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Macroscopic traffic flow optimization on roundabouts

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Abstract: The aim of this paper is to propose an optimization strategy for traffic flow on roundabouts using a macroscopic approach. The roundabout is modeled as a sequence of 2×2 junctions with one mainline and secondary incoming and outgoing roads. We consider two cost functionals: the total travel time and the total waiting time, which give an estimate of the time spent by drivers on the network section. These cost functionals are minimized with respect to the right of way parameter of the incoming roads. For each cost functional, the analytical expression is given for each junction.

Key-words: Road traffic on networks, Macroscopic models, Riemann problem, Optimization, Scalar conservation laws

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Optimisation du trafic routier dans les rond-points

Résumé : Le but de ce travail est l'optimisation du trafic routier dans un rond-point utilisant un modèle macroscopique. Le rond-point est décrit par une séquence de jonctions 2×2 avec une route principale et deux routes secondaires (entrante et sortante). Nous considérons deux fonctions-coût: le temps total de voyage et le temps total d'attente. Elles donnent une estimation du temps que chaque conducteur passe dans la section du résau en question. Ces fonctions-coût sont minimisées par rapport au paramètre de priorité des routes entrantes. Pour chaque fonction-coût, l'expression analytique est calculée au niveau de chaque jonction.

Mots-clés : Réseaux routiers, Modèles macroscopiques, Problème de Riemann, Optimisation, Lois de conservation scalaires

1 Introduction

The first macroscopic traffic flow models are due to the seminal works of Lighthill and Whitham [15] and, independently, Richards [16]. They proposed a fluid dynamic model for traffic flow on an infinite single road, using a non-linear hyperbolic partial differential equation (PDE). This model has, then, been extended to initial boundary value problems in [1] and developed for concave fluxes in [14]. More recently, several authors suggested models for networks, see [4, 5, 7, 8, 12] and references therein. These models consider different type of solutions and some of them have been used for optimization of traffic flow of networks, see for example [3, 2, 6, 10, 11].
In this article, we focus on optimization problems for roundabouts. We consider the model introduced in [7] and extend it to roundabouts. Roundabouts can be modeled as a sequence of 2×2 junctions. In particular, each junction is composed by one mainline and two secondary roads connected through a node. On the mainline we apply a macroscopic approach using the Lighthill-Whitham-Richards (LWR) model, while on the incoming secondary road a buffer of infinite size and capacity is used, whose evolution is described by an ordinary differential equation (ODE) that depends on the difference between the incoming and the outgoing fluxes on the lane. The outgoing secondary road is modeled as a sink. At each junction the Riemann problem is uniquely solved using a right of way parameter for the incoming fluxes, and solutions are constructed via wave-front tracking.
The aim of this paper is to derive the analytical expression of two cost functionals, the Total Travel Time (TTT) and the Total Waiting Time (TWT), and to optimize them through a suitable choice of the right of way parameter. The TTT and the TWT give an estimate of the time spent by drivers in the network sections (TTT) or in the queues at the buffers (TWT). The cost functionals are computed analytically for a simple network consisting of a single 2x2 junction.
This paper is organized as follows. Section 2 describes the junction model and its extension to roundabouts. Section 3 describes in detail the construction of the Riemann Solver at junctions. Finally, in Section 4 we introduce the cost functionals and compute their analytical expression.

2 Mathematical Model

In this work we consider a roundabout joining three roads as illustrated in Figure 1.

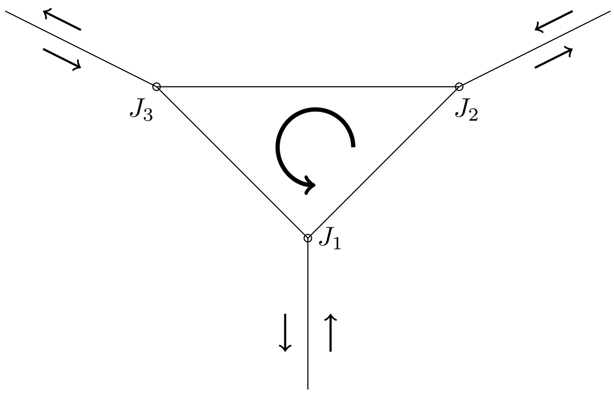
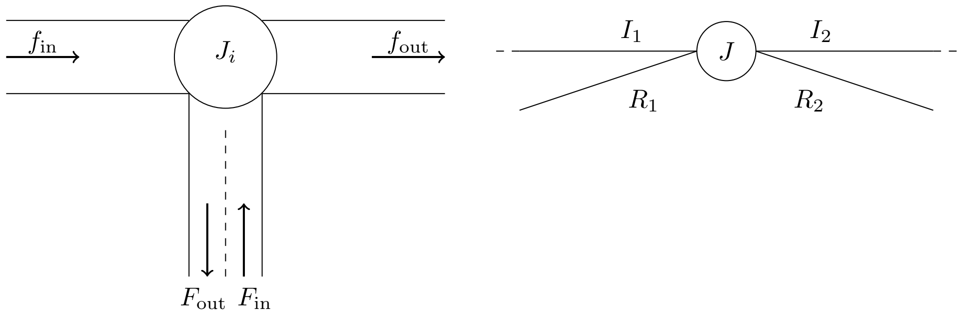


Figure 1: Roundabout considered in the article.

A roundabout can be seen as a periodic sequence of junctions and it can be represented by an oriented graph, in which roads are described by arcs and junctions by vertexes. Each road is modeled by an interval $\mathcal{I}_i = [a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$, $a_i < b_i$ with the possibility of either $a_i = -\infty$ or $b_i = +\infty$. In particular, in our case, each junction can be models as a 2×2 junction, see Figure 2. To recover the behavior of the roundabout periodic boundary conditions are introduced on the



(a) Part of the roundabout. (b) Corresponded junction.

Figure 2: Detail of the network modeled in the article

mainline such that $b_i = a_{i+1}$, $i = 1, 2, 3$. At each junction we will consider the model introduced in [7]: the evolution of the traffic flow in the mainline segments is given by the scalar hyperbolic conservation law:

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \quad (t, x) \in \mathbb{R}^+ \times I_i \quad i = 1, 2, 3, \tag{2.1}$$

where $\rho_i = \rho_i(t, x) \in [0, \rho_{\max}]$ is the mean traffic density, $\rho_{\max}$ the maximal density allowed on the road and the flux function $f : [0, \rho_{\max}] \to \mathbb{R}^+$ is given by following flux-density relation:

$$f(\rho) = \begin{cases} \rho v_{\mathrm{f}} & \text{if } 0 \le \rho \le \rho_{\mathrm{c}}, \\ \dfrac{f^{\max}}{\rho_{\max} - \rho_c}(\rho_{\max} - \rho) & \text{if } \rho_{\mathrm{c}} \le \rho \le \rho_{\max}, \end{cases}$$

with v_{f} the maximal speed of the traffic, $\rho_{\mathrm{c}} = \dfrac{f^{\max}}{v_{\mathrm{f}}}$ the critical density and $f^{\max} = f(\rho_{\mathrm{c}})$ the maximal flux value. Throughout the paper, for simplicity, we will assume $\rho_{\max} = 1$ and $v_{\mathrm{f}} = 1$. In Figure 3 is an example of flux function satisfying these hypotheses.

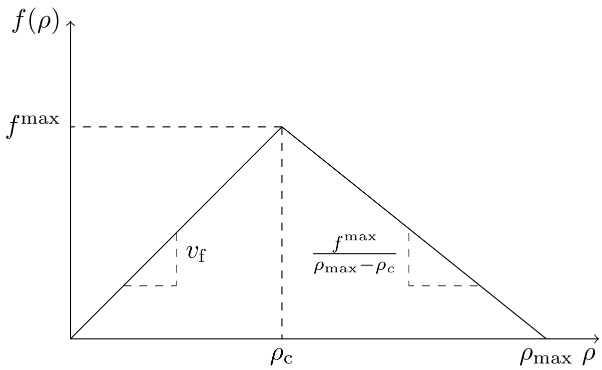


Figure 3: Flux function considered.

The incoming lane of the secondary roads entering the junctions are modeled with a buffer of infinite size and capacity. This choice is made to avoid backward moving shocks at the boundary, for details see [7]. In particular, the evolution of the queue length of each buffer is described by an ODE:

$$\frac{dl_i(t)}{dt} = F_{\text{in}}^i(t) - \gamma_{\text{r}1,i}(t), \quad t \in \mathbb{R}^+ \quad i = 1,2,3, \tag{2.2}$$

where $l_i(t) \in [0, +\infty[$ is the queue length, $F_{\text{in}}^i(t)$ the flux entering the lane and $\gamma_{\text{r}1,i}(t)$ the flux exiting the lane. The outgoing lane is considered as a sink that accepts all the flux coming from the mainline. No flux from the incoming lane is allowed on the outgoing stretch of same road. The Cauchy problem to solve is then:

$$\begin{cases} \partial_t \rho_i + \partial_x f(\rho_i) = 0, & (t,x) \in \mathbb{R}^+ \times I_i, \ i = 1,2,3 \\ \dfrac{dl_i(t)}{dt} = F_{\text{in}}^i(t) - \gamma_{\text{r}1,i}(t), & t \in \mathbb{R}^+, \ i = 1,2,3 \\ \rho_i(0,x) = \rho_{i,0}(x), & \text{on } I_i, \ i = 1,2,3 \\ l_i(0) = l_{i,0} & i = 1,2,3, \end{cases} \tag{2.3}$$

where $\rho_{i,0}(x)$ are the initial conditions and $l_{i,0}$ the initial lengths of the buffers. This will be coupled with an optimization problem at the junctions that gives the distribution of traffic among the roads.

We define the demand $d(F_{\text{in}}^i, l_i)$ of the incoming lane for the secondary roads, the demand function $\delta(\rho_i)$ on the incoming mainline segment at each junction, and the supply function $\sigma(\rho_i)$ on the outgoing mainline segment at each junction as follows.

$$d(F_{\text{in}}^i, l_i) = \begin{cases} \gamma_{\text{r}1,i}^{\max} & \text{if } l_i(t) > 0, \ i = 1,2,3 \\ \min\left(F_{\text{in}}^i(t), \gamma_{\text{r}1,i}^{\max}\right) & \text{if } l_i(t) = 0, \ i = 1,2,3 \end{cases} \tag{2.4}$$

$$\delta(\rho_i) = \begin{cases} f(\rho_i) & \text{if } 0 \le \rho_i < \rho_{\text{c}}, \ i = 1,2,3 \\ f^{\max} & \text{if } \rho_{\text{c}} \le \rho_i \le 1, \ i = 1,2,3 \end{cases} \tag{2.5}$$

$$\sigma(\rho_i) = \begin{cases} f^{\max} & \text{if } 0 \le \rho_i \le \rho_{\text{c}}, \ i = 1,2,3 \\ f(\rho_i) & \text{if } \rho_{\text{c}} < \rho_i \le 1, \end{cases} \tag{2.6}$$

where $\gamma_{\text{r}1,i}^{\max}$ is the maximal flow on the incoming lane $R_{1,i}$, $i = 1,2,3$. Moreover, we introduce $\beta \in [0,1]$ the split ratio of the outgoing lane $R_{2,i}$, and $\gamma_{\text{r}2,i}(t) = \beta f(\rho_i(t,0-))$ $i = 1,2,3$ its flux.

Definition 1 (See [7].) *Consider a junction J with two incoming roads $I_1 =]-\infty, 0[$ and R_1 and two outgoing roads $I_2 =]0, +\infty[$ and R_2. A triple $(\rho_1, \rho_2, l) \in \prod_{i=1}^{2} \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap BV(\mathbb{R})\right) \times \mathbf{W^{1,\infty}}(\mathbb{R}^+; \mathbb{R}^+)$ is an admissible solution to (2.3) if*

1. *ρ_i is a weak solutions on I_i, i.e., $\rho_i : [0, +\infty[\times I_i \to [0,1]$, $i = 1,2$, such that*

$$\int_{\mathbb{R}^+} \int_{I_i} \Big(\rho_i \partial_t \varphi_i + f(\rho_i) \partial_x \varphi_i \Big) dx dt = 0, \quad i = 1,2, \tag{2.7}$$

   for every $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^+ \times I_i)$.

2. *ρ_i satisfies the Kružhkov entropy condition [13] on $(\mathbb{R} \times I_i)$, i.e., for every $k \in [0,1]$ and for all $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^+ \times I_i)$, $t > 0$,*

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^+} \int_{I_i} (|\rho_i - k| \partial_t \varphi_i + \operatorname{sgn}(\rho_i - k)(f(\rho_i) - f(k)) \partial_x \varphi_i) dx dt \\ &\qquad + \textstyle\int_{I_i} |\rho_{i,0} - k| \varphi_i(0,x) dx \ge 0; \quad i = 1,2. \end{aligned} \tag{2.8}$$

3. $f(\rho_1(t,0-)) + \gamma_{\mathrm{r}1}(t) = f(\rho_2(t,0+)) + \gamma_{\mathrm{r}2}(t)$.

4. *The flux of the outgoing mainline* $f(\rho_2(t,0+))$ *is maximum subject to*

$$f(\rho_2(t,0+)) = \min\Big((1-\beta)\delta(\rho_1(t,0-)) + d(F_{\mathrm{in}}(t), l(t)), \sigma(\rho_2(t,0+))\Big), \tag{2.9}$$

and 3.

5. l *is a solution of* (2.2) *for a.e.* $t \in \mathbb{R}^+$.

Remark 1 *A parameter* q *is introduced to ensure uniqueness of the solution, that is,* $q \in]0,1[$ *is a right of way parameter that defines the amount of flux that enters the outgoing mainline from each incoming road. In particular, when the priority applies,* $qf(\rho_2(t,0+))$ *is the flux allowed from the incoming mainline into the outgoing mainline, and* $(1-q)f(\rho_2(t,0+))$ *the flux from the onramp.*

3 Riemann Problem the junction

In this section we recall briefly the costruction of the Riemann Solver at a junction as introduced in [7] and then we apply it to our particular case to recover the expressions of the cost functionals. We will focus only on one junction J. We fix constants $\rho_{1,0}, \rho_{2,0} \in [0,1]$, $l_0 \in [0,+\infty[$, $F_{\mathrm{in}} \in]0,+\infty[$ and a priority factor $q \in]0,1[$. The Riemann problem at J is the Cauchy problem (2.3) where the initial conditions are given by $\rho_{0,i}(x) \equiv \rho_{0,i}$ in I_i for $i = 1,2$. We define the Riemann Solver by means of a Riemann Solver $\mathcal{RS}_{\bar{l}}$, which depends on the instantaneous load of the buffer $\bar{l}$. For each $\bar{l}$ the Riemann Solver $\mathcal{RS}_{\bar{l}}$ is constructed in the following way.

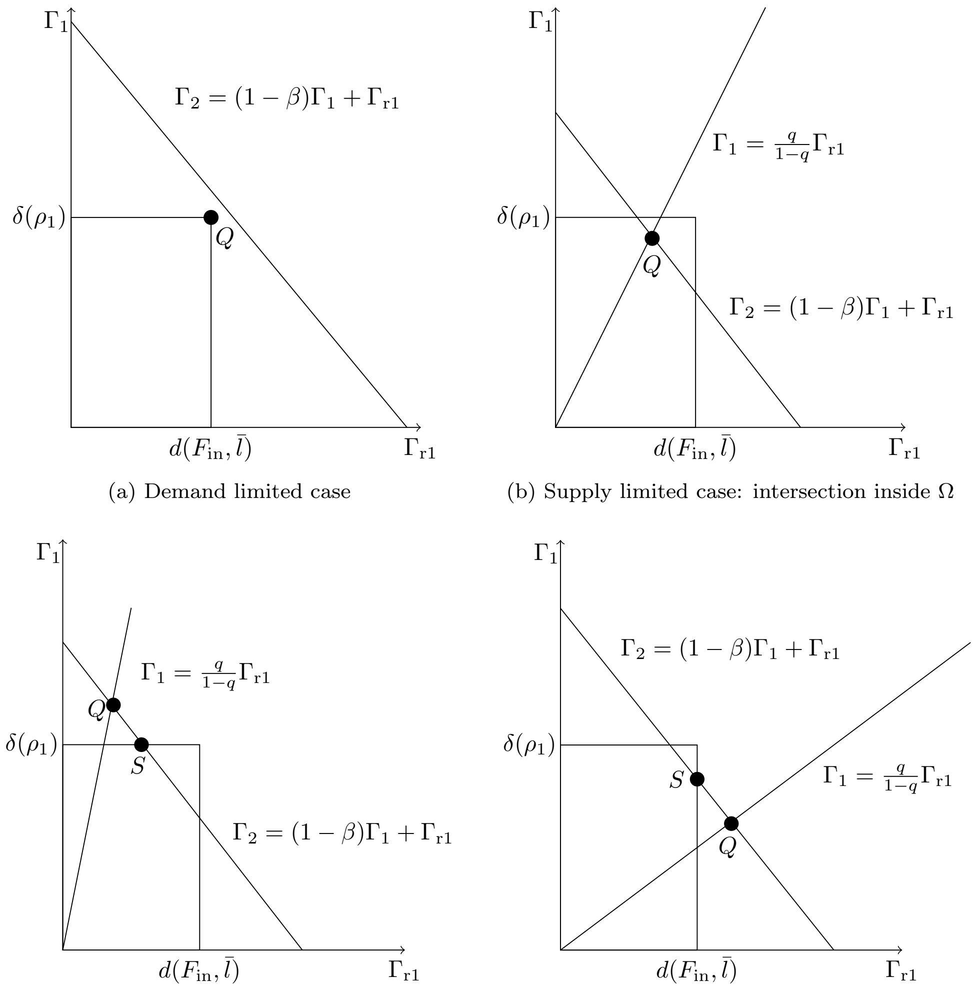
1. Define $\Gamma_1 = f(\rho_1(t,0-))$, $\Gamma_2 = f(\rho_2(t,0+))$, $\Gamma_{\mathrm{r}1} = \gamma_{\mathrm{r}1}(t)$;
2. Consider the space $(\Gamma_1, \Gamma_{\mathrm{r}1})$ and the sets $\mathcal{O}_1 = [0, \delta(\rho_1)]$, $\mathcal{O}_{\mathrm{r}1} = [0, d(F_{\mathrm{in}}, \bar{l})]$;
3. Trace the lines $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$; and $\Gamma_1 = \frac{q}{1-q}\Gamma_{\mathrm{r}1}$;
4. Consider the region

$$\Omega = \Big\{(\Gamma_1, \Gamma_{\mathrm{r}1}) \in \mathcal{O}_1 \times \mathcal{O}_{\mathrm{r}1} : (1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} \in [0, \Gamma_2]\Big\}. \tag{3.1}$$

Different situations can occur depending on the value of Γ_2:

- Demand limited case: $\Gamma_2 = (1-\beta)\delta(\rho_1(t,0-)) + d(F_{\mathrm{in}}, \bar{l})$.
  We set Q to be the point $(\hat{\Gamma}_1, \hat{\Gamma}_{\mathrm{r}1})$ such that $\hat{\Gamma}_1 = \delta(\rho_1(t,0-))$, $\hat{\Gamma}_{\mathrm{r}1} = d(F_{\mathrm{in}}, \bar{l})$ and $\hat{\Gamma}_2 = (1-\beta)\delta(\rho_1(t,0-)) + d(F_{\mathrm{in}}, \bar{l})$, as illustrated in Figure 4(a).

- Supply limited case: $\Gamma_2 = \sigma(\rho_2(t,0+))$.
  We set Q to be the point of intersection of $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$ and $\Gamma_1 = \frac{q}{1-q}\Gamma_{\mathrm{r}1}$. If $Q \in \Omega$, we set $(\hat{\Gamma}_1, \hat{\Gamma}_{\mathrm{r}1}) = Q$ and $\hat{\Gamma}_2 = \Gamma_2$, see Figure 4(b); if $Q \notin \Omega$, we set $(\hat{\Gamma}_1, \hat{\Gamma}_{\mathrm{r}1}) = S$ and $\hat{\Gamma}_2 = \Gamma_2$, where S is the point of the segment $\Omega \cap (\Gamma_1, \Gamma_{\mathrm{r}1}) : (1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$ closest to the line $\Gamma_1 = \frac{q}{1-q}\Gamma_{\mathrm{r}1}$ see Figure 4(c).



(a) Demand limited case

(b) Supply limited case: intersection inside Ω

(c) Supply limited case: intersection outside Ω

Figure 4: Solutions of the Riemann Solver at the junction.

We define the function τ as follows, for details see [9].

Definition 2 *Let $\tau : [0,1] \to [0,1]$ be the map such that:*

- $f(\tau(\rho)) = f(\rho)$ *for every* $\rho \in [0,1]$*;*
- $\tau(\rho) \neq \rho$ *for every* $\rho \in [0,1] \setminus \{\rho_c\}$.

Theorem 1 *Consider a junction J and fix a right f way parameter $q \in]0,1[$. For every $\rho_{1,0}, \rho_{2,0} \in [0,1]$ and $l_0 \in [0,+\infty[$ there exists a unique amdmissible solution $(\rho_1(t,x), \rho_2(t,x), l(t))$ in the sense of Definition 1 satisfying the priority (possibly in an approximate way). Moreover, there exists a unique couple $(\hat{\rho}_1, \hat{\rho}_2) \in [0,1]^2$ such that*

$$\hat{\rho}_1 \in \begin{cases} \{\rho_{1,0}\} \cup]\tau(\rho_{1,0}), 1] & \text{if } 0 \leq \rho_{1,0} \leq \rho_c, \\ [\rho_c, 1] & \text{if } \rho_c \leq \rho_{1,0} \leq 1; \end{cases} \qquad f(\hat{\rho}_1) = \hat{\Gamma}_1, \tag{3.2}$$

and

$$\hat{\rho}_2 \in \begin{cases} [0, \rho_c] & \text{if } 0 \le \rho_{2,0} \le \rho_c, \\ \{\rho_{2,0}\} \cup [0, \tau(\rho_{2,0})[& \text{if } \rho_c \le \rho_{1,0} \le 1; \end{cases} \qquad f(\hat{\rho}_2) = \hat{\Gamma}_2. \tag{3.3}$$

For the incoming road the solution is given by the wave $(\rho_{1,0}, \hat{\rho}_1)$*, while for the outgoing road the solution is given by the wave* $(\hat{\rho}_2, \rho_{2,0})$*. Furthermore, for a.e.* $t > 0$*, it holds*

$$(\rho_1(t, 0-), \rho_2(t, 0+)) = \mathcal{RS}_{l(t)}(\rho_1(t, 0-), \rho_2(t, 0+)).$$

For the proof see [7].

4 Cost Functionals

In this section we define the cost functionals and derive their expressions. We introduce the Total Travel Time (TTT) on the road network and the Total Waiting Time (TWT) on the incoming lanes of the secondary roads, which are defined as follows:

$$TTT(T) = \int_0^T \int_{I_i} \rho(t, x) dx dt + \int_0^T l(t) dt + \int_{I_i} \rho(T, x) dx + l(T) \tag{4.1}$$

$$TWT(T) = \int_0^T l(t) dt + l(T) \tag{4.2}$$

for $T > 0$ that we will take sufficiently big to that the solution is stabilized. Our aim is to derive the explicit form of these cost functionals to study their dependence on the right of way parameter q. Since the solutions of such optimization problems cannot be analytically computed in general, we focus on the case of the junction showed in Figure 2(b) with $I_1 = [-1, 0]$ and $I_2 = [0, 1]$. We suppose that the network and the buffer are empty at $t = 0$ and we assume that the following boundary data are given: f^{in} the inflow on the incoming mainline, f^{out} the outflow on the outgoing mainline and F_{in} the incoming flux of the secondary road. Moreover we assume $F_{\text{in}} \le f^{\max} = \gamma_{\text{r1}}^{\max}$ and $f^{\text{out}} \le f^{\max}$. Now, we can solve the corresponding initial-boundary value problem.
The first step is to compute the demand and supply functions of the roads. We have $\delta(\rho_{1,0}) = 0$, $d(F_{\text{in}}, l) = \min(F_{\text{in}}, \gamma_{\text{r1}}^{\max}) = F_{\text{in}}$ and $\sigma(\rho_{2,0}) = f^{\max}$. Then we can compute Γ_2:

$$\Gamma_2 = \min\Big((1 - \beta)\delta(\rho_{1,0}) + d(F_{\text{in}}, l), \sigma(\rho_{2,0})\Big) = F_{\text{in}}.$$

It is straightforward to see that the problem is demand limited, hence the optimal point is the point at maximal demands. Thus it follows $\hat{\Gamma}_1 = 0$, $\hat{\Gamma}_2 = F_{\text{in}}$ and $\hat{\Gamma}_{\text{r1}} = F_{\text{in}}$, from which we can derive $\hat{\rho}_1 = 0$ and $\hat{\rho}_2 = F_{\text{in}}$. Since we are demand limited we also have $l(t) = 0$. The solution in the $x - t$ plane looks as in Figure 5. The wave produced by the junction problem interacts with the right boundary $x = 1$ at time $t_1 = 1$. Moreover at $x = -1$, the boundary condition enforces the creation of an additional wave at $t = 0$ with speed equal 1. This produces a density $\hat{\rho}_1$ equal at f^{in} which reaches the junction at the same time $t_1 = 1$. The solution looks as in Figure 5.

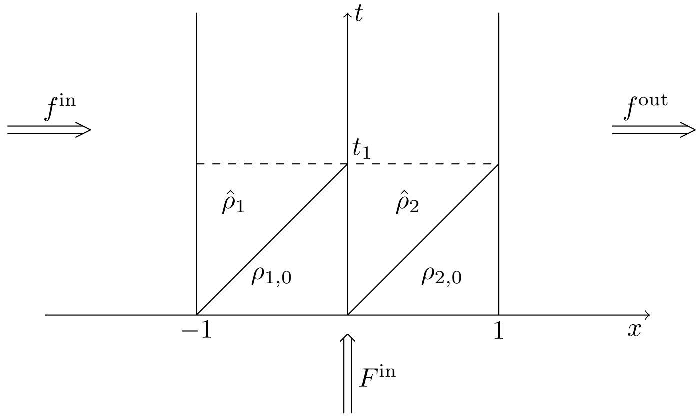


Figure 5: Solution of the initial-boundary value problem for $t \in [0, t_1]$.

At $t_1 = 1$ we solve a new Riemann problem at the junction with initial densities

$$\rho(1, x) = \begin{cases} \hat{\rho}_1 & \text{if } x < 0, \\ \hat{\rho}_2 & \text{if } x > 0. \end{cases}$$

We assume that the splitting ratio $\beta \in (0, 1)$ is fixed and that $0 \le \hat{\rho}_1 < \rho_c$ and $0 \le \hat{\rho}_2 < \rho_c$. The demand and supply functions on the respective roads are $\delta(\hat{\rho}_1) = f^{\text{in}}$, $d(F_{\text{in}}, l_0) = \min(F_{\text{in}}, \gamma_{\text{r1}}^{\max}) = F_{\text{in}}$, $\sigma(\hat{\rho}_2) = f^{\max}$. Computing Γ_2 from these we obtain

$$\Gamma_2 = \min\Big((1 - \beta)\delta(\hat{\rho}_1) + d(F_{\text{in}}, l), \sigma(\hat{\rho}_2)\Big)$$

The case $(1 - \beta)\delta(\hat{\rho}_1) + d(F_{\text{in}}, l) \le \sigma(\hat{\rho}_2)$ is demand limited and thus vehicles enter the junction without restrictions. That is, the roundabout is not congested and hence we do not consider this situation. Since we want to minimize our functionals with respect to the right of way parameter, from now on we will only focus on the supply constrained case and hence, we make the following assumption.

First assumption on data

$$f^{\max} \le (1 - \beta) f^{\text{in}} + F_{\text{in}} \tag{4.3}$$

With this assumption then it is straightforward to compute the value of $\Gamma_2 = f^{\max}$ and hence, $\Gamma_1 = \dfrac{q}{1 - \beta q} f^{\max}$ and $\Gamma_{\text{r1}} = \dfrac{1 - q}{1 - \beta q} f^{\max}$.

Moreover, let us assume the following conditions

A_1) $\Gamma_1 = \dfrac{q}{1 - \beta q} f^{\max} < \delta(\hat{\rho}_1) = f^{\text{in}}$,

A_2) $\Gamma_{\text{r1}} = \dfrac{1 - q}{1 - \beta q} f^{\max} < d(F_{\text{in}}, l) = F_{\text{in}}$.

The following are solutions of the Riemann problem at the junction.

a) If both A_1 and A_2 are satisfied, then $\left(\dfrac{q}{1 - \beta q} f^{\max}, \dfrac{1 - q}{1 - \beta q} f^{\max}, f^{\max}\right)$ is the solution.

b) If $\delta(\rho_1) = f^{\text{in}} \le \dfrac{q}{1 - \beta q} f^{\max}$, then $\big(f^{\text{in}}, f^{\max} - (1 - \beta) f^{\text{in}}, f^{\max}\big)$ is the solution of Riemann problem.

c) If $d(F_{\text{in}}, l) = F_{\text{in}} \leq \dfrac{1-q}{1-\beta q} f^{\max}$, then $\left(\dfrac{f^{\max} - F_{\text{in}}}{1-\beta}, F_{\text{in}}, f^{\max}\right)$ is the solution.

From condition b) at $f^{\text{in}} \leq \dfrac{q}{1-\beta q} f^{\max}$ solving for q we get

$$q \geq q_2 = \frac{f^{\text{in}}}{f^{\max} + \beta f^{\text{in}}} \tag{4.4}$$

Similarly, from condition c) at $d(F_{\text{in}}, l) = F_{\text{in}} = \dfrac{1-q}{1-\beta q} f^{\max}$ solving we have

$$q \leq q_1 = \frac{F_{\text{in}} - f^{\max}}{\beta F_{\text{in}} - f^{\max}} \tag{4.5}$$

To see the relation between q_1 and q_2 with respect to the feasible set we take their difference.

$$\begin{aligned} q_2 - q_1 &= \frac{f^{\text{in}}}{f^{\max} + \beta f^{\text{in}}} - \frac{F_{\text{in}} - f^{\max}}{\beta F_{\text{in}} - f^{\max}} \\ &= \frac{f^{\max}[\underbrace{f^{\max} - (1-\beta) f^{\text{in}} - F_{\text{in}}}_{<0}]}{(f^{\max} + \beta f^{\text{in}})(\underbrace{\beta F_{\text{in}} - f^{\max}}_{<0})}. \end{aligned} \tag{4.6}$$

This holds true because of (4.3) and implies $q_1 \leq q_2$, see Figure 6.

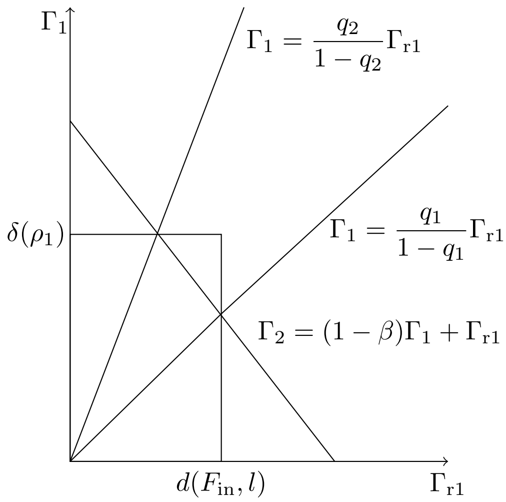


Figure 6: Relationship between q_1 and q_2

We will analyze the different cases that can occur according to the different values of q.

4.1 Case $q_1 < q < q_2$

We solve now the Riemann problem at the junction at t_1. Since we assume that we are supply limited and that $q_1 < q < q_2$ it is straightforward to have the following fluxes at the junction:

$\Gamma_1 = \dfrac{q}{1-\beta q} f^{\max}, \Gamma_{r1} = \dfrac{1-q}{1-\beta q} f^{\max}$ and $\Gamma_2 = f^{\max}$. It follows that

$$\rho_1 = 1 - \frac{(1-f^{\max})q}{1-\beta q} \tag{4.7}$$

and that the wave speed λ is

$$\lambda(\hat{\rho}_1, \rho_1) = \frac{f^{\text{in}}(1-\beta q) - q f^{\max}}{(1-\beta q)(f^{\text{in}}-1) + (1-f^{\max})q} \tag{4.8}$$

The characteristic $x = \lambda(t-1)$ crosses the boundary $x = -1$ at

$$\begin{aligned} t_2 = \frac{\lambda}{1-\lambda} &= 1 - \frac{(1-\beta q)(f^{\text{in}}-1) + (1-\rho_{\text{c}})q}{f^{\text{in}}(1-\beta q) - q f^{\max}} = \\ &= \frac{1-(\beta + f^{\max})q - (1-f^{\max})q}{f^{\text{in}}(1-\beta q) - q f^{\max}} = \frac{1-(1+\beta)q}{f^{\text{in}}(1-\beta q) - q f^{\max}} \end{aligned} \tag{4.9}$$

In the outgoing road $\rho_2 = \rho_{\text{c}}$ which produces a wave with positive speed 1 as can be seen in the 7

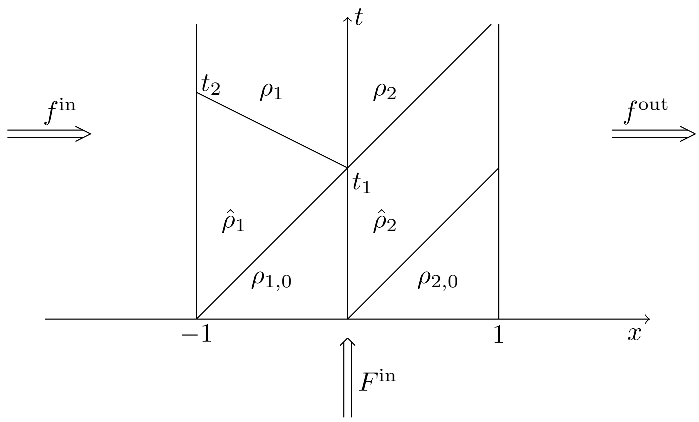


Figure 7: Junction problem at t_1.

The value of the corresponding flux from the buffer is $\Gamma_{r1} = \dfrac{(1-q)f^{\max}}{1-\beta q}$. For all β and $q \in]0,1[$ we have $\dfrac{(1-q)}{1-\beta q} < 1$, so the length of buffer satisfies

$$\dot{l} = F_{\text{in}} - \Gamma_{r1} = F_{\text{in}} - \frac{(1-q)f^{\max}}{1-\beta q} > 0$$

Since $(1-\beta q)F_{\text{in}} > (1-q)f^{\max}$ or $q(f^{\max} - \beta F_{\text{in}}) > f^{\max} - F_{\text{in}}$ then $q > \dfrac{f^{\max} - F_{\text{in}}}{f^{\max} - \beta F_{\text{in}}} = q_1$. Therefore,

$$l(t) = (F_{\text{in}} - \frac{(1-q)f^{\max}}{1-\beta q})(t-1) > 0. \tag{4.10}$$

The length of buffer increases linearly.
Moreover at time t_1 the interaction between the wave in the outgoing road and the boundary at

$x = 1$ can generate an additional wave if $F_{\text{in}} > f^{\text{out}}$. When this is the case, in fact, there is a wave with negative speed which can interact with other waves between $[0, 1]$. Since our purpose is to study the interactions among waves we make the following:
Second assumption on data

$$F_{\text{in}} > f^{\text{out}}. \tag{4.11}$$

When (4.11) is satified there is a wave with negative speed created at $(t_1, 1)$. This new wave creates a density

$$\rho_3 = 1 - \frac{f^{\text{out}}(1 - f^{\max})}{f^{\max}}, \tag{4.12}$$

and has a wave speed $\lambda(\hat{\rho}_2, \rho_3)$

$$\lambda(\hat{\rho}_2, \rho_3) = \frac{F_{\text{in}} - f^{\text{out}}}{\hat{\rho}_2 - \rho_3} = \frac{F_{\text{in}} - f^{\text{out}}}{F_{\text{in}} - 1 + \frac{f^{\text{out}}(1-f^{\max})}{f^{\max}}} = \frac{f^{\max}(F_{\text{in}} - f^{\text{out}})}{(F_{\text{in}} - 1)f^{\max} + (1 - f^{\max})f^{\text{out}}}. \tag{4.13}$$

This wave with equation $x = \lambda(t - 1) + 1$ intersect the characteristic line $x = t - 1$ at a point P as in Figure 8.

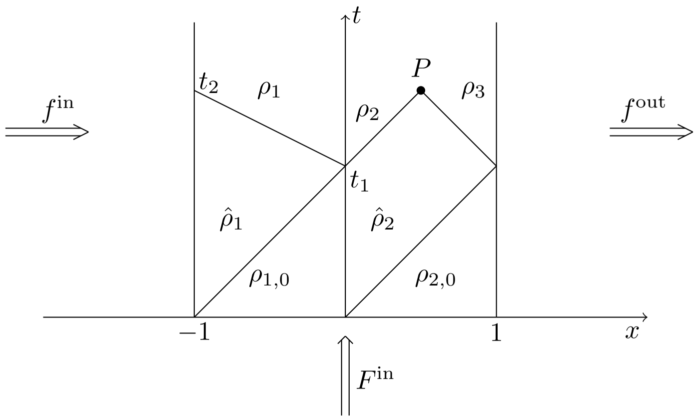


Figure 8: Solution at t_1.

Solving the system we get the coordinates of the point $P = (t_P, x_P)$:

$$\begin{aligned} t_P = t_3 = \frac{\lambda - 2}{\lambda - 1} &= \frac{f^{\max}(F_{\text{in}} - f^{\text{out}}) - 2((F_{\text{in}} - 1)f^{\max} + (1 - f^{\max})f^{\text{out}})}{f^{\max}(F_{\text{in}} - f^{\text{out}}) - ((F_{\text{in}} - 1)f^{\max} + (1 - f^{\max})f^{\text{out}})} = \\ &= \frac{f^{\max}f^{\text{out}} + 2f^{\max} - 2f^{\text{out}} - f^{\max}f^{\text{out}}}{f^{\max} - f^{\text{out}}} \end{aligned} \tag{4.14}$$

$$\begin{aligned} x_P = \frac{1}{1 - \lambda} &= \frac{(F_{\text{in}} - 1)f^{\max} + (1 - f^{\max})f^{\text{out}}}{(F_{\text{in}} - 1)f^{\max} + (1 - f^{\max})f^{\text{out}} - f^{\max}(F_{\text{in}} - f^{\text{out}})} = \\ &= \frac{f^{\max}f^{\text{out}} + f^{\max} - f^{\max}F_{\text{in}} - f^{\text{out}}}{f^{\max} - f^{\text{out}}}. \end{aligned} \tag{4.15}$$

Once we have determined the coordinates of P, we can solve the classical Riemann Problem at P:

$$\begin{aligned} &\partial_t \rho + \partial_x f(\rho) = 0 \\ &\rho(t,x) = \begin{cases} \rho_{\mathrm{c}} & \text{if } x < x_p, \\ \rho_3 & \text{if } x > x_p, \end{cases} \end{aligned} \tag{4.16}$$

where x_p is given by (4.15). Note that $\rho_{\mathrm{c}} < \rho_3$, hence, from the Rankine-Hugoniot jump condition we can compute the characteristic speed of the wave emanating from the point P

$$\lambda(\rho_{\mathrm{c}}, \rho_3) = \frac{f(\rho_3) - f(\rho_{\mathrm{c}})}{\rho_3 - \rho_{\mathrm{c}}} = \frac{f^{\max}}{f^{\max} - 1}. \tag{4.17}$$

The solution of the classical Riemann Problem at P is

$$\rho(t,x) = \begin{cases} \rho_{\mathrm{c}} & \text{if } x < \lambda(t - t_p) + x_p, \\ \rho_3 & \text{if } x > \lambda(t - t_p) + x_p, \end{cases} \tag{4.18}$$

where λ and t_p are respectively as in (4.17) and (4.14). The characteristic line with speed λ starting from P interacts with the junction at $x = 0$. This line intercepts the t-axis at

$$t_4 = t_p - \frac{1}{\lambda} x_p = t_p + \frac{1 - f^{\max}}{f^{\max}} x_p = \frac{(f^{\max})^2 - f^{\max} F_{\mathrm{in}} + f^{\max} - f^{\mathrm{out}}}{f^{\max}(f^{\max} - f^{\mathrm{out}})}. \tag{4.19}$$

The situation at $t = t_4$ looks as in Figure 9

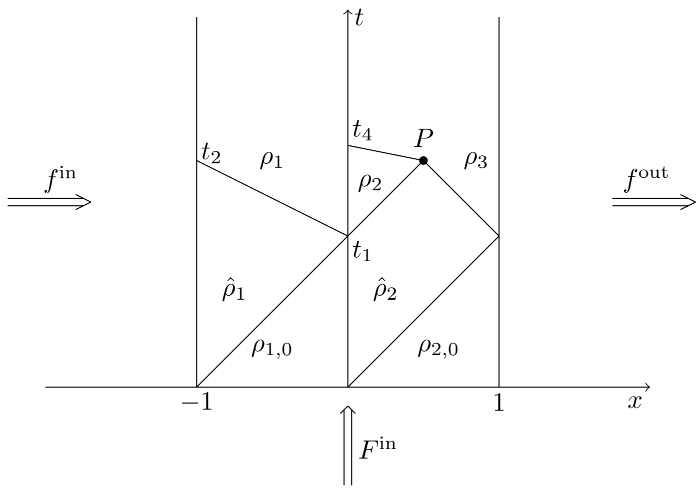


Figure 9: Solution at time $t = t_4$

We can then, solve at t_4 another Riemann problem at the junction:

$$\rho(t_4, x) = \begin{cases} \rho_1 & \text{if } x < 0, \\ \rho_3 & \text{if } x > 0. \end{cases} \tag{4.20}$$

To do so, we compute:

$$d(F_{\mathrm{in}}, l) = \gamma_{\mathrm{r1}}^{\max}, \tag{4.21}$$

$$\delta(\rho_1) = f^{\max}, \tag{4.22}$$

$$\sigma(\rho_3) = f^{\mathrm{out}}. \tag{4.23}$$

In order to limit the complexity of the computation we fix $\gamma_{\mathrm{r}1}^{\max} = f^{\max}$. It follows $\Gamma_2 = \min((1-\beta)\delta(\rho_1)+d(F_{\mathrm{in}},l),\sigma(\rho_3)) = \min((1-\beta)f^{\max}+f^{\max},f^{\mathrm{out}}) = f^{\mathrm{out}}$, $\hat{\Gamma}_1 = \frac{q}{1-\beta q}f^{\mathrm{out}}$ and $\hat{\Gamma}_{\mathrm{r}1} = \frac{1-q}{1-\beta q}f^{\mathrm{out}}$ Since by assumption it holds $f^{\mathrm{out}} < f^{\max}$ the soution of the Riemann problem at the junction lies inside the feasible region as shown in Figure 10.

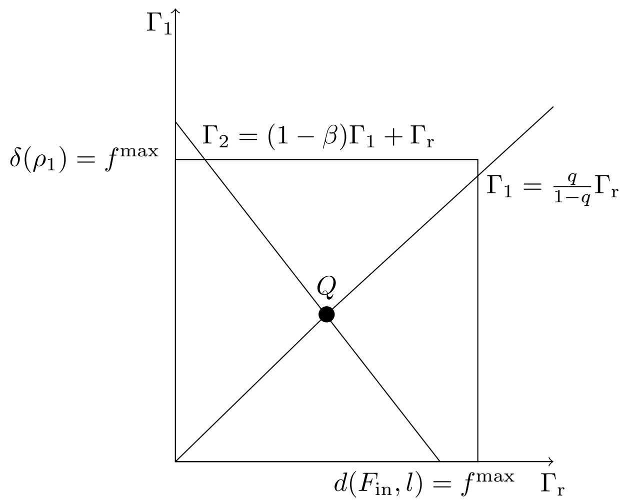


Figure 10: Solution of the Riemann problem at the junction at t_4.

The solution of Riemann Problem at junction is given by

$$(\hat{\Gamma}_1,\hat{\Gamma}_{\mathrm{r}1},\hat{\Gamma}_2) = \left(\frac{q}{1-\beta q}f^{\mathrm{out}}, \frac{1-q}{1-\beta q}f^{\mathrm{out}}, f^{\mathrm{out}}\right)$$

From this we can uniquely recover the corresponding values of the queue length and of the densities of the mainline. Since the flux exiting the secondary road is $\hat{\Gamma}_{\mathrm{r}1} = \frac{1-q}{1-\beta q}f^{\mathrm{out}}$ then the queue length is

$$l(t) = l(t_4) + \left(F_{\mathrm{in}} - \frac{1-q}{1-\beta q}f^{\mathrm{out}}\right)(t-t_4) > 0. \tag{4.24}$$

Thus, the length of the buffer increases linearly.
On the outgoing mainline there is no new wave created since $\hat{\Gamma}_2 = f^{\mathrm{out}} = f(\rho_3)$. And on the incoming mainline we have $\hat{\Gamma}_1 = \frac{q}{1-\beta q}f^{\mathrm{out}} \Rightarrow \frac{q}{1-\beta q}f^{\mathrm{out}} = \frac{1-\check{\rho}_1}{1-\rho_{\mathrm{c}}}f^{\max}$ which gives

$$\check{\rho}_1 = 1 - \frac{(1-f^{\max})qf^{\mathrm{out}}}{(1-\beta q)f^{\max}}. \tag{4.25}$$

On the incoming main road of a roundabout $\rho_{\mathrm{c}} \leq \rho_1 \leq 1$ so $\rho_{\mathrm{c}} \leq \check{\rho}_1 \leq 1$, and the pair $(\rho_1,\check{\rho}_1)$ produces a wave with negative speed on the incoming road of the roundabout. The speed is given by $\lambda(\rho_1,\check{\rho}_1) = \frac{f^{\max}}{f^{\max}-1}$. This characteristic line crosses the left boundary $x=-1$ at a time $t=t_5$ given by

$$t_5 = t_4 - \frac{1}{\lambda} = t_p + \frac{1-f^{\max}}{f^{\max}}(x_p+1) = \frac{f^{\max}f^{\mathrm{in}} - f^{\max}f^{\mathrm{out}} - 2f^{\max} + 2f^{\mathrm{out}}}{f^{\max}(f^{\mathrm{out}}-f^{\max})}. \tag{4.26}$$

Moreover, since $\frac{q}{1-\beta q} f^{\text{out}} < \delta(\rho_1) = f^{\max}$, there is no wave produced by the interaction with the boundary $x = -1$ at time t_5. The solution at t_5 looks like as in Figure 11

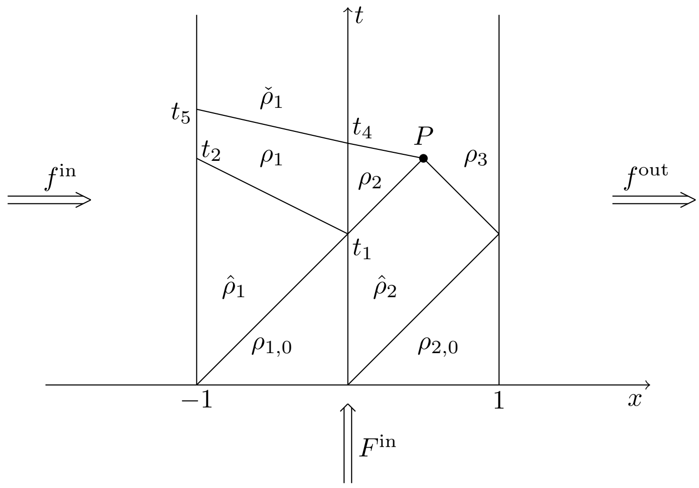


Figure 11: Solution of the problem at t_5

Depending on the priority parameter q, the waves emanating from the junction at time t_1 and t_4 could collide within the region $-1 < x < 0$. The wave starting from $(t_1, 0)$ could be slower than the one created at time t_4 and so they could collide. Solving the system generated by the characteric lines we can find the value of q for which this happens. We find that for $q = \bar{q}$, where $\bar{q}$ is given by

$$\bar{q} = \frac{(1-\beta)\Big(f^{\text{in}} f^{\max} F_{\text{in}} + 2 f^{\text{in}} f^{\max} + 2 f^{\text{in}} f^{\text{out}}\Big) - f^{\max}(\beta f^{\text{in}} f^{\max} + f^{\max} - f^{\text{out}} + f^{\text{in}} f^{\text{out}}(2\beta - 1))}{(f^{\max})^2 (F_{\text{in}} - \beta f^{\text{in}} + \beta - 1 - f^{\text{out}}) + f^{\max} f^{\text{out}} (1 - \beta + \beta f^{\text{in}})} \tag{4.27}$$

the waves collide. We can distinguish two additional cases $q_1 < q < \bar{q}$ and $\bar{q} < q < q_2$

4.1.1 The case $q_1 < q < \bar{q}$

If $q_1 < q < \bar{q}$, the waves do not intersect in the region $-1 < x < 0$ and there are no new waves created. Hence, the study of the problem is concluded.

4.1.2 The case $\bar{q} \leq q < q_2$

As described above, the pair $(\rho_1, \check{\rho}_1)$ produces a wave with negative speed $\lambda = \frac{f^{\max}}{f^{\max} - 1}$ on the incoming main road of the roundabout. This wave interacts with the wave exiting from $(t_1, 0)$ at point Q as in Figure 12

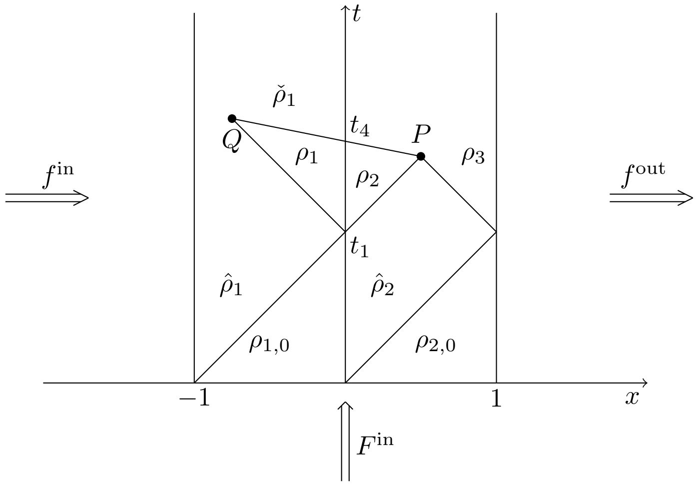


Figure 12: Solution in the case $\bar{q} \leq q < q_2$

The coordinates of the point Q are then

$$t_Q = \frac{f^{\max} t_4((1-\beta q)(f^{\text{in}}-1) + (1-f^{\max})q) + (1-f^{\max})(f^{\text{in}}(1-\beta q) - q f^{\max})}{f^{\max}((1-\beta q)(f^{\text{in}}-1) + (1-f^{\max})q) + (1-f^{\max})(f^{\text{in}}(1-\beta) - q f^{\max})} \tag{4.28}$$

and

$$x_Q = \frac{f^{\max}}{f^{\max}-1}(t_Q - t_4) \tag{4.29}$$

Once we have determined the coordinates of point Q, we can solve classical Riemann Problem at Q.

$$\partial_t \rho + \partial_x f(\rho) = 0 \tag{4.30}$$

$$\rho(t,x) = \begin{cases} \hat{\rho}_1 & \text{if } x < x_Q, \\ \check{\rho}_1 & \text{if } x > x_Q, \end{cases} \tag{4.31}$$

where Q is given by (4.29). By Rankine-Hugoniot jump condition we can derive

$$\begin{aligned} \lambda(\hat{\rho}_1, \check{\rho}_1) = \frac{f(\hat{\rho}_1) - f(\check{\rho}_1)}{\hat{\rho}_1 - \check{\rho}_1} = \frac{f^{\text{in}} - \hat{\Gamma}_1}{\hat{\rho}_1 - \check{\rho}_1} &= \frac{f^{\text{in}} - \dfrac{q}{1-\beta q} f^{\text{out}}}{f^{\text{in}} - \left(1 - \frac{(1-f^{\max}) q f^{\text{out}}}{(1-\beta q) f^{\max}}\right)} = \\ &= \frac{\left(f^{\text{in}}(1-\beta q) - q f^{\text{out}}\right) f^{\max}}{(1-\beta q) f^{\max}(f^{\text{in}}-1) + (1-f^{\max}) q f^{\text{out}}} \end{aligned} \tag{4.32}$$

where λ is the characteristic speed of the wave emanating from the intersection point Q. The solution of the classical Riemann Problem at Q is

$$\rho(t,x) = \begin{cases} \hat{\rho}_1 & \text{if } x < \lambda(t-t_Q) + x_Q, \\ \check{\rho}_1 & \text{if } x > \lambda(t-t_Q) + x_Q, \end{cases} \tag{4.33}$$

where λ and t_Q are respectively as in (4.32) and (4.28). The wave with speed λ starting from Q cross the left boundary $x = -1$ at

$$t_5 = t_Q - \frac{1 + x_Q}{\lambda} \tag{4.34}$$

The complete solution can be seen in Figure 13

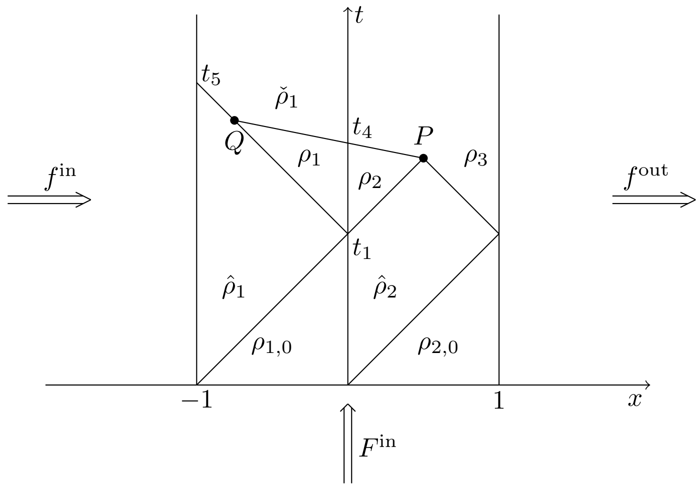


Figure 13: Solution at t_Q

This concludes the analysis of this case.

4.2 Case $q \geq q_2$

In this subsection, we assume that assumption (A1) is not satisfied and give the analysis for the Riemann problem at junction for $t \geq t_1$. The Riemann problem to solve is

$$\partial_t \rho + \partial_x f(\rho) = 0,$$
$$\rho(t,x) = \begin{cases} \hat{\rho}_1 & \text{if } x < 0, \\ \hat{\rho}_2 & \text{if } x > 0. \end{cases} \tag{4.35}$$

The splitting ratio $\beta \in (0,1)$ is fixed and $0 \leq \hat{\rho}_1 < \rho_{\text{c}}$ and $0 \leq \hat{\rho}_2 < \rho_{\text{c}}$ as in the previous section. The demand and supply functions on the corresponded roads are given by $\delta(\hat{\rho}_1) = f(\hat{\rho}_1) = f^{\text{in}}$, $d(F_{\text{in}}, l_0) = \min(F_{\text{in}}, \gamma_{\text{r1}}^{\max}) = F_{\text{in}}$ and $\sigma(\hat{\rho}_2) = f^{\max}$.
From this follows that $\rho_1 = \hat{\rho}_1$ and no wave is created in the incoming mainline. However, on the outgoing link $\rho_2 = \rho_{\text{c}}$ which generates a wave with positive speed. The buffer increases since $l(t) = (F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t-1) > 0$. The wave with positive speed 1 generated at $(t_1, 0)$ interacts with the wave generated from right boundary at point $P = (t_P, x_P)$ at time $t = t_P$ as described in subsection 4.1 under assumption (4.11). At the right boundary $f^{out} = \dfrac{1-\rho_3}{1-f^{max}} f^{max}$, hence one obtain that

$$\rho_3 = 1 - \frac{f^{\text{out}}(1-f^{\max})}{f^{\max}}. \tag{4.36}$$

At this point up to time t_4 the analysis is the same as in the previous case, so we will solve directly the Riemann problem at the junction at t_4 with the new assumption.
The Riemann problem to solve is then

$$\rho(t_4, x) = \begin{cases} \hat{\rho}_1 & \text{if } x < 0, \\ \rho_3 & \text{if } x > 0, \end{cases}$$

coupled at the junction with the following demands and supply functions

$$d(F_{\text{in}}, l) = \gamma_{\text{r1}}^{\max} = f^{\max}, \tag{4.37}$$

$$\delta(\hat{\rho}_1) = f^{\text{in}}, \tag{4.38}$$

$$\sigma(\rho_3) = f^{\text{out}}. \tag{4.39}$$

We can now compute

$$\Gamma_2 = \min\Big((1-\beta)\delta(\hat{\rho}_1) + d(F_{\text{in}}, l), \sigma(\rho_3)\Big) = \min\Big((1-\beta)f^{\text{in}} + f^{\max}, f^{\text{out}}\Big) = f^{\text{out}}$$

and

$$\hat{\Gamma}_1 = \frac{q}{1-\beta q} f^{\text{out}}.$$

According to the value of f^{in} two cases can occur at this point. If $f^{\text{in}} < f^{\text{out}}$ the solution of the Riemann Problem at the junction is given by $(\hat{\Gamma}_1, \hat{\Gamma}_{\text{r1}}, \hat{\Gamma}_2) = (f^{\text{in}}, f^{\text{out}} - (1-\beta)f^{\text{in}}, f^{\text{out}})$. From this it is straightforward to see that $\hat{\rho}_1 = \rho_1$ since $f^{\text{in}} = \hat{\rho}_1$ for $v_{\text{f}} = 1$. No new waves are created. If $f^{\text{in}} > f^{\text{out}}$ the solution of the Riemann problem at the junction becomes

$$(\hat{\Gamma}_1, \hat{\Gamma}_{\text{r1}}, \hat{\Gamma}_2) = \left(\frac{q}{1-\beta q} f^{\text{out}}, \frac{1-q}{1-\beta q} f^{\text{out}}, f^{\text{out}}\right).$$

From this we can uniquely recover the corresponding values of the densities:

$$\check{\rho}_1 = 1 - \frac{(1-f^{\max})q f^{\text{out}}}{(1-\beta q) f^{\text{in}}} \tag{4.40}$$

such that $\rho_{\text{c}} \leq \check{\rho}_1 \leq 1$. The pair $(\hat{\rho}_1, \check{\rho}_1)$ produces a wave with negative speed on the incoming main road of the roundabout, that is

$$\lambda(\hat{\rho}_1, \check{\rho}_1) = \frac{f^{\text{in}} - f^{\text{out}}}{\hat{\rho}_1 - \check{\rho}_1} = \frac{(f^{\text{in}} - f^{\text{out}})(1-\beta q) f^{\text{in}}}{(1-\beta q)((f^{\text{in}})^2 - 1) + (1-rc) f^{\text{out}}}. \tag{4.41}$$

This characteristic line with initial point $(t_4, 0)$ and speed λ crosses the left boundary $x = -1$ when $t = t_5$. That is,

$$t_5 = t_4 - \frac{1}{\lambda} = t_4 - \frac{(1-\beta q)((f^{\text{in}})^2 - 1) + (1-f^{\max}) f^{\text{out}}}{(f^{\text{in}} - f^{\text{out}})(1-\beta q) f^{\text{in}}}. \tag{4.42}$$

Since $\dfrac{q}{1-\beta q} f^{\text{out}} < f^{\text{in}}$, there is no interaction with the boundary $x = -1$ at time t_5. On the outgoing mainline of the roundabout $\hat{\Gamma}_2 = f^{\text{out}} = f(\rho_3)$. Hence no new wave is created on the outgoing link. Hence, the solution looks as in Figure 14.

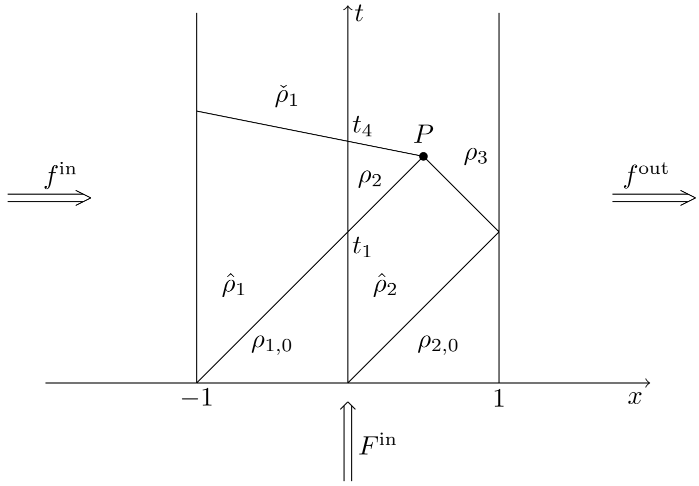


Figure 14: Solution in the case $q \geq q_2$

The flux that leaves the buffer towards the junction is $\hat{\Gamma}_{\mathrm{r1}} = \dfrac{1-q}{1-\beta q} f^{\mathrm{out}}$. Substituting this in the ODE we have $\dot{l} = F_{\mathrm{in}} - \dfrac{1-q}{1-\beta q} f^{\mathrm{out}} > 0$ since $\dfrac{1-q}{1-\beta q} f^{\mathrm{out}} < f^{\mathrm{out}} < F_{\mathrm{in}}$. Integrating it gives

$$l(t) = l(t_4) + \left(F_{\mathrm{in}} - \frac{1-q}{1-\beta q} f^{\mathrm{out}} \right) (t - t_4) > 0. \tag{4.43}$$

Thus, the queue length increase linearly. This complete the analysis for this case.

4.3 The case $q \leq q_1$

In this subsection, we assume that assumption (A2) is not satisfied and give the analysis for the Riemann problem at junction for $t \geq t_1$, as in the subsection 4.2. In order to do so, we compute the demand and supply functions at $t = t_1$

$$d(F_{\mathrm{in}}, l) = F_{\mathrm{in}}, \tag{4.44}$$

$$\delta(\hat{\rho}_1) = f^{\mathrm{in}}, \tag{4.45}$$

$$\sigma(\hat{\rho}_2) = f^{\max}, \tag{4.46}$$

coupled with the following Riemann problem at the junction

$$\rho(t_1, x) = \begin{cases} \hat{\rho}_1 & \text{if } x < 0, \\ \hat{\rho}_2 & \text{if } x > 0. \end{cases}$$

Since we suppose that only assumption A_1 holds true then $\left(\hat{\Gamma}_1, \hat{\Gamma}_{\mathrm{r1}}, \hat{\Gamma}_2\right) = \left(\dfrac{f^{\max} - F_{\mathrm{in}}}{1-\beta}, F_{\mathrm{in}}, f^{\max}\right)$.
From the value of $\hat{\Gamma}_{\mathrm{r1}}$ we can solve the ODE (2.2) and find $l(t) = 0$.
From $\hat{\Gamma}_1$ we can find the new value of the density

$$\rho_1 = \frac{(1-\beta) f^{\max} - (1-\rho_c)(f^{\max} - F_{\mathrm{in}})}{(1-\beta) f^{\max}} = \frac{f^{\max}(f^{\max} - \beta - F_{\mathrm{in}}) + F_{\mathrm{in}}}{(1-\beta) f^{\max}} \tag{4.47}$$

and since $0 \leq \hat{\rho}_1 \leq \rho_c$ then $\rho_c \leq \rho_1 \leq 1$. Moreover, the wave created has a negative speed λ:

$$\lambda(\hat{\rho}_1, \rho_1) = \frac{f^{\text{in}} - \hat{\Gamma}_1}{\hat{\rho}_1 - \rho_1} = \frac{((1-\beta)f^{\text{in}} - (f^{\max} - F_{\text{in}}))f^{\max}}{(1-\beta)f^{\max}(f^{\text{in}} - 1) + (1-\rho_c)(f^{\max} - F_{\text{in}})} \tag{4.48}$$

The characteristic line with speed λ crosses the boundary $x = -1$ at time:

$$\begin{aligned} t_2 = 1 - \frac{1}{\lambda} &= \frac{(1-\beta)f^{\max} - ((1-f^{\max}) + f^{\max})(f^{\max} - F_{\text{in}})}{((1-\beta)f^{\text{in}} - (f^{\max} - F_{\text{in}}))f^{\max}} \\ &= \frac{F_{\text{in}} - \beta f^{\max}}{(f^{\text{in}} - \beta f^{\text{in}} - f^{\max} + F_{\text{in}})f^{\max}} \end{aligned} \tag{4.49}$$

In the outgoing link of the mainline we obtain $\rho_2 = \rho_c$ which produces a wave with positive speed equal to 1 in $x \in [0, 1]$. This wave interacts with a wave coming from the boundary since it is still possible that $F_{\text{in}} > f^{\text{out}}$. In particular, this happens when $q < \dfrac{f^{\max} - f^{\text{out}}}{f^{\max} - \beta f^{\text{out}}}$. We procede now as in the subsection 4.1 and we obtain the coordinates of the point of intersection P, which are given by (4.15) and (4.14). The problem is the same as in subsection 4.2 up to time t_4. At this point we solve again the Riemann problem at the junction

$$\rho(t_4, x) = \begin{cases} \rho_1 & \text{if } x < 0, \\ \rho_3 & \text{if } x > 0. \end{cases}$$

coupled with the following values of demands and supply functions:

$$d(F_{\text{in}}, l) = F_{\text{in}}, \tag{4.50}$$
$$\delta(\rho_1) = f^{\max}, \tag{4.51}$$
$$\sigma(\rho_3) = f^{\text{out}}. \tag{4.52}$$

From this it follows $\Gamma_2 = \min((1-\beta)\delta(\rho_1) + d(F_{\text{in}}, l), \sigma(\rho_3)) = \min((1-\beta)f^{\max} + F_{\text{in}}, f^{\text{out}}) = f^{\text{out}}$, $\hat{\Gamma}_1 = \dfrac{q}{1-\beta q} f^{\text{out}}$ and $\hat{\Gamma}_{r1} = \dfrac{1-q}{1-\beta q} f^{\text{out}}$. From this we can uniquely recover the corresponding value of the densities. In the incoming mainline we have

$$\check{\rho}_1 = 1 - \frac{(1-f^{\max})q f^{\text{out}}}{(1-\beta q)f^{\max}}$$

such that $\rho_c \leq \check{\rho}_1 \leq 1$. This produces a wave of negative speed $\lambda = \dfrac{f^{\max}}{f^{\max} - 1}$ on the incoming road that intersects the boundary $x = -1$ at

$$t_5 = t_p + \frac{1 - f^{\max}}{f^{\max}}(x_P + 1).$$

Since $\dfrac{q}{1-\beta q} f^{\text{out}} < f^{\text{in}} = f^{\max}$, there is no interaction with the boundary $x = -1$ at time t_5. On the outgoing link there is no wave generated since $\hat{\Gamma}_2 = f^{\text{out}} = f(\rho_3)$. Hence, the solution looks as in Figure 15.

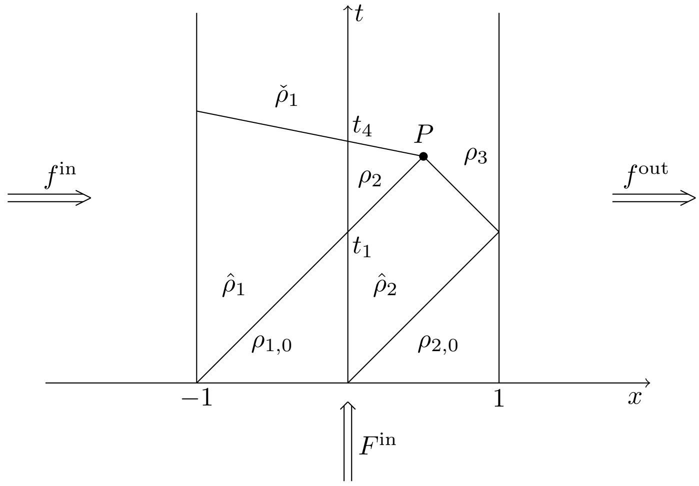


Figure 15: Solution in the case $q \leq q_1$

Last thing to do is to compute the queue length at t_4. To do so, we insert the value of $\hat{\Gamma}_{\text{r1}}$ inside the ODE (2.2) and we solve it obtaining

$$l(t) = \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}}\right)(t - t_4) > 0. \tag{4.53}$$

This concludes our analysis for this case. We are now ready to give the explicit version of the cost functionals for each different case.

4.4 Total Waiting Times and Total Travel Times

- $q_1 \leq q \leq \bar{q}$

  We compute the TWT as follows

$$\begin{aligned} TWT(T, q) = & \int_{t_1}^{t_4} \left(\left(F_{\text{in}} - \frac{(1-q) f^{\max}}{1-\beta q}\right))(t-1)\right) dt \\ & + \int_{t_4}^{T} \left(l(t_4) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}}\right)(t - t_4)\right) dt + l(t_4) \\ & + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}}\right)(T - t_4). \end{aligned} \tag{4.54}$$

Concerning $TTT(T)$, it is given by a constant plus a term depending on the priority, that we denote by $TTT(T,q)$:

$$
\begin{aligned}
TTT(T,q) = & \int\int_{A_1} \hat{\rho}_1(t,x)dtdx + \int\int_{A_2} \rho_1(t,x)dtdx + \int\int_{A_3} \check{\rho}_1(t,x)dtdx \\
& + \int_{t_1}^{t_4} \left(\left(F_{\text{in}} - \frac{(1-q)f^{\max}}{1-\beta q} \right))(t-1) \right) dt \\
& + \int_{t_4}^{T} \left(l(t_4) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}} \right) (t-t_4) \right) dt \\
& + l(t_4) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}} \right) (T-t_4) + \int_{x_1}^{x_2} (\check{\rho}_1 + \rho_3)\, dx
\end{aligned}
\tag{4.55}
$$

where the areas are defined by
$A_1 = \frac{1}{2}(t_2 - 1) = \frac{1}{2}\left(\frac{1-(1+\beta)q}{f^{\text{in}}(1-\beta q) - q f^{\max}} - 1 \right)$
$A_2 = \frac{1}{2}(t_5 + t_4 - t_2 - 1)$
$A_3 = \frac{1}{2}(t_5 - t_4)$
and $T = t_5$, as in Figure 16.

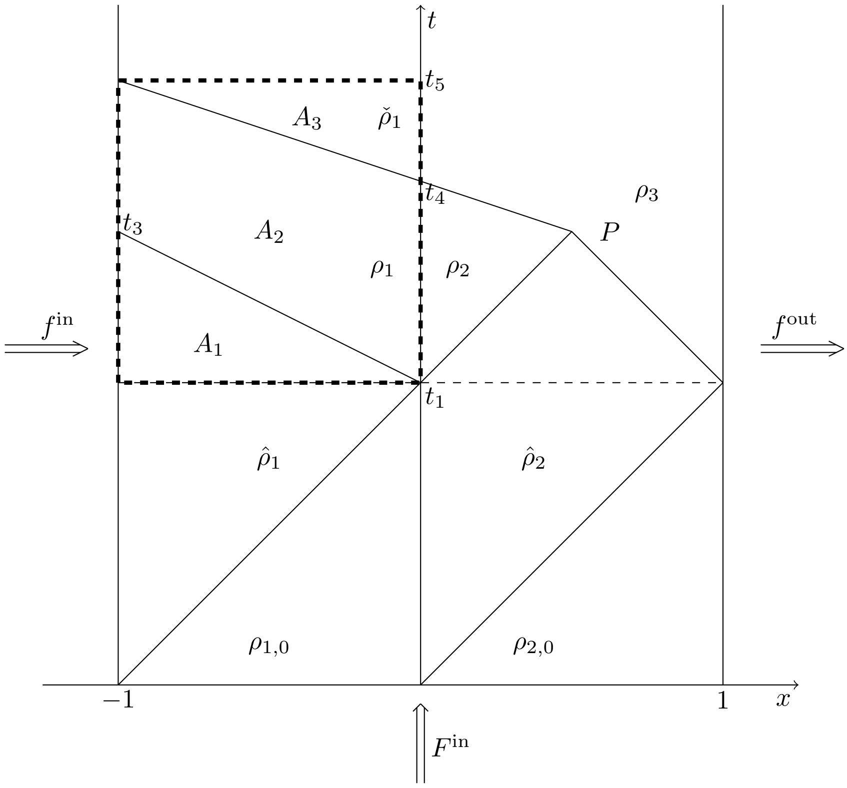


Figure 16: Area of integration in the case $q_1 \leq q \leq \bar{q}$

- $\bar{q} \leq q \leq q_2$

  The TWT is as in (4.54), since it does not depend on the wave interactions but only on the queue length. The $TTT(T)$ is given by the constant term plus

$$\begin{aligned} TTT(T,q) = & \int\int_{A_1+A_2+A_5} \hat{\rho}_1(t,x)dtdx + \int\int_{A_3} \rho_1(t,x)dtdx + \int\int_{A_4+A_6+A_7} \check{\rho}_1(t,x)dtdx \\ & + \int_{t_1}^{t_4} \left(\left(F_{\text{in}} - \frac{(1-q)f^{\max}}{1-\beta q} \right))(t-1) \right) dt + \int_{x_1}^{x_2} (\check{\rho}_1 + \rho_3)\, dx \\ & + \int_{t_4}^{T} \left(l(t_4) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}} \right)(t-t_4) \right) dt \\ & + l(t_4) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q} f^{\text{out}} \right)(T-t_4). \end{aligned} \tag{4.56}$$

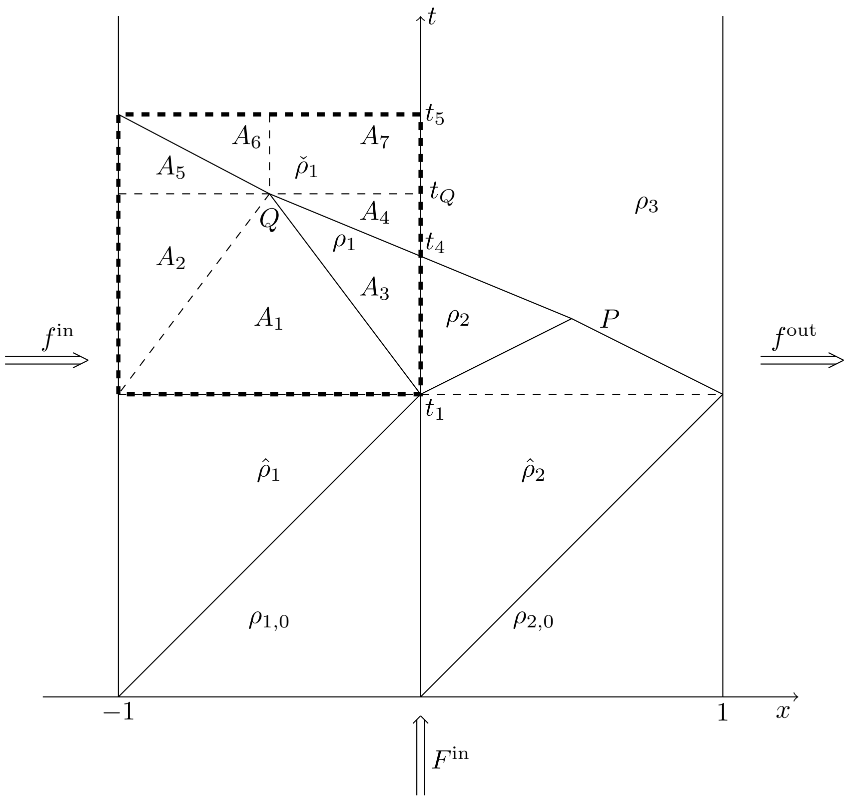


Figure 17: Area of integration in the case $\bar{q} \leq q \leq q_2$

The areas are defined by
$A_1 = \frac{1}{2}(t_Q - 1)$, $A_2 = \frac{1}{2}(t_Q-1)(x_Q+1)$
$A_3 = \frac{1}{2}(x_Q - x_Q t_4)$

$A_4 = \frac{1}{2}(t_Q - t_4)(-x_Q)$
$A_5 = \frac{1}{2}(t_5 - t_Q)(x_Q + 1)$
$A_6 = \frac{1}{2}(t_5 - t_Q)(x_Q + 1)$
$A_7 = (t_5 - t_Q)(-x_Q)$ as in Figure 17.

- $q \geq q_2$

The TWT is given by

$$\begin{aligned} TWT(T,q) = &\int_{t_1}^{t_4} \big(F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t-1)\big)\,dt \\ &+ \int_{t_4}^{T} \left((F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t_4-1) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(t-t_4)\right)dt \\ &+ (F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t_4-1) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(T-t_4). \end{aligned} \tag{4.57}$$

The $TTT(T)$, as usual, is instead calculated by the constant term plus

$$\begin{aligned} TTT(T,q) = &\int\int_{A_1} \hat{\rho}_1(t,x)dtdx + \int\int_{A_2} \check{\rho}_1(t,x)dtdx \\ &+ \int_{t_1}^{t_4} \big(F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t-1)\big)\,dt + \int_{x_1}^{x_2} (\check{\rho}_1 + \rho_3)\,dx \\ &+ \int_{t_4}^{T} \left((F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t_4-1) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(t-t_4)\right)dt \\ &+ (F_{\text{in}} + (1-\beta)f^{\text{in}} - f^{\max})(t_4-1) + \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(T-t_4) \end{aligned} \tag{4.58}$$

and the area for this case are $A_1 = \frac{1}{2}(t_5 - t_4)$ and $A_2 = \frac{1}{2}(t_5 - t_4)$ as shown in the Figure 18.

- $q \leq q_1$

The TWT is computed as

$$TWT(T,q) = \int_{t_4}^{T} \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(t-t_4)dt + \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(T-t_4) \tag{4.59}$$

while the $TTT(T,q)$ is given by

$$\begin{aligned} TTT(T,q) = &\int\int_{A_1} \rho_1(t,x)dtdx + \int\int_{A_2} \check{\rho}_1(t,x)dtdx + \int_{t_4}^{T} \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(t-t_4)dt \\ &+ \left(F_{\text{in}} - \frac{1-q}{1-\beta q}f^{\text{out}}\right)(T-t_4) + \int_{x_1}^{x_2} (\check{\rho}_1 + \rho_3)\,dx \end{aligned} \tag{4.60}$$

where $A_1 = \frac{1}{2}(t_5 - t_4) = A_2$ as shown in Figure 19.

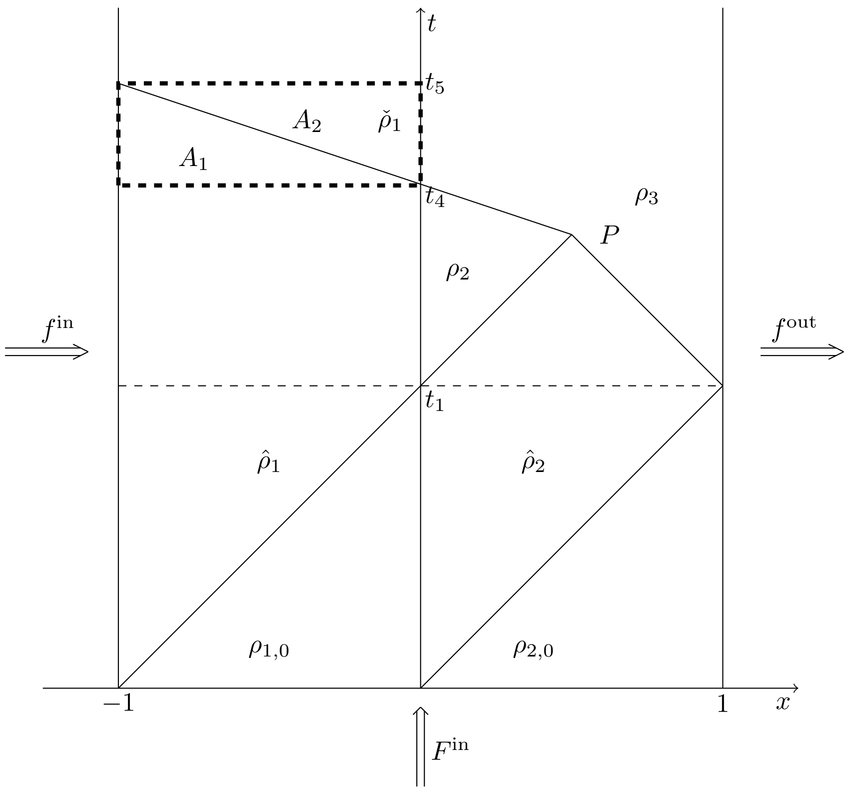


Figure 18: Area of integration in the case $q \geq q_2$

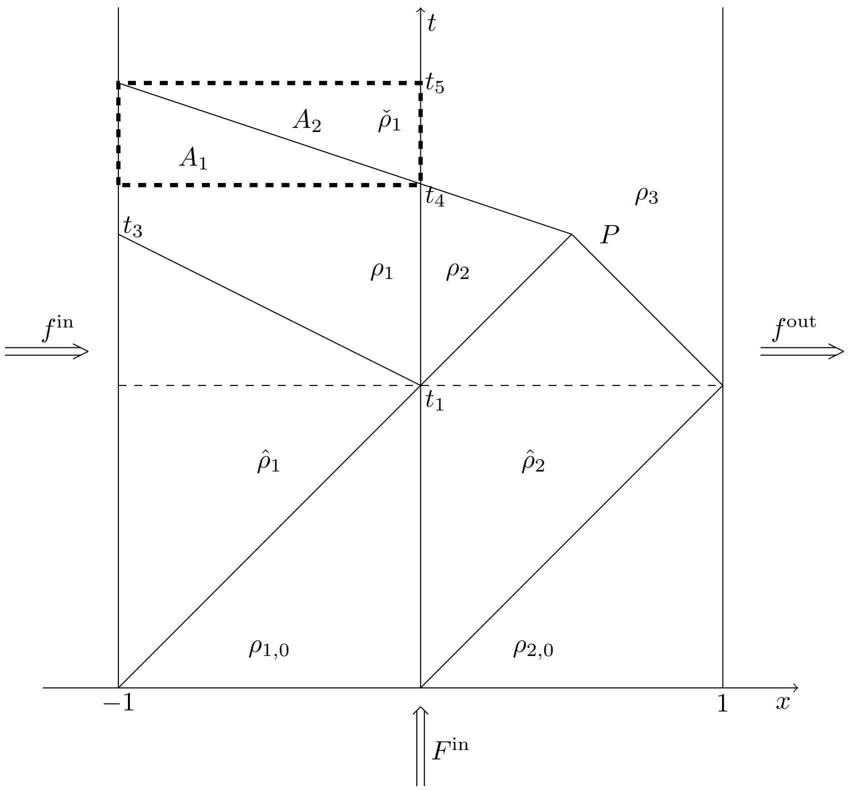


Figure 19: Area of integration in the case $q \leq q_1$

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