Generalized Null Space and Restricted Isometry Properties

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***Abstract*—We propose a theoretical study of the conditions guaranteeing that a decoder will obtain an *optimal* signal recovery from an underdetermined set of linear measurements. This special type of performance guarantee is termed instance optimality and is typically related with certain properties of the dimensionality-reducing matrix M. Our work extends traditional results in sparse recovery, where instance optimality is expressed with respect to the set of sparse vectors, by replacing this set with an arbitrary finite union of subspaces. We show that the suggested instance optimality is equivalent to a generalized null space property of M and discuss possible relations with generalized restricted isometry properties.**

I. Background

Traditional results in sparse recovery relate certain properties of a dimensionality-reducing matrix $\mathbf{M}$, considered as *encoder* to performance guarantees of certain explicit or implicit decoders Δ. A popular family of performance guarantees is coined *instance optimality*: a decoder is *instance optimal* at order k (with respect to $\mathbf{M}$ and two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$) if, for *every* vector $\mathbf{x}$

$$\|\Delta(\mathbf{M}\mathbf{x}) - \mathbf{x}\|_X \leq C\sigma_k(\mathbf{x})_Y \tag{1}$$

where $\sigma_k(\mathbf{x})_Y$ measures the *distance* (in the sense of the Y norm) from $\mathbf{x}$ to the set of k-sparse vectors $\Sigma_k \doteq \{\mathbf{z},\ \|\mathbf{z}\|_0 \leq k\}$.

Cohen *et al* [1] have shown that, for a given $\mathbf{M}$, the existence of an instance optimal decoder is equivalent to a null space property (NSP),

$$\|\mathbf{h}\|_X \leq C'\sigma_{2k}(\mathbf{h})_Y,\ \forall \mathbf{h} \in \text{Ker}\mathbf{M}. \tag{2}$$

However, the corresponding decoder is somewhat inconvenient to handle. To prove the instance optimality of more convenient decoders such as the minimum ℓ_1 norm decoder, one typically exploits restricted isometry properties (RIP) [2]. In recent years, several extensions to the classical RIP have been suggested, e.g. $\mathbf{D}$-RIP [3].

II. Our Work Contribution

Our main goal is to replace Σ_k, which is a finite union of k-dimensional subspaces, by an *arbitrary finite union of low-dimensional subspaces (UoS)* denoted by Σ. Such an extension to general UoS has been considered in a recent paper by Blumensath [4], which suggested generalized RIP for a specific choice of the Y norm. We address the broader class of generalized NSP[1].

One special case of interest covered by our study is that of the cosparse analysis model [5], which has recently drawn considerable attention. In this model the set Σ is the union of low-dimensional spaces associated with the analysis operator $\mathbf{\Omega}$. A popular example is the $\mathbf{\Omega}_{DIF}$ operator, associated to finite differences on the edges of a graph, e.g., a 2D regular grid [5]. This operator generates a UoS Σ covering the family of piecewise constant 2D signals.

We extend the results of Cohen *et al* and derive a more general instance optimality and its equivalent NSP, as stated below,

[1]This work has not been reported yet.

Theorem 1. *Given a dimensionality-reducing matrix $\boldsymbol{M}$, norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and an arbitrary UoS Σ, the following are equivalent:*
I. Instance optimality – there exists some decoder Δ such that

$$\|\Delta(\boldsymbol{M}\boldsymbol{x}) - \boldsymbol{x}\|_X \leq C d\left(\boldsymbol{x}, \Sigma\right)_Y,\ \forall \boldsymbol{x} \tag{3}$$

II. Generalized NSP:

$$\|\boldsymbol{h}\|_X \leq C' d\left(\boldsymbol{h}, \Sigma + \Sigma\right)_Y,\ \forall \boldsymbol{h} \in \mathit{Ker}\boldsymbol{M} \tag{4}$$

Here, $\Sigma + \Sigma = \{\mathbf{z}:\ \mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2,\ \mathbf{z}_1, \mathbf{z}_2 \in \Sigma\}$ and $d(\mathbf{x}, \Sigma)_Y$ is the distance (in the sense of the Y norm) from $\mathbf{x}$ to the set Σ.

A typical choice for X is the ℓ_2 norm and we also focus on this setup. It is well known from [1] that for the special case of $\Sigma = \Sigma_k$ and $Y = \ell_2$, instance optimality (with a constant C independent of the dimension) is only possible if $\mathbf{M}$ does not significantly decrease dimension. Under seemingly "mild" assumptions on Σ, we extend this result to more general UoS. One simple example where the required conditions are trivially met is when Σ is described by the $\mathbf{\Omega}_{DIF}$ analysis operator. In light of the "negative" result we just mentioned, it seems natural to seek for norms $\|\cdot\|_Y$ that can be used in a dimensionality reduction context.

One possible choice for the Y norm is the $\mathbf{M}$-norm, where $\|\mathbf{u}\|_{\mathbf{M}} \doteq \|\mathbf{u}\|_2 + \frac{\|\mathbf{M}\mathbf{u}\|_2}{\sqrt{\alpha}}$. For this specific choice, it was shown in [4] that the *lower* Σ+Σ-*RIP*, expressed as $\alpha\|\mathbf{x}\|_2^2 \leq \|\mathbf{M}\mathbf{x}\|_2^2$, for every $\mathbf{x} \in \Sigma + \Sigma$, implies the existence of a decoder satisfying $\|\Delta(\mathbf{M}\mathbf{x}) - \mathbf{x}\|_2 \leq 2d(\mathbf{x}, \Sigma)_{\mathbf{M}}$. Despite its apparent similarity with the ℓ_2-ℓ_2 instance optimality, this type of instance optimality can actually hold for highly dimensionality-reducing $\mathbf{M}$. Moreover, the M-norm is essentially the best possible norm for instance optimality.

In practice, one would like to exhibit more concrete decoders with the appropriate instance optimality. One approach is through iterative projection algorithms [4] with instance optimality with respect to the $\mathbf{M}$-norm. We intend to explore a different approach – through the minimization of some convex penalty. Our talk will discuss possible choices for the Y norm and their relations with generalized RIP.

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