The intersection problems of parametric curve and surfaces by means of matrix based implicit representations

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Abstract In this paper, we introduce and study a new implicit representation of parametric curves and parametric surfaces . We show how these representations which we will call the matrix implied, establish a bridge between geometry and linear algebra, thus opening the possibility of a more robust digital processing. The contribution of this approach is discussed and illustrated on important issues of geometric modeling and Computer Aided Geometric Design (CAGD) : The curve/curve, curve/surface and surface/surface intersection problems, the point-on-curve and inversion problems, the computation of singularities points.

Keyword Geometric modeling, parametric curves and surfaces, intersection, pencils of matrix, singular points.

1 Introduction

Rational algebraic curves and surfaces can be described in some different ways, the most common being parametric and implicit representations. Parametric representations describe the geometric object as the closed image of a rational map and implicit representations describe it as the zero set of a polynomial equation. Both representations have a wide range of applications in Computer Aided Geometric Design and Geometric Modeling. A parametric representation is much easier for drawing a surface but more difficult for checking if a point lies on a surface whereas an implicit representation is more difficult for drawing a surface but much easier for checking if a point lies on a surface. Implicit representation of parametric curves and parametric surfaces as a matrix has been addressed many times in literature (for example [MC91, SC95, CGZ00, BCD03]). However, it has always been in order to write an

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implicit equation as the determinant of a square matrix. The case of planar curves is particularly well known because one always know how to find such a simple square matrix. We can read the article of T. Sederberg and F. Chen [SC95]who introduced this technique to the problems intersection between plane curves for modeling geometric. The case of parametric surfaces is especially much more difficult because the geometry of their parameterizations becomes richer with the inevitable appearance of base points (these are points where a parameterization is not well defined). In order to find a square matrix whose determinant is an implicit equation, must be restricted to particular classes of parameterizations [CGZ00, BCD03, AC06], which turns out to be very restrictive in practice.

In this paper, we show how, by releasing the constraint matrix square, we can easily form a implicit representation in the form of a matrix for a parametric surface very general. The matrix in question is no longer square, but still allows to characterize the surface: the cancellation a determinant is replaced here by a property of rank drop. In addition, treatment of intersection problems can be reduced to linear algebra computations digital, allowing the operation of robust tools and performing for the approximate calculation, such as the singular value decomposition, calculating eigenvalues and generalized eigenvectors. Note that these implicit representation matrices can be seen as a bridge between the parametric representation of curve, surface and its implicit representation.

This article covers a series of works [BJ03, BC05, BCJ09, BCAS10, BD07, BB10], which led to the notion of implicit representation matrix of a parametric curve or a parametric surface, together with the development of applications for intersection problems in geometric modeling [BBM09, BB10, BB12]. Its content is part of authour's PhD thesis [Ba11].

2 Matrix based implicit representations of parametric surfaces

2.1 Construction of matrix representations

Given a parametric algebraic surface, we briefly recall from [BJ03, BC05] how to build a matrix that *represents* this surface in a sense that we will make explicit. So suppose given a parameterization

$$\begin{aligned} \mathbb{P}^2_{\mathbb{R}} &\xrightarrow{\phi} \mathbb{P}^3_{\mathbb{R}} \\ (s:t:u) &\mapsto (f_1:f_2:f_3:f_4)(s,t,u) \end{aligned}$$

of a surface $\mathbf{S}$ such that $\gcd(f_1,\ldots,f_4) \in \mathbb{R}\setminus\{0\}$. Set $d := \deg(f_i) \geq 1$, $i = 1,2,3,4$ and denote by x,y,z,w the homogeneous coordinates of the projective space $\mathbb{P}^3_{\mathbb{R}}$. Notice that s,t,u are the homogeneous parameters of the surface $\mathbf{S}$ and that an affine parameterization of $\mathbf{S}$ can be obtained by "inverting" one of these parameters; for instance, setting $s' = s/u$ and $t' = t/u$ we get the following affine parameterization of **S**:

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{\phi} \mathbb{R}^3 \\ (s',t') &\mapsto \left(\frac{f_1(s',t',1)}{f_4(s',t',1)}, \frac{f_2(s',t',1)}{f_4(s',t',1)}, \frac{f_3(s',t',1)}{f_4(s',t',1)} \right) \end{aligned}$$

The implicit equation of **S** is a homogeneous polynomial $S(x,y,z,w) \in \mathbb{R}[x,y,z,w]$ of smallest degree such that $S(f_1,f_2,f_3,f_4) = 0$ (observe that it is defined up to multiplication by a nonzero element in $\mathbb{R}$). It is well known that the quantity $\deg(\mathbf{S})\deg(\phi)$ is equal to d^2 minus the number of common roots of f_1,f_2,f_3,f_4 in $\mathbb{P}^2_{\mathbb{R}}$, that are called *base points* of the parameterization ϕ, counted with suitable multiplicities (see for instance [BJ03, Theorem 2.5] for more details). The notation $\deg(\mathbf{S})$ stands for the degree of the surface **S** that is nothing but the degree of the implicit equation of **S**.

The notation $\deg(\phi)$ stands for the degree of the parameterization ϕ (co-restricted to **S**) that, roughly speaking, measures the number of times the surface **S** is drawn by the parameterization ϕ. More precisely, $\deg(\phi)$ is equal to the number of pre-images of a general point on **S** by the parameterization ϕ.

For all non negative integer ν, we build a matrix $\mathtt{M}(\phi)_\nu$ as follows. Consider the set $\mathscr{L}(\phi)_\nu$ of polynomials of the form

$$a_1(s,t,u)x + a_2(s,t,u)y + a_3(s,t,u)z + a_4(s,t,u)w$$

such that

- $a_i(s,t,u) \in \mathbb{R}[s,t,u]$ is homogeneous of degree ν for all $i = 1,\dots,4$,
- $\sum_{i=1}^4 a_i(s,t,u)f_i(s,t,u) \equiv 0$ in $\mathbb{R}[s,t,u]$.

The set $\mathscr{L}(\phi)_\nu$ has a natural structure of $\mathbb{R}$-vector space of finite dimension because each polynomial $a_i(s,t,u)$ is homogeneous of degree ν and that the set of homogeneous polynomials of degree ν in the variables s,t,u is a $\mathbb{R}$-vector space of dimension $\binom{\nu+2}{2}$ with canonical basis the set of monomials $\{s^\nu, s^{\nu-1}t, \cdots, u^\nu\}$. So, denote by $L^{(1)},\dots,L^{(n_\nu)}$ a basis of the $\mathbb{R}$-vector space $\mathscr{L}(\phi)_\nu$; it can be computed by solving a single linear system whose indeterminates are the coefficients of the polynomials $a_i(s,t,u)$, $i = 1,2,3,4$. The matrix $\mathtt{M}(\phi)_\nu$ is then by definition the matrix of coefficients of $L^{(1)},\dots,L^{(n_\nu)}$ as homogeneous polynomials of degree ν in the variables s,t,u. In other words, we have the equality of matrices:

$$\left[s^\nu \; s^{\nu-1}t \; \cdots \; u^\nu \right] \mathtt{M}(\phi)_\nu = \left[L^{(1)} \; L^{(2)} \; \cdots \; L^{(n_\nu)} \right].$$

Notice that we have chosen for simplicity the monomial basis for the $\mathbb{R}$-vector space of homogeneous polynomials of degree ν in s,t,u. However, any other choice, for instance the Bernstein basis, can be made without affecting the result.

For all integer $\nu \geq 2d-2$, the matrix $\mathtt{M}(\phi)_\nu$ is said to be a *representation matrix* of ϕ because it satisfies the following properties under the assumption that the base points of ϕ, if any, form locally a complete intersection, which means that at each base point, the ideal of polynomials (f_1, f_2, f_3, f_4) can be generated by two equations (see [BJ03, Definition 4.8] for more details):

- The entries of $\mathtt{M}(\phi)_\nu$ are linear forms in $\mathbb{R}[x,y,z,w]$.
- The matrix $\mathtt{M}(\phi)_\nu$ has $\binom{\nu+2}{2}$ rows (which is nothing but the dimension of the $\mathbb{R}$-vector space of homogeneous polynomials of degree ν in three variables, here s,t,u) and possesses at least as much columns as rows.
- The rank of $\mathtt{M}(\phi)_\nu$ is $\binom{\nu+2}{2}$.
- When specializing $\mathtt{M}(\phi)_\nu$ at a given point $P \in \mathbb{P}^3_{\mathbb{R}}$, its rank drops if and only if P belongs to $\mathbf{S}$.
- The greatest common divisor of the $\binom{\nu+2}{2}$-minors of $\mathtt{M}(\phi)_\nu$ is equal to the implicit equation of $\mathbf{S}$ raised to the power $\deg(\phi)$.

From a computational point of view, the matrix $\mathtt{M}(\phi)_\nu$ with the smallest possible value of ν has to be chosen. It is rarely a square matrix. Also, notice that the last property given above is never used for computations; our aim is to keep the matrix $\mathtt{M}(\phi)_\nu$ as an implicit representation of $\mathbf{S}$ in place of its implicit equation.

There are many results that lead to enlarge the above family of matrices and to make it available in other contexts. Since a detailed overview of these results is not the main purpose of this paper, we just recall them shortly with appropriate references to the literature:

- The hypothesis on the base points of ϕ can be relaxed. If the base points are locally *almost* complete intersection, meaning that they are locally given by three (and not two) equations, then the above family of matrices can still be constructed and provide a matrix representation of the surface $\mathbf{S}$ plus a certain product of hyperplanes that can be described from the parameterization ϕ. In addition, the bound $2d-2$ for the integer ν can be decreased. See [BC05, BJ03].
- In our setting, ϕ parameterizes what is called a triangular Bezier patch. It turns out that a similar family of matrices $\mathtt{M}(\phi)_\nu$ can be built for parameterizations of tensor product surfaces, and even for any parameterization whose parameter space is a projective toric variety (triangular and tensor product surfaces are particular cases of parameterizations whose parameter space is a projective toric variety). We refer the interested reader to [BD07, BDD09].
- To build the matrices $\mathtt{M}(\phi)_\nu$ we used what is called moving planes, that is to say syzygies of the parameterization ϕ. It is actually possible to build another family of matrices by taking into account moving quadrics, i.e. syzygies associated to the square of the ideal generated by the parameterization of ϕ. In this way, we get a family containing smaller matrices whose entries are either linear or quadratic forms in $\mathbb{R}[x,y,z,w]$. In some sense, they generalize the matrices given in [CGZ00] and [BCD03]. See [BCAS10].

Example 1. The Steiner surface $\mathbf{S}$ of degree 2 parameterized by

$$\phi_1 : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (s^2+t^2+u^2 : tu : st : su)$$

which admits the matrix representation

$$M(x,y,z,w) := \begin{pmatrix} -x & 0 & -y & 0 & -y & y & 0 & z & 0 \\ y & -y & 0 & w & 0 & -x & -y & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & z & 0 & -x \\ w & 0 & 0 & -y & 0 & z & 0 & -y & y \\ 0 & w & 0 & 0 & 0 & z & 0 & 0 & y \\ w & 0 & 0 & 0 & z & 0 & 0 & 0 & y \end{pmatrix}.$$

Example 2. Let **S** be the rational surface of degree 3 which is parametrized by

$$\phi : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : f_2 : f_3 : f_4)$$

where

$$f_1 = s^3 + t^2u, f_2 = s^2t + t^2u, f_3 = s^3 + t^3, f_4 = s^2u + t^2u.$$

Then, a matrix representation of **S** is

$$\begin{pmatrix} 0 & 0 & 0 & w-y & 0 & 0 & z-x \\ w & 0 & 0 & x & w-y & 0 & 0 \\ x-y-z & 0 & 0 & -z & 0 & w-y & 0 \\ 0 & w & 0 & 0 & x & 0 & -y \\ 0 & x-y-z & w & 0 & -z & x & y+z-x \\ 0 & 0 & x-y-z & 0 & 0 & -z & 0 \end{pmatrix}$$

2.2 Points on surface and inversion problem

Suppose given a parameterization ϕ of a parametric surface **S** and a point P in $\mathbb{P}^3$. Denote by $\mathtt{M}(\phi)_\nu$ a matrix representation of ϕ for some integer $\nu \geq \nu_0$. Since its entries are linear forms in the variables s,t,u, one can evaluate $\mathtt{M}(\phi)_\nu$ at P and get a matrix with coefficients in the ground field $\mathbb{R}$. Then, we have that

$$\operatorname{rank}(\mathtt{M}(\phi)_\nu(P)) < \nu + 1 \text{ if and only if } P \in \mathbf{S}.$$

This property answers the point-on curve problem.

If $\operatorname{rank}\mathtt{M}(\phi)_\nu(P) = \operatorname{rank}\mathtt{M}(\phi)_\nu - 1 = \nu$ then P has a unique pre-image $(s_0 : t_0 : u_0)$ by ϕ and moreover, this pre-image can be recovered from the computation of a generator, say $W_P = (w_0, \ldots, w_\nu) \in \mathbb{R}^{\nu+1}$, of the kernel of the transpose of $\mathtt{M}(\phi)_\nu(P)$. Indeed, if $b_0(s,t,u), \ldots, b_\nu(s,t,u)$ is the basis of $\mathbb{R}[s,t,u]_\nu$ that has been chosen to build $\mathtt{M}(\phi)_\nu$, then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$W_P = \lambda\,(b_0(s_0,t_0,u_0), \ldots, b_\nu(s_0,t_0,u_0)).$$

For instance, suppose that $b_i(s,t,u) = s^i t^j u^{\nu-i-j}$, $0 \leq i,j \leq \nu, i+j \leq \nu$ (the usual monomial basis), then $(s_0 : t_0 : u_0) = (w_2 : w_1 : w_0)$.

Obviously, the inversion problems have been tranfered to compute kernel of transposed matrix of $\mathtt{M}(\phi)_\nu(P)$ that exits numerical effective algorithms such as SVD (Singular Value Decomposition).

We also point out that the points $P \in \mathbf{S}$ such that $\operatorname{rank}\mathtt{M}(\phi)_\nu(P) = \operatorname{rank}\mathtt{M}(\phi)_\nu - 1 = \nu$ are precisely the regular points on **S**, that is to say that all the points that do not verify this property are singular points on **S**.

3 Curve/surface Intersection

Suppose given a parametric surface **S** represented by a homogeneous and irreducible implicit equation $S(x,y,z,w) = 0$ in $\mathbb{P}^3_{\mathbb{R}}$ and a rational space curve $\mathscr{C}$ represented by a parameterization

$$\Psi : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}} : (s:t) \mapsto (x(s,t) : y(s,t) : z(s,t) : w(s,t))$$

where $x(s,t), y(s,t), z(s,t), w(s,t)$ are homogeneous polynomials of the same degree and without common factor in $\mathbb{R}[s,t]$.

A standard problem in non linear computational geometry is to determine the set $\mathscr{C} \cap \mathbf{S} \subset \mathbb{P}^3_{\mathbb{R}}$, especially when it is finite. One way to proceed, is to compute the roots of the homogeneous polynomial

$$S(x(s,t), y(s,t), z(s,t), w(s,t)) \tag{1}$$

because they are in correspondence with $\mathscr{C} \cap \mathbf{S}$ through the regular map Ψ. Observe that (1) is identically zero if and only if $\mathscr{C} \cap \mathbf{S}$ is infinite, equivalently $\mathscr{C} \subset \mathbf{S}$ (for $\mathscr{C}$ is irreducible).

Assume that $M(x,y,z,w)$ is a matrix representation of the surface **S**, meaning a representation of the polynomial $S(x,y,z,w)$. By replacing the variables x,y,z,w by the homogeneous polynomials $x(s,t), y(s,t), z(s,t), w(s,t)$ respectively, we get the matrix

$$M(s,t) = M(x(s,t), y(s,t), z(s,t), w(s,t))$$

therefore, we have the following easy property: for all point $(s_0 : t_0) \in \mathbb{P}^1_{\mathbb{R}}$ the rank of the matrix $M(s_0,t_0)$ drops if and only if the point $(x(s_0,t_0) : y(s_0,t_0) : z(s_0,t_0) : w(s_0,t_0))$ belongs to the intersection locus $\mathscr{C} \cap \mathbf{S}$.

It follows that points in $\mathscr{C} \cap \mathbf{S}$ associated to points $(s:t)$ such that $s \neq 0$, are in correspondence with the set of values $t \in \mathbb{R}$ such that $M(1,t)$ drops of rank strictly less than its row and column dimensions i.e. the set of generalized eigenvalues of $M(1,t)$. Now, we present a tenichque of linear algebra which permet us to obtain the regular part of the pencil matrices.

3.1 Linearization of a polynomial matrix

We begin with some notation. Let A and B be two matrices of size $m \times n$ with coefficients in $\mathbb{R}$. We will call a generalized eigenvalue of A and B a value in the set

$$\lambda(A,B) := \{t \in \mathbb{K} : \operatorname{rank}(A - tB) < \min\{m,n\}\}.$$

In the case $m = n$, the matrices A and B have n generalized eigenvalues if and only if $\operatorname{rank}(B) = n$. If $\operatorname{rank}(B) < n$, then $\lambda(A,B)$ can be finite, empty or infinite. Moreover, if B is invertible then $\lambda(A,B) = \lambda(AB^{-1}, I) = \lambda(AB^{-1})$, which is the ordinary spectrum of AB^{-1}.

Suppose given an $m \times n$-matrix $M(t) = (a_{i,j}(t))$ with polynomial entries $a_{i,j}(t) \in \mathbb{R}[t]$. It can be equivalently written as a polynomial in t with coefficients $m \times n$-matrices with entries in $\mathbb{R}$: if $d = \max_{i,j}\{\deg(a_{i,j}(t))\}$ then

$$M(t) = M_d t^d + M_{d-1} t^{d-1} + \ldots + M_0$$

where $M_i \in \mathbb{R}^{m \times n}$.

The generalized companion matrices A, B of the matrix $M(t)$ are the matrices with coefficients in $\mathbb{R}$ of size $((d-1)m+n) \times dm$ that are given by

$$A = \begin{pmatrix} 0 & I_m & \ldots & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & I_m \\ M_0^t & M_1^t & \ldots & \ldots & M_{d-1}^t \end{pmatrix}$$

$$B = \begin{pmatrix} I_m & 0 & \ldots & \ldots & 0 \\ 0 & I_m & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & I_m & 0 \\ 0 & 0 & \ldots & \ldots & -M_d^t \end{pmatrix}$$

where I_m stands for the identity matrix of size m and M_i^t stands for the transpose of the matrix M_i. These companion matrices allows to *linearize* the polynomial matrix $M(t)$ in the sense that there exists two unimodular matrices $E(t)$ et $F(t)$ with coefficients in $\mathbb{C}[t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(t)\,(A - tB)\,F(t) = \left(\begin{array}{c|c} {}^tM(t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \tag{2}$$

Then, we have

$$\operatorname{rank} M(t) \text{ drops} \Leftrightarrow \operatorname{rank}(A - tB) \text{ drops}.$$

We recall t such that $\operatorname{rank}(A-tB)$ drops the generalized eigenvalues of the pencil matrices $A-tB$. So. we transformed the computation of generalized eigenvalues of the matrix polynomial $M(t)$ into the computation of generalized eigenvalues of a pencil of matrices $A-tB$. If the matrices A,B were two square matrices, then we could easily compute their generalized eigenvalues by the QZ-algorithm [GVL96]. Therefore, our next task is to reduce the pencil $A-tB$ into a square pencil that keeps the information we are interested in.

3.2 Extracting the regular part of a non square pencil of matrices

For any couple constant matrices A, B of size $p\times q$, there exist constant invertible matrices P and Q such that the pencil $P(A-tB)Q$ is of the block-diagonal form

$$\operatorname{diag}\{L_{i_1},\ldots,L_{i_s},L^t_{j_1},\ldots,L^t_{j_u},\Omega_{k_1},\ldots,\Omega_{k_v},A'-tB'\}$$

where A',B' are square matrices and B' is invertible. The dimension $i_1,\ldots,i_s,j_1,\ldots,j_u,k_1,..,k_v$ and the determinant of $A'-tB'$ (up to a scalar) are independent of the representation. Where $L_k(t),\Omega_k(t)$ are the two matrices of size $k\times(k+1)$ and $k\times k$ respectively, defined by

$$L_k(t)=\begin{pmatrix} 1 & t & 0 & \ldots & 0\\ 0 & 1 & t & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \ldots & 1 & t & 0\\ 0 & 0 & \ldots & 1 & t\end{pmatrix},$$

$$\Omega_k(t)=\begin{pmatrix} 1 & t & 0 & \ldots & 0\\ 0 & 1 & t & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \ldots & \ldots & 1 & t\\ 0 & 0 & \ldots & 0 & 1\end{pmatrix}.$$

One called this form the Kronecker canonical form of a pencil of matrices (see for instance [Gan66, p. 31-34]).

It is interesting to notice that the above decomposition can be computed within $O(p^2q)$ arithmetic operations. We refer the reader to [BVD88] for a proof, as well as for an analysis of the stability of this decomposition.

Following the ideas developed in [BVD88] and the reduction methods exploited in [Mou05], we now describe an algorithm that allows to remove the Kronecker blocks L_k, L_k^t and Ω_k of the pencil of matrices $A-tB$ in order to extract the regular pencil $A'-tB'$. We also refer the reader to [BBM09] for more details.

We start with a pencil $A-tB$ where A,B are constant matrices of size $p\times q$ with coefficients in a field $\mathbb{R}$. Set $\rho = \operatorname{rank} B$. In the following algorithm, all computational steps are easily realized via the classical LU-decomposition.

- Transform B into its column echelon form; that amounts to determine unitary matrices P_0 and Q_0 such that

$$B_1 = P_0 B Q_0 = [\underbrace{B_{1,1}}_{\rho} \mid \underbrace{0}_{q-\rho}]$$

  where $B_{1,1}$ is an echelon matrix. Then, compute

$$A_1 = P_0 A Q_0 = [\underbrace{A_{1,2}}_{\rho} \mid \underbrace{A_{1,2}}_{q-\rho}]$$

- Transform $A_{1,2}$ into its row echelon form; that amounts to determine unitary matrices P_1 and Q_1 such that

$$P_1 A_{1,2} Q_1 = \left(\frac{A'_{1,2}}{0}\right)$$

  where $A'_{1,2}$ has full row rank while keeping $B_{1,1}$ in echelon form.
  Then we obtain a new pencil of matrices, namely $A_2 - tB_2$.
- Starting from $j=2$, repeat the above steps 1 and 2 for the pencil $A_j - tB_j$ until the $p_j \times q_j$ matrix B_j has full column rank, that is to say until $\operatorname{rank} B_j = q_j$.
- If B_j is not a square matrix, then we repeat the above procedure with the transposed pencil $A_j^t - tB_j^t$.

At last, we obtain the regular pencil $A' - tB'$ where A', B' are two square matrices and B' is invertible. Moreover, we have the

$$\operatorname{rank}(A-tB) \text{ drops} \Leftrightarrow \operatorname{rank}(A'-tB') \text{ drops}.$$

We are now ready to state our algorithm for solving the curve/surface intersection problem:

Algorithm 1: Matrix intersection algorithm

Input: A matrix representation of a surface **S** and a parametrization of a rational space curve $\mathscr{C}$.
Output: The intersection points of **S** and $\mathscr{C}$.
1. Compute the matrix representation M(t).
2. Compute the generalized companion matrices A and B of M(t).
3. Compute the companion regular matrices A' and B'.
4. Compute the eigenvalues of (A', B').
5. For each eigenvalue t_0, the point $P(x(t_0):y(t_0):z(t_0):w(t_0))$ is one of the intersection points.

Remark that this algorithm returns all the points in $\mathscr{C} \cap \mathbf{S}$ except possibly the point $\phi(1:0)$. This latter point can be treated independently.

Example 3. Let $\mathbf{S}$ be the sphere that we suppose given as the image of the parametrization

$$\phi : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : f_2 : f_3 : f_4)$$

where

$$f_1 = s^2 + t^2 + u^2, f_2 = 2su, f_3 = 2st, f_4 = s^2 - t^2 - u^2$$

Let $\mathscr{C}$ be the twisted cubic which is parametrized by

$$x(t) = 1, y(t) = t, z(t) = t^2, w(t) = t^3.$$

The computation of a matrix representation of the sphere $\mathbf{S}$ gives

$$\begin{pmatrix} -y & 0 & z & x+w \\ 0 & -y & -x+w & -z \\ z & x+w & y & 0 \end{pmatrix}.$$

Now, a point P belongs to the intersection of $\mathbf{S}$ and $\mathscr{C}$ if and only if $P = (1 : t : t^2 : t^3)$ and t is one of the generalized eigenvalues of the matrix

$$M(t) = \begin{pmatrix} -t & 0 & t^2 & 1+t^3 \\ 0 & -t & -1+t^3 & -t^2 \\ t^2 & 1+t^3 & t & 0 \end{pmatrix}.$$

As before, we easily compute the eigenvalues and find:

$$t_1 = 0.7373527056, t_2 = -0.7373527056,$$
$$t_3 = 0.5405361044 + 1.031515287i, t_4 = -0.5405361044 - 1.031515287i,$$
$$t_5 = 0.5405361044 - 1.031515287i, t_6 = -0.5405361044 + 1.031515287i.$$

All these eigenvalues have multiplicity 1. They all correspond to one intersection point between $\mathbf{S}$ and $\mathscr{C}$ which has multiplicity 1. By Bezout Theorem, we find here all the intersection points between these two algebraic varieties (all of them are at finite distance).

4 Surface/surface intersection

Computing the intersection between two parametric algebraic surfaces is a fundamental task in Computer Aided Geometric Design. Several methods and approaches have been developed for that purpose. Some of them are based on the use of matrix representations of the objects because they allow to transform geometric operations on the intersection curve into matrix operations. This approach seems to have been

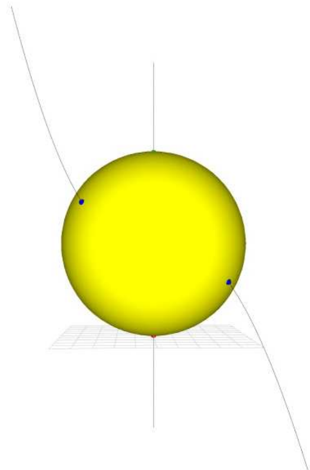

Fig. 1: Intersection of the sphere and the twisted cubic, the axis Oz

first introduced by J. Canny and D. Manocha in their paper [MC91]. Roughly speaking, it consists in representing the implicit equation of one of the two surfaces as the determinant of a certain matrix, necessarily square. Then, instead of using this implicit equation, the matrix itself is used as a representation of this first parametric surface and then a matrix representation of the intersection curve is easily obtained by substituting the implicit variables with the parameterization of the second surface. In this section, we extend the approach of Canny and Manocha about surface/surface intersection for a significantly larger class of parameterizations which have been introduce in Section 2.

Suppose given two distinct parametric surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$. A standard problem in non linear computational geometry is to determine the set $\mathbf{S}_1 \cap \mathbf{S}_2$ which is a curve in $\mathbb{P}^3_{\mathbb{C}}$. As we explained above, one can build a representation matrix of $\mathbf{S}_1$ that we will denote by $M(x,y,z,w)$. Let

$$\Psi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} : (s:t:u) \mapsto (a(s,t,u) : b(s,t,u) : c(s,t,u) : d(s,t,u))$$

be a parameterization of $\mathbf{S}_2$ where $a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)$ are homogeneous polynomials of the same degree and without common factor in $\mathbb{C}[s,t,u]$. By substituting in the matrix $M(x,y,z,w)$ the variables x,y,z,w by the homogeneous polynomials $a(s,t,u), b(s,t,u), c(s,t,u)$, $d(s,t,u)$ respectively, we get the matrix

$$M(s,t,u) := M(\Psi(s:t:u)) = M(a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)).$$

From the properties of the representation matrix $M(x,y,z,w)$, we known that $M(s,t,u)$ has maximal rank ρ (ρ is the number of rows of M). Moreover, for all point $(s_0 : t_0 : u_0) \in \mathbb{P}^2_{\mathbb{R}}$ we have

$$\operatorname{rank}(M(s_0,t_0,u_0)) < \rho \text{ if and only if } \begin{cases} \Psi(s_0:t_0:u_0) \in \mathbf{S}_1 \cap \mathbf{S}_2 \text{ or} \\ (s_0:t_0:u_0) \text{ is a base point of } \Psi. \end{cases} \quad (3)$$

The equivalence (3) shows that the spectrum of the matrix $M(s,t,u)$, that is to say the set

$$\left\{(s_0:t_0:u_0) \in \mathbb{P}^2_{\mathbb{R}} \text{ such that } \operatorname{rank} M(s_0,t_0,u_0) < \rho\right\},$$

yields the intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$ plus the base points of the parameterization Ψ of $\mathbf{S}_2$.

In [BB12] we proved that the spectrum of the matrix $M(s,t,u)$ is an algebraic curve in $\mathbb{P}^2_{\mathbb{R}}$, that is to say is equal to the zero locus of a homogeneous polynomial in $\mathbb{C}[s,t,u]$. In particular, there is no isolated points in the spectrum of $M(s,t,u)$. As a consequence if we use matrix representations to deal with the surface/surface intersection problem, we will end up at some point with a pencil of bivariate and non-square matrices that represents the intersection curve (after dehomogenization). Therefore, in order to be able to handle this intersection curve, for instance to determine its exact topology, it is necessary to extract a pencil of bivariate and square matrices that yields a matrix representation of the intersection curve as a matrix determinant. For that purpose, we develop an algorithm (called ΔW-Decomposition) based on the remarkable work of V. N. Kublanovskaya [KK96, Kub99].

We build two companion matrices $A(t)$ and $B(t)$ which allows to *linearize* the polynomial matrix $M(s,t,1)$ such that the spectrum of the matrix $M(s,t,1)$ coincides the spectrum of the matrix $A(t) - sB(t)$. Then, we provide an algorithm that extracts a square matrix whose determinant represents the intersection locus curve $\mathbf{S}_1 \cap \mathbf{S}_2$. A pencil of polynomial matrices $A(t) - sB(t)$ is equivalent to a pencil of the following form

$$\begin{pmatrix} M_{1,1}(s,t) & 0 & 0 \\ M_{2,1}(s,t) & M_{2,2}(s,t) & 0 \\ M_{3,1}(s,t) & M_{3,2}(s,t) & M_{3,3}(s,t) \end{pmatrix}$$

where the pencil $M_{2,2}(s,t)$ is a regular pencil that corresponds to the intersection locus curve $\mathbf{S}_1 \cap \mathbf{S}_2$.

Now, we get the following algorithm (for more details see [BB12]):

Algorithm 2: Matrix representation of an intersection curve

Input: Two parametric algebraic surfaces S_1 and S_2 such that the parameterization of S_1 has local complete intersection base points.

Output: The intersection curve $S_1 \cap S_2$ represented as a matrix determinant.

1. Compute a matrix representation of S_1, say $M(x,y,z,w)$.

2. Substitute x,y,z,w by the parameterization of S_2 in the matrix M to get a matrix $\mathbb{M}(s,t)$ (set $u=1$).

3. Compute the generalized companion matrices $A(s)$ and $B(s)$ associated to $\mathbb{M}(s,t)$.

4. Return the regular pencil of matrices $M_1(s,t) = A_1(s) - tB_1(s)$.

In comparison with [MC91], our algorithm returns a result of the same type: a determinant matrix representation of the intersection curve, but the class of parameterizations of surfaces for which step 1 can be performed is here dramatically extended. We present an illustrative example.

Example 4. We start with the Steiner surface S_1 parameterized by

$$\phi_1 : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (s^2+t^2+u^2 : tu : st : su)$$

which admits the matrix representation

$$M(x,y,z,w) := \begin{pmatrix} -x & 0 & -y & 0 & -y & y & 0 & z & 0 \\ y & -y & 0 & w & 0 & -x & -y & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & z & 0 & -x \\ w & 0 & 0 & -y & 0 & z & 0 & -y & y \\ 0 & w & 0 & 0 & 0 & z & 0 & 0 & y \\ w & 0 & 0 & 0 & z & 0 & 0 & 0 & y \end{pmatrix}.$$

We want to study the intersection between S_1 and the cubic surface S_2 parameterized by

$$\phi_2 : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (s^3+t^3 : stu : su^2+tu^2 : u^3).$$

As in the previous example, to determine the intersection between $\mathbf{S}_1$ and $\mathbf{S}_2$ we will compute the spectrum of the polynomial matrix

$$M(s,t,u) = \begin{pmatrix} -s^3-t^3 & 0 & -stu & 0 & -stu & stu & 0 & su^2+tu^2 & 0 \\ stu & -stu & 0 & u^3 & 0 & -s^3-t^3 & -stu & 0 & 0 \\ 0 & 0 & u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & -s^3-t^3 \\ u^3 & 0 & 0 & -stu & 0 & su^2+tu^2 & 0 & -stu & stu \\ 0 & u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & 0 & stu \\ u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & 0 & 0 & stu \end{pmatrix}.$$

By dehomogenizing with respect to the variable u, we consider

$$M(s,t) = \begin{pmatrix} -s^3-t^3 & 0 & -st & 0 & -st & st & 0 & s+t & 0 \\ st & -st & 0 & 1 & 0 & -s^3-t^3 & -st & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & s+t & 0 & -s^3-t^3 \\ 1 & 0 & 0 & -st & 0 & s+t & 0 & -st & st \\ 0 & 1 & 0 & 0 & 0 & s+t & 0 & 0 & st \\ 1 & 0 & 0 & 0 & s+t & 0 & 0 & 0 & st \end{pmatrix}.$$

Writing $M(s,t)$ under the form $M(s,t) = M_3t^3 + M_2t^2 + M_1t + M_0$, we obtain the generalized companion matrices of $M(s,t)$:

$$A(s) = \begin{pmatrix} 0 & I_6 & 0 \\ 0 & 0 & I_6 \\ M_0^t & M_1^t & M_2^t \end{pmatrix}, \quad B(s) = \begin{pmatrix} I_6 & 0 & 0 \\ 0 & I_6 & 0 \\ 0 & 0 & -M_3^t \end{pmatrix}.$$

Applying the algorithm that extracts a square matrix for the pencil $A^t(s) - tB^t(s)$, we obtain its regular part $M_1(s,t) = A_1(s) - tB_1(s)$ where

$$A_1(s) = \begin{pmatrix} s & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B_1(s) = \begin{pmatrix} -s^2 & 1 & 0 & 0 & 0 & -s & 0 & 0 & s & 0 & 0 & 0 \\ s^3 & s & s^4 & 1 & 0 & 0 & s^3 & 0 & s^2 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its yields a plane curve of degree 6 whose implicit equation is $\det(M_1(s,t)) = t^2 + 2st + s^2t^2 + 2s^3t^3 - st^5 + s^2 - ts^5$. This plane curve parameterizes $\mathbf{S}_1 \cap \mathbf{S}_2$ through the regular map ϕ_2.

5 Matrix -based implicit representations of parametric curves in space

Let f_0, f_1, f_2, f_3 be homogeneous polynomials in $\mathbb{R}[s,t]$ of the same degree $d \geq 1$ such that their greatest common divisor (GCD) is a non-zero constant in $\mathbb{R}$. Consider the regular map

$$\begin{aligned} \mathbb{P}^1_{\mathbb{R}} &\xrightarrow{\phi} \mathbb{P}^3_{\mathbb{R}} \\ (s:t) &\mapsto (f_0 : f_1 : f_2 : f_3)(s,t). \end{aligned}$$

The image of ϕ is an algebraic curve $\mathscr{C}$ in $\mathbb{P}^3_{\mathbb{R}}$ which is called a *rational* curve.

5.1 Construction of representation matrix

Consider the set of syzygies of $\mathbf{f} := (f_0, f_1, f_2, f_3)$, that is to say the set

$$\mathrm{Syz}(\mathbf{f}) = \left\{ (g_0(s,t), \ldots, g_3(s,t)) \; : \; \sum_{i=0}^{3} g_i(s,t) f_i(s,t) = 0 \right\} \subset \bigoplus_{i=0}^{3} \mathbb{R}[s,t]$$

From a classical structure theorem of commutative algebra called the Hilbert-Burch Theorem (see for instance [Eis, §20.4]), $\mathrm{Syz}(\mathbf{f})$ is known to be a *free* and graded $\mathbb{R}[s,t]$-module of rank 3. Moreover, there exists non-negative integers μ_1, μ_2, μ_3 and 3 vectors of polynomials

$$(u_{i,0}(s,t), u_{i,1}(s,t), u_{i,2}(s,t), u_{i,3}(s,t)) \in \mathrm{Syz}(\mathbf{f}) \subset \mathbb{R}[s,t]^4 \quad (i = 1,2,3) \tag{4}$$

such that

- for all $i \in \{1,2,3\}$, $j \in \{0,1,2,3\}$, $u_{i,j}(s,t)$ is a homogeneous polynomial in $\mathbb{R}[s,t]$ of degree $\mu_i \geq 0$,
- the 3 vectors in (4) form a $\mathbb{R}[s,t]$-basis of $\mathrm{Syz}(\mathbf{f})$,
- $\mu_1 + \mu_2 + \mu_3 = d$,
- For all $j \in \{0,\ldots,3\}$, the determinant of the matrix obtained by deleting the column $(u_{i,j})_{i=1,2,3}$ from the matrix

$$M(s,t) := \begin{pmatrix} u_{1,0}(s,t) & u_{1,1}(s,t) & u_{1,2}(s,t) & u_{1,3}(s,t) \\ u_{2,0}(s,t) & u_{2,1}(s,t) & u_{2,2}(s,t) & u_{2,3}(s,t) \\ u_{3,0}(s,t) & u_{3,1}(s,t) & u_{3,2}(s,t) & u_{3,3}(s,t) \end{pmatrix} \tag{5}$$

  is equal to $(-1)^j c f_j(s,t) \in \mathbb{R}[s,t]$ where $c \in \mathbb{R} \setminus \{0\}$.

A collection of vectors as in (4) that satisfy the above properties is called a μ-basis of the parameterization ϕ. It is important to notice that a μ-basis is far from being unique, but the collection of integers (μ_1, μ_2, μ_3) is unique if we order it. Therefore, in the sequel we will always assume that a μ-basis is ordered so that $0 \leq \mu_1 \leq \mu_2 \leq \mu_3$.

For all integer $i = 1,2,3$ and all integer $\nu \in \mathbb{N}$, consider the matrix $\mathrm{Sylv}_\nu(u_i)$ that satisfies to the identity

$$\left[s^\nu \; s^{\nu-1}t \; \cdots \; st^{\nu-1} \; t^\nu\right] \times \mathrm{Sylv}_\nu(u_i) = \left[s^{\nu-\mu_i}u_i \; s^{\nu-\mu_i-1}tu_i \; \cdots \; st^{\nu-\mu_i-1}u_i \; t^{\nu-\mu_i}u_i\right].$$

It is a $(\nu+1) \times (\nu - \mu_i + 1)$-matrix which usually appears as a building block in well known Sylvester matrices. It follows that the matrix

$$\mathrm{Sylv}_\nu(u_1,u_2,u_3) = \left(\mathrm{Sylv}_\nu(u_1) \middle| \mathrm{Sylv}_\nu(u_2) \middle| \mathrm{Sylv}_\nu(u_3) \right).$$

It has $\nu+1$ rows and $3(\nu+1)-d$ columns. Its entries are *linear forms* in $\mathbb{R}[x_0,\ldots,x_3]$; in particular, it can be evaluated at any point $(x_0 : \cdots : x_3) \in \mathbb{P}^3_{\mathbb{R}}$ and yields a matrix with coefficients in $\mathbb{R}$.

In [BB10], we proved that for all $\nu \geq \mu_3 + \mu_2 - 1$ the matrix $\mathtt{M}(\phi)_\nu := \mathrm{Sylv}_\nu(u_1,u_2,u_3)$ is a *matrix-based representation* of the curve $\mathscr{C}$ i.e

(i) $\mathtt{M}(\phi)_\nu$ is generically full rank, that is to say generically of rank $\nu+1$,
(ii)the rank of $\mathtt{M}(\phi)_\nu$ drops exactly on the curve $\mathscr{C}$.

Of course, in practice the most useful matrix is the smallest one, that is to say $\mathtt{M}(\phi)_{\mu_3+\mu_2-1}$.

Example 5. Let $\mathscr{C}$ be the rational space curve parameterized by

$$\begin{aligned} \mathbb{P}^1_{\mathbb{R}} &\xrightarrow{\phi} \mathbb{P}^3_{\mathbb{R}} \\ (s:t) &\mapsto (s^4 : s^3t : s^2t^2 : t^4). \end{aligned}$$

A μ-basis of $\mathscr{C}$ is given by

$$\begin{aligned} p &= -tx + sy \\ q &= -ty + sz, \\ r &= -t^2 z + s^2 w. \end{aligned}$$

We have $\mu_1 = \mu_2 = 1$, $\mu_3 = 2$ and hence $\mu_3 + \mu_2 - 1 = 2$. Therefore, we obtain the following representation matrix of ϕ:

$$\mathtt{M}(\phi)_2 = \begin{pmatrix} y & 0 & z & 0 & w \\ -x & y & -y & z & 0 \\ 0 & -x & 0 & -y & -z \end{pmatrix}.$$

5.2 Points on curves and inversion problems

Suppose given a parameterization ϕ of a rational curve $\mathscr{C}$ and a point P in $\mathbb{P}^3$. Denote by $\mathtt{M}(\phi)_\nu$ a matrix representation of ϕ for some integer $\nu \geq \mu_3 + \mu_2 - 1$. Since its entries are linear forms in the variables x_0, x_1, x_2, x_3, one can evaluate $\mathtt{M}(\phi)_\nu$ at P and get a matrix with coefficients in the ground field $\mathbb{R}$. Then, we have that

$$\operatorname{rank}\left(\mathtt{M}(\phi)_\nu(P)\right) < \nu + 1 \text{ if and only if } P \in \mathscr{C}.$$

This property answers the point-on curve problem.

If $\operatorname{rank}\mathtt{M}(\phi)_\nu(P) = \operatorname{rank}\mathtt{M}(\phi)_\nu - 1 = \nu$ then P has a unique pre-image $(s_0 : t_0)$ by ϕ and moreover, this pre-image can be recovered from the computation of a generator, say $W_P = (w_0, \ldots, w_\nu) \in \mathbb{R}^{\nu+1}$, of the kernel of the transpose of $\mathtt{M}(\phi)_\nu(P)$. Indeed, if $b_0(s,t), \ldots, b_\nu(s,t)$ is the basis of C_ν that has been chosen to build $\mathtt{M}(\phi)_\nu$, then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$W_P = \lambda \left(b_0(s_0, t_0), \ldots, b_\nu(s_0, t_0)\right).$$

For instance, suppose that $b_i(s,t) = s^i t^{\nu - i}$, $i = 0, \ldots, \nu$ (the usual monomial basis), then $(s_0 : t_0) = (w_1 : w_0)$ if $w_0 \neq 0$, otherwise $(s_0 : t_0) = (1 : 0)$.

We point out that the points $P \in \mathscr{C}$ such that $\operatorname{rank}\mathtt{M}(\phi)_\nu(P) = \operatorname{rank}\mathtt{M}(\phi)_\nu - 1 = \nu$ are precisely the regular points on $\mathscr{C}$, that is to say that all the points that do not verify this property are singular points on $\mathscr{C}$. We will come back again on this property and on the treatment of the singular points on $\mathscr{C}$ in the next section.

Example 6. Suppose that the parameterization ϕ is given by

$$
\begin{aligned}
f_0(s,t) &= 3s^4t^2 - 9s^3t^3 - 3s^2t^4 + 12st^5 + 6t^6, \\
f_1(s,t) &= -3s^6 + 18s^5t - 27s^4t^2 - 12s^3t^3 + 33s^2t^4 + 6st^5 - 6t^6, \\
f_2(s,t) &= s^6 - 6s^5t + 13s^4t^2 - 16s^3t^3 + 9s^2t^4 + 14st^5 - 6t^6, \\
f_3(s,t) &= -2s^4t^2 + 8s^3t^3 - 14s^2t^4 + 20st^5 - 6t^6.
\end{aligned}
$$

A μ-basis for $\mathscr{C}$ is

$$
\begin{aligned}
p &= (s^2 - 3st + t^2)x + t^2y \\
q &= (s^2 - st + 3t^2)y + (3s^2 - 3st - 3t^2)z, \\
r &= 2t^2z + (s^2 - 2st - 2t^2)w.
\end{aligned}
$$

From $\deg(p) = \deg(q) = \deg(r) = 2$, we have $\mu_3 + \mu_2 - 1 = 3$ and hence a matrix representation of $\mathscr{C}$ is given by

$$
\mathtt{M}(\phi)_3 = \begin{pmatrix} x+y & 0 & 3y-3z & 0 & 2z-2w & 0 \\ -3x & x+y & -y-3z & 3y-3z & -2w & 2z-2w \\ x & -3x & y+3z & -y-3z & w & -2w \\ 0 & x & 0 & y+3z & 0 & w \end{pmatrix}.
$$

Let $P = (1:1:1:1) \in \mathbb{P}^3$. Evaluating $\mathtt{M}(\phi)_3$ at P we find that

$$
\mathtt{M}(\phi)_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -3 & 2 & -4 & 0 & -2 & 0 \\ 1 & -3 & 4 & -4 & 1 & -2 \\ 0 & 1 & 0 & 4 & 0 & 1 \end{pmatrix}
$$

is of rank 4 so that P does not lie on $\mathscr{C}$.

If one evaluates the matrix $\mathtt{M}(\phi)_3$ at the point $P = (9:9:9:6) \in \mathbb{P}^3$ we obtain the matrix

$$
\mathtt{M}(\phi)_3(P) = \begin{pmatrix} 18 & 0 & 0 & 0 & 6 & 0 \\ -27 & 18 & -36 & 0 & -12 & 6 \\ 9 & -27 & 36 & -36 & 6 & -12 \\ 0 & 9 & 0 & 36 & 0 & 6 \end{pmatrix}.
$$

which has rank 3. Therefore, P is a smooth point on the curve $\mathscr{C}$. Moreover, the computation of the kernel of the transpose of $\mathtt{M}(\phi)_3(P)$ returns the vector $(1,1,1,1)$. Thus, we deduce that $P = \phi(1:1)$.

5.3 Rank of a representation matrix at a singular point

Let P be a point on $\mathscr{C}$. There exists at least one point $(s_1 : t_1) \in \mathbb{P}^1$ such that $P = \phi(s_1 : t_1)$. Now, let $\mathscr{H}$ be a plane in $\mathbb{P}^3$ passing through P, not containing $\mathscr{C}$ and denote by $H(x,y,z,w)$ an equation (a linear form in $\mathbb{R}[x,y,z,w]$) of $\mathscr{H}$. We have the following degree d homogeneous polynomial in $\mathbb{R}[s,t]$

$$H(f_0(s,t), f_1(s,t), f_2(s,t), f_3(s,t)) = \prod_{i=1}^{d}(t_i s - s_i t) \tag{6}$$

where the points $(s_i : t_i) \in \mathbb{P}^1$, $i = 1, \ldots, d$ are not necessarily distinct. We define the intersection multiplicity of $\mathscr{C}$ with $\mathscr{H}$ at the point P, denoted $i_P(\mathscr{C}, \mathscr{H})$, as the number of points $(s_i : t_i)_{i=1,\ldots,d}$ such that $\phi(s_i : t_i) = P$.

The *multiplicity* $m_P(\mathscr{C})$ of the point P on $\mathscr{C}$ is defined as the minimum of the intersection multiplicity $i_P(\mathscr{C}, \mathscr{H})$ where $\mathscr{H}$ runs over all the hyperplanes not containing $\mathscr{C}$ and passing through the point $P \in \mathscr{C}$, minimum which is reached with a sufficiently generic such $\mathscr{H}$.

Suppose given a representation matrix $\mathtt{M}(\phi)_\nu$ of the curve $\mathscr{C}$ which is built from the μ-basis p, q, r of degree $\mu_1 \leq \mu_2 \leq \mu_3$ correspondence. Its entries are linear forms in $\mathbb{R}[x, y, z, w]$ so that it makes sense to evaluate $\mathtt{M}(\phi)_\nu$ at a point in $\mathbb{P}^3$ to get a matrix $\mathtt{M}(\phi)_\nu(P)$ with entries in $\mathbb{R}$. In [BB10], we prove the property: Given a point P in $\mathbb{P}^3$, for all integer $\nu \geq \mu_2 + \mu_3 - 1$ we have

$$\operatorname{rank} \mathtt{M}(\phi)_\nu(P) = \nu + 1 - m_P(\mathscr{C}),$$

or equivalently $\operatorname{corank} \mathtt{M}(\phi)_\nu(P) = m_P(\mathscr{C})$.

This result provides a stratification of the points in $\mathbb{P}^3$ with respect to the curve $\mathscr{C}$. Indeed, we have that

- if P is such that $\operatorname{rank} \mathtt{M}(\phi)_\nu(P) = \nu + 1$ then $P \notin \mathscr{C}$,
- if P is such that $\operatorname{rank} \mathtt{M}(\phi)_\nu(P) = \nu$ then P is a regular point (i.e. of multiplicity 1) on $\mathscr{C}$,
- if P is such that $\operatorname{rank} \mathtt{M}(\phi)_\nu(P) = \nu - 1$ then P is singular point of multiplicity 2 on $\mathscr{C}$,
- and so on.

Moreover, an immediate consequence is that if P is a singular point on $\mathscr{C}$ then necessarily

$$2 \leq m_P(\mathscr{C}) \leq \mu_2 \text{ or } m_P(\mathscr{C}) = \mu_3. \tag{7}$$

One can read more details in [BB10] for computational singularities aspects of $\mathscr{C}$.

5.4 Curve/curve intersection

Suppose given two rational curves, say $\mathscr{C}_1$ parameterized by

$$\mathbb{P}^1 \xrightarrow{\phi_1} \mathbb{P}^3 : (s:t) \mapsto (f_0 : \cdots : f_3)(s,t) \tag{8}$$

and $\mathscr{C}_2$ parameterized by the regular map

$$\mathbb{P}^1 \xrightarrow{\phi_2} \mathbb{P}^3 : (s:t) \mapsto (g_0 : \cdots : g_3)(s,t). \tag{9}$$

Let $\mathtt{M}(\phi_1)_\nu$ be a representation matrix of $\mathscr{C}_1$ for a suitable integer ν. The substitution in $\mathtt{M}(\phi_1)_\nu$ of the variables x,y,z,w by the homogeneous parameterization of $\mathscr{C}_2$ yields the matrix

$$\mathtt{M}(\phi_1)_\nu(s,t) := \mathtt{M}(\phi_1)_\nu(g_0(s,t),\ldots,g_3(s,t))$$

As a consequence of the properties of a representation matrix, we have the following property: Let $(s_0:t_0)\in\mathbb{P}^1$, then rank $\mathtt{M}(\phi_1)_\nu(s_0,t_0) < \nu+1$ if and only if the point $\phi_2(s_0,t_0)$ belongs to the intersection locus $\mathscr{C}_1\cap\mathscr{C}_2$.

The set $\mathscr{C}_1\cap\mathscr{C}_2$ is in correspondence with the points of $\mathbb{P}^1$ where the rank of $\mathtt{M}(\phi_1)_\nu(s,t)$ drops. By setting $t=1$, the determination of the values of s such that the rank of $\mathtt{M}(\phi_1)_\nu(s,1)$ can be treated at the level of matrices (that is to say without any symbolic computation and in particular without any determinant computations) by using linearization technics and generalized eigenvalues computations. We repeat the algorithm which is presented in Section 3.

Algorithm 3: Intersection of two parametric curves

Input: Two parametric curves $\mathscr{C}_1$ and $\mathscr{C}_2$ given by (8) and (9).
Output: The intersection points of $\mathscr{C}_1$ and $\mathscr{C}_2$.
1. Compute the matrix representation $\mathtt{M}(\phi)_\nu(\phi_1)$ *of* $\mathscr{C}_1$ *for a suitable* ν.
2. Compute the generalized companion matrices A and B of $\mathtt{M}(\phi)_\nu(\phi_1)$.
3. Compute the companion regular matrices A' *and* B'.
4. Compute the eigenvalues of (A',B').
5. For each eigenvalue t_0, $\phi_2(t_0:1)$ *is an intersection point.*

Remark that this algorithm returns all the points in $\mathscr{C}_1\cap\mathscr{C}_2$ except possibly the point $\phi(1:0)$. This latter point can be treated independently.

6 Conclusion

This paper presents a new implicit representation concept of a parametric curve or a parametric surface. This representation is a matrix whose entries are linear forms in the coordinates of $\mathbb{R}^3$. This matrix representation characterizes a curve or a surface by a drop rank property. It is easily to calculate, in addition a useful tool for solving intersection problems. Moreover, its main interest is particularly to transform intersection problems into numerical linear algebra problems for which we have powerful and robust algorithms to solve such as singular value decomposition, calculating generalized eigenvalues or eigenvectors. Thus, in the context more particularly ray tracing on a surface set, this new approach could improve the robustness of the existing methods in particular situations.

All algorithms that we proposed above have been implemented in the software `Maple` that the corresponding files are available at `http://cgi.di.uoa.gr/~thanglb/`

Acknowledgements. An part of this work has been completed while the author worked at Department of Informatics and Telecommunications, University of Athens, Greece under partially supported by Marie-Curie Initial Training Network "SAGA" (ShApes, Geometry, Algebra), FP7-PEOPLE contract PITN-GA-2008-214584.

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