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On disjoint directed cycles with prescribed minimum lengths

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Abstract: In this paper, we show that the k-Linkage problem is polynomial-time solvable for digraphs with circumference at most 2. We also show that the directed cycles of length at least 3 have the Erdős-Pósa Property : for every n, there exists an integer t_n such that for every digraph D, either D contains n disjoint directed cycles of length at least 3, or there is a set T of t_n vertices that meets every directed cycle of length at least 3. From these two results, we deduce that if F is the disjoint union of directed cycles of length at most 3, then one can decide in polynomial time if a digraph contains a subdivision of F.

Key-words: k-linkage, circumference, disjoint directed cycles, subdivision.

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Cycles dirigés disjoints avec des longueurs minimales prescrites.

Résumé : Dans ce rapport, nous montrons que le problème du k-linkage est résoluble en temps polynomial pour les digraphes de circonférence au plus 2. Nous montrons également que les cycles dirigés de longueur au moins 3 possède la Propriété d'Erdős-Pósa : pour tout n, il existe un entier t_n tel que pour tout digraphe D, soit D a n cycles dirigés disjoints de longueur au moins 3, soit il y a un ensemble T d'au plus t_n sommets qui intersecte tous les cycles dirigés de longueur au moins 3. De ces deux résultats, nous déduisons que si F est l'union disjointe de cycles dirigés de longueur au plus 3, alors on peut décider en temps polynomial si un digraphe contient une subdivision de F.

Mots-clés : k-linkage, circonférence, cycles dirigés disjoints, subdivision

1 Introduction

All digraphs considered is this paper are strict, that is they have no loops nor multiple arcs. We rely on [2] for classical notation and concepts.

Let D_1 and D_2 be two digraphs, the disjoint union of D_1 and D_2 is denoted by $D_1 + D_2$. For any digraph D, we set $2D = D + D$ and for any integer n greater than 2, $nD = (n-1)D + D$. In other words, nD is the disjoint union of n copies of D.

A digraph is *connected* if its underlying graph is connected. The *components* of a digraph are the (connected) components of its underlying graph. Hence, a digraph is the disjoint union of its components.

A *linkage* L in a digraph D is a subdigraph which is the disjoint union of directed paths. If $L = P_1 + \cdots + P_k$ and P_i is a directed (x_i, y_i)-path, $1 \leq i \leq k$, then we say that L is a *k-linkage from* $(x_1, \dots, x_k)$ *to* $(y_1, \dots, y_k)$. If $X, Y \subseteq V(D)$ and $\{x_1, \dots, x_k\} \subseteq X$ and $\{y_1, \dots, y_k\} \subseteq Y$, then we say that L is a k-linkage *from X to Y*.

If P is a directed path, then its initial vertex is denoted by $s(P)$ and its terminal vertex by $t(P)$.

1.1 Finding a subdivision of a digraph

A *subdivision of a digraph F*, also called an *F-subdivision*, is a digraph obtained from F by replacing each arc ab of F by a directed (a,b)-path.

It is natural to consider the following decision problem.

F-SUBDIVISION
Input: A digraph D.
Question: Does D contain a subdivision of F?

The analogue problem for undirected graphs can be solved in polynomial time for any undirected graph F. It follows from the Robertson and Seymour algorithm [11] that solves in polynomial time the undirected linkage problem, which is the following.

UNDIRECTED k-LINKAGE
Input: A graph G and $2k$ distinct vertices $x_1, \dots, x_k, y_1, \dots, y_k$.
Question: Is there a k-linkage from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in G?

However, for digraphs, Fortune, Hopcropt and Wyllie [4] showed that the following k-LINKAGE problem is NP-complete for all $k \geq 2$.

k-LINKAGE
Input: A digraph D and $2k$ distinct vertices $x_1, \dots, x_k, y_1, \dots, y_k$.
Question: Is there a k-linkage from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in D?

Using reductions to this problem, Bang-Jensen et al. [1] found many digraphs F for which F-SUBDIVISION is NP-complete. It is in particular the case of every digraph in which every vertex v is *big* (that is such that either $d^+(v) \geq 3$, or $d^-(v) \geq 3$, or $d^+(v) = d^-(v) = 2$). On the other hand, they gave polynomial-time algorithms to solve the problem for many others digraphs. This leads them to conjecturing that there is a sharp dichotomy between NP-complete and polynomial-time solvable instances. According to this conjecture, there are only two kinds of digraphs F: the *hard* ones, for which F-Subdivision is NP-complete, and the *tractable* ones, for which it is polynomial-time solvable. However there is no very clear picture of which graphs should be tractable and which ones should be hard, although some conjectures give some outline. Motivated by directed treewidth and a conjecture of Johnson et al. [7], Seymour (see [1]) posed the following conjecture.

Conjecture 1 (Seymour)**.** If F is a planar digraph with no big vertices, then F-SUBDIVISION is polynomial-time solvable.

Bang-Jensen et al. [1] proposed the following sort of counterpart conjecture.

Conjecture 2 (Bang-Jensen et al. [1])**.** F-Subdivision is NP-complete for every non-planar digraph F.

As an evidence to Conjecture 1, Havet et al. [6] proved it when F has order at most 4. Bang-Jensen et al. [1] also proved that every directed cycle is tractable. In this paper, we are interested in the case when F is the disjoint union of directed cycles. Conjecture 1 restricted to this particular case is the following.

Conjecture 3. If F is a disjoint union of directed cycles, then F-SUBDIVISION is polynomial-time solvable.

Observe that if F is the disjoint union of n directed cycles of lengths $\ell_1, \dots, \ell_n$, then a subdivision of F is the disjoint union of n directed cycles $C_1, \dots, C_n$, each C_i being of length at least ℓ_i. We denote the directed cycle of length ℓ, or *directed ℓ-cycle*, by $\vec{C}_\ell$. A directed cycle of length at least ℓ is called *directed ℓ^+-cycle*.

A particular case of Conjecture 3 is when all the directed cycles of F have the same length.

Conjecture 4. For any two positive integers n and ℓ with $\ell \geq 2$, $n\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable.

In fact, we show in Subsection 2.1 that Conjectures 3 and 4 are equivalent.

An $n\vec{C}_2$-subdivision is the disjoint union of n directed cycles. Therefore Conjecture 4 for $\ell = 2$ can be deduced (See Theorem 13) from the following two theorems.

Theorem 5 (Fortune, Hopcroft and Wyllie [4])**.** *For each fixed k, the k-LINKAGE problem is polynomial-time solvable for acyclic digraphs.*

Theorem 6 (Reed et al. [10])**.** *For every integer $n \geq 0$, there exists an integer t_n such that for every digraph D, either D has a n pairwise-disjoint directed cycles, or there exists a set T of at most t_n vertices such that $D - T$ is acyclic.*

Theorem 6 is a directed analogue of the following theorem due to Erdös and Pósa.

Theorem 7 (Erdős and Pósa [3])**.** *Let n be a positive integer. There exists t_n^* such that for every graph G, either G has n pairwise-disjoint cycles, or there exists a set T of at most t_n^* vertices such that $G - T$ is acyclic.*

More precisely Erdős and Pósa [3] proved that there exist two absolute constants c_1 and c_2 such that $c_1 \cdot n \log n \leq t_n^* \leq c_2 \cdot n \log n$.

We believe that a similar approach may be used to prove Conjecture 4 for all ℓ. We show in Section 2.2 that Conjecture 4 for some ℓ is implied by the two following conjectures for the same ℓ.

The *circumference* of a non-acyclic digraph D, denoted $\mathrm{circ}(D)$, is the length of a longest directed cycle in D. If D is acyclic, then its *circumference* is defined by $\mathrm{circ}(D) = 1$.

Conjecture 8. Let $\ell \geq 2$ be an integer. For any positive integer k, k-LINKAGE is polynomial-time solvable for digraphs with circumference at most $\ell - 1$?

This conjecture already appears as a problem in [1].

Conjecture 9. Let $\ell \geq 2$ be an integer. For every integer $n \geq 0$, there exists an integer $t_n = t_n(\ell)$ such that for every digraph D, either D has a n pairwise-disjoint directed ℓ^+-cycles, or there exists a set T of at most t_n vertices such that $D - T$ is no directed ℓ^+-cycles.

The main results of this paper (Theorems 14 and 16) prove both Conjecture 8 and Conjecture 9 for $\ell = 3$. By virtue of Theorem 13 they imply the following.

Theorem 10. *For any positive integer n, $n\vec{C}_3$-SUBDIVISION is polynomial-time solvable.*

Combined with Lemma 12, this result in turn implies the following.

Corollary 11. *If F is the disjoint union of cycles of length at most 3, then F-SUBDIVISION is polynomial-time solvable.*

2 Reducing to Conjectures 8 and 9

For a digraph D and an integer $\ell \geq 2$, we denote by $\nu_\ell(D)$ the maximum n such that D has n disjoint directed cycles of length at least ℓ, and by $\tau_\ell(D)$ the minimum t such that there exists $T \subseteq V(D)$ with $|T| = t$ meeting all directed cycles of length at least ℓ. Evidently $\nu_\ell(D) \leq \tau_\ell(D)$ and Conjecture 9 states that for every fixed ℓ there exists a function f such that $\tau_\ell(D) \leq f(\nu_\ell(D))$.

2.1 Equivalence of Conjectures 3 and 4

Conjecture 4 is a particular case of Conjecture 3. We now show how Conjecture 3 can be deduced from Conjecture 4.

Lemma 12. *Let F be a disjoint union of n directed cycles, all of length at most ℓ. If $m\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable for all $1 \leq m \leq n$, then F-SUBDIVISION is also polynomial-time solvable.*

Proof. Let n be a positive integer. Assume that $m\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable for any $m \leq n$.

Let $F = \vec{C}_{\ell_1} + \cdots + \vec{C}_{\ell_n}$ with $\ell_1 \leq \cdots \leq \ell_n \leq \ell$. Any F-subdivision is a disjoint union of n directed cycles $\vec{C}_{p_1} + \cdots + \vec{C}_{p_n}$ with $p_1 \leq \cdots \leq p_n$ such that $\ell_i \leq p_i$ for all $1 \leq i \leq n$. The *threshold* of such a subdivision is the largest integer t such that $p_t < \ell$.

For $t = 0$ to n, we check whether there is an F-subdivision with threshold t with the following 'brute force' procedure. We enumerate all possible disjoint unions of directed cycles $U = \vec{C}_{p_1} + \cdots + \vec{C}_{p_t}$ with $p_1 \leq \cdots \leq p_t \leq \ell - 1$ and $\ell_i \leq p_i$ for all $1 \leq i \leq t$. There are at most $O\left(|V(D)|^{(t-1)(\ell-1)}\right)$ such U. For each such U, we check if $D - U$ contains an $(n-t)\vec{C}_\ell$-subdivision (whose union with U would be an F-subdivision with threshold t). This can be done in polynomial time by the hypothesis.

The algorithm is a succession of (at most) $n+1$ polynomial-time procedures, so it runs in polynomial time. □

2.2 Conjectures 8 and 9 imply Conjecture 4

Theorem 13. *Let $\ell \geq 1$ be an integer. If Conjectures 8 and 9 hold for ℓ, then for every positive integer n, $n\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable.*

Proof. Let D be a digraph. Let $t = t_n(\ell)$ with $t_n(\ell)$ as in Conjecture 9. We first check if $\tau_\ell(D) \leq t$. This can be done by brute force, testing for each subset T of $V(D)$ of size t whether it meets all directed ℓ^+-cycles. Such a test can be done by checking whether $D - T$ has circumference $\ell - 1$, that is, has no $\vec{C}_\ell$-subdivision. Since there are $O(|V(D)|^t)$ sets of size t, and $\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable, this can be done in polynomial time.

If no t-subset T meets all directed ℓ^+-cycles, then $\tau_\ell(D) > t$. Therefore, because Conjecture 9 holds for ℓ, D contains an $n\vec{C}_\ell$-subdivision. So we return 'yes'.

If we find a set T of size t that meets all directed ℓ^+-cycles, then $\mathrm{circ}(D - T) \leq \ell - 1$. We use another brute force algorithm which is based on traces.

A *trace* is either a directed ℓ^+-cycle or a linkage. Observe that for any directed ℓ^+-cycle C and any subset Z of $V(D)$, the intersection $C \cap D[Z]$ is a trace. A trace contained in $D[Z]$ is called a *Z-trace*.

Now every ℓ^+-cycle intersects T in a non-empty trace because $\mathrm{circ}(D-T) \leq \ell - 1$. We describe a polynomial-time procedure that, given a set of n pairwise disjoint traces $T_1 \dots, T_n$, checks whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that $T_i = C_i \cap D[T]$ for all $1 \leq i \leq n$. Now since T has size t, there is a bounded number of possible sets of n pairwise disjoint traces T-traces (at most $\binom{t}{n+1}(B_t+1)$, where B_t is the is the number of partitions of a set of size t). Hence running the above procedure for all possible such set of T-traces, we obtain a polynomial-time algorithm that decides whether D contains an $n\vec{C}_\ell$-subdivision.

Let $\mathcal{T} = \{T_1, \dots, T_n\}$ be a set of n pairwise disjoint T-traces. Set $\overline{T} = V(D) \setminus T$. A *trace* is *suitable* if it has at least ℓ vertices, at most t components, and the initial and terminal vertices of all components are in $\overline{T}$.

For each T_i, we shall describe a set $\mathcal{T}_i$ of suitable traces such that a directed ℓ^+-cycle C such that $C \cap T = T_i$ contains at least one trace in $\mathcal{T}_i$. The set $\mathcal{T}_i$ is constructed as follows. Let $\mathcal{U}_i$ be the set of traces that can be obtained from T_i by extending each components P of T_i at both ends by an inneighbour of $s(P)$ and an outneighbour of $t(P)$ in $\overline{T}$. Clearly, $\mathcal{U}_i$ has size at most $|V(D)|^2k$, where k is the number of components of T_i. By construction, each trace of $\mathcal{U}_i$ has its initial and terminal vertices in $\overline{T}$ and has no more components than T_i. Moreover, a directed ℓ^+-cycle C such that $C \cap T = T_i$ contains one trace in $\mathcal{U}_i$. However, the set $\mathcal{U}_i$ might not be our set $\mathcal{T}_i$ because certain traces in it might be to small.

For any trace U, let $g(U)$ be set set of all possibles traces obtained from U by adding one vertex of $\overline{T}$ has outneighbour of a terminal vertex of one component of U. Clearly, $g(U)$ has size at most $k|V(D)|$, where k is the number of components of U, and a directed ℓ^+-cycle C containing U must contains a trace in $g(U)$. Moreover, every trace of $g(U)$ has size $|V(U)|+1$, and no more components than U. Set $g^i(U) = \{U\}$ if i is a non-positive integer and for all positive integer i, define $g^i(U) = \bigcup_{U' \in g^{i-1}(U)} g(U')$. Now the set $\bigcup_{U \in \mathcal{U}_i} g^{\ell - |V(U)|}(U)$ is our desired $\mathcal{T}_i$. Moreover, $\mathcal{T}_i$ is of size at most $t^t \cdot |V(D)|^t$.

To have a polynomial-time procedure to decide whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that $T_i = C_i \cap T$ for all $1 \leq i \leq n$, it suffices to have a procedure that, given an n-tuple $(T'_1, \dots, T'_n)$ of disjoint traces such that $T'_i \in \mathcal{T}_i$, decides whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that T'_i is a subdigraph of C_i for all $1 \leq i \leq n$, and to run it on each possible such n-tuple. Such a procedure can be done as follows. Let $P^i_1, \dots, P^i_{k_i}$ be the components of T'_i. For each n-tuple of circular permutations $(\sigma_1, \dots, \sigma_n)$ of $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_n}$, one checks whether in the digraph D' induced by the vertices of $\overline{T}$ which are not internal vertices of any of the components of the union of the T'_i, if there is a linkage from

$$\left(s(P^1_1), \dots, s(P^1_{k_1}), s(P^2_1), \dots, s(P^2_{k_2}), \ \dots \ , s(P^n_1), \dots, s(P^n_{k_n})\right)$$

to

$$\left(t(P^1_{\sigma_1(1)}), \dots, t(P^1_{\sigma_1(k_1)}), t(P^2_{\sigma_2(1)}), \dots, t(P^2_{\sigma_2(k_2)}), \ \dots \ , t(P^n_{\sigma_n(1)}), \dots, t(P^n_{\sigma_n(k_n)})\right).$$

Now the digraph D' is a subdigraph of $D-T$ and so has circumference at most $\ell - 1$, and the linkage we are looking for has at most t components. Thus each of these instances of $(k_1 + \cdots + k_n)$-LINKAGE can be solved in time $O(|V(D)|^m)$ for some absolute constant m because Conjecture 8 holds for ℓ. □

3 Linkage in digraphs with circumference at most 2

The aim of this section is to prove the following theorem.

Theorem 14. *For each fixed k, the k-*LINKAGE *problem is polynomial-time solvable for digraphs with circumference at most* 2.

We first prove the following lemma.

Lemma 15. *Let $\mathcal{D}$ be a class of digraphs and $\mathcal{S}$ be the class of strong digraphs. If k'-*LINKAGE *is polynomial-time solvable on $\mathcal{D} \cap \mathcal{S}$ for any $k' \le k$, then k-*LINKAGE *is polynomial-time solvable on $\mathcal{D}$.*

Proof. Let D be a digraph in $\mathcal{D}$. Let $\sim$ be the relation defined on $V(D)$ by $u \sim v$ if and only if u and v are in the same strong component. It is clearly an equivalence relation on $V(D)$ with equivalence classes the strong components of D. Let $D/\sim$ be the quotient of D by $\sim$, that is the digraph whose vertices are the strong components of D, and in which there is an arc from a strong component S to another S' if and only if there is an arc of D with tail in S and head in S'. One can also see $D/\sim$ as the digraph obtained by contracting each strong component into a vertex. It is well-known that $D/\sim$ is an acyclic digraph, therefore there is an ordering $S_1,\dots,S_p$ of the strong components such that there is no arc $S_jS_{j'}$ in $D/\sim$ with $j > j'$. This implies that for every $j > j'$, there is no directed (x,y)-path in D with $x \in S_j$ and y in $S_{j'}$. Let $\tilde{D}$ be the digraph $D \setminus \bigcup_{j=1}^{p} A(S_j)$, the digraph whose arcs are those between non-equivalent vertices with respect to $\sim$.

Form a new digraph $\mathbf{D}$ whose vertices are the k-tuples $\mathbf{v} = (v_1,\dots,v_k)$ of distinct vertices of D. For any such k-tuple $\mathbf{v}$, there is a minimum index m such that S_m intersects $\{v_1,\dots,v_k\}$. Let $I = \{i \mid v_i \in S_m\}$. Set $I = \{i_1,\dots,i_{k'}\}$ with $i_1 < i_2 < \cdots < i_{k'}$.

For each k'-tuple $(w_1,w_2,\dots,w_{k'})$ of distinct vertices of $V(D) \setminus \{v_1,v_2,\dots,v_k\}$ such that there exists a k'-tuple $(u_1,u_2,\dots,u_{k'})$ of vertices in $V(S_m)$ such that there is a linkage from $(v_{i_1},v_{i_2},\dots,v_{i_{k'}})$ to $(u_1,u_2,\dots,u_{k'})$ in S_m and u_jw_j is an arc in $\tilde{D}$ for all $1 \le j \le k'$, we put an arc from $\mathbf{v}$ to the k-tuple obtained from it by replacing v_i by w_i for all $i \in I$. We say that such an arc in $\mathbf{D}$ is labelled by S_m.

Observe that there are $O(n^{k'})$ k'-tuples $(u_1,u_2,\dots,u_{k'})$ of $V(S_m)$, and for each of them one can decide in polynomial time whether there is a linkage from $(v_{i_1},v_{i_2},\dots,v_{i_{k'}})$ to $(u_1,u_2,\dots,u_{k'})$ because k'-LINKAGE is polynomial-time solvable on $\mathcal{D}$ by hypothesis. Hence in polynomial time, we can construct the digraph $\mathbf{D}$ which has polynomial size.

We now prove that for any two sets of k distinct vertices $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$, there is a k-linkage from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$ if and only if there is a directed path from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$ in $\mathbf{D}$.

Suppose first that there is a k-linkage $(P_1,\dots,P_k)$ from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$. Since, when $j > j'$, there are no directed (x,y)-paths in D with $x \in S_j$ and y in $S_{j'}$, each P_i goes through the strong components $S_1,\dots,S_p$ in that order, possibly avoiding some. For each $1 \le m \le p$ and each $1 \le i \le k$, let $v_i(m)$ the first vertex in $\bigcup_{j=m}^{p} S_j$ along P_i if $\bigcup_{j=m}^{p} S_j$ and P_i intersect, and $v_i(m) = y_i$ otherwise.

Let $M = \{m_1,\dots,m_r\}$ with $m_1 \le m_2 \le \cdots \le m_r$, be the set of indices m such that $S_m \cap \bigcup_{i=1}^{k} P_i \neq \emptyset$. By definition of $\mathbf{D}$, $\mathbf{v}(m_q)\mathbf{v}(m_{q+1})$ is an arc in $\mathbf{D}$. Thus $\mathbf{v}(m_1)\mathbf{v}(m_2)\dots\mathbf{v}(m_r)$ is a directed path from $(x_1,x_2,\dots,x_k)$ to $(y_1,y_2,\dots,y_k)$ in $\mathbf{D}$.

Suppose now that $\mathbf{D}$ has a directed path $\mathbf{Q}$ from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$ in $\mathbf{D}$. We construct directed walks P_i, $1 \le i \le k$, by the following procedure. At the beginning $P_i = (x_i)$ for all $1 \le i \le k$. For each arc $\mathbf{a} = \mathbf{vw}$ of $\mathbf{Q}$ one after another from the initial vertex to the terminal vertex of $\mathbf{Q}$, we do the following.

Let $I = \{i_1,\dots,i_{k'}\}$ be set of indices i such that $v_i \neq w_i$. By definition of $\mathbf{D}$, there is a strong component S of D, a k'-tuple $(u_1,\dots,u_{k'})$ of disjoint vertices of S, and such that there is a linkage $(R_1,\dots,R_{k'})$ from $(v_{i_1},v_{i_2},\dots,v_{i_{k'}})$ to $(u_1,u_2,\dots,u_{k'})$ in S and $u_jw_{i_j} \in A(\tilde{D})$ for all $1 \le j \le k'$. In that case, we extend each P_{i_j}, $1 \le j \le r$, by appending $R_ju_jw_{i_j}$ at the end of it. Observe that R_j might be a path reduced to the single vertex $v_{i_j} = u_j$.

Observe that in $\mathbf{D}$ an arc labelled by a strong component S_m enters a k-tuple of vertices that all belong to components S_j with $j > m$. In particular, $\mathbf{Q}$ contains at most one arc labelled with any strong component S_m. This implies that each P_i is a directed (x_i,y_i)-path. Combined with the fact that each $(R_1,\dots,R_{k'})$ as defined above is a linkage, it implies that the P_i are disjoint. □

We can easily derive Theorem 14 from Lemma 15.

Proof of Theorem 14. Let $\mathcal{C}_2$ be the class of digraphs with circumference at most 2. A strong digraph D in $\mathcal{C}_2$ is obtained from a tree T by replacing every edge by a directed 2-cycle. Hence there is a k-linkage from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$ in D if and only if there is a k-linkage from $(x_1,\dots,x_k)$ to $(y_1,\dots,y_k)$ in T. Since UNDIRECTED k-LINKAGE is polynomial-time solvable, it follows that k-LINKAGE is polynomial-time solvable on $\mathcal{C}_2 \cap S$. Thus, by Lemma 15, it is polynomial-time solvable on $\mathcal{C}_2$. □

4 Packing directed 3^+-cycles

The aim of this section is to prove by induction on n the following theorem.

Theorem 16. *For every integer $n \geq 0$, there exists an integer t_n such that for every digraph D, either $\nu_3(D) \geq n$ or $\tau_3(D) \leq t_n$.*

Our proof follows the same approach as the one used by Reed et al. [10] to prove Theorem 6. A key ingredient in their proof is that if P is a directed (a,b)-path and Q is a directed (b,a)-path, then $P \cup Q$ contains a directed cycle. However, in such a case, $P \cup Q$ does not necessarily contain a 3^+-cycle. However, $P \cup Q$ does not contain a 3^+-cycle if and only if Q is the converse of P. The *converse* of a directed path $P = (x_1,\dots,x_m)$ is the directed path $(x_m,\dots,x_1)$.

Lemma 17. *Let a and b two distinct vertices, and let P be a directed (a,b)-path and Q be a directed (b,a)-path. Then $P \cup Q$ contains a directed 3^+-cycle if and only if Q is not the converse of P.*

Proof. Clearly, if Q is the converse of P, then $P \cup Q$ contains no directed 3^+-cycle.

Conversely, we prove by induction on the length m of P that if Q is not the converse of P, then $P \cup Q$ contains a directed 3^+-cycle. It holds trivially if $m = 1$. So we may assume that $m \geq 1$. Let $P = (x_0,x_1,\dots,x_m)$. Let y be the penultimate vertex of Q. If $y = x_1$, then $Q - x_0$ is not the converse of $P - x_0$. Hence, by the induction hypothesis, there is a directed 3^+-cycle in $(P - x_0) \cup (Q - x_0)$, and so in $P \cup Q$. Assume now that $y \neq x_1$, then let z be the penultimate vertex in $V(P \cap Q)$ along Q. If $z = x_1$, then $Q[x_1,x_0]$ has length at least 2, and so $(x_0,x_1) \cup Q[x_1,x_0]$ is a directed 3^+-cycle on $P \cup Q$. If $z \neq x_1$, then $P[x_0,z] \cup Q[z,x_0]$ is a directed 3^+-cycle on $P \cup Q$. □

In the proof we will have to make sure that some directed paths are not converse of some others. We follow the lines of the proof of Reed et al. [10] in order to emphasize the extra work required to deal with directed 3^+-cycle.

4.1 Main proof

First replacing 'directed cycle' by 'directed 3^+-cycle' in the proof of Theorem (2.2) of [10], we obtain the following analogue of (2.2) of [10].

Lemma 18. *Let $n \geq 1$ be an integer such that t_{n-1} exists, and let k be an integer. Then there exists an integer $t'(k)$ such that the following holds. Let D be a digraph with $\nu_3(D) < n$, and $\tau_3(D) \geq 2t'(k)$, and let T be a set of size $\tau_3(D)$ such that $D - T$ has no directed 3^+-cycles. For any disjoint subsets $A, B \subseteq T$ with $|A| = |B| = k$, there are distinct vertices $a_1,\dots,a_k$ in A and distinct vertices $b_1,\dots,b_k$ in B, and two k-linkages L_1, L_2 of D, so that*

(i) L_1 links $(a_1,\dots,a_k)$ to $(b_1,\dots,b_k)$,

(ii) L_2 links $(b_1,\dots,b_k)$ to one of $(a_1,\dots,a_k)$, $(a_k,\dots,a_1)$,

(iii) *every directed* 3^+*-cycle of* $L_1 \cup L_2$ *meets* $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.

However, for our purpose we need an extra condition on the two linkages L_1 and L_2. This is in fact why we needed a stronger statement thanLemma 18..

Lemma 19. *Let* $n \geq 1$ *be an integer such that* t_{n-1} *exists, and let* k *be an even integer. Then there exists an integer* $t(k)$ *such that the following holds. Let* D *be a digraph with* $\nu_3(D) < n$, *and* $\tau_3(D) \geq t(k)$, *and let* T *be a set of size* $\tau_3(D)$ *such that* $D - T$ *has no directed* 3^+*-cycles. Then there are distinct vertices* $a_1, \dots, a_k, b_1, \dots, b_k$ *in* T, *and two* k*-linkages* L_1, L_2 *of* D *so that*

(i) L_1 *links* $(a_1, \dots, a_k)$ *to* $(b_1, \dots, b_k)$,

(ii) L_2 *links* $(b_1, \dots, b_k)$ *to one of* $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$,

(iii) *every directed* 3^+*-cycle of* $L_1 \cup L_2$ *meets* $\{a_1, \dots, a_k, b_1, \dots, b_k\}$,

(iv) *no component of* L_1 *is the converse of a component of* L_2.

To prove this lemma, we will need Erdő-Pósa Theorem (Theorem 7) and the following lemma.

Lemma 20. *Let* r *be a positive integer. Let* T *be a tree and* S *a set of at least* $3r-2$ *vertices of* T. *Then there exists a vertex* x *of* T *and two subsets* A *and* B *of* S, *both of size* r *such that every* (A,B)*-path in* T *goes through* x.

Proof. Let E_r be the set of edges e such that both components of $T \setminus e$ have at least r vertices of S. We divide the proof in two cases depending on whether or not E_r is empty.

Assume first that $E_r \neq \emptyset$. Let $e = xy$ be an edge of E_r, and let T_x be the component of $T \setminus e$ containing x and T_y containing y. Both T_x and T_y contain at least r vertices of S. Let A (resp. B) be a set of r vertices of $S \cap V(T_x)$ (resp. $S \cap V(T_y)$). Then every (A,B)-path in T goes through e and so through x.

Assume now that $E_r = \emptyset$.

Claim 20.1. *There exists a vertex* x *such that all components of* $T - x$ *have less than* r *vertices of* S.

Subproof. Let us orient the edges of T as follows. Let $e = uv$ be an edge of T. Since $e \notin E_r$ is empty, exactly one component of $T \setminus e$ contains less than r vertices of S. Without loss of generality, this component is the one containing v. Orient the edge e from u to v. Now every orientation of a tree contains a vertex x with outdegree 0. Consider a component C of $T - x$. It contains exactly one neighbour y of x, and it is precisely the component of $T \setminus xy$ containing y. Thus $|C \cap S| < r$ because the edge is oriented from x to y. Hence all components of $T - x$ have less than r vertices. ◊

Take a vertex x as in the above claim. Let $C_1, \dots, C_m$ be the components of $T - x$. Then $|C_j| \leq r - 1$ for all $1 \leq j \leq m$. Let i be the smallest integer such that $T_i = \bigcup_{j=1}^{i} C_j$ contains at least r vertices of S. Clearly, T_i contains at most $2r-2$ vertices in S, and thus there are least r vertices in $T - T_i$. Let A (resp. B) be a set of r vertices in T_i (resp. $T - T_i$). Then x is in every (A,B)-path. □

Remark 21. The bound $3r-2$ in the above lemma is tight. Indeed consider a tree T with a set S of $3r-3$ leaves and four other vertices x, y_1, y_2 and y_3 such that for every $i \in \{1,2,3\}$, y_i is adjacent to x and $r-1$ leaves. One can check that for every vertex x and two sets A and B of r leaves there is an (A,B)-path avoiding x.

Proof of Lemma 19. Let t_n^* be as in Erdő-Pósa Theorem (Theorem 7); let $r = k + 2t_n^* + 2$; let $t(k) = \max\{3r - 2 + t_n^*, 2t'(r)\}$, let where t' is as in Lemma 18. We claim that $t(k)$ satisfies the lemma.

Let G_2 be the undirected graph with vertex set $V(D)$ in which two vertices x and y are adjacent if and only if $D[\{x,y\}]$ is a directed 2-cycle. To each cycle in G_2 correspond two directed cycles in D, one in each direction. Thus G_2 has less than n disjoint cycles. Hence by Erdő-Pósa Theorem, there is a set $U \subset V(D)$ of size t_n^* such that $G_2 - U$ is acyclic.

Choose $T \subseteq V(D)$ with $|T| = \tau_3(D)$, meeting all directed 3^+-cycles of D. Since $t(k) \geq 3r - 2 + t_n^*$, there is a set S of size $3r-2$ in $T \setminus U$. Since $G_2 - U$ is acyclic, we can extend it into a tree T_2. Hence, by Lemma 20, there exists a vertex x in $V(T_2)$ and two sets A and B in S of size r such that every (A,B)-path in T_2 goes through x. Since $G_2 - U$ is a subgraph of T_2, every (A,B)-path in $G_2 - U$ goes through x.

Since $|T| \geq 2t'(r)$, by Lemma 18, there are distinct vertices $a_1, \dots, a_{k'}$ in A and distinct vertices $b_1, \dots, b_{k'}$ in B, and two k'-linkages L_1', L_2' of D so that

(i) L_1' links $(a_1', \dots, a_{k'}')$ to $(b_1', \dots, b_{k'}')$,

(ii) L_2' links $(b_1', \dots, b_{k'}')$ to one of $(a_1', \dots, a_{k'}')$, $(a_{k'}', \dots, a_1')$,

(iii) every directed 3^+-cycle of $L_1' \cup L_2'$ meets $\{a_1', \dots, a_{k'}', b_1', \dots, b_{k'}'\}$.

For $1 \leq i \leq k'$, let P_i be the component of L_1 with initial vertex a_i' and Q_i the component of L_2 with initial vertex b_i'.

Clearly, if L_2' links $(b_1', \dots, b_{k'}')$ to $(a_{k'}', \dots, a_1')$, then condition (iv) is also verified by L_1' and L_2', because k' is even as k is even. For $1 \leq i \leq k$, set $a_i = a'_{i+(k'-k)/2}$ and $b_i = b'_{i+(k'-k)/2}$, and let $L_1 = \{P_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$ and $L_2 = \{Q_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$. Then $a_1, \dots, a_k, b_1, \dots, b_k$, L_1 and L_2 satisfy the lemma.

Assume now that L_2' links $(b_1', \dots, b_{k'}')$ to one of $(a_1', \dots, a_{k'}')$. At most t_n^* of the P_i intersect U and at most t_n^* of the Q_i intersect U. Thus, since $k' \geq k + 2t_n^* + 1$, there are at least $k+1$ indices i such that both P_i and Q_i do not intersect U. Without loss of generality, we may assume that these indices are $\{1, \dots, k+1\}$. Now for $1 \leq i \leq k+1$, if P_i is the converse of Q_i, then P_i is also a path in $G_2 - U$ and thus it must go through x. Hence there is at most one index i, say $k+1$, such that P_i is the converse of Q_i. Hence $a_1, \dots, a_k, b_1, \dots, b_k$, L_1 and L_2 satisfy the lemma. □

We say a digraph is *divalent* if every vertex has indegree 2 and outdegree 2, or indegree 1 and outdegree 1. In Subsection 4.2 we shall prove the following lemma which is the analogue of Lemma (2.3) of [10].

A pair $\{L_1, L_2\}$ of linkages is *fully intersecting* if each component of L_1 meets each component of L_2, and it is *acyclic* if $L_1 \cup L_2$ has no directed cycles,

Lemma 22. *For every positive integer n, there exists a positive integer k_1 such that for every divalent digraph D, if there is a fully intersecting and acyclic pair of k_1-linkages in D then $\nu_3(D) \geq n$.*

Lemma 22 is proved in Subsection 4.2. We assume it for the moment. We will show how to combine it with Lemma 19 to prove Theorem 16. First we prove the following lemma, which is the analogue of Lemma (2.4) of [10].

Lemma 23. *For every non-negative integer n, there exists a positive integer k so that the following holds. Let D be a digraph, and let $a_1 \dots, a_k, b_1, \dots, b_k$ be distinct vertices of D. Let L_1, L_2 be linkages in D linking $(a_1, \dots, a_k)$ to $(b_1, \dots, b_k)$, and $(b_1, \dots, b_k)$ to one of $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$, respectively, such that no component of L_2 is the converse of L_1. Let every directed 3^+-cycle of $L_1 \cup L_2$ meet $\{a_1, \dots, a_k, b_1, \dots, b_k\}$. Then $\nu_3(D) \geq n$.*

The proof of this lemma is similar to the one of Lemma (2.4) of [10]. However, some extra technical details are required. When reducing to a divalent digraph, we also have to get rid of directed 2-cycles, and Claim 24.1 is now required.

As in [10], we shall also need Ramsey's theorem [9], which can be stated as follows.

Theorem 24 (Ramsey [9])**.** *For all positive integers q,l,r, there exists a (minimum) integer $R_l(r;q)$ so that the following holds. Let Z be a set with $|Z| \geq R_l(r;q)$, let Q be a set with $|Q| = q$, and for each $X \subset Z$ with $|X| = l$ let $f(X) \in Q$. Then there exists $S \subseteq Z$ with $|S| = r$ and there exists $x \in Q$ so that $f(X) = x$ for all $X \subseteq S$ with $|X| = l$.*

Proof of Lemma 23, assuming Lemma 22. Let $n \geq 1$. Let $k' = \max\{k_1, \lceil n/4 \rceil\}$ with k_1 as in Lemma 22. Let $k = 2R_2(4k';9)$ defined as in Theorem 24. We claim that n and k satisfy Lemma 23.

For let $a_1 \dots, a_k, b_1, \dots, b_k, L_1, L_2$ be as in the statement of Lemma 23.

Let $G_i = P_i \cup P_{k+1-i} \cup Q_i \cup Q_{k+1-i}$.

We show by induction on $|E(D)| + |V(D)|$ that $\nu_3(D) \geq n$. If $L_1 \cup L_2 \neq D$, then the result follows immediately by induction, so we may assume that $L_1 \cup L_2 = D$.

Assume that either the arc $e = uv$ belongs to $P_i \cap Q_j$, or the arc uv is in P_i and the arc vu is in Q_j, then consider the graph D' and the two linkages L_1' and L_2' obtained by contracting uv. These two linkages clearly satisfy the hypothesis of Lemma 23 since directed cycles can only be shorten while contracting.

We therefore may assume that every arc of D belongs to exactly one of L_1, L_2, and that D has no directed 2-cycles. In particular, D is divalent and every directed cycle of D meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.

For $1 \leq i \leq k$, let P_i be the component of L_1 with initial vertex a_i, and let Q_i be the component of L_2 with initial vertex b_i.

For $1 \leq h < i \leq k/2$, define $f(\{h,i\})$ as follows. If G_i and G_h are disjoint, let $f(\{h,i\}) = 0$. Otherwise, at least one of the eight digraphs $P_i \cap Q_h$, $P_{k+1-i} \cap Q_h$, $P_i \cap Q_{k+1-h}$, $P_{k+1-i} \cap Q_{k+1-h}$, $Q_i \cap P_h$, $Q_{k+1-i} \cap P_h$, $Q_i \cap P_{k+1-h}$, $Q_{k+1-i} \cap P_{k+1-h}$ is non-null. Number them $1, \dots, 8$ in order; we define $f(\{h,i\}) \in \{0, 1, \dots, 8\}$.

Since $k = 2R_2(4k';9)$, by Theorem 24, there exists $S \subseteq \{1, \dots, \frac{1}{2}k\}$ with $|S| = 4k'$ and x with $0 \leq x \leq 8$ such that $f(\{h,i\}) = x$ for all $h, i \in S$ with $h < i$.

If $x = 0$, then the subdigraphs G_i are pairwise disjoint for all $i \in S$. But as we shall prove in Claim 24.1, each G_i contains a directed 3^+-cycle, and so $\nu_3(D) \geq |S| = 4k' \geq n$.

Claim 24.1. *Each G_i contains a directed 3^+-cycle.*

Subproof. If Q_i is a directed (b_i, a_i)-path, then by assumption Q_i is not the converse of P_i. Thus by Lemma 17 $P_i \cup Q_i$ contains a directed 3^+-cycle, and so G_i also does.

Assume now that Q_i is a directed (b_i, a_{k+1-i})-path, and so Q_{k+1-i} is a directed (b_{k+1-i}, a_i)-path. Q_i contains a directed path R_1 with initial vertex u_1 in P_i and terminal vertex v_1 in P_{k+1-i} whose internal vertices are not in $P_i \cup P_{k+1-i}$. Now Q_{k+1-i} contains a directed path R_2 with initial vertex u_2 in $P_{k+1-i}[v_1, b_{k+1-i}]$ and terminal vertex v_2 in $P_i[a_1, u_1]$ whose internal vertices are not in $P_i \cup P_{k+1-i}$. Observe that u_1, u_2, v_1 and v_2 are all distinct because P_i and P_{k+1-i} are disjoint and Q_i and Q_{k+1-i} are disjoint. Hence the $R_1 \cup P_{k+1-i}[v_1, u_2] \cup R_2 \cup P_i[v_2, u_1]$ is a directed 4^+-cycle in G_i. ◊

Assume now that $x = 1$. Let $S = I \cup J$, where $|I| = k'$, $|J| = 3k'$ and $i < j$ for all $i \in I$ and $j \in J$. Then for all $i \in I$ and all $j \in J$, P_i meets Q_j. There are $2k'$ vertices that are endvertices of paths $P_i, i \in I$, an each of them is an endvertex of at most one $Q_j, j \in J$. Since $|J| \geq 3k'$, there exists $J' \subset J$ with $|J'| = k'$ so that P_i and Q_j have no common endvertex for $i \in I$ and $j \in J'$. Let L_1' be the union of the components $P_i, i \in I$ and L_2' be the union of the components $Q_j, j \in J'$. Now every directed cycle in $L_1' \cup L_2'$ meets $\{a_1 \dots, a_k, b_1, \dots, b_k\}$, and each of $a_1 \dots, a_k, b_1, \dots, b_k$ is incident with at most one arc of $L_1' \cup L_2'$ since P_i and Q_j have no common endvertex for $i \in I$ and $j \in J'$. Hence $L_1' \cup L_2'$ has no directed cycles. We thus have the result by Lemma 22.

The cases $2 \leq x \leq 8$ are similar to the case $x = 1$. □

Proof of Theorem 16, assuming Lemma 22. We prove Theorem 16 by induction on n; we therefore assume that $n \geq 1$ and t_{n-1} exists, and we show that t_n exists. Let k be as in Lemma 23, and let t be as in Lemma 19. We claim that there is no digraph D with $\nu_3(D) < n$ and $\tau_3(D) \geq t$. For suppose that D is such a digraph. By Lemma 19, there exists $a_1 \dots, a_k, b_1, \dots, b_k$ and L_1, L_2 as in Lemma 18, and so $\nu_3(D) \geq n$ by Lemma 23, a contradiction. Thus there is no such D, and consequently t_n exists and $t_n < t$. □

4.2 Proving Lemma 22

In this section, following Section 3 of [10], we show that if a digraph D contains a kind of grid, with some additional paths, then $\nu_3(D)$ is large. We then use this lemma to prove Lemma 22.

Let p, q be positive integers. A (p,q)-*web* in a digraph D is a fully intersecting and acyclic pair (L_1, L_2) of linkages such that L_1 has p components and L_2 has q components.

Let p, q be positive integers. A (p,q)-*fence* in a digraph D is a sequence $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ with the following properties:

(i) $P_1, \dots, P_{2p}$ are pairwise disjoint directed paths of D, and so are $Q_1, \dots, Q_q$;

(ii) for $1 \leq i \leq 2p$ and $1 \leq j \leq q$, $P_i \cap Q_j$ is a directed path (and therefore non-null);

(iii) for $1 \leq j \leq q$, the directed paths $P_1 \cap Q_j, \dots, P_{2p} \cap Q_j$ are in order in Q_j, and the initial vertex of Q_j is in $V(P_1)$ and its terminal vertex is in $V(P_{2p})$;

(iv) for $1 \leq i \leq 2p$, if i is odd then $P_i \cap Q_1, \dots, P_i \cap Q_q$ are in order in P_i, and if i is even then $P_i \cap Q_q, \dots, P_i \cap Q_1$ are in order in P_i.

Let Q_j be a directed (a_j, b_j)-path ($1 \leq l \leq q$); we call $\{a_1, \dots, a_q\}$ the *top* of the fence, and $\{b_1, \dots, b_q\}$ its *bottom*.

The following lemma is the analogue to Lemma (3.1) of [10]. It only differs in the conclusion $\nu_3(D) \geq n$, instead of $\nu(D) \geq n$.

Lemma 25. *For every positive integer n, there are positive integers p, r with the following property. For any $q \geq 2$, let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p,q)-fence in a digraph D, and let there be r disjoint paths in D from the bottom of the fence to the top. Then $\nu_3(D) \geq n$.*

Combining Lemmas (4.4), (4.5) and (4.7) of [10] we directly obtain the following lemma.

Lemma 26. *For all positive integers p, q, there are positive integers p' and q' so that for every digraph G, if D contains a (p', q')-web then it contains a (p,q)-fence.*

In exactly the same way that Reed et al. deduced Lemma (2.3) from Lemmas (3.1), (4.4), (4.5) and (4.7) in [10], one can deduce Lemma 22 from Lemmas 25 and 26.

Hence it only remains to prove Lemma 25.

4.3 Proof Lemma 25

Consider the following lemma (Lemma (3.2)) from [10].

Lemma 27 (Reed et al. [10]). *Let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p,q)-fence in a digraph D, with top A and bottom B. Let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = r$, for $r \leq p$. Then there are directed paths $Q'_1, \dots, Q'_r$ in $P_1, \dots, P_{2p}, Q_1, \dots, Q_q$ so that $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_q)$ is a (p,r)-fence with top A' and bottom B'.*

Remark 28. In the proof of this lemma, the proven (p,r)-fence is a subgraph of the (p,q)-fence, with $A' \subseteq A$ and $B' \subseteq B$. Moreover, if $p \geq 2$, then Q_j has order at least 4 for $1 \leq j \leq q$, because Q_j intersects every P_i $1 \leq i \leq 2p$. So, if $p \geq 2$, Q'_l has size at least 4, for $1 \leq l \leq r$.

We need also an analogue of Lemma (3.3) of [10]:

Lemma 29. *Let $n \geq 1$ be an integer, and let $p \geq 2n$ and $N \geq 2n^2 - 3n + 2$ be integers. For some integer $q \geq 1$ let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p,q)-fence in a digraph D. Let $R_1, \dots, R_N$ be disjoint directed paths of D from the bottom of the fence to the top, so that each R_k has no vertex or arc in $P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q$ except its endvertices. Then $\nu_3(D) \geq n$.*

The proof of this lemma is exactly the same as the one of Lemma (3.3) of [10]. The disjoint directed cycles showed in the proof are of the form $Q'_j R_m$, for Q'_j in a (p,r)-fence $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_q)$, subgraph of the (p,q)-fence $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$. Since $p \geq 2n \geq 2$, by Remark 28 each Q'_j has length at least 2, and so $Q'_j R_m$ has length at least 3. Hence $\nu_3(D) \geq n$.

We prove Lemma 25 by induction on n. The proof is almost identical to the one of Lemma (3.1) in [10]. The only differences are the easy case $n = 1$, for which we need here to take $p = 2$ (instead of $p = 1$) to be sure that the directed cycle is of length at least 3, and the use in place of Lemma (3.3) of its analogue, namely Lemma 29.

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