark is that its n-th square roots form a finite set of $\mathrm{PGL}(3,\mathbb{C})$. So, once ρ' is fixed, the determination of a representation ρ such that $\rho' = \rho_{|\pi_1(M')}$ requires a finite number of choices: we should choose a n-th square root for each $\rho'(\gamma_i^n)$ among a finite number of them. □

6. Examples

In this section we describe exact solutions of the compatibility equations which give all unipotent decorations of the triangulation with two tetrahedra of the figure eight knot's sister manifold. This manifold has one cusp, so is homotopic to a compact manifold whose boundary consits of one torus. In term of theorem 5.2, we are looking to the fiber over $(1,1)$ of the map $z \mapsto (A, A^*)$. We show that besides rigid decorations (i.e. isolated points in the fiber) we obtain non-rigid ones. Namely four 1-parameter families of unipotent decorations.

Among the rigid decorations, one corresponds to the (complete) hyperbolic structure and belongs to $\mathcal{R}(M,\mathcal{T}^+)$. The rigidity then follows from theorem 5.2. At the other isolated points, the rigidity is merely explained by the transversality between $\mathcal{A}(z)$ and $\mathrm{Im}(p \circ F)$, as explained in subsection 5.8.

As for the non-rigid components, their existence shows firstly that rigidity is not granted at all. Moreover the geometry of the fiber over a point in $(\mathbb{C}^*)^2$ appears to be possibly complicated, with intersections of components. The map from the (decorated) representation variety $\mathcal{R}(M,\mathcal{T})$ to its image in the representation variety of the torus turns out to be far from trivial from a geometric point of view.

Let us stress out that these components contain also points of special interest: there are points corresponding to representations with value in $\mathrm{SL}(2,\mathbb{C})$ which are rigid *inside* $\mathrm{SL}(2,\mathbb{C})$, but not anymore inside $\mathrm{SL}(3,\mathbb{C})$.

The analysis of this simple example seems to indicate that basically anything can happen, at least outside of $\mathcal{R}(M,\mathcal{T}^+)$.

6.1. The figure eight sister manifold. This manifold M and its triangulation $\mathcal{T}$ is described by the gluing of two tetrahedra as in Figure 3. Let z_{ij} and w_{ij} be the coordinates associated to the edge ij. We will express all the equations in terms of these edges coordinates (as the face coordinates are monomial in edges coordinates, see (2.2.1)).

The variety $\mathcal{R}(M,\mathcal{T})$ is then given by relations (2.2.2) and (2.2.3) among the z_{ij} and among the w_{ij} plus the faces and edges conditions (2.3.1) and (2.3.2).

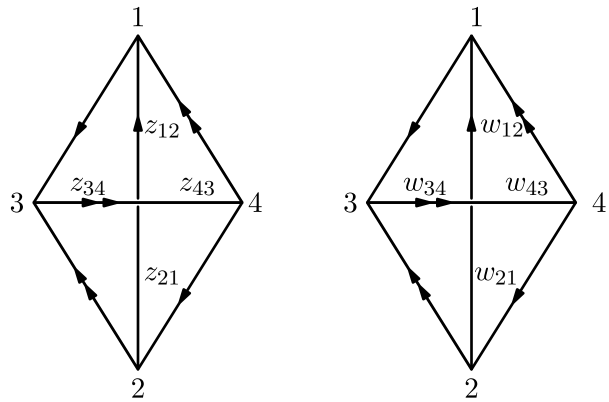

FIGURE 3. The figure eight sister manifold represented by two tetrahedra.

In this case, the edge equations are:

$$(6.1.1) \qquad (L_e) \begin{cases} e_1 := z_{23}z_{34}z_{41}w_{23}w_{34}w_{41} - 1 &= 0, \\ e_2 := z_{32}z_{43}z_{14}w_{32}w_{43}w_{14} - 1 &= 0, \\ e_3 := z_{12}z_{24}z_{31}w_{12}w_{24}w_{31} - 1 &= 0, \\ e_4 := z_{21}z_{42}z_{13}w_{21}w_{42}w_{13} - 1 &= 0. \end{cases}$$

and the face equations are:

$$(6.1.2) \qquad (L_f) \begin{cases} f_1 := z_{21}z_{31}z_{41}w_{12}w_{32}w_{42} - 1 &= 0, \\ f_2 := z_{12}z_{32}z_{42}w_{21}w_{31}w_{41} - 1 &= 0, \\ f_3 := z_{13}z_{43}z_{23}w_{14}w_{34}w_{24} - 1 &= 0, \\ f_4 := z_{14}z_{24}z_{34}w_{13}w_{23}w_{43} - 1 &= 0. \end{cases}$$

Moreover, one may compute the eigenvalues of the holonomy in the boundary torus (see, [1, section 7.3]) taking $A = z_{12}\frac{1}{w_{32}}z_{41}\frac{1}{w_{21}}$, $A^* = \frac{1}{z_{21}}\frac{w_{14}w_{41}}{w_{32}}\frac{1}{z_{14}}\frac{w_{34}w_{43}}{w_{21}}$, $B = z_{31}\frac{1}{w_{14}}z_{42}\frac{1}{w_{23}}$, $B^* = \frac{1}{z_{13}}\frac{w_{23}w_{32}}{w_{14}}\frac{1}{z_{24}}\frac{w_{14}w_{41}}{w_{23}}$ or, equivalently :

$$(6.1.3) \quad (L_{h,A,A^*,B,B^*}) \begin{cases} h_A &:= & w_{32}w_{21}A - z_{12}z_{41} &= 0, \\ h_{A^*} &:= & z_{21}w_{32}z_{14}w_{21}A^* - w_{14}w_{41}w_{34}w_{43} &= 0, \\ h_B &:= & w_{14}w_{23}B - z_{31}z_{42} &= 0, \\ h_{B^*} &:= & z_{13}w_{14}z_{24}w_{23}B^* - w_{23}w_{32}w_{14}w_{41} &= 0. \end{cases}$$

If $A = B = A^* = B^* = 1$ the solutions of the equations correspond to unipotent structures.

6.2. Methods. The computational problem to be solved is the description of a constructible set of $\mathbb{C}^{24}$ defined by the union of the edge equations (L_e), the face equations (L_f), the equations modelizing unipotent structures $(L_{h,1,1,1,1})$ augmented by a set of relations between some of the variables (L_r) and a set of inequations for consistancy (the coordinates are supposed to be different from 0 and 1), with :

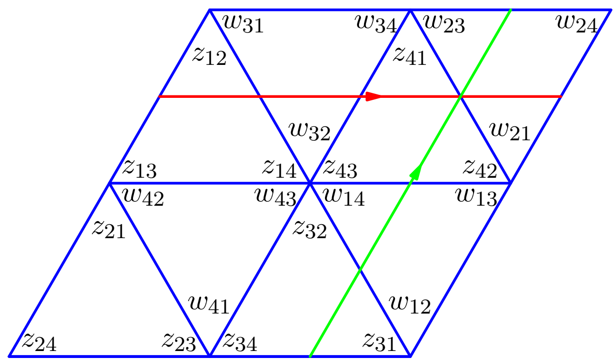

FIGURE 4. The boundary holonomy of the figure eight sister manifold. The red line corresponds to A and the green line to B

(6.2.1)

$$L_r := \begin{cases} w_{13} = \dfrac{1}{1-w_{12}}, & w_{14} = \dfrac{w_{12}-1}{w_{12}}, & w_{23} = \dfrac{w_{21}-1}{w_{21}}, & w_{24} = \dfrac{1}{1-w_{21}}, \\ w_{31} = \dfrac{1}{1-w_{34}}, & w_{32} = \dfrac{w_{34}-1}{w_{34}}, & w_{41} = \dfrac{w_{43}-1}{w_{43}}, & w_{42} = \dfrac{1}{1-w_{43}}, \\ z_{13} = \dfrac{1}{1-z_{12}}, & z_{14} = \dfrac{z_{12}-1}{z_{12}}, & z_{23} = \dfrac{z_{21}-1}{z_{21}}, & z_{24} = \dfrac{1}{1-z_{21}}, \\ z_{31} = \dfrac{1}{1-z_{34}}, & z_{32} = \dfrac{z_{34}-1}{z_{34}}, & z_{41} = \dfrac{z_{43}-1}{z_{43}}, & z_{42} = \dfrac{1}{1-z_{43}}. \end{cases}$$

After a straightforward substitution of the relations L_r in the equations

$$\{e_1, \dots, e_4, f_1, \dots, f_4, h_{A|A=1}, h_{A^*|A^*=1}, h_{B|B=1}, h_{B^*|B^*=1}\},$$

one shows that the initial problem is then equivalent to describing the constructible set defined by a set of 12 polynomial equations

$$\mathcal{E} := \left\{x \in \mathbb{C}^8, P_i(x) = 0,\ i = 1, \dots, 12,\ P_i \in \mathbb{Z}[\mathcal{X}]\right\},$$

in 8 unknowns

$$\mathcal{X} = \{z_{12},\ z_{21},\ z_{34},\ z_{43},\ w_{12},\ w_{21},\ w_{34},\ w_{43}\},$$

and a set of 16 polynomial inequations $\mathcal{I} := \left\{x \in \mathbb{C}^8, u(x) \neq 0, u(x) \neq 1, u \in \mathcal{X}\right\}$.

The Zariski-closure of this constructible set, say $\overline{\mathcal{E} \setminus \mathcal{I}}$ can be computed using direct elimination (many equations are still linear in terms of the variables) and by saturating carefully the resulting ideal by the polynomials $u, u-1, u \in \mathcal{X}$ using classical tools, such as Gröbner bases (see for example [3, chapter 4]).

Using Gröbner bases allows also to compute Hilbert's degrees and dimensions, subsets of algebraically independent variables, as well as Zariski's closures of projections onto some selected coordinates.

For example, the Zariski closure of projection onto the coordinates after some saturations is defined by

$$(w_{34} - z_{43})(-2w_{34}w_{43}z_{43} + w_{34}w_{43} + w_{43}w_{34}z_{43}^2 + z_{43} - w_{43}z_{43}) = 0.$$

Splitting the system with respect to the above factors and reproducing the same kind of computations (saturations, substitutions) iteratively, we finally obtain some 0-dimensional components and four 1-dimensional components.

Each component (0 or 1 dimensional) can be described in the same way: a polynomial P (in one or 2 variables) over $\mathbb{Q}$ such that each coordinate z_{ij} or w_{ij} is an algebraic (over $\mathbb{Q}$) function of the roots of P. In particular, they naturally come in families of Galois conjugates : this is no surprise, as the equations defining $\mathcal{R}(M,\mathcal{T})$ have integer coefficients.

We do not go further in the description of the computations wich will be part of a more general contribution by the last two authors. Let us just mention that the process gives us an exhaustive description of all the components of the constructible set we study. Moreover, the interested reader may easily check that the given solutions verify indeed all the equations.

6.3. Rigid unipotent decorations. We are looking after the isolated points of the set $\mathcal{U} = \{z \in \mathcal{R}(M,\mathcal{T}) \mid A = A^* = B = B^* = 1\}$. There are 4 Galois family of such points

They are described by four irreducible polynomials with integer coefficients in one variable. Two of them are of degree 2 and the other two of degree 8.

The first polynomial is the minimal polynomial of the sixth root of unity $\frac{1+i\sqrt{3}}{2}$. For a root $\omega^\pm = \frac{1\pm i\sqrt{3}}{2}$, the following defines an isolated point in $\mathcal{U}$:

$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega^\pm$$

The solution associated to ω^+ is easily checked to correspond to the hyperbolic structure on M: it is the geometric representation as we called it. The other one is its complex conjugate.

A point of $\mathcal{R}(M,\mathcal{T})$ corresponding to a representation with value inside PU$(2,1)$ (we call such representations CR) with unipotent boundary holonomy was obtained in [4] and is parametrized by the same polynomial, the z and w coordinates being this time given by:

$$z_{12} = z_{21} = -\omega \quad z_{34} = z_{43} = -(\omega^\pm)^2,$$
$$w_{12} = w_{21} = -\omega^2 \quad w_{34} = w_{43} = -\omega^\pm.$$

The two other isolated 0-dimensional components have degree 8 and their minimal polynomial are respectively:

$$P = X^8 - X^7 + 5\,X^6 - 7\,X^5 + 7\,X^4 - 8\,X^3 + 5\,X^2 - 2\,X + 1 = 0$$

and

$$Q(X) = P(1-X) = 0.$$

We do not describe all the z and w coordinates in terms of their roots (for the record, let us mention that z_{43} is directly given by the root). None of these 16 representations are hyperbolic (i.e. inside SL$(2,\mathbb{C})$) nor CR.

6.4. Non-rigid components. The four 1-parameter families of solutions are described as follows: let $\tau^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ be one of the two real roots of $X^2 = X + 1$.

Then the roots $X^2 - XY - Y^2$ define two one-parameter families meeting at $(0,0)$: $X = \tau^\pm Y$. They parametrize four one-parameter families of points of $\mathcal{U}$: $(S_1^\pm)$ and $(S_2^\pm)$. We obtain

$$(S_1^\pm) \quad \begin{cases} z_{12} = w_{12} = \dfrac{X+Y}{X-1}, & z_{21} = w_{21} = 1 + Y \\ z_{34} = w_{34} = \dfrac{X^2 + X - Y}{X(X-1)}, & z_{43} = w_{43} = X. \end{cases}$$

A solution of $S_1^{\pm}$ is CR iff $z_{21} = x + iy$ belongs to the the real circle

$$(x - \tau^{\pm})^2 + y^2 = 1.$$

Among them we obtain only two solutions belonging to $\mathrm{SL}(2, \mathbb{C})$ (and they even belong to $\mathrm{SL}(2, \mathbb{R})$):

$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = 1 + \tau^{\pm}.$$
$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = -\tau^{\pm}.$$

This points are then rigid inside $\mathrm{SL}(2, \mathbb{C})$ but not inside $\mathrm{SL}(3, \mathbb{C})$.

The other two one-parameter families are parametrized as follows

$$(S_2^{\pm}) \quad \begin{cases} z_{12} = w_{21} = 1 + \dfrac{Y}{X} - \dfrac{(X+1)(Y+1)}{X^2 + X - 1} +, & z_{21} = w_{12} = \dfrac{X + Y - 1}{Y - 1}, \\ z_{34} = w_{43} = X + Y, & z_{43} = w_{34} = 1/Y \end{cases}$$

In this case we have

$$\begin{array}{rcl} z_{12}z_{21} = z_{34}z_{43} = w_{12}w_{21} = w_{34}w_{43} & = & \tau^{\pm} + 1, \\ z_{13}z_{31} = z_{24}z_{42} = w_{13}w_{31} = w_{24}w_{42} & = & \tau^{\pm}, \\ z_{14}z_{41} = z_{23}z_{32} = w_{14}w_{41} = w_{23}w_{32} & = & \tau^{\pm} - 1 \end{array}$$

All of these solutions correspond to a CR-spherical structure.

Note that $(S_1^{\pm})$ and $(S_2^{\pm})$ intersect in a solution both hyperbolic and CR-spherical, i.e. inside $\mathrm{SL}(2, \mathbb{R})$:

$$z_{12} = z_{21} = w_{12} = w_{21} = -\tau^{\pm}, z_{34} = z_{43} = w_{34} = w_{43} = \tau^{\pm}.$$

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