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# Projection onto the $k$ -Cosparse Set is NP-Hard

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**Abstract**—We investigate the computational complexity of a problem arising in the context of sparse optimization, namely, the projection onto the set of  $k$ -cosparse vectors w.r.t. some given matrix  $\Omega$ . We show that this projection problem is (strongly) NP-hard, even in the special cases where the matrix  $\Omega$  contains only ternary or bipolar coefficients. Interestingly, this is in stark contrast to the projection onto the set of  $k$ -sparse vectors, which is trivially solved by keeping only the  $k$  largest coefficients.

**Index Terms**—Compressed Sensing, Computational Complexity, Cosparsity, Projection

## I. INTRODUCTION

A central problem in compressed sensing (CS) is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad (\text{P}_0)$$

for a given matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $m < n$ , where  $\|\mathbf{x}\|_0$  denotes the number of nonzero entries in  $\mathbf{x}$ . This problem is known to be strongly NP-hard; the same is true for the variant with  $\mathbf{A}\mathbf{x} = \mathbf{b}$  replaced by  $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \leq \varepsilon$ .

Two related problems arise in signal and image processing, where the unknown signal  $\mathbf{x}$  to be estimated from a low-dimensional observation  $\mathbf{b} = \mathbf{A}\mathbf{x}$  cannot directly be modeled as being sparse. In the most standard approach,  $\mathbf{x}$  is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary  $\mathbf{D}$ , i.e.,  $\mathbf{x} = \mathbf{D}\mathbf{z}$  where the representation vector  $\mathbf{z}$  is sparse. Minimizing  $\|\mathbf{z}\|_0$  s.t.  $\mathbf{A}\mathbf{D}\mathbf{z} = \mathbf{b}$  is obviously also NP-hard.

The alternative *cosparse analysis model* [1] assumes that  $\Omega\mathbf{x}$  has many zeros, where  $\Omega$  is an analysis operator. Typical examples include finite difference operators; they are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\Omega\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad (1)$$

and was also shown to be NP-hard [1, Section 4.1].

A popular approach to solve (P<sub>0</sub>) is the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error  $\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2$  and a hard-thresholding step. In recent adaptations of IHT and related algorithms to the cosparse analysis setting (e.g., [2]), a key step is the projection onto the set of  $k$ -cosparse vectors, as a replacement for hard-thresholding (which is the projection onto the set of  $k$ -sparse vectors).

We show that  $k$ -cosparse projection is strongly NP-hard in general, which contrasts with the synthesis case where the corresponding operation (hard-thresholding) is extremely simple and fast.

## II. COMPLEXITY OF COSPARSE PROJECTION PROBLEMS

Given an  $r \times n$  matrix  $\Omega$  (with  $r > n$ ), an  $n$ -vector  $\omega$ , and a positive integer  $k$ , the (Euclidean) *projection of  $\omega$  onto the set of*

*vectors that are  $k$ -cosparse w.r.t.  $\Omega$*  is formally given by

$$\Pi_{\Omega,k}(\omega) := \arg \min_{\mathbf{z} \in \mathbb{R}^n} \{ \|\omega - \mathbf{z}\|_2 : \|\Omega\mathbf{z}\|_0 \leq k \}. \quad (k\text{-CoSP})$$

Our main result is the following.

**Theorem 1:** Given  $\Omega \in \mathbb{R}^{r \times n}$  ( $r > n$ ),  $\omega \in \mathbb{R}^n$ , and a positive integer  $k \in \mathbb{N}$ , for any  $p \in \mathbb{N} \cup \{\infty\}$ ,  $p > 1$ , it is NP-hard in the strong sense to solve the  $k$ -cosparse  $\ell_p$ -norm projection problem

$$\min_{\mathbf{z} \in \mathbb{R}^n} \{ \|\omega - \mathbf{z}\|_p^q : \|\Omega\mathbf{z}\|_0 \leq k \}, \quad (k\text{-CoSP}_p)$$

where  $q = p$  if  $p < \infty$  and  $q = 1$  if  $p = \infty$ . The problem remains strongly NP-hard even if  $\omega$  has only binary coefficients in  $\{0, 1\}$  (with exactly one entry nonzero) and  $\Omega$  has only ternary or bipolar coefficients in  $\{-1, 0, +1\}$  or  $\{-1, 1\}$ , respectively.

Our proof (see [3] for the details) works with the MIN-ULR<sub>0</sub><sup>−</sup>( $\mathbf{A}, K$ ) problem: Given a matrix  $\mathbf{A} \in \mathbb{Q}^{m \times n}$  and a positive integer  $K \in \mathbb{N}$ , decide whether there exists a vector  $\mathbf{z} \in \mathbb{R}^n$  such that  $\mathbf{z} \neq 0$  and at most  $K$  of the  $m$  equalities in the system  $\mathbf{A}\mathbf{z} = 0$  are violated. This problem was proven to be (strongly) NP-complete even for ternary or bipolar matrices  $\mathbf{A}$  in [4]. We show that one can answer MINULR<sub>0</sub><sup>−</sup>( $\mathbf{A}, K$ ) by at most  $n$  calls to an algorithm solving ( $k$ -CoSP <sub>$p$</sub> ), implying that such a projection algorithm cannot be polynomial in general unless P=NP.

Clearly, if a minimizer was known, we would also know the optimal value of ( $k$ -CoSP <sub>$p$</sub> ). Hence, computing a minimizer is at least as hard as solving ( $k$ -CoSP <sub>$p$</sub> ), and the complexity results of Theorem 1 carry over directly. In particular, we obtain the following result for the usual Euclidean projection:

**Corollary 1:** It is strongly NP-hard to compute  $\Pi_{\Omega,k}(\omega)$ .

In theoretical algorithmic applications of ( $k$ -CoSP), it had so far been assumed that the Euclidean projection problem ( $k$ -CoSP<sub>2</sub>) can be approximated efficiently, see, e.g., [2]. Our results refute this assumption to a certain degree, since NP-hardness in the strong sense implies that no *fully polynomial-time approximation scheme* (FPTAS) can exist unless P=NP. Thus, it remains a challenge to find (practically) efficient *approximation* algorithms for the  $k$ -cosparse projection problem ( $k$ -CoSP <sub>$p$</sub> ), or to establish further (perhaps negative) results concerning its approximability.

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