An Explicit Martingale Version of Brenier's Theorem

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An Explicit Martingale Version of Brenier's Theorem[*]

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Abstract

By investigating model-independent bounds for exotic options in financial mathematics, a martingale version of the Monge-Kantorovich mass transport problem was introduced in [3, 24]. In this paper, we extend the one-dimensional Brenier's theorem to the present martingale version. We provide the explicit martingale optimal transference plans for a remarkable class of coupling functions corresponding to the lower and upper bounds. These explicit extremal probability measures coincide with the unique left and right monotone martingale transference plans, which were introduced in [4] by suitable adaptation of the notion of cyclic monotonicity. Instead, our approach relies heavily on the (weak) duality result stated in [3], and provides, as a by-product, an explicit expression for the corresponding optimal semi-static hedging strategies. We finally provide an extension to the multiple marginals case.

1   Introduction

Since the seminal paper of Hobson [29], an important literature has developed on the topic of robust or model-free superhedging of some path dependent derivative security with payoff ξ, given the observation of the stochastic process of some underlying financial asset, together with a class of derivatives. See [7, 11, 12, 13, 14, 15, 16, 18, 19, 31, 33, 39] and the survey papers of Oblój [40] and Hobson [30]. In continuous-time models, these papers mainly focus on derivatives whose payoff ξ is stable under time change. Then, the key-observation was that, in the idealized context where all $T-$maturity European calls and puts, with all possible strikes, are available for trading, model-free superhedging cost of ξ is closely related to the Skorohod Embedding problem. Indeed, the market prices of all $T-$maturity European calls and puts with all possible strikes allow to recover the marginal distribution of the underlying asset price at time T.

Recently, this problem has been addressed via a new connection to the theory of optimal transportation, see [3, 24, 27, 1, 2, 20, 21]. Our interest in this paper is on the formulation of

a Brenier Theorem in the present martingale context. We recall that the Brenier Theorem in the standard optimal transportation theory states that the optimal coupling measure is the gradient of some convex function which identifies in the one-dimensional case to the so-called Fréchet-Hoeffding coupling [6]. A remarkable feature is that this coupling is optimal for the class of coupling cost functions satisfying the so-called Spence-Mirrlees condition.

We first consider the one-period model. Denote by X, Y the prices of some underlying asset at the future maturities 0 and 1, respectively. Then, the possibility of dynamic trading implies that the no-arbitrage condition is equivalent to the non-emptyness of the set $\mathcal{M}_2$ of all joint measures $\mathbb{P}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying the martingale condition $\mathbb{E}^{\mathbb{P}}[Y|X] = X$. The model-free subhedging and superhedging costs of some derivative security with payoff $c(X, Y)$, given the marginal distributions $X \sim \mu$ and $Y \sim \nu$, is essentially reduced to the martingale transportation problems:

$$\inf_{\mathbb{P} \in \mathbf{M}_2(\mu, \nu)} \mathbb{E}^{\mathbb{P}}[c(X, Y)] \quad \text{and} \quad \sup_{\mathbb{P} \in \mathbf{M}_2(\mu, \nu)} \mathbb{E}^{\mathbb{P}}[c(X, Y)],$$

where $\mathbf{M}_2(\mu, \nu)$ is the collection of all probability measures $\mathbb{P} \in \mathcal{M}_2$ such that $X \sim_{\mathbb{P}} \mu$, $Y \sim_{\mathbb{P}} \nu$. Our main objective is to characterize the optimal coupling measures which solve the above problems. This provides some remarkable extremal points of the convex (and weakly compact) set $\mathbf{M}_2(\mu, \nu)$. In the absence of marginal restrictions, Jacod and Yor [35] (see also Jacod and Shiryaev [34], Dubins and Schwarz [22], for the discrete-time setting) proved that a martingale measure $\mathbb{P} \in \mathcal{M}_2$ is extremal if and only if $\mathbb{P}$-local martingales admit a predictable representation. In the present one-period model, such extremal points of $\mathcal{M}_2$ consist of binomial models. For a specific class of coupling functions c, the extremal points of the corresponding martingale transportation problem turn out to be of the same nature, and our main contribution in this paper is to provide an explicit characterization.

Our starting point is a paper by Hobson and Neuberger [32] who considered the specific case of the coupling function $c(x, y) := |x - y|$, and provided a complete explicit solution of the optimal coupling measure and the corresponding optimal semi-static strategy. In a recent paper, Beiglböck and Juillet [4] address the problem from the viewpoint of optimal transportation. By a convenient extension of the notion of cyclic comonotonicity, [4] introduce the notion of left-monotone transference plan. They also introduce the notion of left-curtain as a left-monotone transference plan concentrated on the graph of a binomial map. The remarkable result of [4] is the existence and uniqueness of the left-monotone transference plan which is indeed a left-curtain, together with the optimality of this joint probability measure for some specific class $\mathcal{C}_{\text{BJ}}$ of coupling payoffs $c(x, y)$. Notice that the coupling measure of [32] is not a left-curtain, and $\mathcal{C}_{\text{BJ}}$ does not contain the coupling payoff $|x - y|$.

As a main first contribution, we provide an explicit description of the left-curtain $\mathbb{P}_*$ of [4]. Then, by using the weak duality inequality,

- we provide a larger class $\mathcal{C} \supset \mathcal{C}_{\text{BJ}}$ of payoff functions for which $\mathbb{P}_*$ is optimal,

- we identify explicitly the solution of the dual problem which consists of the optimal semi-static superhedging strategy,

2

- as a by-product, the strong duality holds true.

Our class $\mathcal{C}$ is the collection of all smooth functions $c : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, with linear growth, such that $c_{xyy} > 0$. We argue that this is essentially the natural class for our martingale version of the Brenier Theorem.

We next explore the multiple marginals extension of our result. In the context of the finite discrete-time model, we provide a direct extension of our result which applies to the context of the discrete monitored variance swap. This answers the open question of optimal model-free upper and lower bounds for this derivative security.

The paper is organized as follows. Section 2 provides a quick review of the Brenier Theorem in the standard one-dimensional optimal transportation problem. The martingale version of the Brenier Theorem is reported in Section 3. We next report our extensions to the multiple marginals case in Section 4. Finally, Section 5 contains the proofs of our main results.

2 The Brenier Theorem in One-dimensional Optimal Transportation

2.1 The two-marginals optimal transportation problem

Let X, Y be two scalar random variables denoting the prices of two financial assets at some future maturity T. The pair (X, Y) takes values in $\mathbb{R}^2$, and its distribution is defined by some probability measure $\mathbb{P} \in \mathcal{P}_{\mathbb{R}^2}$, the set of all probability measures on $\mathbb{R}^2$. For the purpose of the present financial application, the measures have support on $\mathbb{R}^2_+$. For the sake of generality, we consider however the general case.

We assume that $T-$maturity European call options, on each asset and with all possible strikes, are available for trading at exogenously given market prices. Then, it follows from Breeden and Litzenberger [5] that the marginal distributions of X and Y are completely determined by the second derivative of the corresponding (convex) call price function with respect to the strike. We shall denote by μ and ν the implied marginal distributions of X and Y, respectively, $\ell^\mu, r^\mu, \ell^\nu, r^\nu$ the left and right endpoints of their supports, and F_μ, F_ν the corresponding cumulative distribution functions.

By definition of the problem the probability measures μ and ν have finite first moment:

$$\int |x| \mu(dx) + \int |y| \nu(dy) \;\; < \;\; \infty, \tag{2.1}$$

and although the supports of μ and ν could be restricted to the non-negative real line for the financial application, we shall consider the more general case where μ and ν lie in $\mathcal{P}_{\mathbb{R}}$, the collection of all probability measures on $\mathbb{R}$.

We consider a derivative security defined by the payoff $c(X, Y)$ at maturity T, for some upper semicontinuous function $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with linear growth. This condition could be replaced by

$$c(x, y) \le \varphi(x) + \psi(y) \quad \text{for some} \quad \varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R}, \quad \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu). \tag{2.2}$$

The model-independent upper bound for this payoff, consistent with vanilla option prices of maturity T, can then be framed as a Monge-Kantorovich (in short MK) optimal transport problem:

$$P_2^0(\mu, \nu) := \sup_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X, Y)\big] \quad \text{where} \quad \mathcal{P}_2(\mu, \nu) := \big\{\mathbb{P} \in \mathcal{P}_{\mathbb{R}^2} : X \sim_{\mathbb{P}} \mu \text{ and } Y \sim_{\mathbb{P}} \nu\big\},$$

where, for the sake of simplicity, we have assumed a zero interest rate. This can easily be relaxed by considering the forwards of X and Y. Notice that $c(X, Y)$ is measurable by the upper semicontinuity condition on c, and is integrable by the linear growth condition on c together with (2.1) (or the condition (2.2)).

In the original optimal transportation problem as formulated by Monge, the above maximization problem was restricted to the following subclass of measures.

Definition 2.1. *A probability measure* $\mathbb{P} \in \mathcal{P}_2(\mu, \nu)$ *is called a transference map if* $\mathbb{P}(dx, dy) := \mu(dx)\delta_{\{T(x)\}}(dy)$, *for some measurable map* $T : \mathbb{R} \longrightarrow \mathbb{R}$.

The dual problem associated to the MK optimal transportation problem is defined by :

$$D_2^0(\mu, \nu) := \inf_{(\varphi,\psi) \in \mathcal{D}_2^0} \big\{\mu(\varphi) + \nu(\psi)\big\},$$

where, denoting $\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y)$:

$$\mathcal{D}_2^0 := \big\{(\varphi, \psi) : \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu) \text{ and } \varphi \oplus \psi \geq c\big\}.$$

and with $\mu(\varphi) := \int \varphi d\mu$, $\nu(\psi) := \int \psi d\nu$.

The dual problem $D_2^0(\mu, \nu)$ is the cheapest superhedging strategy of the derivative security $c(X, Y)$ using the market instruments consisting of $T-$maturity European calls and puts with all possible strikes. The weak duality inequality

$$P_2^0(\mu, \nu) \leq D_2^0(\mu, \nu)$$

is immediate. For an upper semicontinuous payoff function c, equality holds and an optimal probability measure $\mathbb{P}^*$ for the MK problem P_2^0 exists, see e.g. Villani [43].

Our main interest of this paper is the following one-dimensional version of a result established by Brenier [6], which provides an interesting characterization of $\mathbb{P}^*$ in terms of the so-called Fréchet-Hoeffding pushing forward μ to ν, defined by the map

$$T_* := F_\nu^{-1} \circ F_\mu, \tag{2.3}$$

where F_ν^{-1} is the right-continuous inverse of F_ν:

$$F_\nu^{-1}(t) := \inf\{y : F_\nu(y) > x\}.$$

In particular, the following result relates the MK optimal transportation problem P_2^0 to the original Monge mass transportation problem for a remarkable class of couplings c. This result

4

is more general, in particular the set of measures $\mathbb{P}_T$ induced by a map T pushing forward μ to ν is dense in $\mathcal{P}_{\mathbb{R}^2}$ whenever μ is atomless and we consider compact subsects of $\mathbb{R}^2$. For the purpose of our financial interpretation, this result characterizes the structure of the worst case financial market that the derivative security hedger may face, and characterizes the optimal hedging strategies by the functions φ_* and ψ_* defined up to an irrelevant constant by

$$\varphi_*(x) := c\big(x, T_*(x)\big) - \psi_* \circ T_*(x), \quad \psi'_*(y) := c_y\big(T_*^{-1}(y), y\big), \quad x, y \in \mathbb{R}. \qquad (2.4)$$

Theorem 2.2. (see e.g. [43], Theorem 2.44) *Let c be upper semicontinuous with linear growth. Assume that the partial derivative c_{xy} exists and satisfies the Spence-Mirrlees condition $c_{xy} > 0$. Assume further that μ has no atoms, $\varphi_*^+ \in \mathbb{L}^1(\mu)$ and $\psi_*^+ \in \mathbb{L}^1(\nu)$. Then*
(i) $P_2^0(\mu, \nu) = D_2^0(\mu, \nu) = \int c\big(x, T_*(x)\big)\mu(dx)$,
(ii) $(\varphi_*, \psi_*) \in \mathcal{D}_2^0$, and is a solution of the dual problem D_2^0,
(iii) $\mathbb{P}_*(dx, dy) := \mu(dx)\delta_{T_*(x)}(dy)$ is a solution of the MK optimal transportation problem P_2^0, and is the unique optimal transference map.

Proof. We provide the proof for completeness, as our main result in this paper will be an adaptation of the subsequent argument. First, it is clear that $\mathbb{P}_* \in \mathcal{P}(\mu, \nu)$. Then $\mathbb{E}^{\mathbb{P}_*}[c(X, Y)] \leq P_2^0(\mu, \nu)$. We now prove that

$$(\varphi_*, \psi_*) \in \mathcal{D}_2^0 \quad \text{and} \quad \mu(\varphi_*) + \nu(\psi_*) = \mathbb{E}^{\mathbb{P}_*}[c(X, Y)]. \qquad (2.5)$$

In view of the weak duality $P_2^0(\mu, \nu) \leq D_2^0(\mu, \nu)$, this would imply that $P_2^0(\mu, \nu) = D_2^0(\mu, \nu)$ and that $\mathbb{P}_*$ and (φ_*, ψ_*) are solutions of $P_2^0(\mu, \nu)$ and $D_2^0(\mu, \nu)$, respectively.

Under our assumption that $\varphi_* \in \mathbb{L}^1(\mu)$, $\psi_*^+ \in \mathbb{L}^1(\nu)$, notice that (2.5) is equivalent to:

$$0 = H^0\big(x, T_*(x)\big) = \min_{y \in \mathbb{R}} H^0(x, y), \quad \text{where} \quad H^0 := \varphi_* \oplus \psi_* - c.$$

The first-order condition for the last minimization problem provides the expression of ψ'_* in (2.4), and the expression of φ_* follows from the first equality. Since

$$H_y^0(x, y) \;=\; c_y\big(T_*^{-1}(y), y\big) - c_y(x, y) \;=\; \int_x^{T_*^{-1}(y)} c_{xy}(\xi, y)d\xi,$$

it follows from the Spence-Mirrlees condition that $T_*(x)$ is the unique solution of the first-order condition. Finally, we compute that $H_{yy}^0\big(x, T_*(x)\big) = c_{xy}\big(x, T_*(x)\big)/T'_*(x) > 0$ by the Spence-Mirrlees condition, where the derivatives are in the sense of distributions. Hence $T_*(x)$ is the unique global minimizer of $H(x, .)$. $\qquad \square$

We observe that we may also formulate sufficient conditions on the coupling c so as to guarantee that the integrability conditions $\varphi_*^+ \in \mathbb{L}^1(\mu), \psi_*^+ \in \mathbb{L}^1(\nu)$ hold true. See [43], Theorem 2.44.

Remark 2.3 (Mirror coupling: anti-monotone rearrangement map). (i) *Suppose that the coupling function c satisfies $c_{xy} < 0$. Then, the upper bound $P_2^0(\mu, \nu)$ is attained by the anti-monotone rearrangement map*

$$\overline{\mathbb{P}}_*(dx, dy) := \mu(dx)\delta_{\{\overline{T}_*(x)\}}(dy), \quad \text{where} \quad \overline{T}_*(x) := F_\nu^{-1} \circ \big(1 - F_\mu(-x)\big).$$

To see this, it suffices to rewrite the optimal transportation problem equivalently with modified inputs:

$$\overline{c}(x, y) := c(-x, y), \quad \overline{\mu}(x) := \mu\big((-x, \infty)\big), \quad \overline{\nu} := \nu,$$

so that $\overline{c}$ satisfies the Spence-Mirrlees condition $\overline{c}_{xy} > 0$.
(ii) *Under the Spence-Mirrlees condition $c_{xy} > 0$, the lower bound problem is explicitly solved by the anti-monotone rearrangement. Indeed, it follows from the first part (i) of the present remark that:*

$$\inf_{\mathbb{P} \in \mathcal{P}_2(\mu, \nu)} \mathbb{E}^{\mathbb{P}}\big[c(X, Y)\big] = - \sup_{\mathbb{P} \in \mathcal{P}_2(\mu, \nu)} \mathbb{E}^{\mathbb{P}}\big[-c(X, Y)\big] = -\mathbb{E}^{\overline{\mathbb{P}}_*}\big[-c(X, Y)\big] = \int c\big(x, \overline{T}_*(x)\big)\mu(dx).$$

Remark 2.4. *The Spence-Mirrlees condition is a natural requirement in the optimal transportation setting in the following sense. The optimization problem is not affected by the modification of the coupling function from c to $\overline{c} := c + a \oplus b$ for any $a \in \mathbb{L}^1(\mu)$ and $b \in \mathbb{L}^1(\nu)$. Since $c_{xy} = \overline{c}_{xy}$, it follows that the Spence-Mirrlees condition is stable for the above transformation of the coupling function.*

Example 2.5 (Basket option). *Let $c(x, y) = (x + y - k)^+$, for some $k \in \mathbb{R}$ (see [17, 38] for multi-asset basket options). The result of Theorem 2.2 applies to this example as well, as it is shown in [43] Chapter 2 that the regularity condition $c \in C^{1,1}$ is not needed. The upper bound is attained by the Fréchet-Hoeffding transference map $T_* := F_\nu^{-1} \circ F_\mu$, and the optimal hedging strategy is:*

$$\psi_*(y) = (y - \bar{y})^+, \quad \varphi_*(x) = \big(T_*(x) + x - k\big)^+ - \big(T_*(x) - \bar{y}\big)^+,$$

where $\bar{y}$ is defined by $T_*(k - \bar{y}) = \bar{y}$.

2.2 The multi-marginals optimal transportation problem

The previous results have been extended to the $n-$marginals optimal transportation problem by Gangbo and Święch [25], Carlier [9], and Pass [41]. Let $X = (X_1, \ldots, X_n)$ be a random variable with values in $\mathbb{R}^n$, representing the prices at some fixed time horizon of n financial assets, and consider some upper semicontinuous payoff function $c : \mathbb{R}^n \longrightarrow \mathbb{R}$ with linear growth.

Let $\mu_1, \ldots, \mu_n \in \mathcal{P}_{\mathbb{R}}$ be the corresponding marginal distributions, and $\mu := (\mu_1, \ldots, \mu_n)$. The upper bound market price on the derivative security with a payoff function c is defined by the optimal transportation problem:

$$P_n^0(\mu) := \sup_{\mathbb{P} \in \mathcal{P}_n(\mu)} \mathbb{E}^{\mathbb{P}}\big[c(X)\big], \quad \text{where} \quad \mathcal{P}_n(\mu) := \big\{\mathbb{P} \in \mathcal{P}_{\mathbb{R}^n} : X_i \sim_{\mathbb{P}} \mu_i, 1 \le i \le n\big\}. \quad (2.6)$$

Then, under convenient conditions on the coupling function c (see Pass [41] for the most general ones), there exists a solution $\mathbb{P}_*$ to the MK optimal transportation problem $P_n^0(\mu)$ which is the unique optimal transference map defined by T_*^i, $i = 2, \ldots, n$:

$$\mathbb{P}^*(dx_1, \ldots, dx_n) = \mu_1(dx_1) \prod_{i=2}^{n} \delta_{T_*^i(x_1)}(dx_i), \quad \text{where} \quad T_*^i = F_{\mu_i}^{-1} \circ F_{\mu_1}, \quad i = 2, \ldots, n.$$

The optimal upper bound is then given by

$$P_n^0(\mu) = \int c\big(\xi, T_*^2(\xi), \ldots, T_*^n(\xi)\big) \mu_1(d\xi).$$

3 The Two-Marginals Martingale Transport Problem: Main Results

The main objective of this paper is to obtain a version of the Brenier theorem for the martingale transportation problem introduced by Beiglböck, Henry-Labordère and Penkner [3] and Galichon, Henry-Labordère and Touzi [24]. A result in this direction was first obtained by Hobson and Neuberger [32] and by Beiglböck and Juillet [4]. In contrast with the last reference, our result is an explicit extension of the Fréchet-Hoeffding optimal coupling. We outline in Sections 3.6 and 3.7 the main differences with [4, 32].

3.1 Problem formulation

In the context of the financial motivation of Subsection 2.1, we interpret the pair of random variables X, Y as the prices of the same financial asset at dates t_1 and t_2, respectively, with $t_1 < t_2$. Then, the no-arbitrage condition states that the price process of the tradable asset is a martingale under the pricing and hedging probability measure. We therefore restrict the set of probability measures to:

$$\mathcal{M}_2(\mu, \nu) := \big\{ \mathbb{P} \in \mathcal{P}_2(\mu, \nu) : \mathbb{E}^{\mathbb{P}}[Y|X] = X \big\}.$$

where μ, ν have finite first moment as in (2.1). This set of probability measures is clearly convex, and the martingale condition implies that $\ell^\nu \leq \ell^\mu \leq r^\mu \leq r^\nu$. Throughout this paper, we shall denote

$$\delta F := F_\nu - F_\mu.$$

By a classical result of Strassen [42], $\mathcal{M}_2(\mu, \nu)$ is non-empty if and only if $\mu \preceq \nu$ in sense of convex ordering, i.e.

(i) μ, ν have the same mean: $\int \xi d\delta F(\xi) = 0$,

(ii) and $\int (\xi - k)^+ \mu(d\xi) \leq \int (\xi - k)^+ \nu(d\xi)$, for all $k \in \mathbb{R}$. This condition can also be expressed as:

$$\int_{[k,\infty)} \delta F(\xi) d\xi \leq 0 \quad \text{or, equivalently,} \quad \int_{[-\infty,k)} \delta F(\xi) d\xi \geq 0, \quad \text{for all } k \in \mathbb{R}, \quad (3.1)$$

where the last equivalence follows from the first property (i).

For completeness, we provide in Section 6 some examples of probability measures in $\mathcal{M}_2(\mu, \nu)$ which are commonly using by practitioners in quantitative finance.

Let $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an upper semicontinuous function with linear growth (or the condition (2.2)), representing the payoff of a derivative security. In the present context, the model-independent upper bound for the price of the claim can be formulated as the following martingale optimal transportation problem:

$$P_2(\mu, \nu) := \sup_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}} \big[c(X, Y) \big], \qquad (3.2)$$

Remark 3.1. *When μ and ν have finite second moment, notice that $\mathbb{E}^{\mathbb{P}}[(X - Y)^2] = -\mathbb{E}^{\mathbb{P}}[X^2] + \mathbb{E}^{\mathbb{P}}[Y^2] = \int \xi^2 d\delta F(\xi)$ for all $\mathbb{P} \in \mathcal{M}(\mu, \nu)$. Then, the quadratic case, which is the typical example of coupling in the optimal transportation theory, is irrelevant in the present martingale version.*

We finally report the Kantorovich dual in the present martingale transport problem. Because of the possibility of dynamic trading the financial asset between times t_1 and t_2, the set of dual variables is defined by:

$$\mathcal{D}_2 := \big\{ (\varphi, \psi, h) : \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu), h \in \mathbb{L}^0, \text{ and } \varphi \oplus \psi + h^{\otimes} \geq c \big\}, \qquad (3.3)$$

where $\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y)$, and $h^{\otimes}(x, y) := h(x)(y - x)$. The dual problem is:

$$D_2(\mu, \nu) := \inf_{(\varphi, \psi, h) \in \mathcal{D}_2} \big\{ \mu(\varphi) + \nu(\psi) \big\}, \qquad (3.4)$$

and can be interpreted as the cheapest superhedging strategy of the derivative $c(X, Y)$ by dynamic trading on the underlying asset, and static trading on the European options with maturities t_1 and t_2. Since c has linear growth and μ, ν have finite first-order moments, the weak duality inequality:

$$P_2(\mu, \nu) \leq D_2(\mu, \nu) \qquad (3.5)$$

follows immediately from the definition of both problems. Under suitable conditions on c, [3] proved the strong duality result (i.e. equality holds), and showed the existence of a maximizer $\mathbb{P}_* \in \mathcal{M}_2(\mu, \nu)$ for the martingale transportation problem $P_2(\mu, \nu)$. However, existence does not hold in general for the dual problem $D_2(\mu, \nu)$. An example of non-existence is provided in [3].

8

3.2 Preliminaries

Our objective in this section is to provide explicitly the left-monotone martingale transport plan, as introduced by Beiglböck and Juillet [4].

Definition 3.2. *We say that $\mathbb{P} \in \mathcal{M}_2(\mu, \nu)$ is left-monotone (resp. right-monotone) if there exists a Borel set $\Gamma \subset \mathbb{R} \times \mathbb{R}$ such that $\mathbb{P}[(X, Y) \in \Gamma] = 1$, and for all $(x, y_1), (x, y_2), (x', y') \in \Gamma$ with $x < x'$ (resp. $x > x'$), it must hold that $y' \notin (y_1, y_2)$.*

Our main results hold for probability measures μ, ν satisfying the following restriction.

Assumption 3.3. *The probability measures μ and ν have finite first moments, $\mu \preceq \nu$ in convex order, and μ has no atoms.*

Under this assumption, Theorem 1.5 and Corollary 1.6 of [4] state that there exists a unique left-monotone martingale transport plan $\mathbb{P}_* \in \mathcal{M}_2(\mu, \nu)$, and that the graph of $\mathbb{P}_*$ is concentrated on two maps $T_d, T_u : \mathbb{R} \longrightarrow \mathbb{R}$, $T_d(x) \leq x \leq T_u(x)$ for all $x \in \mathbb{R}$, i.e. $\mathbb{P}_*[Y = T_d(X)] + \mathbb{P}_*[Y = T_u(X)] = 1$.

Remark 3.4. *The condition that F_μ is continuous in Assumption 3.3 implies that δF is upper-semicontinuous, and therefore the local suprema of δF are attained by maximizers.*

For our construction, we introduce the functions:

$$g(x, y) \; := \; F_\nu^{-1}\big(F_\mu(x) + \delta F(y)\big), \quad x, y \in \mathbb{R}, \tag{3.6}$$

where F_ν^{-1} denotes the right-continuous inverse of F_ν, with $F_\nu^{-1} = \infty$ on $(1, \infty)$ and $F_\nu^{-1} = -\infty$ on $(-\infty, 0)$. We also define for a measurable subset $A \in \mathcal{B}_\mathbb{R}$ such that δF is increasing on A:

$$G^A(t, x) := \int_{(-\infty, F_\nu^{-1} \circ F_\mu(x)]} \xi dF_\nu(\xi) - \int_{-\infty}^x \xi dF_\mu(\xi) + \int_{A \cap (-\infty, t]} \big(g(x, \xi) - \xi\big) d\delta F(\xi), \quad t \leq x \in \mathbb{R}. \tag{3.7}$$

In the last integral, notice that $g(x, \xi) - \xi \geq 0$, so that by the increase of δF on A, the integral has a well-defined value in $(-\infty, \infty]$. It will be made clear in Section 5.1 that these functions appear naturally when one imposes that $\mathbb{P}_* \in \mathcal{M}_2(\mu, \nu)$.

Notice that G^A is right-continuous in t, and $G^A(-\infty, \infty) = 0$, a consequence of the fact that μ and ν have the same mean. Our construction uses the following preliminary result, which needs the additional notation:

$$B_0 := \{x \in \mathbb{R} : \delta F \text{ increasing to the right of } x\}, \quad x_0 := \inf B_0,$$

where we say that a function ϕ is increasing (resp. decreasing) to the right of x if for all $\varepsilon_0 > 0$, there exists $\varepsilon \in (0, \varepsilon_0)$ such that $\phi(x + \varepsilon) > \phi(x)$ (resp. $\phi(x + \varepsilon) < \phi(x)$).

Observe that $x_0 = \infty$ if and only if $\mu = \nu$.

Lemma 3.5. *Assume $x_0 < \infty$, let $m \in \mathbb{R}$ be a local maximizer of δF, and consider a Borel subset $A \subset (x_0, m] \cap B_0$. Denote $\bar{A}^m := (x_0, m] \setminus A$, and assume that $\int_{\bar{A}^m} d\phi(\delta F) \geq 0$ for any non-decreasing function ϕ. Then, there exists a unique scalar $t^A(x, m)$ such that, for all $x \geq m$ with $\delta F(x) \leq \delta F(m)$,*

$$t^A(x, m) \in A, \quad \text{and} \quad G^A\big(t^A(x,m)-, x\big) \leq 0 \leq G^A\big(t^A(x,m), x\big).$$

Moreover, $\bar{x}(m) := \inf\{x > m : g\big(x, t^A(x, m)\big) \leq x\}$ satisfies, whenever $\bar{x}(m) < \infty$,

$$\delta F\big(t^A(\bar{x}(m), m)\big) \leq \delta F\big(\bar{x}(m)\big) \leq \delta F\big(t^A(\bar{x}(m)-, m)\big) \tag{3.8}$$
$$\text{and} \quad \delta F \quad \text{is strictly increasing on a right neighborhood of} \quad \bar{x}(m).$$

The proof of this lemma is reported in Subsection 5.1.

3.3 Explicit construction

Our explicit construction requires an additional condition on δF. Let $\mathbf{M}(\delta F)$ denote the collection of all points m such that δF is nondecreasing to the left of m, and decreasing to the right of m:

$$\mathbf{M}(\delta F) \quad := \quad \{m : \delta F'(m-) \leq 0, \text{ and } \delta F \text{ decreasing to the right of } m\}, \tag{3.9}$$

where we recall that δF decreasing to the right of m means that any right neighborhood of m contains a point m' such that $\delta F(m') < \delta F(m)$. Our general characterization of the left monotone transference plan will be obtained in Theorem 3.11 as a limit of explicit monotone transference plans corresponding to an approximating sequence satisfying the following no right accumulation requirement.

Assumption 3.6. $\mathbf{M}(\delta F) \cup \{x_0\}$ *has no right accumulation point.*

Lemma 3.7. *Under Assumption 3.6, the set $\mathbf{M}(\delta F)$ is countable.*

Proof. Under Assumption 3.6, we have $\mathbf{M}(\delta F) = \cup_{n \in \mathbb{N}} \mathbf{M}_n(\delta F)$, where

$$\mathbf{M}_n(\delta F) := \big\{m : \delta F \text{ strictly decreasing on } (m, m + \tfrac{1}{n}], \text{ and } \delta F'(m-) \leq 0\big\}.$$

Then the required result follows from the fact that $\mathbf{M}_n(\delta F)$ is countable. $\qquad \square$

We are now ready for the construction of the left-monotone transference map $\mathbb{P}_*$. We first initialize the construction in Step 0, and continue an iterative construction in the subsequent steps.

Step 0: If $x_0 = -\infty$, we move to Step 1 of the construction. Otherwise, define:

$$T_d(x) = T_u(x) = x \quad \text{for} \quad x \leq x_0. \tag{3.10}$$

If $x_0 = \infty$, i.e. $\mu = \nu$, then this completes the construction of (T_d, T_u). Otherwise, we continue with the following step.

Step 1: By Assumption 3.6, the function δF increases at the right of x_0. Consider the first point of decrease of δF (see Remark 3.4):

$$m_1 := \inf\big\{m > x_0 : \ \delta F \text{ decreasing on } [m, m + \varepsilon) \text{ for some } \varepsilon > 0\big\},$$

By the stochastic dominance $\mu \preceq \nu$, see (3.1), it follows that δF is nondecreasing on $(x_0, m_1]$, $\delta F(m_1) > 0$, and δF is strictly increasing on the set

$$A_1 := (x_0, m_1] \cap B_0.$$

We have $\bar{A}_1^{m_1} = \emptyset$ and we are then in the context of application of Lemma 3.5 with $(m, A) = (m_1, A_1)$. Denoting $x_1 := \bar{x}(m_1)$, we define the maps T_d, T_u on $(-\infty, x_1)$:

$$
\begin{aligned}
T_d(x) \ &= \ T_u(x) \ = \ x \quad \text{for} \quad x_0 < x \leq m_1, \\
T_d(x) := t^{A_1}(x, m_1), \quad &T_u(x) := g\big(x, T_d(x)\big) \quad \text{for} \quad m_1 \leq x < x_1.
\end{aligned}
\tag{3.11}
$$

If $x_1 = \infty$, this completes the construction. See Figure 1 below for such an example. Otherwise, it follows from Lemma 3.5 that δF is strictly increasing at the right of x_1, whenever $x_1 < \infty$. In this case, we continue the construction denoting:

$$B_1 := B_0 \setminus \big\{T_d\big([m_1, x_1)\big) \cup \big[m_1, T_u(x_1)\big)\big\} = B_0 \setminus \big(T_d(x_1), T_u(x_1)\big),$$

where the last equality follows from the fact that T_d is decreasing and $T_d(x) \leq x$, see Remark 3.8.

Step 2: The construction of this step falls in the more general Step i below, and is provided here for the convenience of the reader.

Since $\mu \preceq \nu$ in (3.1), it follows that the set of local maximizers after x_1 is not empty. Recall Assumption 3.6, and let:

$$
\begin{aligned}
m_2 \ &:= \ \inf\big\{m \geq x_1 : \ \delta F \text{ decreasing on } [m, m + \varepsilon) \text{ for some } \varepsilon > 0\big\}, \\
A_2 \ &:= \ (x_1, m_2] \cap B_1 \ = \ (x_0, T_d(x_1)) \cup (x_1, m_2],
\end{aligned}
$$

so that δF is nondecreasing on $[x_1, m_2]$, and strictly increasing on A_2. Moreover $\bar{A}_2^{m_2} = [T_d(x_1), x_1]$ and, since $\delta F(T_d(x_1)) \leq \delta F(x_1)$ by (3.8), we see that $\int_{\bar{A}_2^{m_2}} d\phi(\delta F) \geq 0$ for all nondecreasing function ϕ.

Then, we may apply Lemma 3.5 with $(m, A) = (m_2, A_2)$. Denoting $x_2 := \bar{x}(m_2)$, we may define the maps T_d, T_u on $[x_1, x_2)$:

$$
\begin{aligned}
T_d(x) \ &= \ T_u(x) \ = \ x \quad \text{for} \quad x_1 < x \leq m_2, \\
T_d(x) := t^{A_2}(x, m_2), \quad &T_u(x) := g\big(x, T_d(x)\big) \quad \text{for} \quad m_2 \leq x < x_2.
\end{aligned}
\tag{3.12}
$$

If $x_2 = \infty$, this completes the construction. See Figure 2 below for such an example. Otherwise, it follows from Lemma 3.5 that δF is strictly increasing at the right of x_2, whenever $x_2 < \infty$. In this case, we continue the construction denoting:

$$B_2 := B_1 \setminus \big\{T_d\big([m_2, x_2)\big) \cup \big[m_2, T_u(x_2)\big)\big\}.$$

11

Step i: Suppose that (T_d, T_u) are defined on $(-\infty, x_i)$ for some x_i with δF strictly increasing at the right of x_i, and let a subset $B_i := B_{i-1} \setminus \{T_d([m_i, x_i]) \cup [m_i, x_i]\} \subset B_0$ be given so that by definition, we have

$$G^{A_i}(T_d(x_i), x_i) \geq 0. \tag{3.13}$$

and A_i is obtained iteratively from the previous steps as:

$$A_i = (x_0, m_i] \setminus \big[\cup_{j<i} \{T_d([m_j, x_j)) \cup [m_j, T_u(x_j))\} \big].$$

Since $\mu \preceq \nu$ in (3.1), it follows that the set of local maximizers after x_i is not empty. Recall Assumption 3.6, and let:

$$m_{i+1} := \inf \big\{ m \geq x_i : \delta F \text{ decreasing on } [m, m+\varepsilon) \text{ for some } \varepsilon > 0 \big\},$$

and

$$A_{i+1} := (x_0, m_{i+1}) \cap B_i = (x_0, m_{i+1}] \setminus \big[\cup_{j<i} \{T_d([m_j, x_j)) \cup [m_j, T_u(x_j))\} \big],$$

so that δF is strictly increasing on A_{i+1}.

We observe that $T_d(x_i) \notin [T_d(x_j), T_u(x_j)]$ for any $j < i$, which expresses that our construction provides the left-monotone martingale transport plan, see Definition 3.2. Since $\delta F(T_d(x_i)) \leq \delta F(x_i)$ by (3.8), we have also that $\int_{A_{i+1}^{m_{i+1}}} d\phi(\delta F) \geq 0$ for all nondecreasing function ϕ.

We have thus verified that the conditions of Lemma 3.5 are satisfied by the pair (m_{i+1}, A_{i+1}), and we may then define the maps T_d, T_u on $[x_i, x_{i+1})$ by:

$$\begin{aligned} T_d(x) = T_u(x) = x \quad &\text{for} \quad x_i \leq x \leq m_{i+1}, \\ T_d(x) := t^{A_{i+1}}(x, m_{i+1}), \quad T_u(x) := g(x, T_d(x)) \quad &\text{for} \quad m_{i+1} \leq x < x_{i+1} := \bar{x}(m_{i+1}). \end{aligned} \tag{3.14}$$

If $x_{i+1} = \infty$, the construction is complete. Otherwise, it follows from Lemma 3.5 that δF is strictly increasing at the right of x_{i+1}, whenever $x_{i+1} < \infty$. In this case, we also update:

$$B_{i+1} := B_i \setminus \big\{ T_d([m_{i+1}, x_{i+1})) \cup [m_{i+1}, T_u(x_{i+1})) \big\},$$

and we continue with an additional step.

Case of accumulation: It may happen that the increasing sequence $(m_i)_i$ converges to some $m^1 < \infty$. Then, as the number of steps i tends to infinity, the above construction defines the maps (T_d, T_u) on $(-\infty, m^1)$.

In this case, under Assumption 3.6 which excludes any right accumulation of local maxima, we may start again the construction exactly as in Step i, with $m_{i+1} = m_1^1$. After possibly i steps, this defines $(m_j^1, x_j^1)_{j \leq i}$ which either meets the requirement $x_i^1 = \infty$, or accumulates. Recall that the set $\mathbf{M}(\delta)$ of (3.9) is countable under our Assumption 3.6. Since the set of possible accumulation points m^k is a subset of $\mathbf{M}(\delta F)$, it is at most countable. Then, by transfinite induction,

we relabel the sequence $(m_j^k, x_j^k)_{j,k}$ as a new sequence that we rename $(m_i, x_i)_{i \geq 0}$.

Remark 3.8 (Some properties of T_d). *From the above construction of T_d, we see that*
(i) T_d is right-continuous. Moreover, on each interval (m_i, x_i), it is non-increasing and flat if and only if it reaches an atom of F_ν.
(ii) In general, the restriction of T_d to $\cup_{i \geq 0}(m_i, x_i)$ fails to be non-decreasing. However, for $i \neq j$, we have $T_d\big((m_i, x_i)\big) \cap T_d\big((m_j, x_j)\big) = \emptyset$. Consequently, the right-continuous inverse T_d^{-1} of T_d is well defined.
(iii) Let $I = (a, b) \subset T_d([m_i, x_i])$ be such that δF is flat on I, and δF increases at the right of b and at the left of a. Then, whenever T_d reaches the right endpoint b, it jumps from b to a, i.e. $\Delta T_d\big(T_d^{-1}(b)\big) = a - b$.
(iv) Let x be such that $T_d(x) = T_u(x) = x$. Then, $\{x' \neq x : T_u(x') = x\} = \emptyset$, $\{x' \neq x : T_d(x') = x\} \neq \emptyset$, and reduces to a single point set if $\Delta F_\nu(x) = 0$. Otherwise, if x is an atom of F_ν, the last set has a positive measure under F_μ.

Remark 3.9 (Some properties of T_u). *From the above construction of T_u, we see that*
(i) $T_u([m_i, x_i]) \subset [m_i, x_i]$, and $T_u(x) > x$ for $x \in (m_i, x_i)$ for all i.
(ii) T_u is right continuous with discontinuity points $\{x : \Delta F_\nu(T_d(x)) > 0\}$, recall that F_μ is continuous.
(iii) T_u is nondecreasing, and strictly increasing on the support of μ. The last property will be clear from Theorem 3.10 (ii) below, and implies that the right-continuous inverse T_u^{-1} of T_u is well-defined.

3.4 The left-monotone martingale transport plan

The last construction provides our martingale version of the Fréchet-Hoeffding coupling:

$$T_*(x, dy) := \mathbf{1}_D(x)\delta_{\{x\}}(dy) + \mathbf{1}_{D^c}(x)\big[q(x)\delta_{\{T_u(x)\}}(dy) + (1-q)(x)\delta_{\{T_d(x)\}}(dy)\big], \qquad (3.15)$$

where $x_{-1} = -\infty$, $m_0 := x_0$,

$$D := \cup_{i \geq 0}(x_{i-1}, m_i] \quad \text{and} \quad q(x) := \frac{x - T_d(x)}{T_u(x) - T_d(x)}. \qquad (3.16)$$

Observe that $T_d(x) \leq x \leq T_u(x)$ from our previous construction. Therefore, q takes values in $[0, 1]$.

Theorem 3.10. *Let Assumptions 3.3 and 3.6 hold true. Then,*
(i) the probability measure $\mathbb{P}_*(dx, dy) := \mu(dx)T_*(x, dy)$ is the unique left-monotone transport plan in $\mathcal{M}_2(\mu, \nu)$;
(ii) moreover T_u and T_d solve the following ODEs:

$$d(\delta F \circ T_d) = -(1-q)dF_\mu, \quad d(F_\nu \circ T_u) = qdF_\mu \quad \text{whenever} \quad x \in [m_i, x_i) \quad \text{and} \quad T_d(x) \in \text{int}(A_i).$$

The proof is reported in Section 5.1. The next result characterizes the left-monotone transference map in the case where δF does not satisfy Assumtpion 3.6.

13

Theorem 3.11. *Let Assumption 3.3 hold true, and let* $(\mu_n, \nu)_{n \geq 1} \subset \mathcal{P}_{\mathbb{R}}$ *be such that* $\mu_n \longrightarrow \mu$ *and* $\nu_n \longrightarrow \nu$, *weakly, and* (μ_n, ν_n) *satisfies Assmptions 3.3 and 3.6. For all* $n \geq 1$, *define the corresponding* T^n_* *as in (3.15), and the corresponding* $\mathbb{P}^n_*(dx, dy) := \mu_n(dx)T^n_*(x, dy)$.

Then $\mathbb{P}^n_*$ *converges weakly towards the unique left-monotone transference map.*

Proof. By following the proof of Proposition 2.4 of [3], it follows from Lemma 4.4 p56 in Villani [43] that the sequence $(\mathbb{P}^n_*)_{n \geq 1}$ is weakly compact. Then, after possibly passing to subsequence, $\mathbb{P}^n_* \longrightarrow \hat{\mathbb{P}}$, weakly, for some $\hat{\mathbb{P}} \in \mathcal{M}(\mu, \nu)$. To prove the required result, we shall prove that $\hat{\mathbb{P}}$ is a left-monotone transference map; then, from the uniqueness result of Theorem 1.5 in [4], we may deduce that $\hat{\mathbb{P}}$ does not depend on the chosen subsequence.

Assume to the contrary that $\hat{\mathbb{P}}$ is not left-monotone. Then there exists a support $\hat{\Gamma}$ of $\hat{\mathbb{P}}$ such that

$$(x, y_d), \ (x, y_u), \ (x', y') \in \hat{\Gamma} \ , \ y_d < y_u, \ x' > x, \ \text{and} \ y' \in (y_d, y_u). \tag{3.17}$$

To obtain the required contradiction, we prove below that there exist sequences $(x^n, y^n_d)_n$, $(x^n, y^n_u)_n$, $(x'_n, y'_n)_n$ in a support of $\mathbb{P}^n_*$ such that $(x^n, x'_n) \longrightarrow (x, x')$, and $(y^n_d, y^n_u, y'_n) \longrightarrow (y_d, y_u, y')$. By the left-monotonicity of $\mathbb{P}^n_*$ for all n, we have $y'_n \notin (y^n_d, y^n_u)$, and we obtain by sending $n \to \infty$ that $y' \notin (y_d, y_u)$, contradicting (3.17).

We finally prove that if $(x, y) \in \hat{\Gamma}$, then there exists a sequence (x_n, y_n) and a support of $\mathbb{P}_n$ such that $(x_n, y_n) \longrightarrow (x, y)$. For an arbitrary $\varepsilon > 0$, let φ a continuous function with support in $B_\varepsilon(x, y)$, the open ball centered at (x, y) with radius ε. Then, it follows from the weak convergence of $\mathbb{P}^n_*$ towards $\hat{\mathbb{P}}$ that $\mathbb{E}^{\mathbb{P}^n_*}[\varphi(X, Y)] \longrightarrow \mathbb{E}^{\hat{\mathbb{P}}}[\varphi(X, Y)]$, and the required result follows from the arbitrariness of $\varepsilon > 0$. $\square$

We conclude this subsection by the following remarkable property of T_d.

Proposition 3.12. *Let Assumptions 3.3 and 3.6 hold true. Let* $i \geq 1$ *be such that* δF *is not flat at the left of* m_i. *Then* $T_d(m_i+) = m_i$. *If in addition* F_μ, F_ν *are twice differentiable near* m_i, *then:*

$$T'_d(m_i+) = -1/2 \quad and \quad T''_d(m_i) = +\infty.$$

Proof By construction, we have $T_d(m_i+) = m_i$. Denoting $\varepsilon := x - T_d(x)$, $f_\mu := F'_\mu$, $f_\nu := F'_\nu$, $\delta f := f_\nu - f_\mu$, and recalling that $g(x, x) = x$, we see by direct calculation that

$$g(x, T_d) - x = -\varepsilon \frac{\delta f}{f_\nu}(x) + \frac{\varepsilon^2}{2}\left(\frac{\delta f'}{f_\nu} + \left(\frac{\delta f}{f_\nu}\right)^2 \frac{f'_\nu}{f_\nu}\right)(x) + \circ(\varepsilon^2),$$

$$\delta f \circ T_d(x) = -\varepsilon \delta f'(x) + \circ(\varepsilon).$$

where $\circ$ is a continuous function with $\circ(0) = 0$. Observe that $\delta f > 0$ near m_i by the definition of m_i. Plugging the above expansion in the ODE satisfied by T_d, we see that:

$$T'_d(x) = -\frac{\frac{\delta f}{f_\nu} + \frac{1}{2}\varepsilon\left(\frac{\delta f'}{f_\nu} + \left(\frac{\delta f}{f_\nu}\right)^2 \frac{f'_\nu}{f_\nu}\right) + \circ(\varepsilon)}{1 - \frac{\delta f}{f_\nu} + \frac{1}{2}\varepsilon\left(\frac{\delta f'}{f_\nu} + \left(\frac{\delta f}{f_\nu}\right)^2 \frac{f'_\nu}{f_\nu}\right) + \circ(\varepsilon)} \frac{f_\mu}{\delta f - \varepsilon \delta f' + \circ(\varepsilon)}(x).$$

14

We then take the limit as $x \searrow m_i$, so that $\varepsilon \searrow 0$ and $\delta f(x) \longrightarrow 0$ by the definition of m_i. This leads to $T_d'(x) \longrightarrow -1/2$.

Finally, we compute $T_d''(m_i)$. By the ODE satisfied by T_d and the smoothness of g, it follows that T_d' is differentiable at any $x > m_i$. We then differentiate the ODE satisfied by T_d, and use Taylor expansions. The result follows from direct calculation by sending $x \searrow m_i$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.5 Martingale version of the Brenier Theorem

We next introduce a remarkable triple of dual variables corresponding to a smooth coupling function c. Recall the set D defined in (3.16) on which we have $T_d(x) = T_u(x) = x$, $x \in D$, and the right-continuous inverse functions T_d^{-1}, T_u^{-1} defined in Remark 3.8 (ii) and Remark 3.9 (iii).

The dynamic hedging component h_* is defined up to a constant, on each continuity interval, by:

$$h_*' = \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d} \quad \text{on } D^c, \quad h_* = h_* \circ T_d^{-1} + c_y(.,.) - c_y(T_d^{-1}, .) \quad \text{on } D. \qquad (3.18)$$

The payoff function ψ_* is defined up to a constant on each continuity interval by:

$$\psi_*' = c_y(T_u^{-1}, .) - h_* \circ T_u^{-1} \quad \text{on } D^c, \quad \psi_*' = c_y(T_d^{-1}, .) - h_* \circ T_d^{-1} \quad \text{on } D. \qquad (3.19)$$

The corresponding function φ_* is given by:

$$\begin{aligned} \varphi_*(x) &= \mathbb{E}^{\mathbb{P}_*}\big[c(X, Y) - \psi_*(Y)|X = x\big] \qquad\qquad\qquad\qquad\qquad\qquad (3.20) \\ &= q(x)\big(c(x, .) - \psi_*\big) \circ T_u(x) + \big(1 - q(x)\big)\big(c(x, .) - \psi_*\big) \circ T_d(x), \quad x \in \mathbb{R}. \end{aligned}$$

Finally, we define h_* and ψ_* from (3.18)-(3.19) by imposing that

$$\text{the function} \quad c(., T_u) - \psi_*(T_u) - [c(., T_d) - \psi_*(T_d)] - (T_u - T_d)h \quad \text{is continuous.} (3.21)$$

Theorem 3.13. *Let μ, ν be as in Assumptions 3.3 and 3.6. Assume further that $\varphi_*^+ \in \mathbb{L}^1(\mu)$, $\psi_*^+ \in \mathbb{L}^1(\nu)$, and that the partial derivative of the coupling function c_{xyy} exists and $c_{xyy} > 0$ on $\mathbb{R} \times \mathbb{R}$. Then:*
(i) $(\varphi_*, \psi_*, h_*) \in \mathcal{D}_2$,
(ii) the strong duality holds for the martingale transportation problem, $\mathbb{P}_*$ is a solution of $P_2(\mu, \nu)$, and (φ_*, ψ_*, h_*) is a solution of $D_2(\mu, \nu)$:

$$\int c\big(x, T_*(x, dy)\big)\mu(dx) = \mathbb{E}^{\mathbb{P}_*}\big[c(X, Y)\big] = P_2(\mu, \nu) = D_2(\mu, \nu) = \mu(\varphi_*) + \nu(\psi_*).$$

Remark 3.14 (Mirror coupling: the right-monotone martingale transport plan)**.**
(i) *Suppose that $c_{xyy} < 0$. Then, the upper bound $P_2(\mu, \nu)$ is attained by the right-monotone martingale transport map*

$$\bar{\mathbb{P}}_*(dx, dy) := \bar{\mu}(dx)\bar{T}_*(x, dy),$$

15

where $\bar{T}_*$ is defined as in (3.15) with the pair of probability measures $(\bar{\mu}, \bar{\nu})$:

$$F_{\bar{\mu}}(x) := 1 - F_\mu(-x), \quad and \quad F_{\bar{\nu}}(y) := 1 - F_\nu(-y).$$

To see this, we rewrite the optimal transportation problem equivalently with modified inputs:

$$\bar{c}(x,y) := c(-x,-y), \quad \bar{\mu}\big((-\infty, x]\big) := \mu\big([-x, \infty)\big), \quad \bar{\nu}\big((-\infty, y]\big) := \nu\big([-y, \infty)\big),$$

so that $\bar{c}_{xyy} > 0$, as required in Theorem 3.13. Note that the martingale constraint is preserved by the map $(x, y) \to (-x, -y)$.

(ii) Suppose that $c_{xyy} > 0$. Then, the lower bound problem is explicitly solved by the right-monotone martingale transport plan. Indeed, it follows from the first part (i) of the present remark that:

$$\inf_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big] = - \sup_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[-c(X,Y)\big] = \mathbb{E}^{\bar{\mathbb{P}}_*}\big[c(X,Y)\big] = \int c\big(x, \bar{T}_*(x, dy)\big)\mu(dx).$$

Remark 3.15. *The martingale counterpart of the Spence-Mirrlees condition is $c_{xyy} > 0$. We now argue that this condition is the natural requirement in the present setting. Indeed, the optimization problem is not affected by the modification of the coupling function from c to $\bar{c}(x,y) := c(x,y) + a(x) + b(y) + h(x)(y-x)$ for any $a \in \mathbb{L}^1(\mu)$, $b \in \mathbb{L}^1(\nu)$, and $h \in \mathbb{L}^0$. Since $c_{xyy} = \bar{c}_{xyy}$, it follows that the condition $c_{xyy} > 0$ is stable for the above transformation of the coupling function.*

3.6 Comparison with Beiglböck and Juillet [4]

The notion of left-monotone martingale transport was introduced by Beiglböck and Juillet [4], with an existence and uniqueness result, see Theorem 1.7 and Theorem 6.2.

1. We first show that their conditions on the coupling function fall in the context of our Theorem 3.13:

   - The first class of couplings considered in [4] is of the form $c(x, y) = h(y - x)$ for some differentiable function h whose derivative is strictly concave. Notice that this form of coupling essentially falls under our condition $c_{xyy} > 0$.

   - The second class of couplings considered in [4] is of the form $c(x, y) = \psi(x)\phi(y)$ where ψ is a non-negative decreasing function and ϕ a non-negative strict concave function. This class also essentially falls under our condition that $c_{xyy} > 0$.

2. The proof of [4] does not use the dual formulation of the martingale optimal transport problem. They rather extend the concept of cyclical monotonicity to the martingale context. As a consequence, [4] only provides an existence result and does not contain any explicit characterization of the maps (T_d, T_u) and the optimal semi-static hedging strategy (φ_*, ψ_*, h_*).
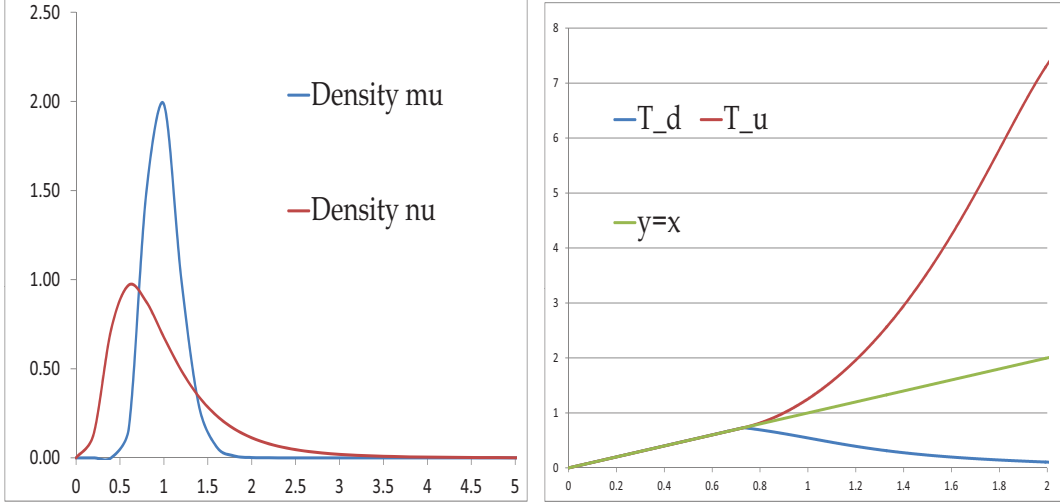
Figure 1: Maps T_d and T_u built from two log-normal densities with variances 0.04 and 0.32. $m_1 = 0.731$.

3. Our left-monotone martingale transport map T_* coincides with the *left-curtain coupling* whose existence (and uniqueness) is stated in Theorem 4.18 of [4].

4. Our construction agrees with the example of two Log-normal distributions $\mu_0 = e^{\mathcal{N}(-\sigma_1^2/2,\sigma_1^2)}$ and $\nu_0 = e^{\mathcal{N}(-\sigma_2^2/2,\sigma_2^2)}$, $\sigma_1^2 < \sigma_2^2$, illustrated in Figure 2 of [4]. By using our construction, we reproduce the left-monotone transference map in Figure 1. Indeed, in this case, $x_0 = -\infty$, δF has a unique local maximizer m_1, which is then the global maximizer of δF, and $x_1 = \infty$. The left-monotone transport plan is explicitly obtained from our construction after Step 1, i.e. no further steps are needed in this case.

Example 3.16. *We provide an example where δF has two local maxima and the construction needs two steps. Let μ and ν be defined by*

$$\mu_1 = \mathcal{N}(1, 0.5) \quad and \quad \nu_1(x) = \frac{1}{3}\Big[\mathcal{N}(1, 2) + \mathcal{N}(0.6, 0.1) + \mathcal{N}(1.4, 0.3)\Big].$$

Clearly μ and ν have mean 1, and $\mu \preceq \nu$. We also immediately check that δF has two local maxima $m_1 = -0.15$ and $m_2 = 0.72$. Figure 2 below reports the maps T_u and T_d as obtained from our construction.

3.7   Comparison with Hobson and Neuberger [32]

Our Theorem 3.13 does not apply to the coupling function $c(x, y) = |x - y|$ considered by Hobson and Neuberger [32]. More importantly, the corresponding maps T_u^{HN} and T_d^{HN} introduced in [32] are both nondecreasing with $T_d^{\text{HN}}(x) < x < T_u^{\text{HN}}(x)$ for all $x \in \mathbb{R}$. So
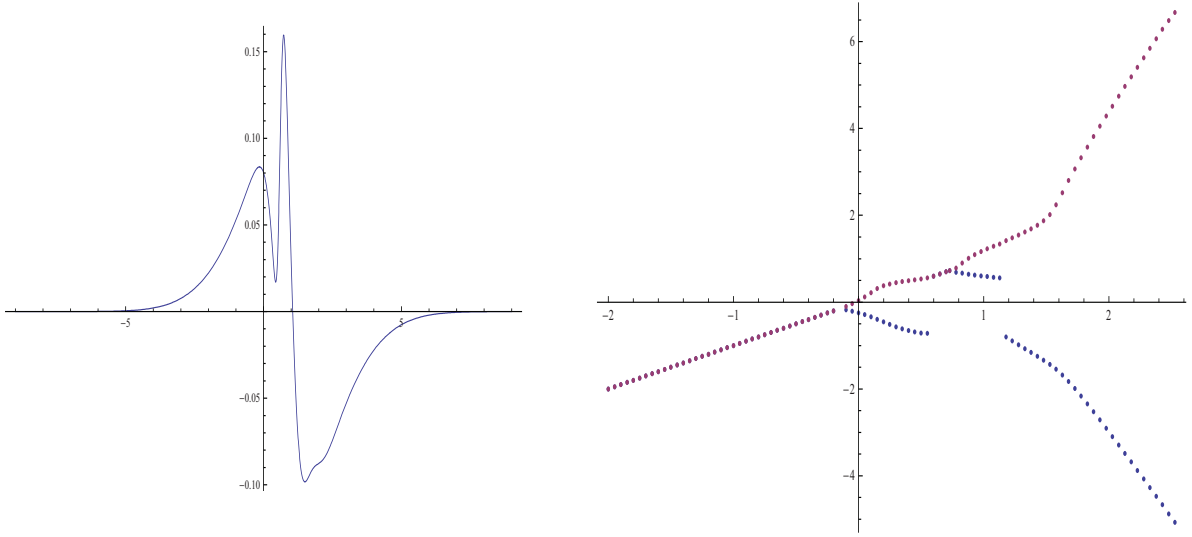
Figure 2: δF has two local maxima (left), and T_d, T_u corresponding to μ_1, ν_1 (right).

our solution (T_d, T_u) is of a different nature and in contrast with the above $(T_d^{\mathrm{HN}}, T_d^{\mathrm{HN}})$, our left-monotone martingale transport map T_* does not depend on the nature of the coupling function c as long as $c_{xyy} > 0$.

However, by following the line of argument of the proof of Theorem 3.13, we may recover the solution of Hobson and Neuberger [32]. As a matter of fact, our method of proof is similar to that of [32], as the dual problem D_2 is exactly the Lagrangian obtained by the penalization of the objective function by Lagrange multipliers.

3.8 Some examples

Example 3.17 (Variance swap)**.** *The coupling in this case is $c(x,y) = \ln^2\left(\frac{y}{x}\right)$ where μ and ν have support in $(0, \infty)$. In particular, it satisfies the requirement of Theorem 3.13 that $c_{xyy} > 0$. Then, the optimal upper bound is given by*

$$P_2(\mu, \nu) = \int_0^\infty \left[q(x) \ln^2\left(\frac{T_u(x)}{x}\right) + (1 - q)(x) \ln^2\left(\frac{T_d(x)}{x}\right) \right] \mu(dx), \qquad (3.22)$$

where q is set to an arbitrary value on D. In Figure 3, we have plotted φ_*, ψ_* and h_* with marginal distributions $\mu_0 = e^{\mathcal{N}(-\sigma_1^2/2, \sigma_1^2)}$ and $\nu_0 = e^{\mathcal{N}(-\sigma_2^2/2, \sigma_2^2)}$, $\sigma_1^2 = .04 < \sigma_2^2 = .32$. We recall that the corresponding maps T_d, T_u are plotted in Figure 1. The expression for ψ_* is

$$\psi'_*(x) = \frac{2}{x} \ln\left(\frac{x}{T_u^{-1}(x)}\right) + 2 \int_{x_0}^{T_u^{-1}(x)} \frac{\ln\left(\frac{T_u(\xi)}{T_d(\xi)}\right)}{\xi(T_u(\xi) - T_d(\xi))} d\xi.$$
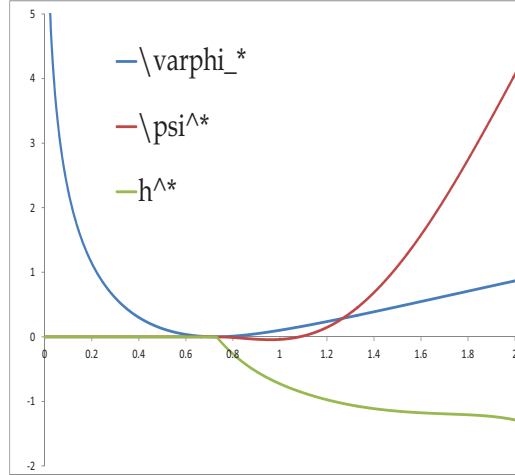
18

Figure 3: Superreplication strategy for a 2-period variance swap given two log-normal densities with variances 0.04 and 0.32.

In particular, $\psi_*''(x) = \frac{2}{x^2}$ for all $x \leq m_1$.

Example 3.18 $(c(x, y) = -\left(\frac{y}{x}\right)^p, \; p > 1,$ and μ, ν have support in $(0, \infty))$. *This payoff function also satisfies the condition of Theorem 3.13 that $c_{xyy} > 0$. The upper bound is*

$$P_2(\mu, \nu) = -\int_0^\infty \left[q(x)\left(\frac{T_u(x)}{x}\right)^p + (1 - q)(x)\left(\frac{T_d(x)}{x}\right)^p \right] \mu(dx).$$

4 The $n-$Marginals Martingale Transport

In this section, we provide a direct extension of our results to the martingale transportation problem under finitely many marginals constraint. Fix an integer $n \geq 2$, and let $X = (X_1, \ldots, X_n)$ be a vector of n random variables denoting the prices of some financial asset at dates $t_1 < \ldots < t_n$. Consider the probability measures $\mu = (\mu_1, \ldots, \mu_n) \in (\mathcal{P}_\mathbb{R})^n$ with $\mu_1 \preceq \ldots \preceq \mu_n$ in the convex order and

$$\int |\xi| \mu_i(d\xi) < \infty \quad \text{and} \quad \int \xi \mu_i(d\xi) = X_0, \quad \text{for all} \; i = 1, \ldots, n.$$

Similar to the two-marginals case, we introduce the set

$$\mathcal{M}_n(\mu) \quad := \quad \left\{ \mathbb{P} \in \mathcal{P}_n(\mu) : X \text{ is a } \mathbb{P}-\text{martingale} \right\},$$

where $\mathcal{P}_n(\mu)$ was defined in (2.6). In the present martingale version, we introduce the one-step ahead martingale transport maps defined by means of the n pairs of maps (T_d^i, T_u^i):

$$T_*^i(x_i, .) \quad := \quad \mathbf{1}_{D_i} \delta_{\{x_i\}} + \mathbf{1}_{D_i^c}\left(q_i(x_i)\delta_{T_u^i(x_i)} + (1 - q_i)(x_i)\delta_{T_d^i(x_i)}\right), \tag{4.1}$$

where $q_i(\xi) := (\xi - T_d^i(\xi))/(T_u^i - T_d^i)(\xi)$ for $\xi \in D_i^c$, and $(D_i, T_d^i, T_u^i)_{i=1,\ldots,n-1}$ are defined as in Subsection 3.3 with the pair (μ_i, μ_{i+1}).

The $n-$marginals martingale transport problem is defined by:

$$P_n(\mu) = \sup_{\mathbb{P} \in \mathcal{M}_n(\mu)} \mathbb{E}^{\mathbb{P}}[c(X)],$$

where the map $c : \mathbb{R}^n \longrightarrow \mathbb{R}$ is of the form

$$c(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} c^i(x_i, x_{i+1})$$

for some upper semicontinuous functions $c^i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with linear growth (or the condition (2.2)), $i = 1, \ldots, n-1$.

The dual problem is defined by

$$D_n(\mu) \quad := \quad \inf_{(u,h) \in \mathcal{D}_n} \sum_{i=1}^{n} \mu_i(u_i),$$

where $u = (u_1, \ldots, u_n)$ with components $u^i : \mathbb{R} \longrightarrow \mathbb{R}$, and $h = (h_1, \ldots, h_{n-1})$ with components $h_i : \mathbb{R}^i \longrightarrow \mathbb{R}$, taken from the set of dual variables:

$$\mathcal{D}_n := \left\{ (u,h) : (u_i)^+ \in \mathbb{L}^1(\mu_i), h_i \in \mathbb{L}^0(\mathbb{R}^i), \text{ and } \oplus_{i=1}^{n} u_i + \sum_{i=1}^{n-1} h_i^{\otimes^i} \geq c \right\}.$$

Here, $\oplus_{i=1}^{n} u_i(x) = \sum_{i \leq n} u_i(x_i)$ and $h_i^{\otimes^i}(x) = h_i(x_1, \ldots, x_i)(x_{i+1} - x_i)$.

Similar to the two-marginals problems, the weak duality inequality $P_n(\mu) \leq D_n(\mu)$ is obvious, and we shall obtain equality in the following result under convenient conditions.

To derive the structure of the optimal hedging strategy, we shall consider the two-marginals (μ_i, μ_{i+1}) problems with coupling functions c^i. By Theorem 3.13, we have for $i = 1, \ldots, n-1$:

$$P_2^i(\mu_i, \mu_{i+1}) := \sup_{\mathbb{P} \in \mathcal{M}(\mu_i, \mu_{i+1})} \mathbb{E}^{\mathbb{P}}[c^i(X,Y)] = \inf_{(\varphi, \psi, h) \in \mathcal{D}_2^i} \{\mu_i(\varphi) + \mu_{i+1}(\psi)\} = \mu_i(\varphi_i^*) + \mu_{i+1}(\psi_i^*),$$

where $\mathcal{D}_2^i$ is defined as in (3.3) with c^i substituted to c, and $(\varphi_i^*, \psi_i^*, h_i^*) \in \mathcal{D}_2^i$ are defined as in (3.18)-(3.19)-(3.20) with c^i substituted to c and (T_u^i, T_d^i) substituted to (T_u, T_d). Finally, we define:

$$u_i^*(x_i) := \mathbf{1}_{\{i<n\}} \varphi_i^*(x_i) + \mathbf{1}_{\{i>1\}} \psi_{i-1}^*(x_i), \quad i = 1, \ldots, n,$$

and $u^* := (u_1^*, \ldots, u_n^*)$, $h^* := (h_1^*, \ldots, h_{n-1}^*)$.

Theorem 4.1. *Suppose $\mu_1 \preceq \ldots \preceq \mu_n$ in convex order, with finite first moment, $\mu_1, \ldots, \mu_{n-1}$ have no atoms, and let Assumption 3.6 hold true for $\delta F = F_{\mu_{i+1}} - F_{\mu_i}$, for all $1 \leq i < n$. Assume further that*

- *c^i have linear growth, that the cross derivatives c_{xyy}^i exist and satisfy $c_{xyy}^i > 0$,*
- *φ_i^*, ψ_i^* satisfy the integrability conditions $(\varphi_i^*)^+ \in \mathbb{L}^1(\mu_i)$, $(\psi_i^*)^+ \in \mathbb{L}^1(\mu_{i+1})$.*

Then, the strong duality holds, the transference map $\mathbb{P}_n^*(dx) = \mu_1(dx_1) \prod_{i=1}^{n-1} T_*^i(x_i, dx_{i+1})$ is optimal for the martingale transportation problem $P_n(\mu)$, and (u^*, h^*) is optimal for the dual problem $D_n(\mu)$, i.e.

$$\mathbb{P}_n^* \in \mathcal{M}_n(\mu), \quad (u^*, h^*) \in \mathcal{D}_n, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}_n^*}[c(X)] = P_n(\mu) = D_n(\mu) = \sum_{i=1}^{n} \mu_i(u_i^*).$$

Proof. Clearly, we have $\mathbb{P}_n^* \in \mathcal{M}_n(\mu)$, which provides the inequality $\mathbb{E}^{\mathbb{P}_n^*}[c(X)] \leq P_n(\mu)$. We next observe that $(u^*, h^*) \in \mathcal{D}_n$ from our construction. Then $D_n(\mu) \leq \sum_{i \leq n} \mu_i(u_i^*) = \mathbb{E}^{\mathbb{P}_n^*}[c(X)]$. The required result follows from the weak duality inequality $P_n(\mu) \leq D_n(\mu)$. □

Remark 4.2. *The optimal lower bound for a coupling function as in Theorem 4.1 is attained by the mirror solution introduced in Remark 3.14.*

Example 4.3 (Discrete monitoring variance swaps)**.** *This is a continuation of our Example 3.17. Suppose that $\mu_1 \preceq \ldots \preceq \mu_n$ have support in $(0, \infty)$ with mean X_0, and let $c(x_1, \ldots, x_n) := \sum_{i=1}^n \left(\ln \frac{x_i}{x_{i-1}} \right)^2$. Then:*

$$P_n(\mu) = \int \left(\ln \frac{\xi}{X_0} \right)^2 \mu_1(d\xi) + \sum_{i=1}^{n-1} \int_0^\infty \left[q_i(\xi) \left(\ln \frac{T_u^i(\xi)}{\xi} \right)^2 + (1 - q_i)(\xi) \left(\ln \frac{T_d^i(\xi)}{\xi} \right)^2 \right] \mu_i(d\xi).$$

This optimal bound depends on all the marginals. The optimal lower bound is attained by our mirror solution, see Remark 4.2.

Remark 4.4. *In particular, their argument holds whenever $c(x, y)$ which satisfies $c(x, x) = 0 = c_y(x, x)$, $(x - y)c_{xy} + c_x > 0$ and our generalized Spence-Mirrlees condition $c_{xyy} > 0$. Note that apart from the last condition, these requirements on c are not preserved by the transformation in Remark 3.15.*

Remark 4.5. *In a related robust hedging problem, Hobson and Klimmek [31], derived an optimal upper bound for a derivative $c(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} c^0(x_i, x_{i+1})$. The difference with our problem above is that they are only given the marginal distribution μ_n for X_n. See also Kahale [36]. We would like to emphasize that [31] assume the variance Kernel c_0 to satisfy the conditions $c^0(x, x) = c_y^0(x, x) = 0$, $(x - y)c_{xy} + c_x > 0$, together with our Spence-Mirrlees condition $c_{xyy} > 0$. In the context of our problem with finitely many given marginals $\mu_1, \ldots, \mu_n$, notice that, apart from the Spence-Mirrlees condition, none of these requirements are preserved by the transformation of Remark 3.15.*

5 Proof of the main results

5.1 Construction of the left-monotone map

This section is devoted to the proof of Theorem 3.10. We first motivate the definition of the maps T_d and T_u through the functions g and G. In this heuristic discussion, we consider the simple case of one single maximizer m_1 with δF strictly increasing before m_1, and we ignore the possible jumps of F_ν.

The first observation about our construction is that for a point $y \in \mathbb{R}$, there are two alternatives:

- either $y \in (-\infty, m_1]$; then $\mathbb{P}_*[Y \in dy] = dF_\mu(y) + \mathbb{E}\big[(1-q)(X)\mathbf{1}_{\{T_d(X)\in dy\}}\big]$, and the requirement that $Y \sim_{\mathbb{P}_*} \nu$ together with the decrease of T_d imply that

$$d(\delta F \circ T_d) = -(1-q)dF_\mu;$$

in particular, in order for T_d to be well-defined, it has to be valued in the domain of increase of δF,

- or $y \in (m_1, \infty)$, then $\mathbb{P}_*[Y \in dy] = \mathbb{E}\big[q(X)\mathbf{1}_{\{T_u(X)\in dy\}}\big]$, and the requirement that $Y \sim_{\mathbb{P}_*} \mu_2$ together with the increase of T_u imply that

$$d(F_\nu \circ T_u) = qdF_\mu.$$

Direct manipulation of these two equations implies that $d\delta F(T_d) = -dF_\mu + dF_\nu(T_u)$. Since $T_d(m_1) = T_u(m_1) = m_1$, this implies that:

$$F_\nu\big(T_u(x)\big) = F_\mu(x) + \delta F\big(T_d(x)\big),$$

i.e. $T_u = g(., T_d)$ as in (3.11), (3.12), and (3.14).

Also, as a consequence of this relation, we see that the requirement $T_u(x) \geq x$ implies that $\delta F(x) \leq \delta F(T_d(x))$. Consequently, the choice of the break point m_1 as the maximizer of δF is necessary.

We next substitute q and T_u in the martingale condition:

$$xdF_\mu = T_u qdF_\mu + T_d(dF_\mu - qdF_\mu) = g(., T_d)d[F_\mu + \delta F(T_d)] - T_d d\delta F(T_d).$$

This implies that, for $x > m_1$:

$$[x - F_\nu^{-1} \circ F_\mu(x)]dF_\mu = d\bigg\{ \int_0^{\delta F(T_d)} F_\nu^{-1}(F_\mu(x) + y)dy \bigg\} - T_d d\delta F(T_d).$$

Integrating from m_1 to x, and using the condition $T_d(m_1) = m_1$, this provides:

$$G(T_d(x), x) - G(m_1, m_1) = 0$$

where

$$
\begin{aligned}
G(t, x) &:= \int_{-\infty}^x [F_\nu^{-1} \circ F_\mu(\xi) - \xi]dF_\mu(\xi) + \int_0^{\delta F(t)} F_\nu^{-1}(F_\mu(x) + y)dy - \int_{(-\infty, t]} \xi d\delta F(\xi) \\
&= \int_{(-\infty, F_\nu^{-1}\circ F_\mu(x)]} \xi dF_\nu(\xi) - \int_{-\infty}^x \xi dF_\mu(\xi) + \int_{(-\infty, t]} [g(x, \xi) - \xi]d\delta F(\xi),
\end{aligned}
$$

in agreement with our definition of G^A in (3.7) for $A = (-\infty, m_1]$. We finally notice by direct computation that $G(m_1, m_1) = 0$, so that T_d must satisfy the equation $G(T_d(x), x) = 0$ for all $x \geq m_1$.

Proof of Lemma 3.5 (i) Since δF is strictly increasing on A, we see that F_ν is strictly increasing in A. Therefore, for $t < m \leq x$, $t \in A$ we have $g(x, t) - t > g(t, t) - t = 0$, implying that $t \longmapsto G^A(t, x)$ is strictly increasing in t on the set A.

We next verify that $G^A(m, x) > 0$ as long as $\delta F(m) > \delta F(x)$. Denoting by d_x the differential with respect to the $x-$variable, we compute by using the conditions on the set A that

$$
\begin{aligned}
d_x G^A(m, x) &= \left(F_\nu^{-1} \circ F_\mu(x) - x + \int_{(-\infty, m]} \partial_x g(x, \xi) \mathbf{1}_A(\xi) d\delta F(\xi) \right) dF_\mu(x) \\
&= \left(g(x, m) - x + \int_{\bar{A}_m} dg(x, \xi) \right) dF_\mu(x) \geq (g(x, m) - x) dF_\mu(x),
\end{aligned}
$$

since $g(x, \xi) = \phi(\delta F(\xi))$ where, for fixed x, the function $y \mapsto \phi(y) := F_\nu^{-1}(\delta F(x) + y)$ is nondecreasing. Since F_μ strictly increases at the right of m, and $G^A(m, m) = 0$, this shows that $G^A(m, x) > 0$ as long as $g(x, m) - x > 0$ or, equivalently, $\delta F(m) > \delta F(x)$.

Then, in order to establish the existence and uniqueness of $t^A(x, m)$, it remains to verify that

$$
\gamma(x) := G^A(-\infty, x) = \int_{(-\infty, F_\nu^{-1} \circ F_\mu(x)]} \xi dF_\nu(\xi) - \int_{-\infty}^x \xi dF_\mu(\xi) < 0 \quad \text{for} \quad \delta F(x) \leq \delta F(m).
$$

Let $\bar{x}_0 := \inf\{x : \delta F(x) > 0\}$. Clearly, $\bar{x}_0 < m \leq x$, and $\gamma = 0$ on $(-\infty, \bar{x}_0)$, $\gamma(r_\nu) = 0$. Moreover, γ is flat on $\mathrm{Supp}(F_\mu)^c$, where $\mathrm{Supp}(F_\mu)$ is a support of F_μ, and we see by direct differentiation that γ is absolutely continuous with respect to μ with:

$$
d\gamma(x) = (F_\nu^{-1} \circ F_\mu(x) - x) dF_\mu(x),
$$

implying that $d\gamma < 0$ at the right of $\bar{x}_0$, by the (strict) convex-order property ($\mu \preceq \nu$) implied by the strict increase of δF on A. Furthermore, let x^* be any possible local maximizer of γ. By the fact that γ is flat off $\mathrm{Supp}(F_\mu)$, we may assume that x^* is either an interior point of $\mathrm{Supp}(F_\mu)$ or x^* is a left accumulation point of $\mathrm{Supp}(F_\mu)$. In both cases, it follows from the first order condition that

$$
F_\nu^{-1}\big(F_\mu(x^*) - \big) \leq x^* \leq F_\nu^{-1}\big(F_\mu(x^*)\big).
$$

If F_ν^{-1} is continuous at the point $F_\mu(x^*)$, then $\delta F(x^*) = 0$, and it follows from the definition of γ that

$$
\gamma(x^*) = \int_{(-\infty, x^*]} \xi d\delta F(\xi) = -\int_{(-\infty, x^*]} (x^* - \xi) d\delta F(\xi).
$$

By the (strict) convex-order property, this implies that $\gamma(x^*) < 0$.

In the alternative case that F_ν^{-1} jumps at the point $F_\mu(x^*)$, notice that F_ν is flat at the right of $F_\nu^{-1} \circ F_\mu(x^*)$, and therefore the conclusion $\gamma(x^*) < 0$ holds true in this case as well. Consequently, $\gamma < 0$ on $(\bar{x}_0, r_\mu)$. Since $x \geq m > \bar{x}_0$, this provides the required strict inequality.

(ii) Suppose $\bar{x}(m) < \infty$. Since δF is strictly increasing on A, the inequalities (3.8) follow from the definition of $\bar{x}(m)$.

It remains to prove that δF strictly increases in a right neighborhood of $\bar{x}(m)$ whenever $\bar{x}(m) < \infty$. By definition, we have $t^A(x,m) > (\delta F)^{-1} \circ \delta F(x)$ on $(m, \bar{x}(m))$, and $t^A(\bar{x}(m), m) \leq (\delta F)^{-1} \circ \delta F(\bar{x}(m))$, where δF^{-1} denotes the inverse function of $\int_{-\infty}^{\cdot} \mathbf{1}_A d\delta F$. We denote $h(x,m) := G^A((\delta F)^{-1} \circ \delta F(x), x)$, and we compute that $d_x h(x,m) = [x - (\delta F)^{-1} \circ \delta F(x)] d\delta F(x)$. Since $x > (\delta F)^{-1} \circ \delta F(x)$ whenever $x > m$, we see that $h(., m)$ decreases down from zero on the right neighborhood of $x = m$ (confirming that $t^A(x,m) > (\delta F)^{-1} \circ \delta F(x)$ near m), and has the same maximum and minimum points as the function δF. Since h must be increasing at a right neighborhood of $\bar{x}(m)$, it follows that δF has the same property. □

Proof of Theorem 3.10 (i) By construction, the probability measure $\mathbb{P}_*$ satisfies the left-monotonicity property of Definition 3.2. In the rest of this proof, we verify that $\mathbb{P}_* \in \mathcal{M}_2(\mu, \nu)$. In particular, by the uniqueness result of Beiglböck and Juillet [4] (Theorem 1.5 and Corollary 1.6), this would imply that $\mathbb{P}_*$ is the unique left monotone transport plan.

First, by the definition of $\mathbb{P}_*$ in (3.15), $X \sim_{\mathbb{P}_*} \mu$, and $\mathbb{E}^{\mathbb{P}_*}[Y|X] = X$. It remains to verify that $Y \sim_{\mathbb{P}_*} \nu$. We argue as in the beginning of Section 5.1 considering separately the following alternatives for any point $y \in \mathbb{R}$:

Case 1: $y = y_d \in D \cap B_0$ corresponds to some point x such that $y_d = T_d(x)$, and we see from the definition of $\mathbb{P}_*$ that:

$$\mathbb{P}_*[Y \in dy_d] = dF_\mu\big(T_d(x)\big) - (1-q)dF_\mu(x) \quad \text{and} \quad dF_\nu\big(T_u(x)\big) = qdF_\mu.$$

Then, $\mathbb{P}_*[Y \in dy_d] = dF_\mu(y_d) - dF_\mu(x) + dF_\nu(T_u(x))$. Since $T_u(x) = g(x, T_d(x))$, this provides $\mathbb{P}_*[Y \in dy_d] = F_\nu(dy)$ by direct substitution.

Case 2: $y = y_u \in D^c$ corresponds to some x such that $y_u = T_u(x)$, and we see from the definition of $\mathbb{P}_*$ that:

$$\mathbb{P}_*[Y \in dy_u] = qdF_\mu(x) = d\delta F(T_d(x)) + dF_\mu.$$

Using again the expression of T_u in terms of T_d, it follows that

$$\mathbb{P}_*[Y \in dy_u] = d\big(F_\nu \circ T_u(x) - F_\mu(x)\big) + dF_\mu(x) = dF_\nu(x).$$

Case 3: At a point of discontinuity of T_u or T_d, the above cases 1 and 2 are immediately adapted to account for the point mass.

Case 4: In the remaining alternative $y \in D \setminus B_0$, we observe that the function δF is flat near y, and there is no $x \neq y$ such that $T_d(x) = y$ or $T_u(x) = y$. Then, it follows from the definition of $\mathbb{P}_*$ that:
$$\mathbb{P}_*[Y \in dy] = dF_\mu(y) = dF_\nu(y).$$

(ii) Differentiating the integral equation defined by G^A at a continuity point of T_d, we see that:

$$
\begin{aligned}
0 &= -\big[F_\nu^{-1} \circ F_\mu(x) - x\big]dF_\mu(x) + \big[g(x, T_d(x)) - F_\nu^{-1} \circ F_\mu(x)\big]dF_\mu(x) \\
&\quad + \big[g(x, T_d(x)) - T_d(x)\big]d\delta F(T_d(x)) \\
&= \big[g(x, T_d(x)) - x\big]dF_\mu(x) + \big[g(x, T_d(x)) - T_d(x)\big]d\delta F(T_d(x)).
\end{aligned}
$$

Since $T_u = g(., T_d)$ this is the required ODE. The ODE for T_u is obtained by using the relation $T_u = g(., T_d)$. □

5.2 The optimal semi-static hedging strategy

We start by following the same line of argument as in the proof of Theorem 2.2 in order to identify the semi-static hedging strategy introduced in (3.18-3.19-3.20). Our objective is then to construct a pair

$$
(\varphi_*, \psi_*, h_*) \in \mathcal{D}_2 \quad \text{such that} \quad \mu(\varphi_*) + \nu(\psi_*) = \mathbb{E}^{\mathbb{P}_*}[c(X, Y)]. \tag{5.1}
$$

This will provide equality in (3.5) with the optimality of $\mathbb{P}_*$ for the optimal transportation problem P_2 and the optimality of (φ_*, ψ_*, h_*) for the dual problem D_2.

By the definition of the dual set $\mathcal{D}_2$, we observe that the requirement (5.1) is equivalent to

$$
\varphi_*(X) + \psi_*(Y) + h_*(X)(Y - X) - c(X, Y) = 0, \quad \mathbb{P}_* - \text{a.s. for some function} \quad h_*, \tag{5.2}
$$

and that the function φ_* is determined by:

$$
\varphi_*(x) = \max_{y \in \mathbb{R}} H(x, y), \quad \text{where} \quad H(x, y) := c(x, y) - \psi_*(y) - h_*(x)(y - x), \quad x, y \in \mathbb{R}. \tag{5.3}
$$

The perfect replication property (5.2), is equivalent to:

$$
\begin{aligned}
\varphi_*(x) &= q(x)(c(x, .) - \psi_*) \circ T_u(x) + (1 - q(x))(c(x, .) - \psi_*) \circ T_d(x), \tag{5.4} \\
h_*(x) &= \frac{(c(x, .) - \psi_*) \circ T_u(x) - (c(x, .) - \psi_*) \circ T_d(x)}{(T_u - T_d)(x)} \quad \text{for} \quad x \in D^c, \tag{5.5}
\end{aligned}
$$

where we observe that we may choose h_* arbitrarily on D.

It remains to determine ψ_* by using the static superhedging condition (5.3). Since T_u and T_d are maximizers in (5.3), it follows from the first-order condition that

$$
\psi_*' \circ T_u(x) = c_y(x, T_u(x)) - h_*(x), \quad \psi_*' \circ T_d(x) = c_y(x, T_d(x)) - h_*(x), \quad x \in D^c, \tag{5.6}
$$

$$
\text{and} \quad \psi_*'(x) = c_y(x, x) - h_*(x) \quad \text{for} \quad x \in D. \tag{5.7}
$$

We now determine h_*. Differentiating (5.5), and using (5.6), we see that for $x \in D^c$:

$$
\begin{aligned}
h_*'(x) &= \frac{d}{dx}\left\{\frac{c(x, T_u) - c(x, T_d)}{T_u - T_d}\right\} + \frac{T_u' - T_d'}{T_u - T_d}\frac{\psi_*(T_u) - \psi_*(T_d)}{T_u - T_d} \\
&\quad + \frac{T_d'\big[c_y(x, T_d) - h_*\big] - T_u'\big[c_y(x, T_u) - h_*\big]}{T_u - T_d}
\end{aligned}
$$

Then, direct calculation leads to the expression of h'_* on D^c reported in (3.18). Since T_d and T_u take values in D and D^c, respectively, and h_* is determined by the last two equations, we see that equation (5.6) determines ψ_* on $\mathbb{R}$. We finally observe that by (5.6) and (5.7), we have for $x \in D$ that $\psi'_*(x) = c_y(T_d^{-1}(x), x) - h_* \circ T_d^{-1}(x) = c_y(x, x) - h_*(x)$, which completes the definition of h_*, up to an irrelevant constant, on D.

5.3 Proof of Theorem 3.13

Following the line of argument of the proof of Theorem 2.2, we see from the weak duality (3.5) that

$$\mathbb{E}^{\mathbb{P}_*}[c(X, Y)] \leq P_2(\mu, \nu) \leq D_2(\mu, \nu).$$

Then, the proof of Theorem 3.13 is completed by the following result.

Lemma 5.1. *Let μ, ν be as in Assumptions 3.3 and 3.6, and suppose that the payoff function c satisfies $c_{xyy} > 0$. Then $\varphi_* \oplus \psi_* + h_*^\otimes \geq c$.*

Proof (i) We first verify that T_u and T_d satisfy the second order condition for a local maximum on D^c. Differentiating (5.6), and using the expression of h'_* in (3.19), it follows from the condition $c_{xyy} > 0$ that, in the distribution sense,

$$H_{yy}(., T_u)T'_u = \left[c_{yy}(., T_u) - \psi''_* \circ T_u\right]T'_u = \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d} - c_{xy}(., T_u) < 0$$

$$H_{yy}(., T_d)T'_d = \left[c_{yy}(., T_d) - \psi''_* \circ T_d\right]T'_d = \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d} - c_{xy}(., T_d) > 0,$$

on D^c. By the nondecrease of T_u and the nonincrease of T_d, this implies that $H_{yy}(., T_u) < 0$ and $H_{yy}(., T_d) < 0$.

(ii) We next show that $y \longmapsto H(., y)$ is increasing before T_d, and decreasing after T_u. In particular, this implies that:

$$\varphi_*(x) = \max_{y \in [T_d(x), T_u(x)]} H(x, y) \quad \text{for all} \quad x \in \mathbb{R}.$$

Set $y := T_u(x)$, let m_i be the local maximum from which $(T_d, T_u)(x)$ is constructed, and consider an arbitrary $y' = T_u(x') > y$ for some $x' > x$. We only report the proof for the case $x' \in (m_j, x_j]$ for some $j \geq i$, the remaining cases are treated similarly. Recalling that $H_y(x, T_u(x)) = 0$, we decompose

$$H_y(x, y') = H_y(x, y') - H_y(x, m_j) + \sum_{i+1}^{j} (A_k + B_k),$$

where

$$A_k := H_y(x, m_k) - H_y(x, x_{k-1}), \quad B_k := H_y(x, x_{k-1}) - H_y(x, m_{k-1} \wedge T_u(x)).$$

26

We next compute from the expression of h_* in (3.18) that:

$$H_y(x, y') - H_y(x, m_j) = \int_{m_j}^{y'} \left[c_{yy}(x, \xi') - \psi''(\xi') \right] d\xi'$$

$$\leq \int_{m_j}^{y'} \left[c_{yy}(x, \xi') - c_{yy}(T_u^{-1}(\xi'), \xi') \right] d\xi'$$

$$= \int_{m_j}^{y'} \int_x^{T_u^{-1}(\xi')} c_{xyy}(\xi, \xi') d\xi d\xi' < 0,$$

where the second inequality follows from the second order condition verified in (i). Similarly, we compute that

$$A_k = \int_{x_{k-1}}^{m_k} \left[c_{yy}(x, \xi') - \psi''(\xi') \right] d\xi'$$

$$\leq \int_{x_{k-1}}^{m_k} \left[c_{yy}(x, \xi') - c_{yy}(T_d^{-1}(\xi'), \xi') \right] d\xi'$$

$$= -\int_{x_{k-1}}^{m_k} \int_x^{T_d^{-1}(\xi')} c_{xyy}(\xi, \xi') d\xi d\xi' < 0,$$

where we used again the second order condition verified in (i). Finally,

$$B_k = \int_{m_{k-1} \vee T_u(x)}^{y} \left[c_{yy}(x, \xi') - \psi_*''(\xi') \right] d\xi'$$

$$\leq \int_{m_{k-1} \vee T_u(x)}^{y} \left[c_{yy}(x, \xi') - c_{yy}(T_u^{-1}(\xi'), \xi') \right] d\xi'$$

$$= -\int_{m_{k-1} \vee T_u(x)}^{y} \int_x^{T_u^{-1}(y')} c_{xyy}(\xi, \xi') d\xi d\xi' < 0.$$

A similar argument also shows that $H_y(x, y') < 0$ for $y' < T_d(x)$.

(iii) We next show that $H(., T_d) = H(., T_u)$. Denote $\delta H := H(., T_u) - H(., T_d)$, and compute:

$$\delta H' := c_x(., T_u) - c_x(., T_d) - (T_u - T_d)h_*'$$
$$+ \left[c_y(., T_u) - \psi_*'(T_u) - h_* \right] T_u' - \left[c_y(., T_d) - \psi_*'(T_d) - h_* \right] T_d'$$

in the distribution sense. By definition of ψ_* and h_*, it follows that $\delta H' = 0$ at any continuity point. Since δH is continuous by our construction, see (3.21), this shows that $\delta H(x) = \delta H(m_i) = 0$, where m_i is the local maximizer from which $(T_d, T_u)(x)$ is defined.

(iv) We finally show that T_u and T_d are global maximizers of $y \longmapsto H(., y)$. Let $x \in D^c$, and denote by m the local maximizer from which $T_d(x)$ and $T_u(x)$ are constructed. For fixed $T = T_u(t) \in (m, T_u(x))$, it follows from similar calculations as in the previous step that

$$\partial_x \{ H(., T_u) - H(., T) \} = c_x(., T_u) - c_x(., T) - (T - T_d)h_*'$$
$$= (T_u - T) \left(\frac{c_x(., T_u) - c_x(., T)}{T_u - T} - \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d} \right) > 0$$

by the condition $c_{xyy} > 0$. Then $H(., T_u) - H(., T) = \int_t^\cdot \partial_x \{H(., T_u) - H(., T)\} > 0$.

By a similar calculation, we also show that $H\big(x, T_d(x)\big) - H(x, T) \geq 0$ for all $T \in (T_d(x), m)$, thus completing the proof that T_d and T_u are global maximizers of $y \longmapsto H(., y)$.

$\square$

6 Complement: Some examples of martingale measures given marginals

6.1 Local volatility model

A first example of a martingale measure fitted to two marginal distributions μ_{t_1} and μ_{t_2}, corresponding to the maturities $t_1 < t_2$, is given by the Dupire local volatility model (in short LV) [23]. We first define an interpolation $(\mu_t)_{t \in [t_1, t_2]}$ which does not violate the no-arbitrage condition, i.e. which obeys to the convex ordering condition. This can be achieved by introducing the implied Black-Scholes accumulated variances $\varpi(t_i, K)$, defined by $\mathrm{BS}\big(K, \varpi(t_i, K)\big) = \mathcal{C}(t_i, K) := \int (\xi - K)^+ \mu_{t_i}(d\xi)$, where BS denotes the Black-Scholes formula

$$\mathrm{BS}(K, v) := X_0 \mathbf{N}\Big(\frac{\ln(X_0/K)}{\sqrt{v}} + \frac{\sqrt{v}}{2}\Big) - K \, \mathbf{N}\Big(\frac{\ln(X_0/K)}{\sqrt{v}} - \frac{\sqrt{v}}{2}\Big),$$

with $\mathbf{N}$ the c.d.f. of the standard Normal distribution, and for $t \in [t_1, t_2]$:

$$\varpi(t, K) = \frac{t_2 - t}{t_2 - t_1} \varpi(t_1, K) + \frac{t - t_1}{t_2 - t_1} \varpi(t_2, K), \ \mathcal{C}(t, K) := \mathrm{BS}\big(K, \varpi(t, K)\big) = \int (\xi - K)^+ \mu_t(d\xi).$$

The Dupire LV model corresponding to this interpolation is defined by the SDE:

$$dX_t = X_t \sigma_{\mathrm{loc}}(t, X_t) dW_t \quad \text{with} \quad \sigma_{\mathrm{loc}}(t, K)^2 := 2 \, \frac{\partial_t \mathcal{C}(t, K)}{\partial_K^2 \mathcal{C}(t, K)},$$

whenever σ_{loc} is well-defined and induces a well-defined weak solution for the above SDE. In general, σ_{loc} is a measure with poor regularity. A rigorous adaptation of this solution, by convenient regularization of σ_{loc}, is provided by Hirsh and Roynette [28], resulting in a new proof of the Kellerer theorem [37].

A natural extension of such a LV model is given by the so-called local stochastic volatility model in which X_t satisfies a non-linear McKean SDE ([26]):

$$dX_t = X_t \, \frac{\sigma_{\mathrm{loc}}(t, X_t)}{\sqrt{\mathbb{E}[a_t^2 | X_t]}} a_t dW_t$$

with a_t a (possibly multi-dimensional) Itô diffusion. Existence and uniqueness for such a non-linear SDE is not at all obvious and still open.

6.2 Local variance Gamma model

A second example, introduced by P. Carr [9], which does not require the construction of a continuous-time implied volatility surface is given by the local variance Gamma model in which the process X_t is defined as a time-homogeneous one-dimensional Itô diffusion $\bar{X}_t$ subordinated by an independent Gamma process Γ_t [9]:

$$X_t \equiv \bar{X}_{\Gamma_t}$$
$$d\bar{X}_t = \sigma(\bar{X}_t)dW_t, \ \bar{X}_0 = X_0$$

The distribution of the Gamma process at time t is a Gamma distribution with density:

$$\mathbb{P}\{\Gamma_t \in ds\} = \frac{\alpha^{\frac{t}{t^*}}}{\Gamma\left(\frac{t}{t^*}\right)} s^{\frac{t}{t^*}-1} e^{-\alpha s}, \ s > 0$$

for some parameters $t^* = t_2 - t_1$, $\alpha = 1/t^*$. The Fokker-Planck PDE reads

$$\frac{1}{2}\sigma(K)^2 \partial_K^2 \mathcal{C}(t_2, K) = \frac{\mathcal{C}(t_2, K) - \mathcal{C}(t_1, K)}{t_2 - t_1}$$

from which we can deduce the local volatility function $\sigma(\cdot)$ from call options valued uniquely at t_1 and t_2. The Dupire infinitesimal calendar spread gets replaced by a discrete calendar spread. Similar to the previous example, a rigorous adaptation of this idea requires a regularization of the above function $\sigma(\cdot)$ as in [28].

6.3 Local Lévy's model

As a last example, we review the local Levy model introduced by Carr, Geman Madan, and Yor [10]. The process X_t is a compensated jump martingale

$$dX_t = \int_{\mathbb{R}} X_{t-} \left(e^x - 1\right) \left(m(dx, dt) - \nu(dx, dt)\right), \quad \nu(dx, dt) = a(t, X_t)k(x)dxdt,$$

where $m(dx, dt)$ is the counting measure with compensator ν. The analogue of the Dupire formula is

$$\partial_t \mathcal{C} = \int_0^\infty \partial_y^2 \mathcal{C}(t, y) y a(t, y) \psi\left(\ln \frac{K}{y}\right) dy,$$

with the double tail ψ of the Lévy measure $k(x)$ given by

$$\psi(t, z) = \mathbf{1}_{\{z<0\}} \int_{-\infty}^z e^x \int_{-\infty}^x k(u)dudx + \mathbf{1}_{\{z>0\}} \int_z^\infty e^x \int_x^\infty k(u)dudx.$$

References

[1] Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W., Temme, J. : *A trajectorial interpretation of Doobs martingale inequalities.* Ann. Appl. Probab., 2012. To appear.

[2] Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W. : *A Model-free Version of the Fundamental Theorem of Asset Pricing and the Super-Replication Theorem.* Preprint 2013.

[3] Beiglböck, M., Henry-Labordère, P., Penkner, F. : *Model-independent Bounds for Option Prices: A Mass-Transport Approach*, preprint arXiv:1106.5929, Finance and Stochastics, to appear (2013).

[4] Beiglböck, M., Juillet, N. : *On a problem of optimal transport under marginal martingale constraints*, preprint (Aug. 2012), arXiv:1208.1509.

[5] Breeden, D.T., Litzenberger, R.H. : *Prices of state-contingent claims implicit in options prices*, J. Business, 51, 621651 (1978).

[6] Brenier, Y. : *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C. R. Acad. Sci. Paris Sci. Paris Série I Math., 305(19): 805-808, 1987.

[7] Brown, H., Hobson, D., Rogers, L.C.G. : *Robust hedging of barrier options.* Math. Finance, 11(3):285314, 2001.

[8] Carlier, G. : *On a class of multidimensional optimal transportation problems*, Journal of convex analysis, Vol. 10 (2003), No. 2, 517-529.

[9] Carr, P. : *Local variance Gamma option pricing model*, presentation, IBCI conference, Paris (April 2009).

[10] Carr, P. , Geman, H., Madan, D.B., Yor, M. : *From local volatility to local Lévy models*, Quantitative Finance, Oct. 2004.

[11] Cousot, L. : *Necessary and suffcient conditions for no static arbitrage among european calls.* Courant Institute, New York University, 2004.

[12] Cousot, L. : *Conditions on option prices for absence of arbitrage and exact calibration.* Journal of Banking and Finance, 31(11):33773397, 2007. Risk Management and Quantitative Approaches in Finance.

[13] Cox, A. M. G., Hobson, D., Oblój, J. : *Pathwise inequalities for local time: applications to Skorokhod embeddings and optimal stopping.* Ann. Appl. Probab., 18(5):18701896, 2008.

[14] Cox, A. M. G., Oblój, J. : *Robust hedging of double touch barrier options.* SIAM J. Financial Math., 2:141182, 2011.

[15] Cox, A. M. G., Obłój, J. : *Robust pricing and hedging of double no-touch options*. Finance Stoch., 15(3):573605, 2011.

[16] Cox, A. M. G., Wang, J. : *Root's Barrier: Construction, Optimality and Applications to Variance Options*. Ann. Appl. Prob., to appear, 2012.

[17] d'Aspremont A., El Ghaoui, L. : *Static arbitrage bounds on basket option prices*. Mathematical Programming, Series A 106(3), pp. 467-489, July 2006.

[18] Davis, M.H.A., Hobson, D. : *The range of traded option prices*. Math. Finance, 17(1):114, 2007.

[19] Davis, M.H.A., Obłój, J., Raval, V. : *Arbitrage Bounds for Prices of Options on Realized Variance*. Math. Finance, to appear., 2012.

[20] Dolinsky, Y., Soner, H.M. : *Robust hedging and martingale optimal transport in continuous time*, preprint (Aug. 2012), arXiv:1208.4922v1.

[21] Dolinsky, Y., Soner, H.M. : *Robust Hedging under Proportional Transaction Costs*. Preprint, 2013.

[22] Dubins, L.E., Schwarz, G. : *On extremal martingale distributions*, Proc. Fifth Berkeley Symp. on Math. Statist. and Prob., Vol. 2, Pt. 1 (Univ. of Calif. Press, 1967), 295-299.

[23] Dupire, B. : *Pricing with a Smile*, Risk Magazine 7, 18-20 (1994).

[24] Galichon, A., Henry-Labordère, P., Touzi, N. : *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options*, http://ssrn.com/abstract=1912477, Annals of Applied Probability, to appear (2013).

[25] Gangbo, W., Święch, A. : *Optimal maps for the multidimensional Monge-Kantorovich prooblem*, Comm. Pure Appl. Math., 51 (1) : 23-45, 1998.

[26] Guyon, J., Henry-Labordère, P. : *Being particular about calibration*, Available at SSRN: http://ssrn.com/abstract=1885032, Risk magazine, Jan. (2012).

[27] Henry-Labordère, P., Obłój, J., Spoida, P., Touzi, N. : *Maximum maximum of martingales given marginals*. Preprint 2013.

[28] Hirsch, F., Roynette, B. : *A new proof of Kellerer's theorem*, ESAIM: Probability and Statistics, Vol.16, Sept. 2012, pp 48-60.

[29] Hobson, D. : *Robust hedging of the lookback option*, Finance and Stochastics, 2, pp329-347, 1998.

[30] Hobson, D. : *The Skorokhod embedding problem and model-independent bounds for option prices*. In Paris-Princeton Lectures on Mathematical Finance 2010, volume 2003 of Lecture Notes in Math., pages 267318. Springer, Berlin, 2011.

[31] Hobson, D., Klimmek, M. : *Model independent hedging strategies for variance swaps,* to appear in Finance and Stochastics (2012).

[32] Hobson, D., Neuberger, A. : *Robust bounds for forward start options,* Mathematical Finance, Vol 22, Issue 1, pp 31-56 (Jan. 2012).

[33] Hobson, D., Pedersen,J.L. : *The minimum maximum of a continuous martingale with given initial and terminal laws.* Ann. Probab., 30(2):978999, 2002.

[34] Jacod, J., Shiryaev, A.N. : *Local martingales and the fundamental asset pricing theorems in the discrete-time case.* Finance and Stochastics 2, 259-273 (1998).

[35] Jacod, J., Yor, M. : *Etude des solutions extrémales et représentation intégrale des solutions pour certains problèmes de martingales,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 38 (1977), no. 2, 83-125.

[36] Kahale, N. : *Model-independent lower bound on variance swaps,* preprint (2009).

[37] Kellerer, H.G. : *Markov-Komposition und eine Anwendung auf Martingale,* Math. Ann., 198, 99122 (1972).

[38] Laurence, P., Wang, T.H. : *Sharp upper and lower bounds for basket options,* Applied mathematical Finance, 12 (3): 253-282, 2005.

[39] Oberhauser, H., Dos Reis, G. : *Root's Barrier, Viscosity Solutions of Obstacle Problems and Reflected FBSDEs.* Preprint 2013.

[40] Oblój, J. : *The Skorokhod embedding problem and its offspring.* Probab. Surv., 1:321390, 2004.

[41] Pass, B. : *Uniqueness and Monge solutions in the multi-marginal optimal transportation problem.* SIAM J. Math. Anal. 43 (2011) 2758-2775.

[42] Strasse,, V. *The existence of probability measures with given marginals.* Ann. Math. Statist., 36:423-439, 1965.

[43] Villani, C. : *Optimal Transportation, Old and New.* Springer, Series Grundlehren der mathematischen Wissenschaften, Vol. 338 (2009).