Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations

Zhiping Rao, Antonio Siconolfi, Hasnaa Zidani

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Abstract

We establish a comparison principle for a Hamilton–Jacobi–Bellman equation, more appropriately a system, related to an infinite horizon problem in presence of an interface. Namely a low dimensional subset of the state variable space where discontinuities in controlled dynamics and costs take place. Since corresponding Hamiltonians, at least for the subsolution part, do not enjoy any semicontinuity property, the comparison argument is rather based on a separation principle of the controlled dynamics across the interface. For this, we essentially use the notion of ε–partition and minimal ε–partition for intervals of definition of an integral trajectory.

Key words. Optimal control, discontinuous dynamics and cost, comparison principle, Hamilton-Jacobi equations, transmission conditions, viscosity solutions.

AMS subject classifications. 49L20, 49J15, 35F21

1 Introduction

This paper is devoted to the analysis of an infinite horizon problem in presence of an interface, specific object of the investigation being detection of an appropriate Hamilton–Jacobi–Bellman (HJB) equation and deduction of comparison results for it. By interface we mean a low–dimensional subset of the state variable space where discontinuities in controlled dynamics and costs take place.

We assume the state variable space $\mathbb{R}^d$ to be partitioned in two disjoint open sets Ω_1, Ω_2 plus their common boundary, the interface, that we denote by Γ and take of class C^2 without requiring any connectedness condition, see Section 3.1. Each open region is associated with a compact control set, say A_i, $i = 1, 2$, and controlled dynamics and costs, denoted by f_i, ℓ_i respectively, are defined on $\Omega_i \times A_i$, while the interface is assumed control theoretically void, namely not supporting any dynamics or cost. Appropriate transmission conditions are assumed on Γ. An integrated system is then built by performing on Γ suitable convex combinations of f_i and ℓ_i. To do that, we follow the model recently proposed by Barles–Briani–Chasseigne in [5], see Section 3.1.

Due to the discontinuous setting across the interface, the proof of comparison result must be rethought from scratch since doubling variable method as well as regularization by means of sup/inf convolutions are not suitable. Other results available in the literature on discontinuous Hamilton–Jacobi equations, see for instance [9], [12], [18], [27] [28], are not of help for our model.

Apart comparison principle, related relevant issues have been for the moment left aside. For instance in [5] controls are divided between regular and singular, according to the behavior of associated velocities on the interface, and correspondingly, two different value functions are analyzed. This approach is certainly interesting and capable of promising developments, but we do not follow it and just consider the value function associated to all controls of the integrated system. Another crucial point that we do not have

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considered, is to provide stability results adapted to the setting. In [5] some initial results are given in this respect, but some substantial progress is still needed to produce a well defined theoretical frame.

The field of Hamilton-Jacobi equations with discontinuous coefficients is of growing interest both from theoretical viewpoint and for applications. It appears in the modelization of several physical problems, such as the problem of *ray light propagation in an inhomogeneous medium with discontinuous refraction index* [28, 24]. The subject is also interesting for applications in Hybrid Control Theory [8]. In this area, the caracterization of the value function and the comparison principle for the corresponding HJB equation are usually studied under restrictive assumptions, called trasversality assumptions, that prevent the optimal trajectories from complex interactions with interfaces [29, 4, 6, 20]. We consider the present paper as a first step in the direction of the study of some hybrid systems without transversality conditions.

Three papers have been particularly influential for our work, apart the already cited [5], we would like to mention Bressan–Hong [10], which has been, as far as we know, the first paper on the subject and where the relevance of HJB tangential equations, namely equations posed on the interfaces, is pointed out. Third reference is Barnard–Wolenski [11], which has attracted our attention on the fact that admissible curves of integrated system are actually integral trajectory of an essential, somehow hidden, dynamics and have showed the effectiveness of Filippov Approximation Theorem in this context, see Section 2.2. Our topic is also related, at least for difficulties to be tackled, with studies of Hamilton–Jacobi equations in domains with junctions or on networks, see [22, 1, 25].

Regarding our hypotheses, we consider systems as general as possible in the open regions and, as already pointed out, assume specific transmission conditions just on Γ. More precisely, we assume a sort of permeability of the interface, see **(H3)(i)**, namely the possibility to go from Γ to any of the two open regions following admissible trajectories. This is unavoidable if we want the value function to be continuous. We moreover require some controllability, but just of tangential type on Γ, see **(H3)(ii)**, implying that all subsolutions of HJB are Lipschitz–continuous when restricted to the interface.

We emphasize that no coercivity requirements on the Bellman Hamiltonians related to systems in Ω_i are assumed. Theses are actually quite onerous from a control theoretic viewpoint and implies Lipschitz continuity of subsolutions on the whole space, which simplifies to some extent the analysis. To the best of our knowledge, all comparison results holding for HJB equations in presence of some sort of interface, junctions or posed on networks have been established to date assuming coercivity of corresponding Hamiltonians.

Our last assumption is the convexity, at any point of the interface, of the set of all admissible velocities/costs, see **(H4)**. In our understanding, this is actually the less satisfactory and more technical requirement, we crucially exploit it to prove a regularity result for an augmented dynamics on Γ, see Theorem A.1. Same assumption appears in [5] and [11]. The use of relaxed controls will hopefully allow to weaken it or at least clarify its meaning in relation to the model.

The detection of the HJB equation appropriate to the setting clearly poses problems only on the interface, since in each open region it is natural to choose the usual Hamiltonian associated to f_i, ℓ_i and discount factor λ, denoted in what follows by H_i, $i = 1, 2$. To justify our formulation on Γ, we start noticing that the presence of a maximum in the formula for Hamiltonians of Bellman type produces a lack of symmetry between super and subsolution condition. The subsolution condition is indeed more demanding since the corresponding inequality must hold for any control. It is understandingly crucial to determine the effective controls acting on the interface in our model, and it is here that the essential dynamics comes into play. Our analysis, in fact, suggests, see Lemma 3.3, that the relevant controls we are searching, are those corresponding to tangential velocities, see also [11]. A similar remark is also done, but not systematically exploited, in [5].

We take for the supersolution part on Γ the Bellman Hamiltonian corresponding to all control in A, which turns out to be equal to $\max\{H_1, H_2\}$. This is the Hamiltonian for supersolutions indicated by Ishii's theory, the reference frame for discontinuous Hamilton–Jacobi equations. However the Hamiltonian provided by the same theory for subsolutions, namely $\min\{H_1, H_2\}$, does not seem well adapted to our setting since it does not take into any special account controls corresponding to tangential velocities.

We consider for subsolutions the Hamiltonian of Bellman type with controls associated to tangential velocities, accordingly the corresponding equation is restricted on the interface, which means that viscosity tests take place at local constrained maximizers with constraint Γ, or test functions can be possibly just defined on Γ, see formula (HJB) in Section 3.2. Same Hamiltonian also appears in [5], the difference is that in our case to satisfy such a tangential equation is the unique condition we impose on subsolutions on Γ, and not an additional one.

This is in our opinion the most relevant new point in the paper. It deeply changes the nature of the system because now equations pertaining to subsolutions are completely separated in the three regions of the partition. This requires, first, some compatibility conditions, otherwise there is no hope to get comparison results. Secondly, comparison must be based not on semicontinuity property of the Hamiltonian, that we do not have, at least for the subsolution part, but on a separation principle of the controlled dynamics of the integrated system, related to the partition, we will explain later on.

Our results are the following: we show that the value function is a bounded continuous solution of the previously described HJB equation, we are able to compare lower semicontinuous supersolutions with upper semicontinuous subsolution, which are in addition continuous at any point of Γ. We deduce, as a consequence of the previous properties, uniqueness of value function as solution. See Theorems 3.5, 3.6, 3.7. Note that the extra condition on subsolutions for comparison is a mild one, just a continuity requirement at any point of Γ.

The methodology we use is of dynamical type, see [15], [16]. Namely, instead of directly working with viscosity test functions, we get the comparison by first establishing optimality properties for sub/supersolution, or invariance of the hypograph of any subsolution and epigraph of any supersolutions with respect to an augmented controlled dynamics defined in $\mathbb{R}^d \times \mathbb{R}$. See Theorems 5.1 and 6.1.

When trying to apply these techniques to the problem under investigation, the main difficulty we find in our discontinuous setting is the lack of Lipschitz–continuity of the controlled multivalued dynamics. This regularity is in fact essential to exploit Strong Invariance Theorem, see Theorem 2.8, for the subsolution analysis, and plays also a role for supersolutions since allows using Weak Invariance Theorem, see Theorem 2.7, in spite dynamics not being convex–valued, via Relaxation Theorem. We are nevertheless able to show, see Theorem A.1, that the controlled dynamics possess the right regularity separately in the open regions of the partition and on the interface.

If one could accordingly divide in disjoint intervals the times where trajectories of the integrated system, or at least those relevant for the infinite horizon problem, lie in different regions of the partition, then this partial regularities could be glued together, possibly with the help of some cutoff function, allowing arguing as in the usual case without interface. The obstruction in doing that comes from the presence of trajectories y, which cannot by no means ruled out a priori, such that the set of accumulation points of

$$\partial \{t \mid y(t) \in \Omega_1 \cup \Omega_2\};$$

is nonempty. Around these times the curve may wildly oscillates among the regions of partition. This time set is usually termed in Hybrid Control Theory after the Greek philosopher Zeno of Elea.

To overcome the obstacle, we resort to a weak separation principle, which is explained in Appendix B, and specifically in Proposition B.3. It basically says that, given a positive parameter ε, the interval of definition of any trajectory can be decomposed in such a way that if in a given subinterval I the curve intersects the interface then it stays outside it for a time set with measure less than ε, and it lies for all t in a suitable neighborhood of Γ. This allows, exploiting Filippov Approximation Theorem and controllability assumptions on Γ, to approximate it by trajectories without Zeno times, for which the above described gluing technique can be applied.

The paper is organized as follows: in Section 2 we expose the basic tools of our analysis with particular reference to invariance results for multivalued vector fields, Filippov Approximation Theorem ad differential properties of the interface. Section 3 makes precise the setting with assumptions, definition of the involved dynamics, and statements of main results. More important, it is written down the Hamilton–Jacobi–Bellman equation we propose for our model, see (HJB). In Section 4 we study the continuous character of the value function, while Sections 5, 6 represent the core of the paper with results showing the relationship between being sub/supersolution to (HJB) and enjoying sub and superoptimality properties. In Section 7 we provide the demonstrations of main theorems. Finally, the first appendix is about augmented dynamics with the proof of its Lipschitz character on the interface, and the second is centered on the notion of ε partitions for a given curve, minimal ε–partitions and the introduction of an index related to it, see Definition B.4, on which induction arguments of Sections 5, 6 are based.

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2 Preliminaries

2.1 Multifunction and invariance properties for differential inclusions

In this subsection we collect some well known results about solutions of differential inclusions and invariance properties for them, see [15], [16]. Given a multifunction Z defined in some open subset of $\mathbb{R}^n$, we will say that a curve y is an *integral curve* or a *trajectory* of Z if it satisfies

$$\dot{y}(t) \in Z(y(t)) \quad \text{for a.e. } t.$$

We say that Z has *linear growth* if there exists a positive constant C with

$$|q| \le C\,(|x|+1) \quad \text{for any } x \text{ where } Z \text{ is defined, any } q \in Z(x).$$

Unless otherwise specified, all the multifunctions appearing throughout the paper will have linear growth and will be either upper semicontinuous with compact convex values or locally Lipschitz–continuous (namely, Lipschitz continuous in any bounded subset) with compact values. In both settings we are guaranteed that for any given initial datum there exist trajectories of Z attaining it at $t = 0$, and any such trajectories can be extended on $[0, +\infty)$ if Z is defined in the whole of $\mathbb{R}^n$, or, in general until they reach the boundary of the set where Z is defined.

Definition 2.1. (Normal cone) *Given a closed subset $\mathcal{C} \subset \mathbb{R}^n$ and $x \in \partial\mathcal{C}$, we define the normal cone to $\mathcal{C}$ at x as*

$$\{p \in \mathbb{R}^n : \ \exists\, \varepsilon > 0 \text{ such that } proj_{\mathcal{C}}(x + \varepsilon\, p) = x\},$$

where $proj_{\mathcal{C}}$ stands for the projection on $\mathcal{C}$. Notice that the previous relation still holds for any positive quantity less than ε. Up to reducing ε, we can also suppose that x is the unique projection point of $x + \varepsilon\, p$. We will write $p \perp \mathcal{C}$ at x to mean that p is a nonzero vector of the normal cone at x, notice that, given $x \in \partial\mathcal{C}$, the set of nonzero normal vectors can be empty.

We are going to apply results about normal cones, in combination with forthcoming invariance properties, taking as closed sets epigraphs of lower semicontinuous and hypographs of upper semicontinuous functions. We denote by $\mathcal{E}p(\cdot)$, $\mathcal{H}p(\cdot)$, respectively, these entities. Even if we just mention it in the proof of Proposition 5.4, we recall, using our terminology and notations, a result of [26] which is crucial for matching nonzero normal vectors to epi/hypgraphs and differentials of viscosity test functions.

Proposition 2.2. *Let w be a lower (resp. upper) semicontinuous function. Assume that $(p, 0)$ is a nonzero normal vector to $\mathcal{E}p(w)$ (resp. to $\mathcal{H}p(w)$) at some point $(x_0, w(x_0))$, then there are sequences $(x_k, w(x_k))$, (p_k, s_k), with $s_k \neq 0$ and $(p_k, s_k) \perp \mathcal{E}p(w)$ (resp. $(p_k, s_k) \perp \mathcal{H}p(w)$), such that*

$$(x_k, w(x_k)) \to (x_0, w(x_0)) \quad \textit{and} \quad (p_k, s_k) \to (p, 0).$$

In what follows we consider a closed set $\mathcal{C}$, and a multifunction Z defined in some open set containing $\mathcal{C}$. We recall her below some definitions and properties from non-smooth analysis.

Definition 2.3. (Weak tangential condition) *We say that Z satisfies the* weak tangential condition *on $\mathcal{C}$ if*

$$p \cdot q \le 0 \quad \textit{for any } x \in \partial\mathcal{C},\ p \perp \mathcal{C} \textit{ at } x, \textbf{ some } q \in Z(x).$$

The condition is clearly empty at any x at which no nonzero normal vectors to $\mathcal{C}$ do exist.

Definition 2.4. (Strong tangential condition) *We say that Z satisfies the* strong tangential condition *on $\mathcal{C}$ if*

$$p \cdot q \le 0 \quad \textit{for any } x \in \partial\mathcal{C},\ p \perp \mathcal{C} \textit{ at } x, \textbf{ any } q \in Z(x).$$

Definition 2.5. (Weak invariance) *We say that $\mathcal{C}$ is* weakly (forward) invariant *for Z if for any $x \in \mathcal{C}$ there exists an integral curve y of Z with $y(0) = x$ such that $y(t) \in \mathcal{C}$ for all positive t.*

Definition 2.6. (Strong invariance) *We say that $\mathcal{C}$ is* strongly (forward) invariant *for Z if for any $x \in \mathcal{C}$, any integral curve y of Z with $y(0) = x$, one has $y(t) \in \mathcal{C}$ for all positive t.*

Theorem 2.7. *Assume Z to be convex compact valued, upper semicontinuous and to have linear growth. If Z satisfies the weak tangential condition on $\mathcal{C}$, then $\mathcal{C}$ is weakly invariant with respect to it.*

Theorem 2.8. *Assume Z to be compact valued, locally Lipschitz continuous and to have linear growth. If Z satisfies the strong tangential condition on $\mathcal{C}$, then $\mathcal{C}$ is strongly invariant with respect to it.*

2.2 Filippov Approximation Theorem

An essential tool in our analysis will be Filippov Approximation Theorem, which provides an estimate of how far a given curve, say y, is from some integral trajectory of a Lipschitz multifunction Z in terms of the distance to $Z(y(t))$ of $\dot{y}(t)$. Reference for it is [14], [3], however, since the original formulation is local in time, and we instead need a global result, we modify the proof assuming some additional invariance with respect to a given closed subset, according to our setting in Sections 5, 6

We first introduce the *reachable set* for a given multifunction Z in $\mathbb{R}^n$, $B \subset \mathbb{R}^n$, $T > 0$. We consider all points reached from some initial set not only in the prescribed time T, but in any time shorter than it, as well.

$$\mathcal{R}_Z(B,T) = \bigcup_{t \in [0,T]} \{x \in \mathbb{R}^n \mid \exists \text{ traj.} y \text{ of } Z \text{ with } y(0) \in B, \, y(t) = x\}. \tag{2.1}$$

If B reduces to a singleton, say $\{x_0\}$, we will simply write $\mathcal{R}_Z(x_0, T)$.

If Z has linear growth then it is an immediate consequence of Gronwall Lemma that $\mathcal{R}_Z(B,T)$ is bounded for any bounded subset B, any $T > 0$.

Theorem 2.9. *Let $\mathcal{C}$ be a closed subset of $\mathbb{R}^n$, and $\mathcal{C}_\natural$ an open neighborhood of $\mathcal{C}$. Let y be a curve defined in some interval $[0,T]$ such that*

- **(i)** $y(0) \in \mathcal{C}$.
- **(ii)** $y([0,T]) \subset \mathcal{C}_\natural$.

Let Z be a multifunction defined in $\mathcal{C}_\natural$ with

- **(iii)** *Z is locally Lipschitz–continuous, compact valued and has linear growth.*
- **(iv)** *$\mathcal{C}$ is strongly invariant for Z.*

Then there exists a trajectory y_* of Z defined in $[0,T]$, contained in $\mathcal{C}$, and with $y_*(0) = y(0)$, such that

$$|y_*(t) - y(t)| \leq e^{L\,t} \int_0^T d(\dot{y}, Z(y))\, ds \quad \textit{for any } t \in [0,T],$$

where L is the Lipschitz constant of Z in some bounded open neighborhood of $\mathcal{R}_Z(y(0),T)$ contained in $\mathcal{C}_\natural$. (Note that $\mathcal{R}_Z(y(0),T)$ is indeed bounded, being Z with linear growth, and is in addition contained in $\mathcal{C}$ because of the invariance condition **(iv)***.)*

Proof. We denote by B a bounded open neighborhood of $\mathcal{R}_Z(y(0),T)$ in $\mathcal{C}_\natural$ where Z has Lipschitz constant L, and by ρ, P positive constants with

$$\mathcal{R}_Z(y(0),T) + B(0,\rho) \subset B \tag{2.2}$$

and

$$|q| < P \quad \text{for } q \in Z(x), \, x \in B \cup y([0,T]).$$

All the curves starting at $y(0)$ with (a.e.) velocity less than P are contained in B for $t \in [0, t_0]$, where $t_0 = \min\left\{T, \frac{\rho}{P}\right\}$. We construct by recurrence a sequence of curves of this type as follows: we set $y_0 = y$ and for $k \geq 1$ define

$$Z_k(t) = \{q \in Z(y_{k-1}(t)) \mid |q - \dot{y}_{k-1}(t)| = d(\dot{y}_{k-1}(t), Z(y_{k-1}(t)))\} \quad \text{for a.e. } t \in [0, t_0]$$

Since this multifunction is measurable, see [14], we extract a measurable selection denoted by f_k. We then define y_k in $[0, t_0]$ as the curve determined by $\dot y(t) = f_k(t)$, for a.e. t and $y_k(0) = y(0)$. We set

$$d_Z = \int_0^{t_0} d(\dot y(s), Z(y(s))\, ds.$$

We have for a.e. $t \in [0, t_0]$

$$\begin{aligned} |\dot y_1(t) - \dot y(t)| &= d(\dot y(t), Z(y(t)) \\ |y_1(t) - y(t)| &\leq d_Z \end{aligned}$$

and for $k \geq 1$

$$\dot y_{k+1}(t) \in Z(y_k(t)) \tag{2.3}$$

$$|\dot y_{k+1}(t) - \dot y_k(t)| = d(\dot y_k(t), Z(y_k(t)) \leq L\, |y_k(t) - y_{k-1}(t)| \tag{2.4}$$

$$|y_{k+1}(t) - y_k(t)| \leq L \int_0^t |y_k(s) - y_{k-1}(s)|\, ds. \tag{2.5}$$

We deduce for any $t \geq 0$:

$$|y_2(t) - y_1(t)| \leq L \int_0^t |y_1(s) - y(s)|\, ds \leq L\, d_Z\, t$$

and

$$|y_{k+1}(t) - y_k(t)| \leq L \int_a^t |y_k(s) - y_{k-1}(s)|\, ds \leq d_Z\, \frac{L^k\, t^k}{k!}.$$

It is straightforward to deduce from this information, see [14], that y_k uniformly converge to a trajectory $\overline{y}$ of Z in $[0, t_0]$ satisfying the assertion with t_0 in place of T.

If $t_0 < T$ then using the same argument as above we show that $\overline{y}$ can be extended, still satisfying the assertion, in the interval $[0, t_1]$, where $t_1 = \min\left\{T, 2\, \frac{\rho}{P}\right\}]$. To do that, we exploit that any curve defined in $[t_0, t_1]$, taking the value $\overline{y}(t_0)$ at t_0 and with velocity less than P is contained in B. This is in turn true because of (2.2) and $\overline{y}(t_0) \in \mathcal{R}_Z(y([0, T], T)$. The proof is then concluded because we can iterate the argument till we reach T. □

Following [11], we use at least part of the previous argument to deduce a property for Lipschitz–continuous multifunctions possessing convex values, to be used in the proof of Theorem 5.2.

Corollary 2.10. *We assume Z to be defined in an open set B of $\mathbb{R}^n$ and to be locally Lipschitz–continuous, compact convex valued. For any $x_0 \in B$, $q_0 \in Z(x_0)$, there is a C^1 integral curve y_* of Z, defined in some interval $[0, T]$, with $y_*(0) = x_0$, $\dot y_*(0) = q_0$.*

Proof. We set $y(t) = x_0 + q_0\, t$, $t \in [0, T]$, for T small enough. It comes from assumptions that the correspondence

$$t \mapsto \{q \in F(y(t)) \mid |q - \dot y(t)| = d(\dot y(t), Z(y(t)))\}$$

defined in $[0, T]$ is univalued and continuous, furthermore it takes the value q_0 at $t = 0$. It follows that the curve y_1, defined as in the proof of Theorem 2.9, is of class C^1 and satisfies $y_1(0) = x_0$, $\dot y_1(0) = q_0$, same properties hold true for any of the y_k. Following Theorem 2.9, we see that both y_k, $\dot y_k$ uniformly converge, up to a subsequence, as $k \to +\infty$. The limit curve satisfies the claim. □

2.3 The interface

We partition $\mathbb{R}^d$ as

$$\mathbb{R}^d = \Omega_1 \cup \Omega_2 \cup \Gamma,$$

where Ω_1, Ω_2 are two nonempty open disjoint subsets and

$$\Gamma = \partial\Omega_1 = \partial\Omega_2$$

is a C^2 hypersurface (not necessarily connected), namely an imbedded submanifold of $\mathbb{R}^d$ of codimension 1, here *embedded* simply means that the submanifold topology is the relative topology inherited by $\mathbb{R}^d$. We will refer to it throughout the paper as *the interface*. We make the arbitrary choice of defining the signed distance from Γ looking at it as boundary of Ω_2. Namely:

$$g(x) = \begin{cases} d(x,\Gamma) & x \in \overline{\Omega}_1, \\ -d(x,\Gamma) & x \in \Omega_2. \end{cases} \tag{2.6}$$

It is clear that, at any $x \in \Gamma$, $Dg(x)$ is the unit normal vector of Γ pointing outside Ω_2 and inside Ω_1. We denote by $\mathcal{T}_\Gamma(x)$, $\mathcal{T}^*_\Gamma(x)$ tangent and cotangent space, respectively, at any $x \in \Gamma$, the *cotangent bundle* $\mathcal{T}^*\Gamma$ is made up by all the pairs (x,p) with $x \in \Gamma$ and $p \in \mathcal{T}^*_\Gamma(x)$. We indicate by $d_\Gamma(\cdot)$ the Riemannian distance on Γ induced by the Euclidean metric of $\mathbb{R}^d$, which is given by any pair x, z of Γ by

$$d_\Gamma(x,z) = \inf \left\{ \int_0^1 |\dot{y}|\, ds \mid y : [0,1] \to \Gamma,\ y(0) = x,\ y(1) = z \right\}.$$

It is clearly finite in each connected component of Γ. We will use the following well known facts:

- **(i)** Γ has countably many connected component.
- **(ii)** There is an open neighborhood $\Gamma_\natural$ of Γ in $\mathbb{R}^d$ where the projection on Γ is of class C^1.
- **(iii)** The signed distance g is of class C^2 in $\Gamma_\natural$.
- **(iv)** Given a connected component Γ_0 of Γ and $x \in \Gamma_0$, the function $d_\Gamma(x,\cdot)$ is of class C^2 in $\Gamma_0 \setminus \{x\}$. Moreover d_Γ is locally equivalent in Γ_0 to Euclidean distance. Namely for any compact subset Θ of Γ_0 there is $N > 1$ with
$$|x - z| \le d_\Gamma(x,z) \le N\,|x - z| \quad \text{for any } x,\ z \text{ in } \Theta.$$
- **(v)** For any pair of points belonging to the same connected component of Γ, say Γ_0, there is a minimal geodesic for d_Γ of class C^1 linking them, namely such curve lies in Γ_0 and its Euclidean length realizes the Riemannian distance.

Item (i) directly comes from paracompactness of Γ, second item is a consequence of ε–neighborhood Theorem, see [19]. The third comes from the fact that proj_Γ appears in the derivative of distance, see [21] and [17, Remark 5.6]. Item(iv) basically depends on the fact that for any point of Γ the differential of the exponential map is the identity at 0. For the last one we exploit that any connected component of Γ is complete because is closed in $\mathbb{R}^d$ and invoke Hopf–Rinow Theorem.

Some examples of partitions are given in Figure 1. In fig.1(a), Ω_1 is the union of spheres with the same radius and located at a same distance each to another, the interface Γ is the union of the balls' boundaries. In fig.1(b), the interface is the union of vertical lines. In fig.1(c), the domain Ω_2 is union of balls that are disjoint but closer and closer when going to infinity. In this example, the interface is the union of the boundaries of the balls.

Remark 2.11. *Being Γ an embedded submanifold of $\mathbb{R}^d$, any point of it belonging, say, to the connected component Γ_0, must have a neighborhood U in $\mathbb{R}^d$ with $U \cap \Gamma \subset \Gamma_0$. This implies: first, that only a finite number of connected component of Γ can intersect a given compact subset of $\mathbb{R}^d$ and, second, that for any connected component Γ_0 of Γ the set*

$$\Gamma_0^\natural := \{x \in \Gamma_\natural \mid proj_\Gamma(x) \in \Gamma_0\}$$

is a connected component of $\Gamma_\natural$.

3 Definition of the setting and main results

3.1 Setting of the problem

As explained in the introduction, we have two separate different dynamics together with cost functions f_1, ℓ_1, f_2, ℓ_2 respectively defined in the open regions Ω_1 and Ω_2. We assume that the discount factor is the same, say $\lambda > 0$. We provide notations, along with some specifications:

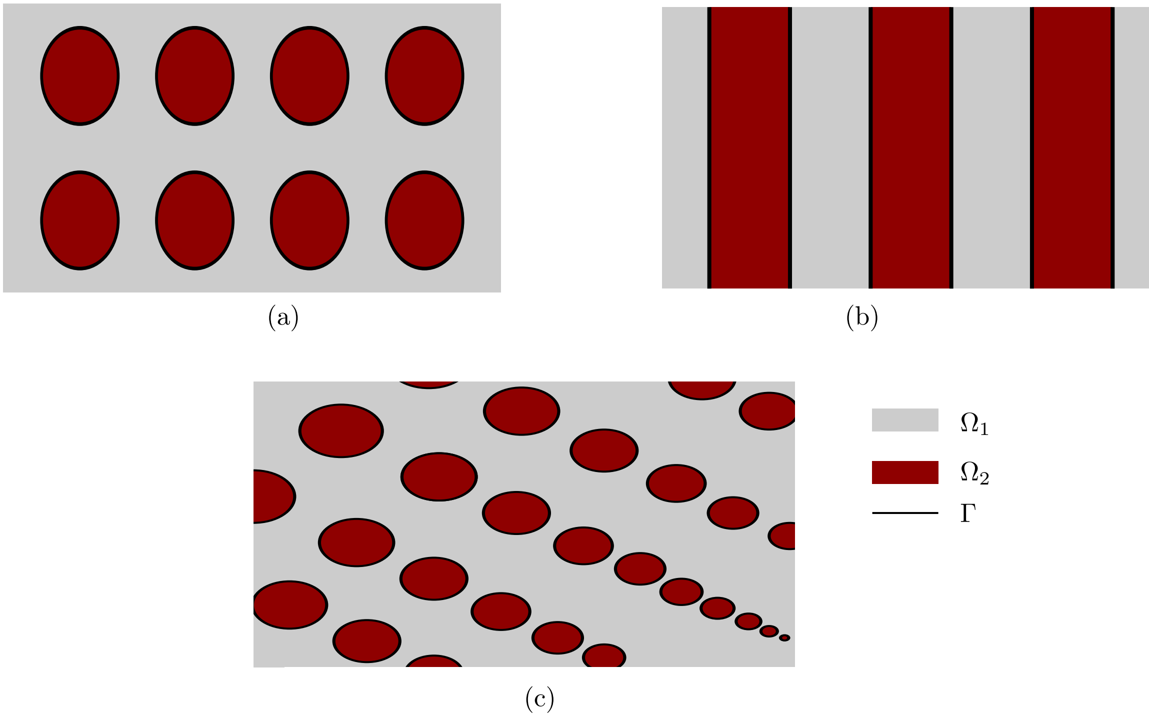

Figure 1: Some examples of partitions that be considered within the framework of this paper.

Interface The interface Γ $(= \partial\Omega_1 = \partial\Omega_2)$ is a C^2 hypersurface of $\mathbb{R}^d$.

Control sets A_i is a compact subset of $\mathbb{R}^m$, for some $m \in \mathbb{N}$.

Controlled dynamics $f_i : \overline{\Omega}_i \times A_i \to \mathbb{R}^d$.

Costs $\ell_i : \overline{\Omega}_i \times A_i \to \mathbb{R}$.

Here the index i takes the values 1, 2 and $\overline{\Omega}_i$ indicates the closure of Ω_i. Note that both f_i and ℓ_i, $i = 1, 2$ are defined up to the interface. The conditions we are assuming on the two systems in Ω_i, $i = 1, 2$, are fairly standard.

(H1) $f_i(x, \text{a})$ is continuous in both arguments and Lipschitz continuous in x, uniformly with respect to a.

(H2) $\ell_i(x, \text{a})$ is continuous in both arguments and bounded.

We will denote by M, L positive constants satisfying

$$|\ell_i(x, \text{a})| \le M \qquad \text{for } i = 1, 2,\ x \in \overline{\Omega}_i,\ \text{a} \in A_i. \tag{3.1}$$

$$|f_i(x, \text{a}) - f_i(z, \text{a})| \le L\,|x - z| \qquad \text{for } i = 1, 2,\ x,\ z \in \overline{\Omega}_i,\ \text{a} \in A_i. \tag{3.2}$$

Following [5], we introduce the control set

$$A := A_1 \times A_2 \times [0, 1]. \tag{3.3}$$

We consider A_1, A_2 subsets of A identifying them with $A_1 \times A_2 \times \{0\}$ and $A_1 \times A_2 \times \{1\}$, respectively. In our model admissible controls depends on the state variable. We set

$$A(x) = \begin{cases} A_i & \text{for } x \in \Omega_i,\ i = 1, 2, \\ A & \text{for } x \in \Gamma. \end{cases}$$

The three components representation (3.3) allows to univocally associate, to any control, cost and dynamics by performing convex combinations. More specifically, we define velocities and costs for the integrated system, when $x \in \mathbb{R}^d$, $(\mathrm{A}_1, \mathrm{A}_2, \mu) \in A(x)$, by

$$f(x, \mathrm{A}_1, \mathrm{A}_2, \mu) = \mu f_1(x, \mathrm{A}_1) + (1-\mu) f_2(x, \mathrm{A}_2), \tag{3.4}$$

$$\ell(x, \mathrm{A}_1, \mathrm{A}_2, \mu) = \mu \ell_1(x, \mathrm{A}_1) + (1-\mu) \ell_2(x, \mathrm{A}_2). \tag{3.5}$$

Note that f and ℓ restricted to $\Omega_i \times A_i$ gives back f_i, ℓ_i.

We proceed introducing the transmission conditions of dynamics and costs on the interface on which our analysis is based. The first is a controllability condition which, loosely speaking, is divided in a tangential and normal part with respect to Γ.

(H3)(i) For $i = 1, 2$, any $x \in \Gamma$, there is A, B in A_i with $Dg(x) \cdot f_i(x, \mathrm{A}) > 0$ and $Dg(x) \cdot f_i(x, \mathrm{B}) < 0$, where g is defined as in (2.6).

(H3)(ii) There exists $R > 0$ such that for any $x \in \Gamma$

$$\big\{f(x, \mathrm{A}) \mid \mathrm{A} \in A\big\} \supset B_R \cap \mathcal{T}_\Gamma(x).$$

Secondly, we require convexity of costs and admissible velocities. It will be specifically used in the proof of Theorem A.1.

(H4) For any $x \in \Gamma$ the set $\big\{(f(x, \mathrm{A}), \ell(x, \mathrm{A})) \mid \mathrm{A} \in A\big\}$ is convex.

Remark 3.1. *Condition* **(H3)(i)** *can be equivalently expressed saying that for any point of the interface there are admissible displacements of the two systems pointing strictly inward and outward Ω_1 and Ω_2.*

Assumption **(H4)** *means: Given $x \in \Gamma$, A, B in A, $\rho \in [0, 1]$ there exist $\mathrm{C} \in A$ with*

$$\rho\,(f(x, \mathrm{A}), \ell(x, \mathrm{A})) + (1-\rho)\,(f(x, \mathrm{B}), \ell(x, \mathrm{B})) = (f(x, \mathrm{C}), \ell(x, \mathrm{C})).$$

Unless differently stated, all above conditions will be in place throughout the paper. Dynamics of the integrated system is given by the multivalued vector field

$$F(x) = \{f(x, \mathrm{A}) \mid \mathrm{A} \in A(x)\} \quad \text{for any } x \in \mathbb{R}^d.$$

Clearly F is Lipschitz–continuous in Ω_1 and Ω_2, but just upper semicontinuous on the whole of $\mathbb{R}^d$, in addition it has linear growth and possess compact, but in general non convex values, therefore existence of integral trajectories for any positive times is not in principle guaranteed. However it can be deduced from transmission conditions **(H3)**, for instance by (ii) any integral curve reaching the interface can be extended on $[0, +\infty)$ in a sliding mode along it.

We apply Filippov Implicit Function Lemma to show that for any trajectory y defined in $[0, +\infty))$ of F, there is a measurable selection $\alpha(t)$ of $t \mapsto A(y(t))$ with

$$\dot{y}(t) = f(y(t), \alpha(t)) \quad \text{for a.e. } t. \tag{3.6}$$

For this, we give a quite general version of it as a factorization result for measurable maps, see [23].

Theorem 3.2. *Let I be an interval of and $\Upsilon : I \to \mathbb{R}^d \times \mathbb{R}^d$ be a measurable function. Let K be a closed subset of $\mathbb{R}^d \times \mathbb{R}^m$ and $\Psi : K \to \mathbb{R}^d \times \mathbb{R}^d$ be continuous. Assume that $\Upsilon(I) \subset \Psi(K)$, then there is a measurable function $\Phi : I \to K$ with*

$$\Psi \circ \Phi(t) = \Upsilon(t) \quad \textit{for a.e. } t \in I.$$

Now we set in the previous statement $I = [0, +\infty)$, $K = \{(x, \mathrm{A}) \mid x \in \mathbb{R}^d, \mathrm{A} \in A(x)\}$, $\Upsilon(t) = (y(t), \dot{y}(t))$, $\Psi(x, \mathrm{A}) = (x, f(x, \mathrm{A}))$. It is apparent that with theses choices assumptions of Theorem 3.2 are in place, in particular the condition $\Upsilon(I) \subset \psi(K)$ is equivalent of y being integral trajectory of F. We get existence of a measurable $\Phi : [0, +\infty) \to K$, $\Phi(t) = (\Phi_1(t), \Phi_2(t))$ with

$$(y(t), \dot{y}(t)) = f(\Phi_1(t), f(\Phi_1(t), \Phi_2(t)).$$

We deduce that $\Phi_1(t) = y(t)$ and $\Phi_2(t) = \alpha(t)$ is the admissible control we were looking for.

The pairs trajectory/control (α, y) related as in (3.6) will be called *admissible*. We are in position to define value function of integrated system, which is the main character of our investigation:

$$v(x) := \inf \left\{ \int_0^{+\infty} e^{-\lambda s} \ell(y(s), \alpha(s))\, ds \mid (\alpha, y) \text{ satisfies (3.6) with } y(0) = x \right\}. \tag{3.7}$$

It is convenient to single out controls corresponding to tangential displacements on Γ putting

$$A_\Gamma(x) = \{a \in A \mid f(x, \mathrm{A}) \in \mathcal{T}_\Gamma(x)\} \quad \text{for any } x \in \Gamma$$

and accordingly

$$F_\Gamma(x) = \{f(x, \mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x)\} = F(x) \cap \mathcal{T}_\Gamma(x) \quad \text{for any } x \in \Gamma.$$

It is a consequence of assumption **(H3)** that $A_\Gamma(x)$ and $F_\Gamma(x)$ are nonempty for any $x \in \Gamma$. We finally define the augmented dynamics:

$$G(x,\xi) = \{(f(x,\mathrm{A}), \lambda\xi - \ell(x,\mathrm{A})) \mid \mathrm{A} \in A(x)\} \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R} \tag{3.8}$$

$$G_\Gamma(x,\xi) = \{(f(x,\mathrm{A}), \lambda\xi - \ell(x,\mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x)\} \quad (x,\xi) \in \Gamma \times \mathbb{R}. \tag{3.9}$$

The multifunction G is upper semicontinuous and possess linear growth, in addition we see from its very definition that the diameter of G is locally bounded in $\mathbb{R}^d \times \mathbb{R}$. This property will be exploited in Theorems 5.5 and 6.1.

In the sequel, we shall need to strengthen the assumptions on costs to establish a Lipschitz–continuity property for the tangential augmented dynamics G_Γ, see Appendix A.

(H5) The cost functions ℓ_i are locally Lipschitz continuous, with respect to the first argument in $\overline{\Omega}_i$, uniformly for a varying on A_i. This implies that ℓ is locally Lipschitz–continuous in Γ, uniformly with respect to $a \in A$.

We will get rid of the extra condition **(H5)** in the proof of main theorems by approximating costs just satisfying **(H2)** with their x–partial sup/inf–convolutions. To define the sup–convolution of ℓ_i we set for $n \in \mathbb{N}$, $x \in \overline{\Omega}_i$, $\mathrm{A} \in A_i$, $i = 1, 2$

$$\ell_i^n(x, \mathrm{A}) = \max \left\{ \ell_i(z, \mathrm{A}) - \frac{n}{2} |z - x|^2 \mid z \in \overline{\Omega}_i \right\}. \tag{3.10}$$

The following properties hold, see [13]:

- $\ell_i^n(\cdot, \mathrm{A})$ is bounded and locally Lipschitz–continuous in $\overline{\Omega}_i$, uniformly with respect to A.
- ℓ_i^n locally uniformly converges to ℓ_i, as n goes to infinity in $\overline{\Omega}_i \times A_i$.

The inf-convolutions are defined as in (3.10) by replacing max and - by min and $+$, respectively. They enjoy the same two properties listed above for sup-convolutions. We denote them by ℓ_n^i.

Next lemma shows the relevance of velocities tangential to Γ for the dynamics given by F. This will reflect on the formulation of Hamilton–Jacobi–Bellman equation adapted to our setting in next subsection.

Lemma 3.3. *Let y be an integral trajectory of F, then $\dot{y}(t) \in F_\Gamma(y(t))$ for a.e. t with $y(t) \in \Gamma$.*

Proof. Set $J = \{t \mid y(t) \in \Gamma\}$, we assume J of positive measure, otherwise there is nothing to show. Let t be a non isolated time in J where y is differentiable. Points of this kind make up a set of full measure in J being the isolated times of J countable. Denote by t_n a sequence of J converging to t. We have

$$F(y(t)) \ni \dot{y}(t) = \lim_n \frac{y(t_n) - y(t)}{t_n - t}$$

and, being $y(t_n)$ in Γ

$$\lim_n \frac{y(t_n) - y(t)}{t_n - t} \in \mathcal{T}_\Gamma(y(t)).$$

This proves the assertion. □

3.2 Hamilton–Jacobi–Bellman equation and statement of main results

We introduce the Hamiltonians appearing in the differential problem we are interested on. For any $(x,p) \in \overline{\Omega}_i \times \mathbb{R}^d$ we set

$$H_i(x,p) = \max\{-p \cdot f_i(x, \mathrm{A}_i) - \ell_i(x, \mathrm{A}_i) \mid \mathrm{A}_i \in A_i\}.$$

Note that for $x \in \Gamma$

$$\max\{H_1(x,p), H_2(x,p)\} = \max\{-p \cdot f(x, \mathrm{A}) - \ell(x, \mathrm{A}) \mid \mathrm{A} \in A\}.$$

For any $(x,p) \in \Gamma \times \mathbb{R}^d$ we set

$$H_\Gamma(x,p) = \max\{-p \cdot f(x, \mathrm{A}) - \ell(x, \mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x)\}.$$

We consider the problem

$$\text{(HJB)} \qquad \begin{cases} \lambda u(x) + H_i(x, Du) = 0 & \text{in } \Omega_i,\ i = 1,2, \\ \lambda u(x) + \max\{H_1(x,Du), H_2(x,Du)\} \geq 0 & \text{at any } x \in \Gamma, \\ \lambda u(x) + H_\Gamma(x, Du) \leq 0 & \text{on } \Gamma. \end{cases}$$

The first equation is in the usual viscosity sense, for the second we require the supersolution inequality to hold at any point of the interface for any viscosity test function in $\mathbb{R}^d$. The third is an equation restricted on the interface, accordingly tests take place at local constrained maximizers with constraint Γ, or test functions can be possibly just defined on Γ.

As a consequence of assumption **(H3)(ii)**, we have

Proposition 3.4. *Any bounded upper semicontinuous subsolution to the third equation of* (HJB) *in* Γ *is locally Lipschitz-continuous on* Γ.

Proof. This is the usual argument which holds for subsolutions of equations with coercive Hamiltonians. Some adaptation is just required since the problem is posed in an hypersurface. By **(H3)(ii)**

$$\lim_{\substack{|p| \to +\infty \\ p \in \mathcal{T}^*_\Gamma(x)}} H_\Gamma(x,p) = +\infty \quad \text{uniformly in } \Gamma.$$

Being our subsolution, say u, bounded we deduce

$$|Du| \leq C \quad \text{on } \Gamma \text{ for a suitable } C \tag{3.11}$$

again, this must be understood in the viscosity sense on Γ, we will consider test functions defined on Γ, with differentials in the cotangent bundle of Γ. Now fix a connected component Γ_0 of Γ, $x_0 \in \Gamma_0$ and $C' > C$. The function

$$u(x) - u(x_0) - C' \, d_\Gamma(x_0, x)$$

attains maximum in Γ_0. If it is strictly positive then corresponding maximizers are different from x_0 and $C' \, d_\Gamma(x_0, \cdot)$ is an admissible test function for (3.11) at any of them, which is impossible because

$$C' \, |D d_\Gamma(x_0, x)| \geq C' > C \quad \text{for all } x \in \Gamma_0.$$

Therefore maximum in object must be zero, and, being x_0 an arbitrary point of Γ_0, we deduce

$$|u(x) - u(z)| \leq C \, d_\Gamma(x,z) \quad \text{for any } x,\ z \text{ in } \Gamma_0,$$

which in turns implies that u is locally Lipschitz–continuous in Γ_0, being d_Γ and the Euclidean distance are locally equivalent in Γ_0. The full assertion, namely local Lipschitz continuity in Γ and not just on connected components, just comes from the fact that any compact subset of $\mathbb{R}^d$ intersects only a finite number of connected components of Γ, see Remark 2.11, and these components are at a positive distance apart. □

Note that in (HJB) equations pertaining to subsolutions are completely separated in the three regions of partition. It is patent that to get comparison results some compatibility condition must be introduced. It actually takes a quite simple form, namely we will consider subsolutions that, beyond being upper semicontinuous outside Γ, and locally Lipschitz when restricted to Γ, according to previous result, are also continuous at any point of the interface.

The main results of this paper are the following:

Theorem 3.5. *Under assumptions* **(H1)**–**(H4)** *the value function is bounded and continuous in* $\mathbb{R}^d$*, locally Lipschitz–continuous on* Γ*. It is, in addition, solution to* (HJB)*.*

Theorem 3.6. *We assume* **(H1)**–**(H4)***. Let* w *and* u *be a bounded lower semicontinuous supersolution and a bounded upper semicontinuous to* (HJB)*, respectively. Assume, in addition, that* u *is continuous at any point of* Γ*. Then* $u \leq w$ *in* $\mathbb{R}^d$*.*

By combining the previous theorems, we finally get:

Theorem 3.7. *Under assumptions* **(H1)**–**(H4)** *the value function is the unique bounded continuous solution of* (HJB) *in* $\mathbb{R}^d$*.*

4 Value function

Here we prove the part of Theorem 3.5 relative to the continuity properties of value function, the remainder will be postponed to Section 7. We point out, as starting remark, that straightforwardly the value function satisfies the *dynamical programming principle*, see [7]. This principle is the combination of two notions of optimality that will be also important in our deduction and will be specifically studied for our model, with some adaptation, in Sections 5, 6. We recall the definitions.

Definition 4.1. (Superoptimality) *We say that a lower semicontinuous function* w *satisfies the superoptimality property if*

$$w(x) \geq \inf \left\{ e^{-\lambda\, t} w(y(t)) + \int_0^t e^{-\lambda\, s} \ell\,(y(s), \alpha(s))\; ds \mid (\alpha, y) \text{ satisfies } (3.6) \text{ with } y(0) = x \right\}.$$

for any $x \in \mathbb{R}^d$*,* $t \in [0, +\infty)$

Definition 4.2. (Suboptimality) *We say that a upper semicontinuous function* u *satisfies the suboptimality property if*

$$u(x) \leq \inf \left\{ e^{-\lambda\, t} u(y(t)) + \int_0^t e^{-\lambda\, s} \ell\,(y(s), \alpha(s))\; ds \mid (\alpha, y) \text{ satisfies } (3.6) \text{ with } y(0) = x \right\}.$$

for any $x \in \mathbb{R}^d$*,* $t \in [0, +\infty)$*.*

We first prove a lemma on the behavior of controlled dynamics around the interface, which is direct consequence of the controllability conditions **(H3)(i)**. It will be used in the remainder of this section, as well as in Section 6 about supersolution properties and superoptimality.

Lemma 4.3. *Given any compact subset of* Γ*, say* Θ*, there exist in correspondence positive constants* r *and* S *such that if* $x \in \Omega_i \cap (\Theta + B(0, r))$*,* $i = 1, 2$*, we can find two trajectories* $\overline{y}$*,* $\underline{y}$ *of* F *and* $\overline{T}$*,*$\underline{T}$ *less than* $S\,|g(x)|$ *with*

$$\overline{y}(0) = x, \quad \overline{y}(\overline{T}) \in \Gamma, \quad \overline{y}([0, \overline{T})) \subset \Omega_i \tag{4.1}$$

$$\underline{y}(0) \in \Gamma, \quad \underline{y}(\underline{T}) = x, \quad \underline{y}((0, \underline{T}]) \subset \Omega_i. \tag{4.2}$$

A remark is preliminary to the proof.

Remark 4.4. *Controlled vector fields* f_i *can be extended to* $(\Omega_i \cup \Gamma_\natural) \times A_i$ *by setting*

$$f_i(x, a) = f_i(proj_\Gamma(x), a)$$

The extended f_i *are continuous in both arguments and locally Lipschitz–continuous when first variable varies in* $\Gamma_\natural$*. Accordingly, the related multivalued maps* $x \mapsto f_i(x, A_i)$ *are locally Lipschitz–continuous in* $\Gamma_\natural$*. We are going to use these properties in the forthcoming proof of the lemma.*

Proof. (*of Lemma 4.3*) We prove that the assertion for $i = 1$. The functions

$$\begin{array}{rcl} x & \mapsto & \min\{Dg(x) \cdot f_1(x, \text{A}) \mid \text{A} \in A_1\} \\ x & \mapsto & \max\{Dg(x) \cdot f_1(x, \text{A}) \mid \text{A} \in A_1\} \end{array}$$

are continuous in $\Gamma_\natural$ and, in force of assumption **(H3)(i)**, the first is moreover strictly negative and the latter strictly positive; they consequently keep same sign in $\Theta + B(0, \rho) \subset \Gamma_\natural$ for a suitable $\rho > 0$. We deduce that for an appropriate choice of $C > 0$ the set–valued functions

$$\begin{array}{rcl} \underline{F}(x) & = & \{f_1(x, \text{A}) \mid Dg(x) \cdot f_i(x, \text{A}) \leq -C\} \\ \overline{F}(x) & = & \{f_1(x, \text{A}) \mid Dg(x) \cdot f_i(x, \text{A}) \geq C\} \end{array}$$

both takes nonempty, which is the important fact, compact values in $\Theta + B(0, \rho)$. They are, in addition, upper semicontinuous. However, since in general they do not possess better continuity properties and are not convex–valued, we are not guaranteed of existence of solutions to the corresponding differential inclusions. For this reason we pass to relaxed problems and apply later Relaxation Theorem. The differential inclusions

$$\dot{y} \in \overline{\text{co}}\, \underline{\text{F}}(\text{y}) \tag{4.3}$$

$$\dot{y} \in -\overline{\text{co}}\, \overline{\text{F}}(\text{y}) \tag{4.4}$$

posed in $\Theta + B(0, \rho)$, admit in fact solutions for any initial point, being the right hand–side multifunctions upper semicontinuous with convex compact nonempty values. Further, if y is one of these solutions and $[0, T)$ its maximal interval of definition, with $T < +\infty$, then

$$\lim_{t \to T} y(t) \in \partial(\Theta + B(0, \rho)). \tag{4.5}$$

We set $S = \frac{2}{C}$ and $r > 0$ with

$$r < \min\left\{\frac{\rho\, C}{4\, M_0}, \frac{\rho}{3}\right\}, \tag{4.6}$$

where M_0 is a constant estimating from above the norm of any element of $f_1(x, A_1)$, for x varying in $\Theta + B(0, \rho)$.

Given $x \in (\Theta + B(0, r)) \cap \Omega_1$, let y be an integral curve of (4.3) starting at x, we denote by $[0, T)$ its maximal interval of definition. If $T \leq S\, g(x)$ then, taking into account (4.6) and that $g(x) \leq r$, we have

$$d(y, \Theta) \leq |y(t) - x| + r < t\, M_0 + \frac{\rho}{3} < \frac{\rho}{2} + \frac{\rho}{3}$$

for any $t \in [0, T)$, which is in contrast with (4.5). Consequently $T > S\, g(x)$ must hold, then we have

$$g(y(S\, g(x))) = g(x) + \int_0^{S\, g(x)} Dg(y) \cdot \dot{y}\, ds \leq g(x) - C\, S\, g(x) < 0,$$

so that $y(S\, g(x)) \in \Omega_2$. Curve y is also a trajectory of the relaxed dynamics $\overline{\text{co}}\, \text{f}_1(\text{x}, \text{A}_1)$, and, being $f_1(x, A_1)$ Lipschitz–continuous in $\Theta + B(0, \rho)$, see Remark 4.4, it can uniformly approximated in $[0, S\, g(x)]$ by integral curves of $f_1(x, A_1)$ with same initial point, thanks to Relaxation Theorem [**?**, 2]. There thus exists one of such trajectory, say $\overline{y}$, satisfying $\overline{y}(0) = x$, $\overline{y}(S\, g(x)) \in \Omega_2$, so that the first exit time of it from Ω_1, say $\overline{T}$, is less than $S\, g(x)$. The curve $\overline{y}$ in $[0, \overline{T}]$ satisfies (4.1). Same argument, with slight adaptations, shows the existence of an integral curve $\underline{y}$ of $-F_1$ and $\underline{T} < S\, g(x)$ with

$$\underline{y}(0) = x, \quad \underline{y}(\underline{T}) \in \Gamma, \quad \underline{y}([0, \underline{T})) \subset \Omega_1.$$

We then prove (4.2) by considering

$$t \mapsto \underline{y}(\underline{T} - t) \quad \text{in } [0, \underline{T}].$$

The proof for $i = 2$ is the same, up to obvious adjustements. $\square$

The main result of the section is:

Theorem 4.5. *Under assumptions* **(H1)**–**(H4)** *the value function v is bounded and continuous in $\mathbb{R}^d$. It is moreover locally Lipschitz continuous on Γ.*

Proof. We divide the proof in several steps:

(1) Local Lipschitz–continuity on Γ

This property is easily obtained using suboptimality of v plus assumption **(H3)(ii)** and local equivalence of Riemannian and Euclidean distance in any connected component of the interface.

(2) Continuity at any point of Γ

Taking into account that v, restricted on the interface, is continuous, according to previous step and Remark 2.11, it is enough to show

$$v(x_n) \to v(x_0) \quad \text{for any } x_0 \in \Gamma,\, x_n \to x_0,\, x_n \in \Omega_i \text{ for any } n,\, i = 1 \text{ or } 2.$$

By applying Lemma 4.3 with $\Theta = \{x_0\}$, we see that for a suitable $S > 0$ and n large enough there exist positive sequences T_n, $\hat{T}_n$ satisfying $T_n \leq S|g(x_n)|$, $\hat{T}_n \leq S|g(x_n)|$ for any n, and admissible trajectories y_n, $\hat{y}_n$, defined in $[0, T_n]$, $[0, \hat{T}_n]$, respectively, with $y_n([0, T_n)) \subset \Omega_i$, $\hat{y}_n([0, \hat{T}_n)) \subset \Omega_i$, corresponding to controls α_n, $\hat{\alpha}_n$ respectively, such that

$$y_n(0) = x_n, \quad y_n(T_n) =: z_n \in \Gamma$$
$$y_n(0) =: \hat{z}_n \in \Gamma, \quad \hat{y}_n(\hat{T}_n) = x_n.$$

Since all supports of such curves is contained in some compact set, their velocities are equibounded, so that

$$z_n \to x_0 \quad \text{and} \quad \hat{z}_n \to x_0 \quad \text{as } n \to +\infty. \tag{4.7}$$

By suboptimality and boundedness condition on ℓ_i we have

$$v(x_n) \;\leq\; \int_0^{T_n} e^{-\lambda s}\, \ell_i(y_n, \alpha_n)\, ds + e^{-\lambda T_n}\, v(z_n) \leq M\, S\, |g(x_n)| + v(z_n) \tag{4.8}$$

$$v(\hat{z}_n) \;\leq\; \int_0^{\hat{T}_n} e^{-\lambda s}\, \ell_i(\hat{y}_n, \hat{\alpha}_n)\, ds + e^{-\lambda \hat{T}_n}\, v(x_n) \leq M\, S\, |g(x_n)| + v(x_n) \tag{4.9}$$

where M is defined as in (3.1). Putting together (4.7), (4.8), (4.9), we derive

$$\limsup v(x_n) \leq \lim v(z_n) = v(x_0)$$
$$\liminf v(x_n) \geq \lim v(\hat{z}_n) = v(x_0),$$

which shows the assertion.

(3) Final part: continuity of v in $\mathbb{R}^d$.

We consider a bounded subset B of Ω_i. We will prove that, given any $\varepsilon > 0$, a $\delta > 0$ can be determined with

$$v(x_1) - v(x_0) < 4\,\varepsilon \quad \text{for any pair of elements } x_0,\, x_1 \text{ of } B \text{ with } |x_0 - x_1| < \delta. \tag{4.10}$$

This fact, combined with previous steps, will fully give the assertion. We then fix ε and in correspondence some entities we need in the proof. We select $T_\varepsilon > 1$ such that

$$\int_{T_\varepsilon}^{+\infty} e^{-\lambda s}\, |\ell(y, \alpha)|\, ds < \varepsilon. \tag{4.11}$$

for any admissible pair (α, y). We denote by K a compact set containing the support of any integral curve of F, starting at B, and defined in $[0, T_\varepsilon]$, and by $\nu(\cdot)$ an uniform continuity modulus for both ℓ_i in $K \times A_i$ and v in $\Gamma \cap K$. We assume, to simplify notation, that M, besides bounding cost, also bounds the velocities in $F(x)$, when x varies in K. Finally, we denote by r, S the constants provided by Lemma 4.3 with $\Gamma \cap K$ in place of Θ. We take δ with

$$\delta\, e^{L\, T_\varepsilon} \;\leq\; \min\left\{r, \frac{\varepsilon}{M\, S}\right\} \tag{4.12}$$

$$\nu\left(\delta\,(1 + M\, S)\, e^{L\, T_\varepsilon}\right) \;\leq\; \frac{\varepsilon}{T_\varepsilon} < \varepsilon. \tag{4.13}$$

Let x_0, x_1 be a pair of elements of B with $|x_0 - x_1| < \delta$. Let α_0 be an ε–optimal control for $v(x_0)$ and y_0 the corresponding trajectory starting at x_0. We denote by T_0 its first exit time from Ω_i. We consider the problem

$$\begin{cases} \dot y_1 = & f_i(y_1, \alpha_0) \\ y_1(0) = & x_1 \end{cases}$$

in $\Omega_i \times (0, T_0)$. Let $[0, T_1)$ be the maximal interval of definition of the solution. If $T_1 < T_0$ then such solution can be extended in $[0, T_1]$ and $y_1(T_1) \in \Gamma$. We set $T = \min\{T_0, T_1\}$. We clearly have

$$|y_1(t) - y_0(t)| \leq \delta\, e^{L\,t}\, |x_1 - x_0| \quad \text{for any } t \in [0, T].$$

If $T \geq T_\varepsilon$ then the interface does not enter in the deduction of the estimate (4.10), which goes as in the usual case.

If instead $T < T_\varepsilon$, we have $|y_1(T) - y_0(T)| < r$ by (4.12), and at least one between $y_1(T)$ and $y_0(T)$ belongs to Γ, say $y_0(T) \in \Gamma$ to fix our ideas. By Lemma 4.3, there is an integral curve of the controlled dynamics f_i joining $y_1(T)$ to a point $z \in \Gamma$ in a time less or equal $S\, g(y_1(T))$. We deduce

$$\begin{aligned} v(y_1(T)) &\leq S\, g(y_1(T))\, M + v(z) \leq S\, M\, \delta\, e^{L\,T} + v(z) \\ |y_0(T) - z| &\leq |y_0(T) - y_1(T)| + |y_1(T) - z| \leq \delta\, e^{L\,T} + S\, M\, \delta\, e^{L\,T}. \end{aligned}$$

We deduce from this estimate and (4.12), (4.13)

$$\begin{aligned} v(y_1(T)) &\leq \varepsilon + v(z) \\ |v(y_0(T)) - v(z)| &\leq \varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} v(x_1) - v(x_0) &\leq \int_0^T |\ell_1(y_1(s)) - \ell_1(y_0(s))|\, ds + e^{-L\,T} \big(v(y_1(T)) - v(y_0(T))\big) + \varepsilon \\ &\leq \varepsilon + \big(v(y_1(T)) - v(z)\big) + |v(y_0(T)) - v(z)| + \varepsilon \leq 4\,\varepsilon \end{aligned}$$

as desired. If instead $y_1(T) \in \Gamma$ then we apply Lemma 4.3 considering an admissible trajectory from some point of Γ to $y_0(T)$ to get the same conclusion. □

5 Subsolutions and suboptimality

In this section we aim at showing:

Theorem 5.1. *We assume* **(H1)**–**(H5)***. A bounded upper semicontinuous function u, which is, in addition, continuous at any point of Γ, is subsolution to* (HJB) *if and only if $\mathcal{H}p(u)$ is strongly invariant for G, or, equivalently, u satisfies the suboptimality property.*

In Section 6.1 we will get rid of **(H5)** and prove the same statement only assuming **(H1)**–**(H4)**. We start by the implication:

Theorem 5.2. *We assume* **(H1)**–**(H5)***. If u is a bounded upper semicontinuous function satisfying the suboptimality property then it is a subsolution to* (HJB)*.*

Proof. Outside the interface there is nothing new, so we focus on $x_0 \in \Gamma$ where u admits a C^1 viscosity test function from above, say ψ, with x_0 local constrained maximizer of $u - \psi$ on Γ, we also assume $\psi(x_0) = u(x_0)$. We aim at proving

$$\lambda u(x_0) + \max\{-D\psi(x_0) \cdot f(x_0, \mathrm{A}) - \ell(x_0, \mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x_0)\} \leq 0. \tag{5.1}$$

By Theorem A.1 and Corollary A.2, recall that we are assuming **(H5)**, the multifunction G_Γ, suitably extended outside the interface, is locally Lipschitz–continuous in $\Gamma_\natural$. Therefore, given $\mathrm{A}_0 \in A_\Gamma(x_0)$, we can apply Corollary 2.10 to find a C^1 integral curve of G_Γ, say (y, ζ), in $[0, T]$, for some $T > 0$, with $(y(0), \zeta(0)) = (x_0, u(x_0))$, $(\dot y(0), \dot\zeta(0)) = (f(x_0, \mathrm{A}_0), \lambda\, u(x_0) - \ell(x_0, \mathrm{A}_0))$. Clearly $y(t) \in \Gamma$ for any t and there is an admissible control α such that for all $t \in [0, T]$

$$\begin{aligned} \dot y(t) &= f(y(t), \alpha(t))) \\ \dot\zeta(t) &= \lambda\, \zeta(t) - \ell(y(t), \alpha(t))) \end{aligned}$$

in addition $t \mapsto \ell(y(t), \alpha(t)))$ is continuous and its limit, as $t \to 0$, is $\ell(x_0, \mathrm{A}_0)$. Because of the suboptimality of u, $\psi(x_0) = u(x_0)$ and $y(t) \in \Gamma$ for any t, we have for $t \in [0, T]$

$$u(x_0) \leq e^{-\lambda\, t}\, \psi(y(t)) + \int_0^t e^{-\lambda\, s}\, \ell(y, \alpha)\, ds$$

and consequently

$$\frac{\psi(x_0) - e^{-\lambda\, t}\, \psi(y(t))}{t} \leq \frac{1}{t} \int_0^t e^{-\lambda\, s}\, \ell(y, \alpha)\, ds.$$

This implies, passing at the limit for $t \to 0$ and exploiting the aforementioned continuity properties of cost in t

$$\lambda\, u(x_0) - D\psi(x_0) \cdot f(x_0, \mathrm{A}_0) \leq \ell(x_0, \mathrm{A}_0).$$

This concludes the proof because A_0 has been selected arbitrarily in $A_\Gamma(x_0)$. □

For the converse implication some preliminary material is needed. We derive a first invariance result for the hypograph of u on Γ through Theorem 2.8 and the local Lipschitz–continuous character of G_Γ.

Proposition 5.3. *Let u be an upper semicontinuous subsolution to* (HJB)*, then $\mathcal{H}p(u) \cap \big(\Gamma \times \mathbb{R}\big)$ is strongly invariant for G_Γ.*

Proof. In view of Theorem 2.8 and Corollary A.2, we have just to check that G_Γ satisfies the strong tangential condition on $\mathcal{H}p(u) \cap \big(\Gamma \times \mathbb{R}\big)$. Being the interior of such set empty, this condition must be satisfied at any of its points. If $(x_0, \xi_0) \in (\mathrm{int}\ \mathcal{H}p(u)) \cap \big(\Gamma \times \mathbb{R}\big)$ then any nonzero normal vector at it has the form $(p, 0)$ with p normal to Γ at x_0, then the strong tangential condition comes from the fact that $F_\Gamma(x_0) \subset \mathcal{T}_\Gamma(x_0))$.

If instead $(x_0, \xi_0) \in (\partial \mathcal{H}p(u)) \cap \big(\Gamma \times \mathbb{R}\big)$ then $\xi_0 = u(x_0)$ since u is continuous in Γ. We consider $(p, s) \perp \mathcal{H}p(u) \cap \big(\Gamma \times \mathbb{R}\big)$ at $(x_0, u(x_0))$ and pick $\varepsilon > 0$ such that

$$(x_0 + \varepsilon\, p, u(x_0) + \varepsilon\, s) \text{ has } (x_0, u(x_0)) \text{ as unique projection on } \mathcal{H}p(u) \cap \big(\Gamma \times \mathbb{R}\big). \tag{5.2}$$

We divide the argument according on whether s is vanishing or strictly positive. In the first instance, we reach the sought conclusion arguing as in the first step provided that p is normal to Γ at x_0. We show by contradiction that $s = 0$ and p not normal is impossible because of the Lipschitz–continuity of u on Γ. We take $q \in \mathcal{T}_\Gamma(x_0)$ with

$$c := p \cdot q > 0,$$

and consider a regular curve y defined in some small interval $[0, T]$ and lying on Γ with

$$y(0) = x_0 \qquad \text{and} \qquad \dot{y}(0) = q,$$

we moreover denote by L_u a Lipschitz constant for u in a bounded subset of Γ containing the support of y. We have for t small enough

$$\begin{aligned} |y(t) - (x_0 + \varepsilon\, p)|^2 + |u(y(t) - u(x_0)|^2 \quad &\leq \quad (1 + L_u^2)\, |y(t) - x_0|^2 - 2\, \varepsilon\, (y(t) - x_0) \cdot p + \varepsilon^2\, |p|^2 \\ &\leq \quad o(t) - c\, \varepsilon\, t + \varepsilon^2\, |p|^2, \end{aligned}$$

in contrast with (5.2), recall that $s = 0$. The case $s > 0$ is left, we can assume $s = 1$. The ball of $\mathbb{R}^d \times \mathbb{R}$ centered at $(x_0 + \varepsilon\, p, u(x_0) + \varepsilon)$ and with radius $\varepsilon\, \sqrt{|p|^2 + 1}$ is locally at $(x_0, u(x_0))$ the graph of a smooth function, say ψ, with $-D\psi(x_0) = p$, which is viscosity test function from above to u at x_0 with Γ as constraint. This implies, being u subsolution to (HJB)

$$(p, 1) \cdot (f(x, a), \lambda w(x_0) - \ell(x_0, a)) = \lambda w(x_0) - D\psi(x_0) \cdot f(x, a) - \ell(x_0, a)) \leq 0$$

for any $a \in A_\Gamma(x_0)$, concluding the proof. □

Next result is about an invariance property for G outside Γ. For this we essentially exploit the continuity condition of u on the interface. This is actually the unique point where such a condition enters into play. Assumption **(H5)** is not used here.

Proposition 5.4. *Let u, (y,η) be an upper semicontinuous subsolution to* (HJB)*, which is, in addition, continuous at any point of Γ, and an integral curve of G defined in an interval $[a,b]$, respectively. Assume that $(y(a),\eta(a)) \in \mathcal{H}p(u)$, and*

$$y(t) \notin \Gamma \quad \textit{for } t \in (a,b).$$

Then

$$(y(t),\eta(t)) \in \mathcal{H}p(u) \quad \textit{for } t \in [a,b].$$

Proof. Given $\rho > 0$, we consider a Lipschitz–continuous cutoff function $\phi_\rho : [0,+\infty) \to [0,1]$ with

$$\phi_\rho(0) = 0 \quad \text{for } s \in [0, \tfrac{\rho}{2}], \quad \phi_\rho(s) = 1 \quad \text{for } s \geq \rho$$

and define

$$G_\rho(x,\xi) = \phi_\rho(|g(x)|)\, G(x,\xi) \quad \text{for any } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}.$$

The multifunction G_ρ is locally Lipschitz–continuous in the whole $\mathbb{R}^d \times \mathbb{R}$ and reduces to $\{0\}$ in a suitable neighborhood of $\Gamma \times \mathbb{R}$.

We claim that $\mathcal{H}p(u)$ is strongly invariant for G_ρ. It is enough, in view of Theorem 2.8, to check strong tangential condition for $\mathcal{H}p(u)$ with respect to G_ρ, and, in addition to check it for $(x_0,\xi_0) \in \partial\mathcal{H}p(u)$ with x_0 outside Γ or even far enough from it, where the images of G are different from $\{0\}$. We then consider $(x_0,\xi_0) \in \partial\mathcal{H}p(u)$ and $x_0 \in \Omega_i$, $i = 1$ or 2, with $(p,s) \perp \mathcal{H}p(u)$ at it. The argument is well known, we sketch it for reader's convenience. If $\xi_0 = u(x_0)$ and $s > 0$, so that we can assume $s = 1$, then we find a smooth viscosity test function from above ψ to u at x_0 with $D\psi(x_0) = -p$. Given $(f_1(x_0,a), \lambda\, u(x_0) - \ell_1(x_0,a))$, we exploit that u is subsolution of (HJB) to get

$$(p,1) \cdot \phi_\rho(g(x_0)\,(f(x,a), \lambda w(x_0) - \ell(x_0,a)) = \phi_\rho(g(x_0)\big(\lambda w(x_0) - D\psi(x_0) \cdot f_1(x,a) - \ell_1(x_0,a))\big) \leq 0.$$

In the case where $s = 0$ and /or $\xi_0 > u(x_0)$, we basically use Proposition 2.2 to get similar estimate. The claim is in the end proved.

Now consider a curve y as in the statement, with $y((a,b)) \subset \Omega_i$. If $y(a) \notin \Gamma$ then

$$(y,\eta)([a,b-\varepsilon]) \cap \Gamma = \emptyset \quad \text{for any } \varepsilon > 0$$

then

$$\min\{g(y(t)) \mid t \in [a,b-\varepsilon]\} = \rho,$$

for some $\rho = \rho(\varepsilon) > 0$, so that (y,η) is a trajectory of G_ρ and by the first part of the proof,

$$(y(t),\eta(t)) \in \mathcal{H}p(u) \quad \text{for } t \in [a,b-\varepsilon]\ .$$

Taking into account that $\mathcal{H}p(u)$ is closed, we get the assertion sending ε to 0. If, on the contrary, $y(a) \in \Gamma$, we exploit that u is continuous at $y(a)$ and $(y(a),\eta(a)) \in \mathcal{H}p(u)$ to find for any $\varepsilon > 0$ small a $\delta_\varepsilon > 0$ satisfying

$$(y(a+\varepsilon), \eta(a+\varepsilon) - \delta_\varepsilon) \in \mathcal{H}p(u), \tag{5.3}$$

and $\delta_\varepsilon \to 0$ as ε goes to 0. Being the support of (y,η), for $t \in [a+\varepsilon, b-\varepsilon]$, compact and disjoint from Γ, we can argue as above to deduce from (5.3)

$$(y(t), \eta(t) - \delta_\varepsilon\, e^{\lambda\,(t-a-\varepsilon)}) \in \mathcal{H}p(u) \quad \text{for } t \in [a+\varepsilon, b-\varepsilon]\ .$$

We thus get the assertion passing at the limit for $\varepsilon \to 0$. □

In the forthcoming proof it is couched the crucial induction argument on the index $\jmath_\varepsilon$, see the notion of ε–partition introduced in Appendix B. It will be also employed, with suitable adaptations, to demonstrate the main results about superoptimality in next section.

Theorem 5.5. *We assume* **(H1)**–**(H5)***. Let u be a bounded upper semicontinuous subsolution to* (HJB)*, which is, in addition, continuous at any point of Γ, then $\mathcal{H}p(u)$ is strongly invariant for G.*

Proof. We consider a trajectory (y, η) of G with $(y(0), \eta(0)) \in \mathcal{H}p(u)$ in the interval $[0, T]$, for $T > 0$. We can assume that $y([0, T]) \cap \Gamma \neq \emptyset$, otherwise the curve should be contained in $\mathcal{H}p(u)$ in force of Proposition 5.4. We select a compact set $K_0 \subset \mathbb{R}^d$ containing in its interior the reachable set

$$\mathcal{R}_{F_\Gamma}(y([0, T]) \cap \Gamma, T).$$

(See (2.1) for the definition of the reachable set.) We introduce some constants that will appear in the forthcoming estimates.

- M_u, L_u is an upper bound for $|u|$ in $\mathbb{R}^d$ and a Lipschitz constant for u in $(K_0 \cap \Gamma) \times \mathbb{R}$, respectively, see Theorem 3.5.
- P estimates from above the diameter of $G(x, \xi)$ for $(x, \xi) \in \mathcal{R}_G(y[0, T] \times (\eta([0, T]) \cup [-M_u, M_u])), T)$.
- L_G is a Lipschitz constant for G_Γ (suitably extended outside the interface, see (A.11)) in $(K_0 \cap \Gamma_\natural) \times \mathbb{R}$.

We break down the argument into slices depending on a positive integer index and prove the result by induction. We prove an inequality depending a parameter ε devoted to be infinitesimal for any compact portion of (y, η), assuming a condition on the cardinality of ε–minimal partitions. We consider ε so small that any integral curve of F defined in some compact interval and with support contained in K_0 and any ε–partition related to it satisfy the weak separation principle stated in Proposition B.3. This is the precise statement of the sequence, depending upon $k \in \mathbb{N}$, of properties we are going to demonstrate by induction:

$(\mathbf{P_k})$ For any interval $[a, b] \subset [0, T]$ such that $\jmath_\varepsilon(y; a, b) \leq k$, one has

$$\eta(b) - \exp(\lambda\,(b - a))\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp((L_G + \lambda)\,(b - a))\,P\,\varepsilon \leq u(y(b)),$$

where $[\cdot]^+$ stands for the positive part.

We first show $(\mathbf{P_2})$. We fix $[a, b] \subset [0, T]$ with $\jmath_\varepsilon(y; a, b) = 2$, and modify the component $\eta(t)$ of our curve in $[a, b]$ setting

$$\zeta(t)) := \eta(t) - [\eta(a) - u(y(a))]^+\, e^{\lambda\,(t-a)}. \tag{5.4}$$

The curve (y, ζ) is still a trajectory of G in $[a, b]$, but now the initial datum at $t = a$ satisfies

$$\zeta(a) = \eta(a) - [\eta(a) - u(y(a))]^+ \in \mathcal{H}p(u). \tag{5.5}$$

Since $\zeta(a)$ is either equal to $\eta(a)$ or to $u(y(a))$, then

$$y([a, b]) \times \zeta([a, b]) \subset \mathcal{R}_G(y[0, T] \times (\eta([0, T]) \cup [-M_u, M_u]), T). \tag{5.6}$$

We divide the proof according on whether $y((a, b)) \cap \Gamma$ is empty or not. In the first instance by Proposition 5.4, and (5.5) the modified curve is contained in $\mathcal{H}p(u)$, and so

$$\zeta(b) = \eta(b) - e^{\lambda\,(b-a)}\,[\eta(a) - u(y(a))]^+ \leq u(y(b))$$

which implies the claimed inequality. In the second case $y(a)$ and $y(b)$ belong to the interface and

$$|\{t \in [a, b] \mid y(t) \notin \Gamma\}| < \varepsilon, \tag{5.7}$$

in addition by Lemma 3.3

$$(\dot{y}(t), \dot{\zeta}(t)) \in G_\Gamma(y(t), \zeta(t)) \qquad \text{for a.e. } t \in (a, b) \setminus J, \tag{5.8}$$

where J the time set appearing in (5.7). On the other side, bearing in mind (5.6) and that (y, ζ) is an integral curve of G, we deduce from the very definition of P

$$d((\dot{y}(t), \dot{\zeta}(t)), G_\Gamma(y(t), \zeta(t))) < P \qquad \text{for a.e. } t \in J. \tag{5.9}$$

Combining (5.7), (5.8), (5.9), we finally obtain

$$\int_a^b d((\dot{y}(s), \dot{\zeta}(s)), G_\Gamma((y(s)), \zeta(s)))\,ds \leq \varepsilon\,P. \tag{5.10}$$

By the assumption on ε, $y([a,b])$ is contained in $\Gamma_\natural$. We can then apply Theorem 2.9 with $\mathcal{C} = \Gamma \times \mathbb{R}$, $\mathcal{C}_\natural = \Gamma_\natural \times \mathbb{R}$, and the multifunction $Z = G_\Gamma$, taking into account that L_G is a Lipschitz constant of G_Γ in $(K_0 \cap \Gamma_\natural) \times \mathbb{R}$, and this set clearly contains a bounded open neighborhood of $\mathcal{R}_{G_\Gamma}((y(a), \zeta_0(a)), b-a))$, as prescribed in that theorem. We get the existence of an integral curve (z_0, ζ_0) of G_Γ, defined in $[a,b]$, with $(z_0(a), \zeta_0(a)) = (y(a), \zeta(a))$, satisfying by (5.10)

$$
\begin{aligned}
&(5.11) \qquad & |z_0(b) - y(b)| &\leq \exp\big(L_G\,(b-a)\big)\,\varepsilon\,P \\
&(5.12) \qquad & |\zeta_0(b) - \zeta(b)| &\leq \exp\big(L_G\,(b-a)\big)\,\varepsilon\,P.
\end{aligned}
$$

Since $(z_0(a), \zeta_0(a)) \in \Gamma \times \mathbb{R}$ then by Proposition 5.3 and (5.5)

$$
\zeta_0(b) \leq u(z_0(b)). \tag{5.13}
$$

By Lipschitz–continuity of subsolution u, we derive from (5.11)

$$
u(y(b)) + L_u\,\exp\big(L_G\,(b-a)\big)\,\varepsilon\,P \geq u(z_0(b)), \tag{5.14}
$$

and taking also into account (5.12), (5.13), we get

$$
\zeta(b) - \exp\big(L_G\,(b-a)\big)\,\varepsilon\,P \leq u(y(b)) + L_u\,\exp\big(L_G\,(b-a)\big)\,\varepsilon\,P.
$$

Recalling the definition of $\zeta(t)$ given in (5.4) we further obtain

$$
\eta(b) - e^{\lambda\,(b-a)}\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp\big(L_G\,(b-a)\big)\,\varepsilon\,P \leq u(y(b)),
$$

and, replacing L_G in the second exponential by $L_G + \lambda$, which is larger, we reach the sought inequality, ending the proof of $(\mathbf{P_2})$.

Given $k \in \mathbb{N}$, $k \geq 2$, we now assume $(\mathbf{P_2}), \cdots, (\mathbf{P_k})$ to hold and prove $(\mathbf{P_{k+1}})$. Taking $\jmath(y;a,b) = k+1$, we denote by

$$
\{t_1 = a, \cdots, t_{k+1} = b\}
$$

a minimal ε–partition of $[a,b]$ related to y, by Proposition B.5 there are two positive constant ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ satisfying

$$
\jmath_{\varepsilon_1}(y; a, t_k) = k \quad \text{and} \quad \jmath_{\varepsilon_2}(y; t_k, b) = 2.
$$

By inductive step

$$
\begin{aligned}
u(y(t_k)) &\geq \eta(t_k) - \exp(\lambda\,(t_k - a))\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp((L_G + \lambda)\,(t_k - a))\,P\,\varepsilon_1 \\
u(y(b)) &\geq \eta(b) - \exp(\lambda\,(b - t_k))\,[\eta(t_k) - u(y(t_k))]^+ - (1 + L_u)\,\exp((L_G + \lambda)\,(b - t_k))\,P\,\varepsilon_2
\end{aligned}
$$

Replacing in the second inequality of above the estimate of $[\eta(t_k) - u(y(t_k))]^+$ provided in the first one, we get

$$
\begin{aligned}
u(y(b)) &\geq \eta(b) - \exp(\lambda\,(b - t_k))\,\big(\exp(\lambda\,(t_k - a)\,[\eta(a) - u(y(a))]^+ + (1 + L_u)\,\exp((L_G + \lambda)\,(t_k - a))\,P\,\varepsilon_1\big) \\
&- \quad (1 + L_u)\,\exp((L_G + \lambda)\,(b - t_k))\,P\,\varepsilon_2 \\
&\geq \eta(b) - \exp(\lambda\,(b - a))\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp((L_G + \lambda)\,(b - a)\,P\,(\varepsilon_1 + \varepsilon_2).
\end{aligned}
$$

This finish the proof by induction. We apply the property so far established to (y, η) in the whole of $[0,T]$. Taking into account that $[\eta(0) - u(y(0)]^+ = 0$ by assumption, that ε can be arbitrarily small and the error in $(\mathbf{P_k})$ goes to 0 as $\varepsilon \to 0$, we deduce $(y(T), \eta(T)) \in \mathcal{H}p(u)$. This completes the argument, being T arbitrary. $\square$

By putting together Theorems 5.2 and 5.5 we get Theorem 5.1 and end the section.

6 Supersolutions and superoptimality

Here we aim at showing an approximate superoptimality principle for bounded supersolution of (HJB). In this section we assume **(H5)**, in Section 7 we provide a generalization of the result only exploiting **(H1)**–**(H4)**.

Theorem 6.1. *We assume* **(H1)**– **(H5)**. *Let w be a bounded lower semicontinuous supersolution of* (HJB) *and $M_w > 0$ with $|w| < M_w$ in $\mathbb{R}^d$. Given $x_0 \in \mathbb{R}^d$ and positive constants T_0 and δ, there exists (y, α) admissible with $y(0) = x_0$ such that*

$$w(x_0) \geq \int_0^T e^{-\lambda s}\, \ell(y, \alpha)\, ds - e^{-\lambda T}\, M_w - \delta \quad \textit{for some } T \in (T_0, 4\, T_0 + 1).$$

We will use the following invariance property for epigraphs of supersolutions, which can be straightforwardly obtained as in the usual non partitioned case:

Proposition 6.2. *Let w be a lower semicontinuous supersolution to* (HJB)*, then $\mathcal{E}p(w)$ is weakly invariant for $\overline{\mathrm{co}}\, \mathrm{G}$.*

The difficulty in deducing Theorem 6.1 from Proposition 6.2 in presence of an interface is that, as usual, we do not have Lipschitz–continuity of the multivalued vector field on the whole $\mathbb{R}^d \times \mathbb{R}$, and this prevents us from directly applying Relaxation Theorem to approximate curves of the relaxed dynamics, see [2]. We break the arguments in two parts and use Relaxation Theorem for the portions of curves far from the interface and Filippov Approximation Theorem for those more close to Γ. The two parts will be glued together by exploiting the controllability conditions of **(H3)**, This qualitative idea will be made precise through the notion of ε–partition introduced in Appendix B and reasoning by induction on $\jmath_\varepsilon$.

Proof. (*of Theorem 6.1*) By Proposition 6.2 there is an integral curve (y_0, η_0) of $\overline{\mathrm{co}}\, \mathrm{G}$ taking the value $(x_0, w(x_0))$ at $t = 0$, defined in $[0, 2\, T_0]$, and lying in $\mathcal{E}p(w)$, if $y_0([0, 2\, T_0]) \cap \Gamma = \emptyset$ then the assertion can be obtained. as in the usual non partitioned case, using Relaxation Theorem. We then assume this intersection to be nonempty and select a compact set $K_0 \subset \mathbb{R}^d$ containing in its interior

$$\mathcal{R}_{F_\Gamma}(y_0([0, 2\, T_0]) \cap \Gamma, 2\, T_0).$$

The reachable set $\mathcal{R}_{F_\Gamma}(\cdot, \cdot)$ has been defined in (2.1). We recall, see Remark 2.11, that there is only a finite number of connected components of Γ intersecting K_0, say $\Gamma_1, \cdots, \Gamma_n$ for some $n \in \mathbb{N}$.

We proceed introducing some quantities we will use in the forthcoming estimates:

- P estimates from above the diameter of $G(x, \xi)$ for $(x, \xi) \in y_0([0, 2\, T_0]) \times \eta_0([0, 2\, T_0])$.
- $N > 1$ express the equivalence of Euclidean distance and d_Γ in $K_0 \cap \Gamma_i$, $i = 1, \cdots, n$, namely

$$|x - z| \leq d_\Gamma(x, z) \leq N\, |x - z| \quad \text{for } x,\ z \text{ in } K_0 \cap \Gamma_i.$$

- L_G is a Lipschitz constant for G_Γ in $(K_0 \cap \Gamma_\natural) \times \mathbb{R}$.

We finally recall that R is the constant related to the controllability condition on the interface, stipulated in **(H3)(ii)**. We define

$$C = \frac{N}{R} \exp(2\, L_G\, T_0)\, P \tag{6.1}$$

Being the proof quite long, we divide it in several steps to make reading easier.

(1) Argument by induction

We make a slicing of the argument as in Theorem 5.5 and then make an inductive reasoning. We consider ε so small that any integral curve of F defined in some compact interval and with support contained in K_0 and any ε–partition related to it satisfy the weak separation principle stated in Proposition B.3.

($\mathbf{P_k}$) Given an interval $[a_0, b_0] \subset [0, 2\, T_0]$ such that $\jmath_\varepsilon(y_0; a_0, b_0) = k$, there exists, for any $\xi_0 \in \mathbb{R}$, a trajectory of G defined in some interval $[a, b]$ with

- $(y(a), \eta(a)) = (y_0(a), \xi_0)$.
- $C\, \varepsilon + 2\, (b_0 - a_0) > b - a > \frac{b_0 - a_0}{2}$.
- $\eta(b) + \exp(\lambda\, (b - a))\, [\eta_0(a_0) - \xi_0]^+ + P\, \left(1 + \frac{N}{R}\, Q(\xi_0, b - a)\right)\, \exp((L_G + \lambda)\, (b - a))\, \varepsilon \geq \eta_0(b_0)$.

- $y(b) = y_0(b_0)$ whenever $y_0(b_0) \in \Gamma$.

The function $Q(\cdot,\cdot)$ has been defined in (A.12).

(2) Proving $(\mathbf{P_2})$ when $y_0([a_0,b_0]) \cap \Gamma = \emptyset$

We first show $(\mathbf{P_2})$ assuming $y_0([a_0,b_0])$ contained in one of the two open region of the partition, say Ω_i. Since G is locally Lipschitz–continuous in $\Omega_i \times \mathbb{R}$, and y is at a positive distance from the interface, we find in this case by Relaxation Theorem, for any given ρ, an integral curve $(y,\overline{\eta})$ of G, defined in $[a_0,b_0]$, with $(y(a_0),\overline{\eta}(a_0)) = (y_0(a_0),\eta_0(a_0))$ and

$$|y(t) - y_0(t)| + |\overline{\eta}(t) - \eta_0(t)| < \rho \quad \text{for } t \in [a_0,b_0]. \tag{6.2}$$

By Filippov Implicit Function Lemma, y is an integral trajectory in $[a_0,b_0]$ of $f_i(y,\alpha)$ for some admissible control α. We denote by η the solution of $\dot{\eta} = \lambda\,\eta - \ell_i(y,\alpha)$ taking the value ξ_0 at $t = a_0$, then

$$\overline{\eta}(b_0) - \eta(b_0) \leq \exp(\lambda\,(b_0 - a_0))\,(\eta_0(a_0) - \xi_0)$$

and consequently

$$\begin{aligned} \eta_0(b_0) &\leq |\overline{\eta}(b_0) - \eta_0(b_0)| + (\overline{\eta}(b_0) - \eta(b_0)) + \eta(b_0) \\ &\leq \rho + \exp(\lambda\,(b_0 - a_0))\,[\eta_0(a_0) - \xi_0]^+ + \eta(b_0), \end{aligned}$$

which proves the assertion with $[a,b] = [a_0,b_0]$, being ρ arbitrary.

(3) Proving $(\mathbf{P_2})$ when $y_0((a_0,b_0)) \cap \Gamma = \emptyset$ and $y_0(a_0)$, $y_0(b_0)$ possibly in Γ

We now assume $y_0((a_0,b_0)) \subset \Omega_i$, and both $y_0(a_0)$ and $y_0(b_0)$ to be in Γ. We again apply Relaxation Theorem in a slightly reduced interval to stay away from Γ. We find, for any $\rho > 0$ sufficiently small, an integral curve $(y,\overline{\eta})$ of G in $[a_0+\rho, b_0-\rho]$ with $(y(a_0+\rho),\overline{\eta}(a_0+\rho)) = (y_0(a_0+\rho),\eta_0(a_0+\rho))$ and

$$|y(t) - y_0(t)| + |\overline{\eta}(t) - \eta_0(t)| < \rho \quad \text{for } t \in [a_0+\rho, b_0-\rho]. \tag{6.3}$$

We have

$$\begin{aligned} |y(a_0+\rho) - y_0(a_0)| &\leq |y(a_0+\rho) - y_0(a_0+\rho)| + |y_0(a_0) - y_0(a_0+\rho)| \\ &\leq \rho + O(\rho) = O(\rho) \end{aligned} \tag{6.4}$$

and the same inequality holds for $|y(b_0-\rho) - y_0(b_0)|$, therefore, bearing in mind that $y_0(a_0)$ and $y_0(b_0)$ are on the interface, we have

$$|g(y(a_0+\rho))| = O(\rho) \quad \text{and} \quad |g(y(b_0-\rho))| = O(\rho).$$

We can thus apply Lemma 4.3 to continuously extend y in $[a_0+\rho-t_1, b_0-\rho+t_2]$, for some t_1, t_2 positive, through concatenation with other trajectories of F in such a way that

$$\begin{aligned} & t_1 = O(\rho),\ t_2 = O(\rho) \\ & y(a_0+\rho-t_1) \text{ and } y_0(a_0) \text{ belong to the same connected component of } \Gamma \\ & y(b_0-\rho+t_2) \text{ and } y_0(b_0) \text{ belong to the same connected component of } \Gamma \end{aligned} \tag{6.5}$$

We proceed considering a geodesics on Γ linking $y_0(a_0)$ to $y(a_0+\rho-t_1)$ and $y(b_0-\rho+t_2)$ to $y_0(b_0)$. We parametrize it with constant velocity R in intervals $[a_0+\rho-t_1-t'_1, t_1+\rho-t_1]$, $[b_0-\rho+t_2, t_2-\rho+t_2+t'_2]$, respectively, for appropriate $t'_1 \geq 0$, $t'_2 \geq 0$. By assumption **(H3)(ii)** these curves are admissible for the controlled dynamics, and we employ it to further extend y by concatenation in $[a_0+\rho-t_1-t'_1, b_0-\rho+t_2+t'_2]$.

The next step is to estimate t'_1, t'_2. We actually make explicit calculations just for t'_1, being those for t'_2 identical. We preliminarily calculate using (6.4), (6.5)

$$\begin{aligned} |y(a_0+\rho-t_1) - y_0(a_0)| &\leq |y(a_0+\rho-t_1) - y(a_0+\rho)| + |y(a_0+\rho) - y_0(a_0)| \\ &\leq O(\rho) + O(\rho) = O(\rho). \end{aligned}$$

Being d_Γ locally equivalent to the Euclidean distance, this implies

$$d_\Gamma(y(a_0+\rho-t_1), y_0(a_0))) \leq O(\rho)$$

and, taking into account that the geodesics have been parametrized with velocity R, we get

$$t'_1 \leq \frac{O(\rho)}{R} = O(\rho), \quad t'_2 = O(\rho). \tag{6.6}$$

We set

$$a = a_0 + \rho - t_1 - t'_1, \quad b = b_0 - \rho + t_2 + t'_2.$$

The curve y in $[a, b]$ is altogether an integral curve of F and so it is in correspondence with an admissible control α. By construction we have

$$y(a) = y_0(a_0) \quad \text{and} \quad y(b) = y_0(b_0). \tag{6.7}$$

We denote by η, for $t \in [a, b]$, the solution of $\dot{\eta} = \lambda\, \eta - \ell(y, \alpha)$ with

$$\eta(a) = \xi_0, \tag{6.8}$$

then, bearing in mind (6.5), (6.6)

$$\begin{aligned}
\eta_0(a_0 + \rho) - \eta(a_0 + \rho) &\leq |\eta_0(a_0 + \rho) - \eta_0(a_0)| + \eta_0(a_0) - \eta(a_0 + \rho) \\
&\leq O(\rho) + \eta_0(a_0) - \exp(\lambda\, (t_1 + t'_1)) \left(\xi_0 - \int_a^{a_0+\rho} \exp(-\lambda\, (t - a))\, \ell(y, \alpha)\, dt \right) \\
&\leq O(\rho) + \eta_0(a_0) + \exp(\lambda\, (t_1 + t'_1)) \left(- \xi_0 + M\, (t_1 + t'_1) \right) \\
&\leq O(\rho) + (1 - \exp(\lambda\, (t_1 + t'_1))\, \xi_0 + [\eta_0(a_0) - \xi_0]^+ = O(\rho) + [\eta_0(a_0) - \xi_0]^+.
\end{aligned}$$

This implies, taking into account that $\overline{\eta}(a_0 + \rho) = \eta_0(a_0 + \rho)$ and $b_0 - a_0 - 2\,\rho \leq b - a$

$$\begin{aligned}
\overline{\eta}(b_0 - \rho) - \eta(b_0 - \rho) &\leq \exp(\lambda\, (b_0 - a_0 - 2\,\rho))\, (\overline{\eta}(a_0 + \rho) - \eta(a_0 + \rho)) \\
&\leq O(\rho) + \exp(\lambda\, (b - a))\, [\eta_0(a_0) - \xi_0]^+.
\end{aligned}$$

By this last inequality, (6.3), (6.5), (6.6), we get

$$\begin{aligned}
\eta_0(b_0) &\leq |\eta_0(b_0 - \rho) - \eta_0(b_0)| + |\overline{\eta}(b_0 - \rho) - \eta_0(b_0 - \rho)| + (\overline{\eta}(b_0 - \rho) - \eta(b_0 - \rho)) + |\eta(b) - \eta(b_0 - \rho)| + \eta(b) \\
&\leq O(\rho) + \rho + O(\rho) + \exp(\lambda\, (b - a))\, [\eta_0(a_0) - \xi_0]^+ + Q(\xi_0, b - a)\, O(\rho) + \eta(b) \\
&= O(\rho) + \exp(\lambda\, (b - a))\, [\eta_0(a_0) - \xi_0]^+ + Q(\xi_0, b - a)\, O(\rho) + \eta(b).
\end{aligned}$$

We recall that the function $Q(\cdot, \cdot)$ is defined in Appendix A. Taking into account the above formula, the fact that ρ can be chosen arbitrarily small and (6.7), (6.8), the assertion is proved. The above argument can be easily adapted to the case where just one of the two extremal points $y_0(a_0)$, $y_0(b_0)$ belongs to the interface. Notice that if $y_0(a_0) \notin \Gamma$ then a can be taken equal to a_0 and similarly $b = b_0$ whenever $y_0(b_0) \notin \Gamma$. The proof of this part is therefore concluded.

(4) Proving $(\mathbf{P_2})$ when $y_0((a_0, b_0)) \cap \Gamma \neq \emptyset$

Since $\jmath_\varepsilon(y_0; a_0, b_0) = 2$

$$|\{t \in [a_0, b_0] \mid y_0(t) \notin \Gamma\}| < \varepsilon. \tag{6.9}$$

The proof goes along the same lines of the corresponding part in Theorem 5.5. We first get

$$\int_{a_0}^{b_0} d((\dot{y}_0(s), \dot{\eta}_0(s)), G_\Gamma((y_0(s)), \eta(s)))\, ds \leq \varepsilon\, P$$

and then, by the assumption on ε, that $y_0([a_0, b_0])$ is contained in $\Gamma_\natural$. We apply Theorem 2.9 with $\mathcal{C} = \Gamma \times \mathbb{R}$, $\mathcal{C}_\natural = \Gamma_\natural \times \mathbb{R}$, and the multifunction $Z = G_\Gamma$, taking into account that L_G is a Lipschitz constant of G_Γ in $(K_0 \cap \Gamma_\natural) \times \mathbb{R}$ which contains a bounded open neighborhood of $\mathcal{R}_{G_\Gamma}((y_0(a_0), \eta_0(a_0)), b_0 - a_0))$, as prescribed in that theorem. We get the existence of an integral curve $(y, \overline{\eta})$ of G_Γ defined in $[a_0, b_0]$ and contained in the interface with

$$(y(a_0), \overline{\eta}(a_0)) = (y_0(a_0), \eta_0(a_0)) \tag{6.10}$$

and

$$(6.11) \qquad |y(b_0) - y_0(b_0)| \leq \exp\big(L_G\,(b_0 - a_0)\big)\,\varepsilon\,P$$

$$(6.12) \qquad |\overline{\eta}(b_0) - \eta_0(b_0)| \leq \exp\big(L_G\,(b_0 - a_0)\big)\,\varepsilon\,P.$$

Since $y(b_0)$ and $y_0(b_0)$ are in the same connected component of Γ, say Γ_j, we can extend y in some interval $[a_0, b_0 + t_2]$, for a suitable $t_2 \geq 0$ by concatenation with a geodesics in Γ_j joining $y(b_0)$ to $y_0(b_0)$, parametrized with constant velocity R. Since $y(b_0)$, $y_0(b_0)$ are in K_0

$$d_\Gamma(y(b_0), y_0(b_0)) < N\,|y(b_0) - y_0(b_0)| < N\,\exp\big(L_G\,(b_0 - a_0)\big)\,\varepsilon\,P,$$

which, in turn, implies

$$t_2 < \frac{N}{R}\,\exp\big(L_G\,(b_0 - a_0)\big)\,\varepsilon\,P.$$

We set $a = a_0$ and $b = b_0 + t_2$. Recalling the definition of C given in (6.1) and the above estimate of t_2, we have

$$(6.13) \qquad C\,\varepsilon + (b_0 - a_0) \geq b - a \geq b_0 - a_0.$$

The curve y so extended in $[a, b]$ is an integral curve of F_Γ and so it is in correspondence with an admissible control α, in addition it satisfies

$$(6.14) \qquad y(b) = y(b_0 + t_2) = y_0(b_0)$$

We denote by η, for $t \in [a, b]$, the curve identified by $\dot{\eta} = \lambda\,\eta - \ell(y, \alpha)$ and

$$(6.15) \qquad \eta(a) = \xi_0.$$

By (6.10), (6.15) we have

$$\overline{\eta}(b_0) - \eta(b_0) \leq \exp(\lambda\,(b_0 - a_0))\,[\eta_0(a_0) - \xi_0]^+.$$

We finally gather information from (6.12) and the above formula to get

$$\begin{aligned}
\eta_0(b_0) &\leq |\eta_0(b_0) - \overline{\eta}(b_0)| + \overline{\eta}(b_0) - \eta(b_0) + |\eta(b_0) - \eta(b_0 + t_2)| + \eta(b_0 + t_2) \\
&\leq \exp\big(L_G\,(b_0 - a_0)\big)\,\varepsilon\,P + \exp(\lambda\,(b_0 - a_0))\,[\eta_0(a_0) - \xi_0]^+ \\
&+ \frac{N}{R}\,\exp\big(L_G\,(b_0 - a_0)\big)\,P\,Q(\xi_0, b - a)\,\varepsilon + \eta(b).
\end{aligned}$$

Therefore, using (6.13)

$$(6.16) \qquad \eta_0(b_0) \leq P\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,\exp\big(L_G\,(b - a)\big)\,\varepsilon + \exp(\lambda\,(b - a))\,[\eta_0(a_0) - \xi_0]^+ + \eta(b).$$

We claim that (y, η) satisfies all the properties in $(\mathbf{P_2})$. In fact, such curve is an integral curve of G by construction, and satisfies the basic estimate because of (6.16). Moreover $y(b) = y(b_0 + t_2) = \overline{y}(b_0 + t_2) = y_0(b_0)$ by (6.14), the condition at $t = a = a_0$ is also satisfied thanks to (6.10). Finally (6.13) gives the desired estimate on $b - a$ in terms of $b_0 - a_0$, C and ε.

(5) Proving $(\mathbf{P_{k+1}})$

We assume $(\mathbf{P_2}), \cdots, (\mathbf{P_k})$ to hold and $\jmath(y_0; a_0, b_0) = k + 1$. The idea is to exploit Proposition B.5, we denote by

$$\{t_1 = a_0, \cdots, t_{k+1} = b_0\}$$

a minimal ε–partition of $[a_0, b_0]$ related to y_0, then there are two positive constant ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ satisfying

$$\jmath_{\varepsilon_1}(y_0; a, t_k) = k \quad \text{and} \quad \jmath_{\varepsilon_2}(y_0; t_k, b) = 2.$$

By inductive step there are two integral curves (y_1, η_1) ad (y_2, η_2) of G, defined in intervals $[a_1, b_1]$, $[a_2, b_2]$, respectively, enjoying the following properties:

(i) $(y_1(a_1), \eta_1(a_1)) = (y_0(a_0), \xi_0)$ and $(y_2(a_2), \eta_2(a_2)) = (y_0(t_k), \eta_1(b_1))$

(ii) $C\,\varepsilon_1 + 2\,(t_k - a_0) > b_1 - a_1 > \frac{t_k - a_0}{2}$ and $C\,\varepsilon_2 + 2\,(b_0 - t_k) > b_2 - a_2 > \frac{b_0 - t_k}{2}$

(iii) $\eta_1(b_1) + \exp(\lambda\,(b_1 - a_1))\,[\eta_0(a_0) - \xi_0]^+ + P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b_1 - a_1)\right)\,\exp((L_G + \lambda)\,(b_1 - a_1))\,\varepsilon_1 \geq \eta_0(t_k)$.

(iv) $\eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,[\eta_0(t_k) - \eta_1(b_1)]^+ + P\,\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2 \geq \eta_0(b_0)$.

(v) $y_1(b_1) = y_0(t_k)$ because $y_0(t_k) \in \Gamma$, see the definition of ε–partition.

(vi) $y_2(b_2) = y_0(b_0)$ if $y_0(b_0) \in \Gamma$

We set $a = a_1$, $b = b_1 + b_2 - a_2$ and define a curve in $[a, b]$ by setting

$$\begin{cases} (y(t), \eta(t)) = (y_1(t), \eta_1(t)) & \text{for } t \in [a_1, b_1] \\ (y(t), \eta(t)) = (y_2(t + a_2 - b_1), \eta_2(t + a_2 - b_1)) & \text{for } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

Notice that (y, η) is continuous because of items (i), (v), and it is an integral curve of G being the concatenation of two of such curves. It attains the value $(y_0(a_0), \xi_0)$ at a thanks to (i), inequalities

$$C\,\varepsilon + 2\,(b_0 - a_0) > b - a > \frac{b_0 - a_0}{2}$$

hold by (ii), and the condition at $t = b$, in case $y_0(b_0)$ is on Γ, is satisfied by (vi). Finally we combine estimates in (iii) and (iv) to get

$$\begin{aligned}
\eta_0(b_0) &\leq \eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,[\eta_0(t_k) - \eta_1(b_1)]^+ + P\,\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2 \\
&\leq \eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,\big\{\exp(\lambda\,(b_1 - a_1))\,[\eta_0(a_0) - \xi_0]^+ \\
&+ P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b_1 - a_1)\right)\,\exp((L_G + \lambda)\,(b_1 - a_1))\,\varepsilon_1\big\} \\
&+ P\,\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2
\end{aligned}$$

By Lemma A.5

$$\begin{aligned}
Q(\eta_1(b_1), b_2 - a_2) &\leq Q(\xi_0, b - a) \\
Q(\xi_0, b_1 - a_1) &\leq Q(\xi_0, b - a).
\end{aligned}$$

We plug these relations in the previous estimates to get

$$\begin{aligned}
\eta_0(b_0) &\leq \eta(b) + \exp(\lambda\,(b - a))[\eta_0(a_0) - \xi_0]^+ + P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\exp((\lambda + L_G)\,(b - a))\,\varepsilon_1 \\
&+ P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,\exp((L_G glued + \lambda)\,(b_2 - a_2))\,\varepsilon_2 \\
&\leq \eta(b) + \exp(\lambda\,(b - a))[\eta_0(a_0) - \xi_0]^+ + P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,\exp((L_G + \lambda)\,(b - a))\,\varepsilon.
\end{aligned}$$

This segment of the proof is then complete.

(6) Final part

We fix $\delta > 0$ and ε with

$$\begin{aligned}
C\,\varepsilon &< 1 \\
P\,\left(1 + \frac{N}{R}\,Q(w(x_0), 4\,T_0 + 1)\right)\,\exp((L_G + \lambda)\,T)\,\varepsilon &< \delta.
\end{aligned}$$

Owing to the above part of the proof, we find a trajectory (y, η) of G defined in some interval $[a, b]$ of length $b - a =: T \in (T_0, 4\,T_0 + 1)$ such that $(y(a), \eta(a)) = (x_0, w(x_0))$ and

$$\eta(b) + \delta > \eta(b) + P\,\left(1 + \frac{N}{R}\,Q(w(x_0), 4\,T_0 + C\,\varepsilon)\right)\,\exp((L_G + \lambda)\,T)\,\varepsilon \geq \eta_0(2\,T_0).$$

It is not restrictive to assume $[a,b]=[0,T]$. Taking into account that (y_0,η_0) is contained in $\mathcal{E}p(w)$ we further obtain

$$\eta(T)+\delta \geq \eta_0(2\,T_0) \geq w(y_0(2\,T_0)) \geq -M_w. \tag{6.17}$$

Since y is an integral curve of F, there is a corresponding admissible control α such that

$$\eta(T)=e^{\lambda\,T}\left(w(x_0)-\int_0^T e^{-\lambda\,t}\,\ell(y,\alpha)\,dt\right).$$

Plugging this relation in (6.17) we find

$$e^{\lambda\,T}\left(w(x_0)-\int_0^T e^{-\lambda\,t}\,\ell(y,\alpha)\,dt\right) \geq -\delta-M_w$$

and finally

$$w(x_0) \geq \int_0^T e^{-\lambda\,t}\,\ell(y,\alpha)\,dt - e^{-\lambda\,T}(M+\delta).$$

This proves the assertion. □

7 Proof of main results

We start by providing the announced generalizations of Theorems 5.1, 6.1 without assuming **(H5)**.

Theorem 7.1. *We assume* **(H1)**–**(H4)**. *A bounded upper semicontinuous function u, which is, in addition, continuous at any point of Γ, is subsolution to* (HJB) *if and only if $\mathcal{H}p(u)$ is strongly invariant for G, or, equivalently, u satisfies the suboptimality property.*

Proof. We consider, for $n \in \mathbb{N}$, the x–partial sup–convolutions of ℓ_i, and the corresponding Hamiltonians H_i^n, H_Γ^n obtained by replacing in H_i, H_Γ ℓ_i by ℓ_i^n. We also define ℓ^n through formula (3.5) with ℓ_i^n in place of ℓ_i. Since

$$\ell_i^n(x,\mathrm{A}) \geq \ell_i(x,\mathrm{A}) \quad \text{for any } x \in \overline{\Omega}_i,\ \mathrm{A} \in A_i \tag{7.1}$$

we deduce

$$H_i^n \leq H_i \quad \text{and} \quad H_\Gamma^n \leq H_\Gamma. \tag{7.2}$$

In addition H_i^n, H_Γ^n pointwise monotonically converge, as n goes to infinity, to H_i in $\overline{\Omega}_i \times \mathbb{R}^d$ and H_Γ in $\Gamma \times \mathbb{R}^d$, respectively. This in turn implies local uniform convergence by Dini Theorem.

(1) suboptimality $\Rightarrow$ subsolution.

By (7.1) u satisfies the suboptimality property with integral cost ℓ replaced b ℓ^n, for any n. Taking into account the Lipschitz character of ℓ_i^n, we deduce from Theorem 5.2 that u is subsolution to (HJB) with H_i, H_Γ replaced by H_i^n, H_Γ^n; passing at the limit for $n \to +\infty$ and bearing in mind the aforementioned convergence of Hamiltonians, we get the assertion.

(2) subsolution $\Rightarrow$ suboptimality.

Because of (7.2), if u is a subsolution to (HJB), then it is also subsolution of the Hamilton–Jacobi equation obtained by replacing in (HJB) H_i, H_Γ by H_i^n, H_Γ^n, respectively, for any n.

Since the ℓ_i^n are locally Lipschitz–continuous in x, uniformly in the control variable, then by Theorem 5.5

$$u(x) \leq e^{-\lambda\,t}u(y(t)) + \int_0^t e^{-\lambda\,s}\ell^n\left(y(s),\alpha(s)\right)\,ds \tag{7.3}$$

for any t, n, $x \in \mathbb{R}^d$, (α, y) admissible with $y(0) = x$. We deduce from local uniform convergence of ℓ^n to ℓ in $\cup_i(\Omega_i \times A_i) \cup (\Gamma \times A)$, corresponding convergence of the integrals in (7.3), so that we get at the limit

$$u(x) \le e^{-\lambda t} u(y(t)) + \int_0^t e^{-\lambda s} \ell\,(y(s), \alpha(s))\; ds,$$

which shows that u satisfies the suboptimality problem with respect to the original system. This ends the proof. □

Theorem 7.2. *We assume* **(H1)**– **(H4)**. *Let w be a bounded lower semicontinuous supersolution of* (HJB) *and $M_w > 0$ with $|w| < M_w$ in $\mathbb{R}^d$. Given $x_0 \in \mathbb{R}^d$ and positive constants T_0 and δ, there exists (y, α) admissible with $y(0) = x_0$ such that*

$$w(x_0) \ge \int_0^T e^{-\lambda s}\, \ell(y, \alpha)\, ds - e^{-\lambda T}\, M_w - \delta \quad \textit{for some } T > T_0.$$

Proof. We follow the same lines of second implication of the previous theorem, with some difference in the final estimate. We define ℓ_n as in (3.5) with ℓ_n^i in place of ℓ_i, and then consider Hamiltonians H_n^i, H_n^Γ obtained by replacing ℓ_i, ℓ with ℓ_n^i, ℓ_n, respectively. Notice that ℓ_n locally uniformly converge to ℓ in $\cup_i(\Omega_i \times A_i) \cup (\Gamma \times A)$.

Since $H_n^i \ge H_i$, $H_n^\Gamma \ge H_\Gamma$, then w is still supersolution to (HJB) with H_n^i, H_n^Γ substituting H_i, H_Γ. Therefore, by Theorem 6.1 there exist (y_n, α_n) admissible with $y_n(0) = x_0$, $T_n > T_0$ such that

$$w(x_0) \ge \int_0^{T_n} e^{-\lambda s} \ell_n(y_n, \alpha_n)\, ds - e^{-\lambda T_n}\, M_w - \frac{\delta}{2}. \tag{7.4}$$

In addition $T_n < 4\,T_0 + 1$ for any n, then, being y_n integral curves of F starting at x_0, there exists a compact subset of $\mathbb{R}^d$ containing $y_n([0, T_n])$ for any n. Consequently, by local uniform convergence of ℓ_n to ℓ

$$\max_{[0,T_n]} |\ell_n(y_n(t), \alpha_n(t)) - \ell(y_n(t), \alpha_n(t))| \to 0 \quad \text{as } n \text{ goes to infinity}$$

and for n suitably large we find

$$\int_0^{T_n} e^{-\lambda s} \ell_n(y_n, \alpha_n)\, ds \ge \int_0^{T_n} e^{-\lambda s} \ell(y_n, \alpha_n)\, ds - \frac{\delta}{2}$$

which implies by (7.4)

$$w(x_0) \ge \int_0^{T_n} e^{-\lambda s} \ell(y_n, \alpha_n), ds - e^{-\lambda T_n}\, M_w - \delta.$$

This concludes the proof. □

Proofs of the main results we proceed giving are founded on previous theorems.

Proof. (*of Theorem 3.5*) The continuity properties of value function v have already been established in Theorem 4.5. The proof that it is supersolution to (HJB) is straightforward, so we omit it. Finally, satisfying the dynamical programming principle, v is also subsolution in force of Theorem 7.1. □

Remark 7.3. *In order to compare our results with those of [5] we point out that the value function also solves the problem*

$$\begin{cases} \lambda u(x) + H_i(x, Du) = 0 & \textit{in } \Omega_i,\ i = 1, 2 \\ \lambda u(x) + \max\{H_1(x, Du), H_2(x, Du)\} \ge 0 & \textit{at any } x \in \Gamma \\ \lambda u(x) + \min\{H_1(x, Du), H_2(x, Du)\} \le 0 & \textit{at any } x \in \Gamma \end{cases} \tag{7.5}$$

Here there are not tangential equations, and the inequalities must be understood in the viscosity sense with test functions in $\mathbb{R}^d$. In fact we can directly prove, as in the usual non partitioned case, that if $x \in \Gamma$, $\mathrm{A} \in A_1$

and $Dg(x) \cdot f_1(x, \mathrm{A}) > 0$, or, in other terms $f_1(x, \mathrm{A})$ points strictly inside Ω_1, then for any C^1 viscosity test function for above ψ to v at x

$$\lambda\, v(x) - D\psi(x) \cdot f_1(x, \mathrm{A}) - \ell_1(x, \mathrm{A}) \leq 0.$$

If instead $\mathrm{B} \in A_2$ and $Dg(x) \cdot f_2(x, \mathrm{B}) < 0$ then

$$\lambda\, v(x) - D\psi(x) \cdot f_2(x, \mathrm{B}) - \ell_2(x, \mathrm{B}) \leq 0.$$

Now assume by contradiction that the value function is not subsolution of the last equation in the above problem at x. This means that there is $\mathrm{A} \in A_1$, $\mathrm{B} \in A_2$ with

$$\begin{aligned} \lambda\, v(x) - D\psi(x) \cdot f_1(x, \mathrm{A}) - \ell_1(x, \mathrm{A}) &> 0 \\ \lambda\, v(x) - D\psi(x) \cdot f_2(x, \mathrm{B}) - \ell_2(x, \mathrm{B}) &> 0 \end{aligned}$$

then by the above considerations $Dg(x) \cdot f_1(x, \mathrm{A}) \leq 0$ and $Dg(x) \cdot f_2(x, \mathrm{B}) \geq 0$, therefore there is $\mu \in [0, 1]$ with $Dg(x) \cdot (\mu\, f_1(x, \mathrm{A}) + (1 - \mu)\, f_2(x, \mathrm{B})) = 0$ and clearly

$$\lambda\, v(x) - D\psi(x) \cdot (\mu\, f_1(x, \mathrm{A}) + (1 - \mu)\, f_2(x, \mathrm{B})) - (\mu\, \ell_1(x, \mathrm{A}) + (1 - \mu)\, \ell_2(x, \mathrm{B})) > 0,$$

which is in contradiction with v being solution to (HJB).

Proof. (*of Theorem 3.6*) let w, u, x_0 be a a bounded lower semicontinuous supersolution, a bounded upper semicontinuous subsolution continuous at any point of Γ, and a point of $\mathbb{R}^d$, respectively. We take a common upper bound M_0 for $|w|$, $|u|$, $|v|$ in $\mathbb{R}^d$. We aim at proving

$$w(x_0) \geq v(x_0) \geq u(x_0), \tag{7.6}$$

which gives the assertion being x_0 arbitrary in $\mathbb{R}^d$. We fix $\varepsilon > 0$ and thereafter δ, T_0 with

$$2\, e^{-\lambda\, T_0}\, M_0 + \delta < \varepsilon. \tag{7.7}$$

We recall that v satisfies the dynamical programming principle, and invoke Theorem 7.2 for w, to get for a suitable pair (y, α) admissible with $y(0) = x_0$ and $T > T_0$

$$\begin{aligned} v(x_0) &\leq \int_0^T \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, v(y(T)) \\ w(x_0) &\geq \int_0^T \ell(y, \alpha)\, ds - e^{-\lambda\, T}\, M_0 - \delta. \end{aligned}$$

We deduce

$$w(x_0) \geq v(x_0) - 2\, e^{-\lambda\, T}\, M_0 - \delta$$

and, taking into account (7.7)

$$w(x_0) \geq v(x_0) - \varepsilon. \tag{7.8}$$

Similarly, we invoke Theorem 7.1 for u and use again dynamical programming principle for v to get for a suitable pair (y, α) admissible with $y(0) = x_0$

$$\begin{aligned} v(x_0) &\geq \int_0^{T_0} \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, v(y(T)) - \delta \\ u(x_0) &\leq \int_0^{T_0} \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, u(y(T)). \end{aligned}$$

We deduce

$$v(x_0) \geq u(x_0) - 2\, e^{-\lambda\, T_0}\, M_0 - \delta$$

and, taking into account (7.7)

$$v(x_0) \geq u(x_0) - \varepsilon. \tag{7.9}$$

Relations (7.8) and (7.9) imply (7.6) since ε is arbitrary. □

Proof. (*of Theorem 3.7*) The assertion is a direct consequence of Theorem 3.6 and 3.5. □

A Appendix A: Augmented dynamics

We start stating and proving the main result of this appendix assuming **(H5)**. Condition **(H4)** also plays an essential role in here.

Theorem A.1. *Under the assumptions* **(H1)**–**(H5)** *the multifunction G_Γ is Lipschitz continuous on $\Theta \times \mathbb{R}$, for any compact subset Θ of G.*

Proof. We aim at showing the existence of $\overline{L} > 0$ such that for any pair (x,ξ), (z,η) in $\Theta \times \mathbb{R}$ and $\big((f(x,\text{A}), \lambda\xi - \ell(x,\text{A})\big) \in G_\Gamma(x,\xi)$ there exists a control $\text{D} \in A_\Gamma(z)$ satisfying

$$\big|f(x,\text{A}) - f(z,\text{D})\big| + \big|\lambda\,\xi - \ell(x,\text{A}) - \lambda\,\eta + \ell(z,\text{D})\big| \le \overline{L}\,\big(|x-z| + |\xi - \eta|\big). \tag{A.1}$$

It is not restrictive to prove the above inequality for $|x-z|$ small, therefore since there are just a finite number of connected components of Γ intersecting Θ, see Remark 2.11, and such components are at a positive distance apart, we can assume, without loosing generality, that Θ is in addition connected.

We set

$$C = \min\left\{-\max_{x\in\Theta}\min_{q\in F(x)} Dg(x)\cdot q, \min_{x\in\Theta}\max_{q\in F(x)} Dg(x)\cdot q\right\}.$$

Note that C is strictly positive because of assumption **(H3)(i)**. Being g of class C^2 and Θ connected, Dg is Lipschitz–continuous in Θ. We can assume that the constant L appearing in (3.2) is also a Lipschitz constant for ℓ, and Dg in Θ, and that the constant M appearing in (3.1) estimates from above $|Dg|$ in Θ and $|f|$ in $\Theta \times A$.

We take

$$|x-z| < \frac{C}{3\,L\,M}. \tag{A.2}$$

In force of Lipschitz continuity in the state variable of f_i, ℓ, on Γ, we have

$$|f(x,\text{A}) - f(z,\text{A})| + |\lambda\xi + \ell(x,\text{A}) - \lambda\eta - \ell(z,\text{A})| < (2\,L+\lambda)\big(|x-z| + |\xi-\eta|\big). \tag{A.3}$$

We first assume $Dg(z)\cdot f(z,\text{A})$ strictly positive. By the very definition of C there is $\text{B} \in A$ with

$$Dg(x)\cdot f(x,\text{B}) = -C, \tag{A.4}$$

being $-3\,M\,L|x-z| > -C$ by (A.2), we can take, in force of **(H4)**, $\text{C} \in A$ such that $\big(f(x,\text{C}), \ell(x,\text{C})\big)$ lies in the segment joining $\big(f(x,\text{A}), \ell(x,\text{A})\big)$ to $\big(f(x,\text{B}), \ell(x,\text{B})\big)$ and satisfies

$$Dg(x)\cdot f(x,\text{C}) = -3\,M\,L|x-z|. \tag{A.5}$$

We claim that $Dg(z)\cdot f(z,\text{C})$ is negative. With the usual trick of adding–subtracting the same quantity, we can write it as

$$\big(Dg(z) - Dg(x)\big)\cdot f(z,\text{C}) + Dg(x)\cdot(f(z,\text{C}) - f(x,\text{C})) + Dg(x)\cdot f(x,\text{C}).$$

Taking into account (A.5), and estimating term by term, we actually get

$$Dg(z)\cdot f(z,\text{C}) \le (M\,L + M\,L - 3\,M\,L)\,|x-z| < 0.$$

Since signs of $Dg(z)\cdot f(z,\text{A})$ and $Dg(z)\cdot f(z,\text{C})$ are opposite, there is, by **(H4)**, $\text{D} \in A_\Gamma(z)$ such that $\big(f(z,\text{D}), \ell(z,\text{D})\big)$ lies in the segment joining $\big(f(z,\text{A}), \ell(z,\text{A})\big)$ to $\big(f(z,\text{C}), \ell(z,\text{C})\big)$. We proceed showing that estimate (A.1) takes place for such a D and a suitable $\overline{L}$.

To start with, we exploit that the function

$$\varphi : (p,\sigma) \mapsto |p - f(x,\text{A})| + |\sigma - \ell(x,\text{A})|$$

is convex, to obtain

$$\varphi(f(z,\text{D}), \ell(z,\text{D})) \le \max\big\{\varphi(f(z,\text{A}), \ell(z,\text{A})), \varphi(f(z,\text{C}), \ell(z,\text{C}))\big\}. \tag{A.6}$$

First term under maximum in above formula is easily estimated using Lipschitz–continuity of f and ℓ, we have

$$|f(z,\mathrm{A}) - f(x,\mathrm{A})| + |\ell(z,\mathrm{A}) - \ell(x,\mathrm{A})| < 2\,L\,|x-z|. \tag{A.7}$$

We proceed providing estimate for the second one. We first exploit the relation

$$\big(f(x,\mathrm{A}) - f(x,\mathrm{C}), \ell(x,\mathrm{A}) - \ell(x,\mathrm{C})\big) = \rho\,\big(f(x,\mathrm{A}) - f(x,\mathrm{B}), \ell(x,\mathrm{A}) - \ell(x,\mathrm{B})\big),$$

which holds for some ρ positive. Owing to (A.4), (A.5), we can make precise that $\rho = \frac{3\,M\,L}{C}\,|x-z|$, which, in turn, implies

$$|f(x,\mathrm{A}) - f(x,\mathrm{C})| = \frac{3\,M\,L}{C}\,|x-z|\,|f(x,\mathrm{A}) - f(x,\mathrm{B})| \tag{A.8}$$

$$|\ell(x,\mathrm{A}) - \ell(x,\mathrm{C})| = \frac{3\,M\,L}{C}\,|x-z|\,|\ell(x,\mathrm{A}) - \ell(x,\mathrm{B})|. \tag{A.9}$$

Since

$$\big|f(x,\mathrm{A}) - f(x,\mathrm{B})\big| \le 2\,M, \quad \big|\ell(x,\mathrm{A}) - \ell(x,\mathrm{B}))\big| \le 2\,M$$

we derive from (A.8), (A.9)

$$\big|f(x,\mathrm{A}) - f(x,\mathrm{C})\big| + \big|\ell(x,\mathrm{A}) - \ell(x,\mathrm{C})\big| < \frac{12\,M^2\,L}{C}\,|x-z|,$$

exploiting this estimate and the inequality

$$\begin{aligned}\big|f(x,\mathrm{A}) - f(z,\mathrm{C})\big| + \big|\ell(x,\mathrm{A}) - \ell(z,\mathrm{C})\big| &\le \big|f(x,\mathrm{A}) - f(x,\mathrm{C})\big| + \big|f(x,\mathrm{C}) - f(z,\mathrm{C})\big| \\ &+ \big|\ell(x,\mathrm{A}) - \ell(x,\mathrm{C})\big| + \big|\ell(x,\mathrm{C}) - \ell(z,\mathrm{C})\big|\end{aligned}$$

we finally yield

$$\big|f(x,\mathrm{A}) - f(z,\mathrm{C})\big| + \big|\ell(x,\mathrm{A}) - \ell(z,\mathrm{C})\big| < \left(\frac{12\,M^2\,L}{C} + 2\,L\right)\,|x-z|. \tag{A.10}$$

This immediately implies, taking into account (A.6), (A.7)

$$\big|f(x,\mathrm{A}) - f(z,\mathrm{D})\big| + \big|\ell(x,\mathrm{A}) - \ell(z,\mathrm{D})\big| < \left(\frac{12\,M^2\,L}{C} + 2\,L\right)\,|x-z|)$$

and consequently

$$\big|f(x,\mathrm{A}) - f(z,\mathrm{D})\big| + \big|\lambda\,\xi - \ell(x,\mathrm{A}) - \lambda\,\eta + \ell(z,\mathrm{D})\big| < \left(\frac{12\,M^2\,L}{C} + 2\,L + \lambda\right)\,\big(|x-z| + |\xi-\eta|\big),$$

showing the assertion in case $Dg(z)\cdot f(z,\mathrm{A}) > 0$. Same estimate is actually obtained, adapting the argument of above, if $Dg(z)\cdot f(z,\mathrm{A}) < 0$. Finally, If $Dg(z)\cdot f(z,\mathrm{A}) = 0$, then $\big(f(z,\mathrm{A}), \lambda\,\eta - \ell(z,\mathrm{A})\big) \in G_\Gamma(z,\eta)$ and

$$\begin{aligned}\big|f(x,\mathrm{A}) - f(z,\mathrm{A})\big| + \big|\lambda\,\xi - \ell(x,\mathrm{A}) - \lambda\,\eta + \ell(z,\mathrm{A})\big| &< (2\,L + \lambda)\,\big(|x-z| + |\xi-\eta|\big) \\ &< \left(\frac{12\,M^2\,L}{C} + 2\,L + \lambda\right)\,\big(|x-z| + |\xi-\eta|\big).\end{aligned}$$

This finish the proof. □

We extend G_Γ in $\Gamma_\natural \times \mathbb{R}$ by setting

$$G_\Gamma(x,\xi) = G_\Gamma(\mathrm{proj}_\Gamma(x),\xi). \tag{A.11}$$

Exploiting the Lipschitz–continuity of the projection on the interface for points in $\Gamma_\natural$, we deduce from the previous theorem:

Corollary A.2. *The multifunction G_Γ, extended as in* (A.11)*, is Lipschitz–continuous in $B \times \mathbb{R}$, for any bounded subset B of $\Gamma_\natural$.*

Proof. We denote by L_0 a positive quantity which is at the same time Lipschitz constant for proj_Γ in B and for G_Γ in $\mathrm{proj}_\Gamma(B) \times \mathbb{R}$. Given (x_1, ξ_1), (x_2, ξ_2) in $B \times \mathbb{R}$, $(q_1, \sigma_1) \in G_\Gamma(x_1, \xi_1) = G_\Gamma(\mathrm{proj}_\Gamma(x_1), \xi_1)$ there exists $(q_2, \sigma_2) \in G_\Gamma(\mathrm{proj}_\Gamma(x_2, \xi_2) = G_\Gamma(x_2, \xi_2)$ with

$$|(q_1, \sigma_1) - (q_2, \sigma_2)| \leq L_0 \, |\mathrm{proj}_\Gamma(x_1), \xi_1) - \mathrm{proj}_\Gamma(x_2), \xi_2)| \leq L_0^2 \, |(x_1, \xi_1) - (x_2, \xi_2)|.$$

This proves the assertion. □

Remark A.3. *It worths pointing out that G_Γ, extended as above indicated, has nothing to do with the control system outside the interface, nevertheless to have it defined in a neighborhood of Γ is important to apply Filippov Approximation Theorem, as done in Theorems 5.5, 6.1.*

We proceed defining a function $Q : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ via

$$Q(\xi, T) = \lambda \, e^{\lambda T} \left(|\xi| + M \left(T + \frac{1}{\lambda}\right) \right), \tag{A.12}$$

where M is defined as in (3.1). The following fact justifies the introduction of Q.

Lemma A.4. *Let (y, η) be an integral curve of G defined in some interval $[a, b]$, then*

$$|\dot{\eta}(t)| \leq Q(|\eta(a)|, b - a) \quad \textit{for a.e. } t \in (a, b).$$

Proof. For a.e. t and a suitable $\mathrm{A} \in A(y(t))$ we have:

$$\begin{aligned} |\dot{\eta}(t)| &= |\lambda \, \eta(t) - \ell(y(t), \mathrm{A})| \leq \lambda \, \exp(\lambda \, (t - a)) \left(|\eta(a)| + M \, (t - a)\right) + M \\ &\leq \lambda \, e^{\lambda \, (b-a)} \left(|\eta(a)| + M \left(b - a + \frac{1}{\lambda}\right) \right) = Q(\eta(a), b - a). \end{aligned}$$

□

We record a couple of elementary properties of function Q.

Lemma A.5.

(i) *$Q(\xi, T_1) < Q(\xi, T_2)$ for any ξ and $T_1 < T_2$.*

(ii) *Let (y, η) is a trajectory of G defined in some interval $[a, b]$, then*

$$Q(\eta(t), b - t) \leq Q(\eta(a), b - a) \quad \textit{for any } t \in (a, b).$$

Proof. First item is patent. The second one comes from the following computation:

$$\begin{aligned} Q(\eta(t), b - t) &= \lambda \, e^{\lambda \, (b-t)} \left(|\eta(t)| + M \left(b - t + \frac{1}{\lambda}\right) \right) \\ &\leq \lambda \, \exp(\lambda \, (b - t)) \left(\exp(\lambda \, (t - a)) \left(|\eta(a)| + M \, (t - a)\right) + M \left(b - t + \frac{1}{\lambda}\right) \right) \\ &= \lambda \, \exp(\lambda \, (b - a)) \left(|\eta(a)| + M \, (t - a)\right) + \lambda \, \exp(\lambda \, (b - t)) \, M \left(b - t + \frac{1}{\lambda}\right) \\ &\leq \lambda \, \exp(\lambda \, (b - a)) \left(|\eta(a)| + M \left(b - a + \frac{1}{\lambda}\right) \right) = Q(\eta(a), b - a). \end{aligned}$$

□

B Appendix B: ε–partitions

Given a curve y defined in some compact interval $[a, b]$, we define the *event set* as

$$E_y = \partial \{t \in [a, b] \mid y(t) \in \Omega_1 \cup \Omega_2\};$$

this terminology, we have adapted from hybrid control theory, reflects the fact that at such times something memorable happens, namely the possible passage from one basic phase of the life of the curve to another, theses are the times when y lies in one of the open sets Ω_1, Ω_2, or it is sliding along the interface.

In the special case where E_y is made of isolated points, and so it is finite being the interval of definition compact, then such phases follow one another in a well ordered and separated way, there is in fact a finite partition of $[a, b]$ with points of E_y such that in the interior of any interval the curve is in Ω_1 or Ω_2 or Γ.

As pointed out in the introduction, this nice frame is messed up in presence of accumulation points of E_y, also called Zeno times. However we point out in this section that for any $\varepsilon > 0$ a partition of $[a, b]$ keeping some separation property among different phases can be defined also if the Zeno set is nonempty, up to time sets of 1–dimensional measure less than ε. These are the ε–partitions mentioned in the title of the section. We adopt the following terminology:

A *partition* of $[a, b]$ is any finite strictly increasing sequence of times $\{t_1, \cdots, t_k\}$ with $t_1 = a$, $t_k = b$.

An *interval of the partition* is any interval with two subsequent elements of the partition as endpoints.

Definition B.1. **(ε–partition)** *Given $\varepsilon > 0$, and a curve y defined in $[a, b]$, a partition of $[a, b]$ will be called $\varepsilon-$* partition related to y *provided the following conditions hold:*

(i) *All points of it, except possibly a and b, belong to E_y.*

(ii) *Given the (possibly empty) family*

(B.1) $$\mathcal{I} = \{\textit{open intervals } I \textit{ of the partition with } y(I) \cap \Gamma \neq \emptyset\}$$

then all the endpoints of intervals in $\mathcal{I}$ belong to E_y.

(iii) $\sum_{I \in \mathcal{I}} |I \setminus \{t \in [a, b] \mid y(t) \in \Gamma\}| < \varepsilon$.

Notice that item (ii) of the previous definition is about the status of endpoints a and b. It is equivalent of requiring

$$y(t) \notin \Gamma \quad \text{for } t \in (a, t_2) \text{ whenever } y(a) \notin \Gamma$$

and same property, *mutatis mutandis*, for b.

Proposition B.2. *Given a curve y in $\mathbb{R}^d$ defined in some compact interval $[a, b]$ and $\varepsilon > 0$, there exists an ε–partition of $[a, b]$ related to y.*

Proof. We set

(B.2) $$J = \{t \in (a, b) \mid y(t) \in \Omega_1 \cup \Omega_2\}.$$

If $J = (a, b)$ or $J = \emptyset$, then we simply take the partition $\{a, b\}$ to prove the assertion. In the other cases, J being open is the disjoint union of a countable family of open intervals. Being its measure finite we can find a finite subfamily

(B.3) $$\{J'_1, \cdots, J'_h\} \quad \text{for some } h \in \mathbb{N}$$

with

(B.4) $$\left| \bigcup_{l=1}^{h} J'_l \right| = \sum_{l=1}^{h} |J'_l| > |J| - \varepsilon.$$

We set for $l = 1 \dots h$

$$a_l = \begin{cases} \max \{t \leq \inf J'_l \mid y(t) \in \Gamma\} & \text{if the set under the max is nonempty} \\ a & \text{otherwise} \end{cases}$$

and

$$b_l = \begin{cases} \min\{t \geq \sup J'_l \mid y(t) \in \Gamma\} & \text{if the set under the min is nonempty} \\ b & \text{otherwise} \end{cases}$$

We define new open intervals by

$$J_l = (a_l, b_l) \quad \text{for } l = 1, \cdots, h.$$

We further set

$$\begin{aligned} J_{00} &= \big(a, \min\{t \in [a,b] \mid y(t) \in \Gamma\}\big) \quad (J_{00} = \emptyset \text{ if } y(a) \in \Gamma) \\ J_0 &= \big(\max\{t \in [a,b] \mid y(t) \in \Gamma\}, b\big) \quad (J_0 = \emptyset \text{ if } y(b) \in \Gamma) \end{aligned}$$

By construction $\bigcup_{l=1}^h J'_l \subset \bigcup_{l=1}^h J_l \subset J$, therefore by (B.4)

$$\text{(B.5)} \qquad \left| \bigcup_{l=1}^h J_l \cup J_{00} \cup J_0 \right| > |J| - \varepsilon.$$

We consider the family of enlarged intervals plus J_{00}, J_0. We claim that two of such intervals either coincide or are disjoint. Take first J_m, J_n for some $1 \leq m \neq n \leq h$, assume, to fix our ideas

$$\text{(B.6)} \qquad \sup J'_n \leq \inf J'_m$$

(recall that $J'_m \cap J'_n = \emptyset$), if $J_m \cap J_n \neq \emptyset$ then $b_n > a_m$ but this implies, by the very definition of a_m and taking into account that $y(J_m) \cap \Gamma = \emptyset$, that $b_n \geq \sup J'_m$ which in turn gives $b_n \geq b_m$; being the opposite inequality direct consequence of (B.6), we finally get $b_n = b_m$. Arguing similarly we also prove equality of right endpoints, under the assumption of nonempty intersection, and show the claim for J_m, J_n.

Now, assume $J_{00} \neq \emptyset$ and take any $m \in \{1, \cdots, h\}$, if $a_m > a$, then $a_m \geq \min\{t \in [a,b] \mid y(t) \in \Gamma\}$, and the quantity in the right hand–side is the right endpoint of J_{00}. This shows $J_{00} \cap J_m = \emptyset$. If, on the contrary, $a_m = a$, then since $J_m \subset J$ then $b_m = \min\{t \in [a,b] \mid y(t) \in \Gamma\}$, which shows $J_{00} = J_m$.

Similarly, if $J_0 \neq \emptyset$ one proves that either it coincides with J_l, for some $l = 1, \cdots, h$ or it is disjoint with any of them. Finally, J_0, J_{00} are disjoint by their very definition. The claim is then fully proved.

Therefore, up to removing copies, and possibly empty intervals, and reindexing, we end up with a family $\{J_1, \cdots, J_k\}$, for some $k \in \mathbb{N}$, of disjoint open intervals all contained in J , satisfying by (B.5)

$$\text{(B.7)} \qquad \left| \bigcup_{l=1}^k J_l \right| = \sum_{l=1}^h |J_l| > |J| - \varepsilon.$$

Consider the partition given by all their endpoints, suitably indexed, plus a and b, it enjoys conditions (i), (ii) of the definition of ε–partition, which actually justifies the previous construction. Take $\mathcal{I}$ as defined in (B.1). If $I \in \mathcal{I}$ then $I \cap \cup_l J_l = \emptyset$ and so

$$\left(\bigcup_{I \in \mathcal{I}} I \right) \cap J \subset J \setminus \left(\bigcup_{l=1}^k J_l \right).$$

From this we derive

$$\sum_{I \in \mathcal{I}} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| = \sum_{I \in \mathcal{I}} |I \cap J| = \left| \left(\bigcup_{I \in \mathcal{I}} I \right) \cap J \right| \leq \left| J \setminus \bigcup_{l=1}^k J_l \right| < \varepsilon$$

which gives the assertion. □

We proceed deducing that when ε is small with the respect to the velocity of the curve in object, then a sort of *weak separation principle* holds for any ε–partition. We emphasize that the size of such an ε does not depend on the length of intervals but just on velocities. In next proposition we state this property just for integral trajectories of F, since these are the curves we are interested on.

Proposition B.3. *Given a compact subset K_0 of $\mathbb{R}^d$, there is $\varepsilon_0 > 0$ such that for any integral curve y of F defined in some compact interval $[a,b]$, with $y([a,b]) \subset K_0$ one has:*

If I is a closed interval of an ε–partition of $[a,b]$ related to y with $\varepsilon < \varepsilon_0$, then the two (mutually non–exclusive) possibilities hold

$$\textit{either} \quad y(I) \subset \Omega_i, \ i = 1,2, \quad \textit{or} \quad y(I) \subset \Gamma_\natural.$$

Proof. We denote by M_0 a constant estimating from above $|f|$ in $\cup_i((K_0 \cap \Omega_i) \times A_i) \cup ((K_0 \cap \Gamma) \times A)$.

If $y(I) \cap \Gamma = \emptyset$ then $y(I) \subset \Omega_i$ for a suitable choice of i. If instead $y(I) \cap \Gamma \neq \emptyset$, we take $t_0 \in I$, and consider a time neighborhood I_0 of t_0 of radius ε and so of measure $2\,\varepsilon$. if I_0 contains a endpoint of I then $y(I_0) \cap \Gamma \neq \emptyset$ by item (ii) in the definition of ε–partition, same conclusion is reached in force of item (iii), if instead $I_0 \subset I$. Summing up: there is $t_1 \in [a,b]$ with $y(t_1) \in \Gamma$, $|t_1 - t_0| < \varepsilon$, therefore

$$|y(t_1) - y(t_0)| \leq M_0\,|t_1 - t_0| < M_0\,\varepsilon. \tag{B.8}$$

Being $\Gamma_\natural$ open and K_0 compact there is $\delta > 0$ with $(\Gamma \cap K_0) + B(0,\delta) \subset \Gamma_\natural$. Taking into account (B.8) and that the support of y is contained in K_0, it is enough, for proving the assertion, to take $\varepsilon_0 < \frac{\delta}{M_0}$.

□

We attach to any curve defined in a compact interval a natural number, namely the smallest cardinality of an ε–partition related to the curve. Loosely speaking, its size captures, when ε varies, how complicated is the behavior of the curve around the interface Results in Sections 5 and 6, on which, in turn, the main comparison theorem is based, are obtained by means of an inductive argument on this index.

Definition B.4. (minimal ε–partition) *We say that an ε–partition is* minimal *if there are no ε–partitions of $[a,b]$ for y with less elements. We denote the cardinality of any such ε–minimal partition by $\jmath_\varepsilon(y;a,b)$.*

We point out for later use a sort of additive property of the index $\jmath_\varepsilon$.

Proposition B.5. *Given $\varepsilon > 0$, consider an ε–minimal partition $\{t_1 = a, t_2, \cdots t_k = b\}$ with $k = \jmath_\varepsilon(y;a,b) > 2$. For any $1 < h < k$, there exist two positive constants ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that*

$$\jmath_{\varepsilon_1}(y;a,t_h) = h \quad \textit{and} \quad \jmath_{\varepsilon_2}(y;t_h,b) = k - h + 1.$$

Proof. Basically there is nothing to prove, we just exploit the very definition of ε–minimal partition and additivity of measure. We define $\mathcal{I}$ as in (B.1) and set

$$\begin{aligned} \mathcal{I}_1 &= \{I \in \mathcal{I} \mid I \subset [a,t_h]\} \\ \mathcal{I}_2 &= \{I \in \mathcal{I} \mid I \subset [t_h,b]\}, \end{aligned}$$

clearly

$$\sum_{I \in \mathcal{I}_1} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| + \sum_{I \in \mathcal{I}_2} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| < \varepsilon,$$

and we can thus find ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that

$$\sum_{I \in \mathcal{I}_i} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| < \varepsilon_i \quad \text{for } i = 1,2.$$

This shows that $\{t_1 \cdots, t_h\}$ is an ε_1–partition for y in $[a,t_k]$ and $\{t_h \cdots, t_k\}$ an ε_2–partition in $[t_h,b]$. We claim that both these partitions are minimal. In fact, if there were an ε_1–minimal partition of $[a,t_k]$ with less than h elements that the union of it with $\{t_{h+1}, \cdots, t_k\}$ should yield an $(\varepsilon_1 + \varepsilon_2 = \varepsilon)$–partition of the whole of $[a,b]$ with less then k element, which is contrast with $\jmath(y;a,b) = k$. Same conclusion is reached denying $\jmath_{\varepsilon_2}(y;t_h,b) = k - h + 1$. This proves the claim, which, in turn, immediately implies the assertion. □

References

[1] Y. Achdou, F. Camilli, A. Cutrì AND N. Tchou, *Hamilton-Jacobi equations on networks*, to appear on Nonlinear Differential Equation and Application.

[2] J.-P. Aubin AND A. Cellina, *Differential inclusions*, vol. 264 of Comprehensive studies in mathematics, Springer, Berlin, Heidelberg, New York, Tokyo, 1984.

[3] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston,. Basel, Berlin.

[4] F. Bagagiolo and K. Danieli. *Infinite horizon optimal control problems with multiple thermostatic hybrid dynamics*, Nonlinear Analysis: Hybrid Systems, **6**, pp. 824–838 (2012).

[5] G. Barles, A. Briani AND E. Chasseigne, *A Bellman approach for two-domains optimal control problems in* $\mathbb{R}^N$, ESAIM:COCV, 2013.

[6] G. Barles, S. Dharmatti and M. Ramaswamy, *Unbounded Viscosity Solutions of Hybrid Control Systems*, ESAIM: COCV, **16(1)**, pp. 176–193 (2010).

[7] M. Bardi AND I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 1997.

[8] M.S. Branicky, V.S. Borkar and S.K. Mitter, *A Unified Framework for Hybrid Control: Model and Optimal Control Theory*. IEEE Transactions On Automatic Control, **43**, pp. 31–45 (1998).

[9] A. Briani AND A. Davini, *Monge solutions for discontinuous Hamiltonians*, ESAIM. Control, Optimisation and Calculus of Variations, **11**, pp. 229-251 (2005).

[10] A. Bressan AND Y. Hong, *Optimal control problems for control systems on stratified domains*, Network and Heterogeneous Media, **2**, pp. 313-331 (2007).

[11] R.C. Barnard AND P.R. Wolenski, *The minimal time function on stratified domains*, preprint.

[12] F. Camilli and A. Siconolfi, *Hamilton-Jacobi equations with measurable dependence on the state variable*, Advances in Differential Equations, **8**, pp. 733-768 (2003).

[13] P. Cannarsa AND C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control* Progress in nonlinear differential equations and their applications, Birkhauser Verlag, Basel–Boston–Berlin, 2004.

[14] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial Mathematics, 1990.

[15] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern AND P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer-Verlag, New York, 1997.

[16] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern AND P.R. Wolenski, *Qualitative properties of trajectories of control systems: a survey*, J. Dynamical and Contr. Sys. 1 (1995).

[17] M.C. Delfour and J.P. Zolésio, *Shape analysis via oriented distance functions*, J. of Functional Analysis **123**, pp. 129-201 (1994).

[18] Y. Giga, P. Górka and P. Rybka, *A comparison principle for Hamilton-Jacobi equations with discontinuous Hamiltonians*, Proc. Amer. Math. Soc. **139**. pp. 1777-1785 (2011).

[19] V. Guillemin and A. Pollack , *Differential topology*, Prentice–Hall, London, 1974.

[20] G. Granato and H. Zidani, *Reachability of delayed hybrid systems using level-set methods*. Submitted, 2012.

[21] S. Krantz and H. Park, *Distance to* C^k *hypersurfaces* J. Diff, Eq. **40**, pp. 116–120 (1981).

[22] C. Imbert, R. Monneau AND H. Zidani, *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, ESAIM-COCV, 2012.

[23] J. McShane and R. B. Warfield, Jr., *On Filippov's implicit functions lemma*, Proc. Amer. Math. Soc. **18**, pp. 41-47 (1967).

[24] R.T. Newcomb and J. Su. *Eikonal equations with discontinuities.* Differential and Integral equations, **8**, pp. 1947–1960 (1995).

[25] Z. Rao and H. Zidani, *Hamilton-Jacobi-Bellman Equations on Multi-Domains*, In Control and Optimization with PDE Constraints, International Series of Numerical Mathematics, vol. 164, Birkhäuser Basel, (2013).

[26] R. T. Rockafellar, *Proximal subgradients, marginal values, and augmented Lagrangian in nonconvex optimization*, Math. Oper. Res. **6**, pp. 424-436 (1981).

[27] P. Soravia, *Boundary Value Problems for Hamilton-Jacobi Equations with Discontinuous Lagrangian*, Indiana Univ. Math. J. **51**, pp. 451–477 (2002).

[28] P. Soravia, *Degenerate eikonal equations with discontinuous refraction index*, ESAIM Control Optim. Calc. Var. **12** (2006).

[29] H. Zhang and M.R. James. *Optimal Control of Hybrid Systems and a Systems of Quasi-Variational Inequalities*, SIAM Journal of Control Optimization, **45**, p. 722–761 (2005).