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Abstract. We consider both a cooperative as well as non-cooperative admission into an M/M/1 queue. The only information available is a signal that sais whether the queue size is smaller than some L or not. We first compute the globally optimal and the Nash equilibrium stationary policy as a function of L. We compare the performance to that of full information on the queue size. We identify the L that optimizes the equilibrium performance.

1 Introduction

This paper is devoted to revisitting the problem of whether an arrival should queue or not in an M/M/1 queue. This is perhaps the first problem to be studied in optimal control of queues, going back to the seminal paper of Pinhas Naor [1]. Naor considered an M/M/1 queue, in which a controller has to decide whether arrivals should enter a queue or not. The objective was to minimize a weighted difference between the average expected waiting time of those that enter, and the acceptance rate of customers. Naor then considers the individual optimal threshold (which can be viewed as a Nash equilibrium in a non-cooperative game between the players) and shows that it is also of a threshold type with a threshold $L' > L$. Thus, under individual optimality, arrivals that join the queue wait longer in average. Finally, he showed that there exists some toll such that if it is imposed on arrivals for joining the queue then the threshold value of the individually optimal policy can be made to agree with the social optimal one. Since this seminal work of Naor there has been a huge amount of research that extend the model. More general interarrival and service times have been considered, more general networks, other objective functions and other queueing disciplines, see e.g. [2–10] and references therein.

In the original work of Naor, the decision maker(s) have full state information when entering the system. However, the fact that a threshold policy is optimal implies that for optimally controlling arrivals we only need partial information - we need a signal to indicate whether the queue exceeds or not the threshold value L. The fact that this much simpler information structure is sufficient for obtaining the same performance as in the full information case motivates us to study the performance of threshold policy and related optimization issues.

We first consider the socially optimal control policy for a given (non-necessarily optimal) threshold value L. When L is chosen non-optimally then the optimal policy for the partial information problem does not anymore coincide with the policy with full information.

We then study the individual optimization problem with the same partial information: a signal (red) if the queue length exceeds some value L and a green signal otherwise.

For both the social and the individual optimization problems we show that the following structure holds: either whenever the signal is green all arrivals are accepteed with probability 1, or whenever the signal is red all arrivals are rejected with probability 1.

We note that by using this signalling approach instead of providing full state information, users cannot choose any threshold policy with parameter different than L. Thus, in the individual optimization case, one could hope that by determining the signalling according to the value L that optimizes the socially optimal problem (in case of full information), one would obtain the socially optimal performance. We show that this is not the case, and determine the value L for which the reaction of the users optimizes the system performance. We compare this to the performance in case of full information.

2 The model

Assume an M/M/1 queue where the admission rate is $\underline{\lambda}$ for $i \geq L$ and is otherwise $\overline{\lambda}$. Let μ be the service rate and set $\overline{\rho} = \overline{\lambda}/\mu$ and $\underline{\rho} = \underline{\lambda}/\mu$. We shall make the standard stability assumption that $\underline{\rho} < 1$. The balance equations are given

$$\mu\pi(i+1, L) = \lambda\pi(i, L)$$

where $\lambda = \underline{\lambda}$ for $i > L$ and is otherwise given by $\lambda = \overline{\lambda}$. The solution of these equations give

$$\pi(i, L) = \pi(0, L)\overline{\rho}^i$$

for $i \leq L$ and otherwize

$$\pi(i, L) = \pi(L, L)\underline{\rho}^{i-L}. \tag{1}$$

Hence

$$\pi(0,L) = \frac{1}{\sum_{i=0}^{L-1} \overline{\rho}^i + \overline{\rho}^L \sum_{i=0}^{\infty} \underline{\rho}^i}$$

$$= \frac{1}{\frac{1-\overline{\rho}^L}{1-\overline{\rho}} + \frac{\overline{\rho}^L}{1-\underline{\rho}}}$$

Thus

$$\pi(L,L) = \frac{1-\underline{\rho}}{1-\left(\frac{(1-\underline{\rho})(1-\overline{\rho}^{-L})}{1-\overline{\rho}}\right)} \tag{2}$$

Assume that an arrival receives the information on whether the size of the queue exceeds $L-1$ or not. If it does not exceeds we shall say that it receives a "green" signal denoted by G, and otherwise a red one (R). The conditional state probabilities given the signals are denoted by

$$\pi(i,L|R) = (1-\underline{\rho})\underline{\rho}^{i-L}$$

for $i \geq L$, and is otherwise zero. The conditional tail distribution is

$$P(I > n|R) = \underline{\rho}^{n+1-L}$$

for $n \geq L$, and is otherwise 1. Thus

$$E(I|R) = (L-1) + \frac{1}{(1-\underline{\rho})} \tag{3}$$

For a green light we have:

$$\pi(i,L|G) = \frac{1-\overline{\rho}}{1-\overline{\rho}^L}\overline{\rho}^i$$

for $0 \leq i < L$ and is otherwise zero. Hence the tail probabilities are

$$P(I > n|G) = \frac{\overline{\rho}^{n+1} - \overline{\rho}^L)}{1-\overline{\rho}^L}$$

for $n < L$, and is otherwise 0. Hence

$$E(I|G) = \frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right) \tag{4}$$

3 The partially observed control problem

We assume that $\nu < \underline{\lambda}$ is the rate of some uncontrolled Poisson flow. In addition there is an independent Poisson arrival flow of intensity ζ. We restrict to stationary policies, i.e. policies that are only function of the observation. A policy is thus a set of two probabilities: q_s where s is either R or G. q_s is the probability of accepting an arrival when the signal is s. For a given policy, we obtain the framework of the previous section with

$$\underline{\lambda} = \nu + \zeta q_R, \quad \overline{\lambda} = \nu + \zeta q_G.$$

Our goal is to minimize over $\mathbf{q}$

$$J_{\mathbf{q}} = E_{\mathbf{q}}[I] - \gamma T_{acc}(\mathbf{q}) = \sum_{s=G,R} P_{\mathbf{q}}(s)(E_{\mathbf{q}}[I|s] - \gamma T_{acc}(\mathbf{q}))$$

where

$$T_{acc} = \overline{\lambda} * P(G) + \underline{\lambda} * P(R) = \mu[P(R)(\underline{\rho} - \overline{\rho}) + \overline{\rho}]$$

and $P(R) = P(I \geq L)$ is given by

$$P(R) = \pi(L, L)\frac{1}{1-\underline{\rho}} \tag{5}$$

$$E[I] = E[I|R]*P(R)+E[I|G]*P(G) = (E[I|R]-E[I|G])*P(R)+E[I|G],$$

$$= \Big(((L-1) + \frac{1}{\underline{\rho}^L(1-\underline{\rho})}) - (\frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right))\Big) \times$$

$$\frac{(1-\overline{\rho})\overline{\rho}^L}{(1-\underline{\rho}) + \overline{\rho}^L(\underline{\rho}-\overline{\rho})} + \Big(\frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right)\Big)$$

The expression obtained for $J_{\mathbf{q}}$ is lower semi-continuous in the policy $\mathbf{q} = \{q_s, s = G, R\}$. Hence a minimizing policy $\mathbf{q}^*$ exists.

Lemma 1. *Assume that $\nu > 0$. If $\rho_R \geq 1$ then for any L $E[I]$ is infinite. In particular, if $\underline{\rho} \geq 1$ then for any L and any q, $E[I]$ is infinite.*

Proof. The expected queue length $E[I_t]$ at any time t and for any L is bounded from below by the one obtained by $E[I'_t] - L$ where I'_T is the queue size obtained when replacing q_G with $q_G = q_R$. $E[I'_t]$ correspondns to an M/M/1 queue with a workload $\rho \geq 1$ which is known to have infinite expectation. ■.

3.1 The structure of optimal policies

Figure 1 shows the values of the two component of the vector ρ corresponding to the optimal policy. We assume that ν and λ are such that $\overline{\rho} = 0.8$ and $\underline{\rho} = 0.3$. We further took $\mu = 1$, $\gamma = 15$, for 4 different values of the threshold L.

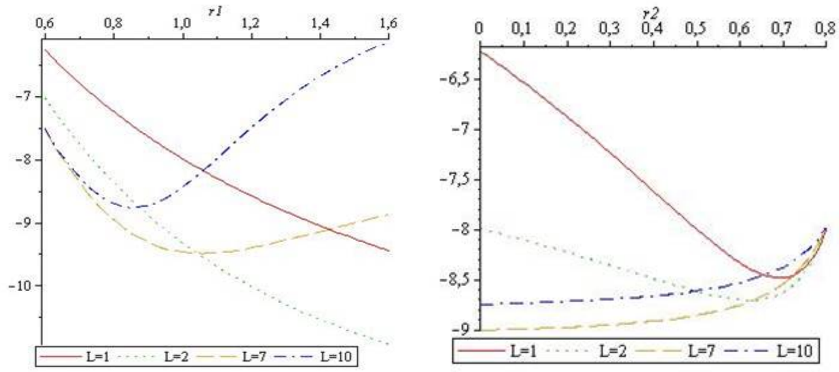

Fig. 1. The performance of different policies for several values of L

We observe the following structure: for any L, the optimal vector ρ satisfies the following property: whenever the minimum cost is achieved at an interior point for one of the components of ρ, then it is achieved on the boundary for the other component. More precisely, the optimal ρ satisfies either $\rho_2 = (r2) = \underline{\rho}$ or $\rho_1 = (r1) = \overline{\rho}$. We shall next prove this structure for the partially observable control problem.

Theorem 1. *Assume that $0 < \nu/\mu < 1$. Then there is a unique optimal stationary strategy and it has the following property: either $q^*(G) = 1$ or $q^*(R) = 0$.*

Proof. Let $\mathbf{q}$ be optimal. We first show that $\alpha > 0$ where $\alpha := \mu - (\nu + q_R)\zeta)$. Indeed, if it were not the case then we would have $\rho \geq 1$ so by the previous Lemma, the queue length and hence the cost would be infinite. But then $\mathbf{q}$ cannot be optimal since the cost can be made finite by choosing $q_R) = 0$.

Assume that an optimal policy $\mathbf{q}$ does not have the structure stated in the Theorem. This would imply that q_R can be further decreased and q_G increased. In particular, one can perturb $\mathbf{q}$ in that way so that T_{acc} is unchanged. More precisely, note first that T_{acc} is monotone increasing in both q_R) and in q_G. Hence

$$T_{acc}(1, q(R)) \geq T_{acc}(\mathbf{q}) \geq T_{acc}(q_G), 0),$$

Hence, if $T_{acc}(1, 0) < T_{acc}(\mathbf{q})$ then there is some $\mathbf{q_2}$ such that

$$T_{acc}(\mathbf{q_2}) = T_{acc}(\mathbf{q})$$

where either

$$\mathbf{q_2} := (1, q_R^2), \quad \text{or} \quad \mathbf{q_2} := (q_G^2, 0)$$

We have

$$P_{\mathbf{q_2}}(I = 0) = 1 - T_{acc}(\mathbf{q_2}) = 1 - T_{acc}(\mathbf{q})$$

(e.g. from Little's Theorem). From rate balance arguments it follows that

$$P_{\mathbf{q_2}}(I = i) = (1 - T_{acc})(\mathbf{q_2})\overline{\rho}_2^i \qquad \text{for } i \leq L. \tag{6}$$

Hence

$$P_{\mathbf{q_2}}(I \geq i) < P_{\mathbf{q}}(I \leq i) \tag{7}$$

for $i \geq L$. Thus

$$P_{\mathbf{q2}}(R) < P_{\mathbf{q}}(R).$$

By combining this with (1) it follows that

$$P_{\mathbf{q_2}}(I \geq i) = P_{\mathbf{q_2}}(R)\underline{\rho}(\mathbf{q_2})^{i-L} \leq P_{\mathbf{q_2}}(R)\underline{\rho}(\mathbf{q_2})^{i-L} \leq = P_{\mathbf{q_2}}(I \geq i)$$

Hence (7) holds for all i. Taking the sum over i we thus obtain that

$$E_{\mathbf{q_2}}[I] < E_{\mathbf{q}}[I].$$

Since T_{acc} are the same under $\mathbf{q}$ and $\mathbf{q_2}$, it follows that $J_{\mathbf{q_2}} < J_{\mathbf{q}}$. Hence $\mathbf{q}$ is not optimal, which contradicts the assumption in the beginning of the proof. This establishes the structure of optimal policies. ■

3.2 Optimizing the signal

Here we briefly discuss the case of choosing L so as to minimize $J_{\mathbf{q}}$ not only with respect to $\mathbf{q}$ but also with respect to the value L of the threshold.

To that end we first consider the problem of minimizing J over all stationary policies in case that full state information is available. This is a Markov decision process and an optimal policy is known to exists within the pure stationary policies. Moreover, a direct extension of the proof in [1] can be used to show that the structure of the optimal policy is of a threshold type: accept all arrivals as long as the state is below a threshold and reject all controlled arrivals otherwise. Note however that this policy makes use only of the information available also in our cases, i.e. of whether the state exceeds L or not.

We conclude that the problem of optimizing $J_{\mathbf{q}}$ over both L and $\mathbf{q}$ has an optimal pure threhold policy i.e. with $q_R = 0$ and $q_{G=1}$, or in other words $\mathbf{q} = (1, 0)$.

The optimal L for our problem can therefore be computed by minimizing $J_{\mathbf{q}}$ over pure threshold policies. In Figure (2) we compute this optimal L for $\mu = 1$, $\eta = 0.01$, $\lambda = 0.98$ and $\gamma = 1, 5, 10, 15, 20$. and obtain $L = 5$ for $\gamma = 20$.

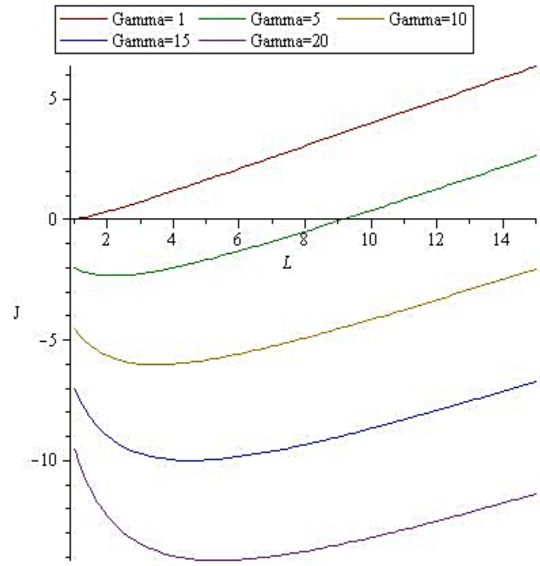

Fig. 2. The optimal policy for several values of L and γ

4 The game problem

We again assume that there is some uncontrolled flow ν and a flow of strategic players with intensity ζ. All users receive the signal G or R as before, and we restrict to polices as in the control case.

Assume that an arrival has a reward $\psi > 0$ for being processed in the queue, and a waiting cost of $E[W|s]$ where W is the waiting time. Note that $E[W|s] = E[I|s]/\mu$.

Let $Y(P)$, where $P = P(s)$, $s = R, G$ be the set of best responses of an individual if all the rest use $P(s), s = R, G$, and the system is in the corresponding steady state.

Then q is an equilibrium strategy if and only if $q \in Y(q)$. Note that the cost $J(q, P)$ corresponding to a strategy q of a player, when all others play P satisfies the following in order to be a best response to P: for each s, if $q(s)$ is not pure (is not 0 or 1) then at s, any other probability q' is also a best response.

The cost for a user for entering when the signal is s given that the strategy of other users is $\mathbf{q} = (q_G, q_R)$ is given by

$$V_{\mathbf{q}}(s) = E_{\mathbf{q}}[W|s] - \gamma = E_{\mathbf{q}}[I|s]/\mu - \gamma \tag{8}$$

It is zero if it does not enter. Here, $E_{\mathbf{q}}[I, s]$ are given by

$$E_{\mathbf{q}}(I|R) = (L-1) + \frac{1}{(1-\rho)} \tag{9}$$

where $\mathbf{q} = (1, q_R)$ and where $\rho = (\nu + \zeta q_R)/\mu$, and

$$E_{\mathbf{q}}(I|G) = \frac{1}{1-\rho^L}\left(\frac{(\rho^L - \rho)}{\rho - 1} - (L-1)\rho^L\right) \tag{10}$$

where $\mathbf{q} = (q_G, 0)$ and where $\rho = (\nu + \zeta q_G)/\mu$. (the derivations of the above are as in (3) and (4), respectively.

4.1 Structure of equilibrium

The following gives the structure of equilibria policies.

Theorem 2. *(1) The equilibrium policy is to enter for any signal if and only if* $V_{(1,1)}(R) \leq 0$
(2) The equilibrium is of the form $\mathbf{q} = (1, q_R)$ *where* $q_R \in (0, 1)$ *if and only if* $V_{(1,1)}(R) > 0 > V_{(1,0)}(R)$*. In this case, the equilibrium is given by the* $q = (1, q_R)$ *where* q_R *is the solution of* $V_{(1,q_R)} = 0$ *where* V_q *is given in (8).*
(3) The equilibrium is of the form $\mathbf{q} = (q_G, 0)$ *where* $q_G \in (0, 1]$ *if and only if* $V_{(1,0)}(G) \geq 0$*. In this case, the equilibrium is given by the* $\mathbf{q} = (q_G, 0)$ *where* q_G *is the solution of* $V_{(q_G,0)} = 0$ *and where* $V_{\mathbf{q}}$ *is given in (8).*

Proof. Follows directly from continuity of the expected queue length with respect to $\mathbf{q}$ and from the fact that $V_{(q_G,q_R)}$ is strictly monotone increasing in both arguments. We establish the continuity in the Appendix using an approach that does not require the exact explicit form of $E[I]$ and thus will be useful when attempting to generalize the results to other models (such as the case of more than a single server). ■.

4.2 Numerical Examples

We consider here as an example the parameters $\gamma = 20$, $\mu = 1$, $\lambda = 0.98$ and $\zeta = 0.01$. For all L condition (1) of Theorem 2 does not hold, so (1,1) is not an equilibrium. condition (2) of the Theorem holds for $L \leq 20$. In that case, the equilibrium is given by $(1, q_R)$ where q_R is given in Fig 3. The value at equilibrium is given in Figure 4 for the case of the signal G and is otherwize zero for all $L \leq 20$. For the case of $L > 20$ we have the opposite, i.e. $V_R = 0$. V_G is given by $E[I|G] - \gamma$ where $E[I|G]$ is expressed in (4).

Let L^* denote one plus the largest value L for which $V_{1,0} < 0$. L^* thus separates case (2) and (3) in Theorem 2. Then L^* equals the smallest integer greater than or equal to $\gamma\mu$. In our case it is given by 20 as is seen in Figure 4. For every $L > L^*$ we know that, in fact, $q_R = 0$. Indeed, a red signal in that case would mean that the queue exceeds size $\gamma\mu$ and thus if the individual entered, its expected waiting time would exceed γ. For $L < L^*$ we know that $q_G = 1$ since the expected time of an admitted customer would be smaller than γ. It is then easy to see that for $L = L^*$, the pure threshold policy with parameter L^* is a pure (state dependent) equilibrium in the game with full information.

4.3 Optimizing the signal

We are interested here in finding the L for which the induced equilibrium gives the best system performance. We plot the system performance J at equilibrium as a function of L in Figure 5.

The optimal L is seen to equal 20 and the corresponding performance measures at equilibrium are $J = -14.13$ and $T_{acc} = 0.83$. We conclude that the policy for which the social cost is minimized has the same performance as the full state information equilibrium policy.

If we take the $L = 5$ which we had computed for optimizing the system performance, and use it in the game setting, we obtain $T_{acc} = 0.95$ and $J^* = -3.49$. which indeed gives a much worse performance than the performance under the $L = 20$.

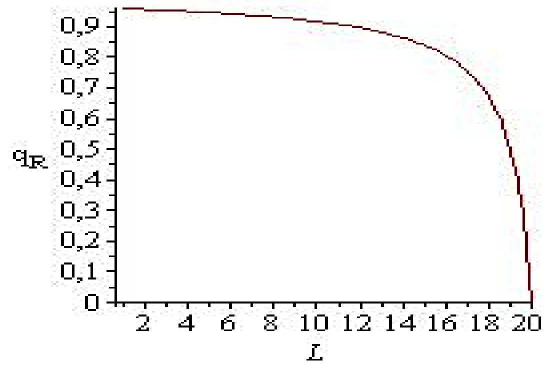

Fig. 3. Equilibrium action q_R as a function of L

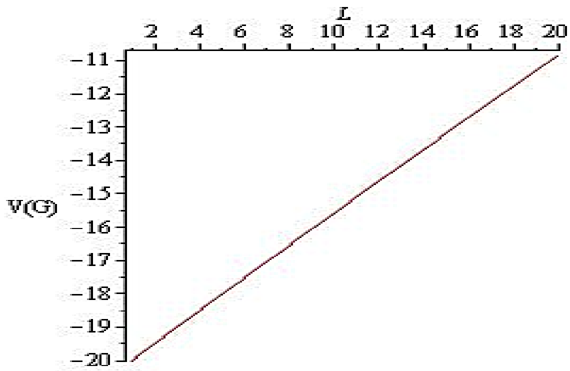

Fig. 4. Equilibrium value V_G for signal G as a function of L. We used case (2) in Theorem 2 and the results are therefore valid only for $L \leq 20$.

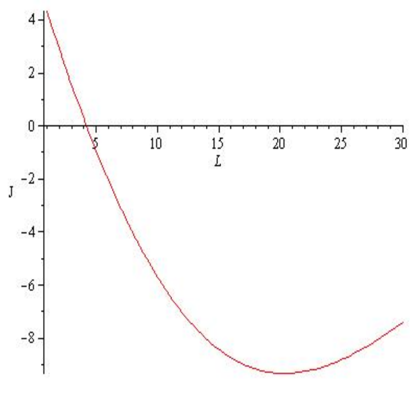

Fig. 5. The social value J at equilibrium as a function of L

5 Appendix: Uniform f-geometric ergodicity and the continuity of the Markov chain

We show continuity of the expected queue length in $\mathbf{q}$ for $q_{\mathbf{R}}$ restricted to some closed interval for which the corresponding value of ρ_R is smaller than 1. (Due to Lemma 1 there exists indeed such an interval such that any policy for which q_R is not in the interval cannot be optimal.

We show that the Markov chain is f-Geometric Ergodic and then use Lemma 5.1 from [11].

Consider the Markov chain embedded at each transition in the queue size. Thus for $I \geq \max(L, 1)$, with probability β the event is a departure and otherwise it is an arrival, where

$$\beta := \frac{\mu}{\mu + \nu + q_R \zeta}.$$

Note $\alpha > 0$ implies that $\beta > 1/2$ (α is defined in the proof of Theorem 1).

Define $f(i) = \exp(\gamma i)$, for any $I \geq \max(L, 1)$,

$$E[f(I_{t+1}) - f(I_t)|I_t = i] = \beta \exp[\gamma(i-1)] + (1-\beta)\exp[\gamma(i+1)] - \exp(\gamma i)$$
$$= f(i)\Delta \quad \text{where} \quad \Delta = \beta z^{-1} + (1-\beta)z - 1$$

and where $z := \exp(-\gamma)$. Note that $\Delta = 0$ at

$$z_{1,2} = \frac{1 \pm \sqrt{1 - 4\beta(1-\beta)}}{2(1-\beta)} = \{1, \frac{\beta}{1-\beta}\}$$

Thus $\Delta < 0$ for all γ in the interval $\left(0, \log\left(\frac{\beta}{1-\beta}\right)\right)$ (which is nonempty since we showed that $1 > \beta > 1/2$). We conclude that for any γ in that interval, f is a Lyapunov function and the Markov chain is γ-geometrically ergodic uniformly in $\mathbf{q}$.

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