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De nombreux jeux impliquant deux joueurs dans un graphe ont été étudiés en théorie des graphes : *Gendarmes et voleur, Ange et Démon, Observeur et surfeur, Dominants universels*, etc. Outre la capture d'un fugitif ou la lutte contre le feu, ces jeux ont aussi des applications dans les réseaux de télécommunications car, d'une part, ils permettent de mieux appréhender les structures des réseaux, et d'autre part, ils permettent de modéliser et d'étudier des problèmes de ces réseaux (e.g., problème de cache dans l'internet). Dans tous ces jeux, chaque joueur contrôle des jetons sur les sommets du graphe et selon les jeux, les joueurs peuvent: déplacer des jetons le long des arêtes du graphe, ajouter/supprimer des jetons, etc. Dans ce travail, nous proposons une approche générale en définissant un jeu qui constitue, entre autre, une relaxation fractionnaire de tous les jeux mentionnés ci-dessus. Pour ce jeu générique, nous montrons qu'il existe un algorithme en temps polynomial, en le nombre de sommets du graphe et le nombre de maximum de tours de jeu autorisés, pour décider si un des joueurs a une stratégie gagnante. Cet algorithme permet de calculer une stratégie efficace (approximation à un facteur $\log n$ près), gagnante avec forte probabilité, pour le problème de cache.

Keywords: Graph, Surveillance, Fractional Surveillance, Approximation

1   Introduction

In graph theory, various *combinatorial games* played on graphs have been studied. On the one hand, they provide a better understanding of graph structures which helped to design efficient algorithms to solve problems in telecommunication networks [KLNS12]. On the other hand, such games provide nice theoretical models for telecommunication network problems themselves [FGJM+12]. In this paper, we present a new approach, based on fractional relaxation of linear programming, for studying these problems. In particular, it leads to a polynomial-time algorithm that computes lower bound for these problems.

Combinatorial games. A *combinatorial game* refers to a turn-by-turn game played by two players, C and R, on a (di)graph. In graph theory, many such games have been studied and share several properties.

In *Cops and Robbers* games [AF84], Player C controls a team of $k \in \mathbb{N}^*$ cops and R controls one robber. C starts by choosing vertices of G to put its cops, then R chooses a vertex of G to put its robber. Then, turn-by-turn, first C may move each of its cops along one edge, then R may do the same with its robber. C wins if its cops are able to capture the robber, i.e. if a cop occupies the same vertex as the robber. The objective of R is to avoid capture indefinitely. A *winning strategy* for C is a function that describes the next move for C to win. It assigns to every *state* (positions of the cops and of the robber) the moves that must be done by the cops to ensure the capture of the robber whatever its moves may be. The main question is to decide if there is a *winning strategy* for C using a predetermined number of cops. The structural characterization of graphs for which C with k cops wins has led to many studies [BN11] and applications in telecommunication networks [KLNS12]. Unfortunately, deciding the smallest k such that C wins in a graph is PSPACE-complete [Mam12].

Another example of such a game has been defined to model the web-page caching and, more generally, prefetching problems [FGJM+12]. In the *surveillance game*, C first marks a node where R must place its token. Then, turn-by-turn, C marks at most $k \in \mathbb{N}^*$ nodes and R may move its token along an edge. C wins if it can mark all nodes before R reaches an unmarked vertex. Computing the smallest k such that C has a winning strategy is PSPACE-complete [FGJM+12].

[†]A full version of this paper can be found in [GNPS13].

The idea of a fractional game and results presented in this work can be applied to many other combinatorial games such as variants of Cops and Robber games [Mam12], *Angels and Devils* [BCG82], *Eternal Dominating Sets* [BCG+04] and *Eternal Vertex Cover* [KM11]. All these games can be described in an unified way. Initially, both players place some tokens, then, turn-by turn, first C plays then $\mathcal{R}$ plays. The goal of C is to reach some particular state while $\mathcal{R}$ wants to perpetually avoid it.

Fractional games. In order to unify and provide some solutions for these combinatorial games, we define the notion of *Fractional Combinatorial Game*. Specifically, we propose a framework for a combinatorial game where C and $\mathcal{R}$ are allowed to use fractions of tokens on the vertices of a graph of order n.

For instance, in a 4-node cycle C_4, it is easy to see that 2 cops are necessary to capture one robber in the integral (classical) game. On the other hand, if C initially places half-cops on three nodes of C_4 (i.e., using $3/2$ cops in total), we can prove that C can capture one splittable robber (exercise left to the reader).

Our results. We define the notion of a *fractional game* in which both players can split their tokens, of *semi-fractional game* in which only C can split its tokens and of *integral games* in which tokens are not splittable. In particular, all above mentioned games are integral games.

The main characteristic of these fractional games is that they satisfy some *convexity* property. Thanks to this property, we are able to design an algorithm that, given a fractional game in a n-node graph, decides whether there is a strategy for C to win in $t \in \mathbb{N}$ turns and, if C can win, that computes a corresponding winning strategy. This algorithm is polynomial in t and in *the size of the game*. In all above mentioned games, the size of the game is polynomial in n and, therefore, our algorithm can decide in time polynomial in n if C can win the fractional game in a number of turns polynomial in n.

Our second result is that, under a weak additional assumption which is satisfied in all games mentioned, C wins the fractional game in $t \geq 0$ turns if and only if it wins the semi-fractional game in t turns. Since if C wins the integral game, then it wins the semi-fractional game, this implies that C wins the integral game only if it wins the fractional game. Therefore, our algorithm computes lower bounds for the optimization problems corresponding to the integral games.

Lastly, in the surveillance game, we show an integral strategy, that wins against $\mathcal{R}$ with high probability. Moreover, the expected cost of this integral strategy is in $O(\log n)$ factor from the fractional strategy.

2 Fractional Games (Fractional Surveillance Game)

We define a family of *Fractional Combinatorial Games* in a n-node graph G (with vertex-set $V(G) = \{1, \ldots, n\}$) in which two players turn-by-turn modify a real vector representing their positions on G (the state of the game). In a Fractional Combinatorial Game, the moves allowed for each player are described as "convex" operators on vectors in $\mathbb{R}^{O(n)}$. The game is also defined by various convex sets of $\mathbb{R}^{O(n)}$: the set $\mathcal{W}_C$ of *winning states*, the set I of *initial states*, the set $\mathcal{V}$ of *valid states*. C wins the game if it reaches a state in $\mathcal{W}_C$ before $t \in \mathbb{N}$ turns. $\mathcal{R}$ wins otherwise or if it reaches a state not in $\mathcal{V}$.

Due to space restrictions, and for an easier description, we focus on the *Surveillance Game*. We describe the fractional variant of this game and the main guidelines of our algorithm to solve it.

The amount of tokens belonging to C on each vertex i of G is represented as a vector $c \in \mathbb{R}^n_+$ in which c_i is the amount of tokens on vertex i. Similarly, the amount of tokens of $\mathcal{R}$ is represented as a vector $r \in \mathbb{R}^n_+$. A *state* of the game is a pair $(c, r) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

At each turn, C is allowed to place fractions of marks on nodes of G but must place at most k marks in total. The set of possible moves for C is the set $\mathcal{X}_C(k)$ of vectors $x \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n x_i \leq k$. In other words, during its turn, C can move the current state of the game (c, r) to a state $(c + x, r)$ with $x \in \mathcal{X}_C(k)$. The set of possible moves for $\mathcal{R}$ is the set $\Delta_{\mathcal{R}}$ of matrices $\delta \in [0, 1]^n \times [0, 1]^n$ such that it satisfies these two restrictions: if (i, j) is not an arc of G then $\delta_{i,j} = 0$ and $\sum_{i=0}^n \delta_{i,j} = 1$ for all $j \in V(G)$. In other words, during its turn, $\mathcal{R}$ can move the game from state (c, r) to the state $(c, \delta r)$ with $\delta \in \Delta_{\mathcal{R}}$ and if δ is a matrix in $\Delta_{\mathcal{R}}$, then $\delta_{i,j}$ represents the amount of tokens of $\mathcal{R}$ that moves from vertex j to vertex i during a move of $\mathcal{R}$.

The *fractional surveillance game* played on a digraph G with initial state (c, r) is a turn-by-turn game starting with player C where C, during its turn, moves the current state (c, r) to $(c + x, r)$ with $x \in \mathcal{X}_C(k)$ and $\mathcal{R}$, during its turn, moves the current state (c, r) to $(c, \delta r)$ with $\delta \in \Delta_{\mathcal{R}}$. The game is won by $\mathcal{R}$ as soon as there exists $i \in V(G)$ such that $r_i > c_i$ in the current state of the game. Otherwise, C wins when $c_i \geq 1$ for all $i \in V(G)$.

The *fractional surveillance number*, $\text{fsn}(G,v)$, is the minimum k such that C can always win the fractional surveillance game against R regardless of its moves where the initial state (c^v, r^v) is given by $c_v = r_v = 1$ and for all other vertices i we have $c_i = r_i = 0$. Let a *round* be composed of one turn of C and one turn of R. One simple remark is that the game is over in at most $\lceil (|V(G)| - 1)/k \rceil$ rounds. This is a direct consequence from the fact that, at each step that is not the last, C can use all its k marks.

The integral surveillance game is the restriction of the fractional game when the states of the game are vectors of $\mathbb{N}^n \times \mathbb{N}^n$, the moves of C are vectors of $\mathbb{N}^n$ and the moves of R are $\{0,1\}_{n \times n}$ matrices.

3   Polynomial-time Algorithm for the Fractional Game

The main ideas to design a polynomial-time algorithm are as follows. We use a set of linear inequality (i.e., a linear program) to describe the set C_t of states from which C can win in $t > 0$ rounds. Then, in polynomial time, we compute the linear program describing C_{t+1}. More precisely, we prove:

Theorem 1. *For any digraph G and initial vertex $v \in V(G)$, there is a polynomial-time (in n) algorithm that decides if C can win a fractional game. If the answer is positive, our algorithm computes the corresponding winning strategy for C.*

Let R_0 be the set of winning states for C, i.e., the set of states (c,r) with $c_i \geq 1$, for any $i \leq n$. Let $i > 0$ and C_i (resp., R_i) be the set of the states (c,r) such that, starting from (c,r) and if the first move is done by C (resp., by R), C can win the game, in at most i rounds and using at most k marks per turn.

By induction on $i > 0$, we build a description of C_i from the one of R_{i-1}, then a description of R_i is obtained from the one of C_i. These descriptions have size (number of variables and inequalities) polynomial in n. This process is repeated until a description of C_F is obtained. Finally, C wins starting from v if and only if $(c^v, r^v) \in C_F$ which can be checked in polynomial time.

Obtaining C_t from R_{t-1}. Assume that R_{t-1} is described by a linear program with polynomial number of variables and inequalities. Note that $R_0 = \{(c,r) \in \mathbb{R}_+^{2n} \mid \forall i \leq n, c_i \geq 1, \text{ and } \sum_{i \leq n} r_i = 1\}$. It is not difficult to prove that:

Lemma 2. *The state (c,r) belongs to $C_t \Leftrightarrow c_i \geq r_i$ for all $i \in V(G)$ and there is $x \in X_C(k)$ such that $(c+x, r)$ belongs to R_{t-1}.*

Roughly, Lemma 2 states that C can win in t rounds if and only if C can do a move ($\exists x \in X_C(k)$) to reach a state $(c+x, r) R_{t-1}$ in which C wins in $t-1$ rounds when R starts playing.

Let L_R be the linear program describing R_{t-1}. Assume that L_R has (among others) variables c_i^{t-1} and r_i, $1 \leq i \leq n$, that represent the state of the game. To describe C_t, we add to L_R new variables c_i^t, $i \leq n$, describing the configuration of C at this turn and variables x_i ($i \leq n$) representing the amount of marks that must be added at node i by C at this turn. Then we add the following inequalities $x_i \geq 0$, $c_i^t \geq 0$, $c_i^t + x_i = c_i$, $c_i^t \geq r_i$ for all $i \in V(G)$ and $\sum_{i=1}^n x_i \leq k$. Clearly, a vector satisfying L_C is such that $(c_1^t, \ldots, c_n^t, r_1, \ldots, r_n) \in C_t$. Moreover, the number of new variables and inequalities is polynomial in n.

Obtaining R_t from C_t. In what follows, to obtain a linear program describing R_t assume that there is a linear program describing C_t that has size polynomial in n. It is not difficult to prove that:

Lemma 3. *The state (c,r) belongs to $R_t \Leftrightarrow c_i \geq r_i$ for all $i \in V(G)$ and for all $\delta \in \Delta_R$ we have that $(c, \delta r)$ belongs to C_t.*

That is, a state (c,r) is in R_t if for every possible move δ of R the state $(c, \delta r)$ satisfies all inequalities describing C_t. In other words, $(c,r) \in R_t$ means that whatever the move of R is, it reaches a state in C_t where C wins in t rounds.

A state (c,r) is in R_t if for every possible move δ of R the state $(c, \delta r)$ satisfies all inequalities describing C_t. Let $A_i(c,r) \leq b_i$ be an inequality in the linear program describing C_t. The main tool used in the proof of Lemma 3 is that there are specific elements $\delta_i \in \Delta_R$ such that if $A_i(c, \delta_i r) \leq b_i$ then $A_i(c, \delta r) \leq b_i$ for all other elements $\delta \in \Delta_R$. In other words, for each inequality describing C_t there is a "best" move for R in order to violate this inequality. Another important property of these specific elements is that they can be found in time polynomial in n.

The linear program for $\mathcal{R}_t$ is obtained by taking each inequality, $A_i(c,r) \le b_i$, describing C_t and rewriting it in the following manner. Let (B_1, B_2) be equal to A_i. We can rewrite $A_i(c,r) \le b_i$ as $B_1 c + B_2 r \le b_i$. Then, in the linear program for $\mathcal{R}_t$, the inequality $A_i(c,r) \le b_i$ is replaced by $B_1 c + B_2 \delta_i r \le b_i$.

4 Integral Surveillance and Integral Winning Strategy

Let the *semi-fractional surveillance game* be the game where the states are given by vectors in $\mathbb{R}^n_+ \times \mathbb{N}^n$, the moves of C by vectors in $\mathbb{R}^n_+$ and the moves of $\mathcal{R}$ by $\{0,1\}_{n \times n}$ matrices. In other words, only $\mathcal{R}$ is not allowed to split itself. Let $\mathrm{sfsn}(G,v)$ be the minimum number of marks such that C always wins regardless of the moves made by $\mathcal{R}$ in the semi-fractional surveillance game.

Clearly, C wins in the semi-fractional game only if C wins in the fractional game (because $\mathcal{R}$ is less powerful in the semi-fractional game). On the other hand, C wins in the integral game only if C wins in the semi-fractional game (since C is less powerful in the integral game). The "best" move for C that we identified in the previous section appears to be integral. It follows that:

Theorem 4. *For any digraph G and $v \in V(G)$ we have:* $\mathrm{fsn}(G,v) = \mathrm{sfsn}(G,v) \le \mathrm{sn}(G,v)$.

This is a direct result from the fact that among the specific elements for inequality $A_i(m,s) \le b_i$ describing C_t there is always one δ_i that is integral. This roughly means that the best strategy for $\mathcal{R}$ is to, during its turn, either move all its tokens on a vertex to one of its neighbors or to pass its turn.

Theorem 5. *If C wins the fractional surveillance game with k marks in an n-node graph, then C wins the Surveillance Game with high probability if it is allowed to use $O(k \log n)$ marks.*

To show how C can win with high probability, let (c,r) be the current state of the game before each turn of C. If x_i is the amount of marks that a fractionary C uses in $i \in V(G)$ when the initial state is given by (c,r), then the integral C repeats $O(\log n)$ times a test with probability x_i; if any of these tests gives a positive result then C marks vertex i. Then, by follwing the proof of the $O(\log n)$-approximation for set cover in [Vaz04], this strategy has the desired properties.

5 Conclusion

We think that such an approach can lead to approximation algorithms for other games compatible with this framework. For instance, this approach could lead to new insights into solving the Meyniel's conjecture ($O(\sqrt{n})$ cops can capture one robber in any n-node graph). An important question is to know whether the games related to tree-decompositions can fit this framework.

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