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# Enumerating the edge-colourings and total colourings of a regular graph\*

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## Abstract

In this paper, we are interested in computing the number of edge colourings and total colourings of a connected graph. We prove that the maximum number of  $k$ -edge-colourings of a connected  $k$ -regular graph on  $n$  vertices is  $k \cdot ((k-1)!)^{n/2}$ . Our proof is constructive and leads to a branching algorithm enumerating all the  $k$ -edge-colourings of a connected  $k$ -regular graph in time  $O^*((k-1)!)^{n/2}$  and polynomial space. In particular, we obtain an algorithm to enumerate all the 3-edge-colourings of a connected cubic graph in time  $O^*(2^{n/2}) = O^*(1.4143^n)$  and polynomial space. This improves the running time of  $O^*(1.5423^n)$  of the algorithm due to Golovach et al. [12]. We also show that the number of 4-total-colourings of a connected cubic graph is at most  $3 \cdot 2^{3n/2}$ . Again, our proof yields a branching algorithm to enumerate all the 4-total-colourings of a connected cubic graph.

## 1 Introduction

We refer to [6] for standard notation and concepts for graphs. In this paper, all the considered graphs are loopless, but may have parallel edges. A graph with no parallel edges is said to be *simple*. Let  $G$  be a graph. We denote by  $n(G)$  the number of vertices of  $G$ , and for each integer  $k$ , we denote by  $n_k(G)$  the number of degree  $k$  vertices of  $G$ . Often, when the graph  $G$  is clearly understood, we abbreviate  $n(G)$  to  $n$  and  $n_k(G)$  to  $n_k$ . Throughout the paper, we are concerned with connected graphs.

Graph colouring is one of the classical subjects in graph theory. See for example the book of Jensen and Toft [14]. From an algorithmic point of view, for many colouring type problems, like vertex colouring, edge colouring and total colouring, the existence problem asking whether an input graph has a colouring with an input number of colours is NP-complete. Even more, these colouring problems remain NP-complete when the question is whether there is a colouring of the input graph with a fixed (and greater than 2) number of colours [11, 13, 19].

Exact algorithms to solve NP-hard problems are a challenging research subject in graph algorithms. Many papers on exact exponential time algorithms have been published in the last decade. One of the major results is the  $O^*(2^n)$ -time inclusion-exclusion algorithm to compute the chromatic

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number of a graph found independently by Björklund, Husfeldt [3] and Koivisto [16], see [5]. This approach may also be used to establish a  $O^*(2^n)$ -time algorithm to count the  $k$ -colourings and to compute the chromatic polynomial of a graph. It also implies a  $O^*(2^m)$ -time algorithm to count the  $k$ -edge-colourings and a  $O^*(2^{n+m})$ -time algorithm to count the  $k$ -total-colourings of a given graph.

Since edge colouring and total colouring are particular cases of vertex colouring, a natural question is to ask if faster algorithms than the general one may be designed in these cases. For instance, very recently Björklund et al. [4] showed how to detect whether a  $k$ -regular graph admits a  $k$ -edge-colouring in time  $O^*(2^{(k-1)n/2})$ .

The existence problem asking whether a graph has a colouring with a fixed and small number  $k$  of colours also attracted a lot of attention. For vertex colourability the fastest algorithm for  $k = 3$  has running time  $O^*(1.3289^n)$  and was proposed by Beigel and Eppstein [2], and the fastest algorithm for  $k = 4$  has running time  $O^*(1.7272^n)$  and was given by Fomin et al. [10]. They also established algorithms for counting  $k$ -vertex-colourings for  $k = 3$  and 4. The existence problem for a 3-edge-colouring is considered in [2, 17, 12]. Kowalik [17] gave an algorithm deciding if a graph is 3-edge-colourable in time  $O^*(1.344^n)$  and polynomial space and Golovach et al. [12] presented an algorithm counting the number of 3-edge-colourings of a graph in time  $O^*(3^{n/6}) = O^*(1.201^n)$  and exponential space. Golovach et al. [12] also showed a branching algorithm to enumerate all the 3-edge-colourings of a connected cubic graph in time  $O^*(25^{n/8}) = O^*(1.5423^n)$  and polynomial space. In particular, this implies that every connected cubic graph of order  $n$  has at most  $O(1.5423^n)$  3-edge-colourings. They give an example of a connected cubic graph of order  $n$  having  $\Omega(1.2820^n)$  3-edge-colourings. In Section 2, we prove that a connected cubic graph of order  $n$  has at most  $3 \cdot 2^{n/2}$  3-edge-colourings and give an example reaching this bound. Our proof can be translated into a branching algorithm to enumerate all the 3-edge-colourings of a connected cubic graph in time  $O^*(2^{n/2}) = O^*(1.4143^n)$  and polynomial space. Furthermore, we extend our result proving that every  $k$ -regular connected graph of order  $n$  admits at most  $k \cdot ((k-1)!)^{n/2}$   $k$ -edge-colourings. This improves the bound derived from the maximal number of perfect matchings of a graph with a given degree sequence given by Alon and Friedland [1]. Also, similarly 3-edge-colourings, we derive a branching algorithm to enumerate all the  $k$ -edge-colourings of a connected  $k$ -regular graph in time  $O^*(((k-1)!)^{n/2})$  and polynomial space.

Regarding total colouring, very little has been done. Golovach et al. [12] showed a branching algorithm to enumerate the 4-total-colourings of a connected cubic graph in time  $O^*(2^{13n/8}) = O^*(3.0845^n)$ , implying that the maximum number of 4-total-colourings in a connected cubic graph of order  $n$  is at most  $O^*(2^{13n/8}) = O^*(3.0845^n)$ . In Section 3, we lower this bound to  $3 \cdot 2^{3n/2} = O(2.8285^n)$ . Again, our proof yields a branching algorithm to enumerate all the 4-total-colourings of a connected cubic graph in time  $O^*(2.8285^n)$  and polynomial space.

## 2 Edge colouring

A (*proper*) *edge colouring* of a graph is a colouring of its edges such that two adjacent edges receive different colours. An edge colouring with  $k$  colours is a  *$k$ -edge-colouring*. We denote by  $c_k(G)$  the number of  $k$ -edge-colourings of a graph  $G$ .

## 2.1 General bounds for $k$ -regular graphs

In this section, we are interested in computing the number of  $k$ -edge-colourings of  $k$ -regular connected graphs. We start by computing exactly the number of 3-edge-colourings of the cycles.

**Proposition 1.** *Let  $C_n$  be the cycle of length  $n$ .*

$$c_3(C_n) = \begin{cases} 2^n + 2, & \text{if } n \text{ is even,} \\ 2^n - 2, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* By induction on  $n$ . It is easy to check that  $c_3(C_2) = c_3(C_3) = 6$ .

Let  $C_n = (v_1, v_2, \dots, v_n, v_1)$ . Let  $A$  be the set of 3-edge-colourings of  $C_n$  such that  $c(v_{n-1}v_n) \neq c(v_1v_2)$  and  $B$  the set of 3-edge-colourings of  $C_n$  such that  $c(v_{n-1}v_n) = c(v_1v_2)$ . The 3-edge-colourings of  $A$  are in one-to-one correspondence with those of  $C_{n-1}$  and the pair of colourings of  $B$  agreeing everywhere except on  $v_nv_1$  are in one-to-one correspondence with the 3-edge-colourings of  $C_{n-2}$ . Thus  $c_3(C_n) = c_3(C_{n-1}) + 2c_3(C_{n-2})$ . Hence, if  $n$  is even, then  $c_3(C_n) = 2^{n-1} - 2 + 2(2^{n-2} + 2) = 2^n + 2$ , and if  $n$  is odd, then  $c_3(C_n) = 2^{n-1} + 2 + 2(2^{n-2} - 2) = 2^n - 2$ .  $\square$

Let us now present our method which is based on a classical tool:  $(s, t)$ -ordering.

**Definition 2.** Let  $G$  be a graph and  $s$  and  $t$  be two distinct vertices of  $G$ . An  $(s, t)$ -ordering of  $G$  is an ordering of its vertices  $v_1, \dots, v_n$  such that  $s = v_1$  and  $t = v_n$ , and for all  $1 < i < n$ ,  $v_i$  has a neighbour in  $\{v_1, \dots, v_{i-1}\}$  and a neighbour in  $\{v_{i+1}, \dots, v_n\}$ .

**Lemma 3** (Lempel et al. [18]). *A graph  $G$  is a 2-connected graph if, and only if, for every pair  $(s, t)$  of vertices, it admits an  $(s, t)$ -ordering.*

In fact, Lempel et al. established Lemma 3 only for simple graphs but it can be trivially extended to graphs since replacing all the parallel edges between two vertices by a unique edge does not change the connectivity.

**Theorem 4.** *Let  $G$  be a 2-connected subcubic graph. Then  $c_3(G) \leq 3 \cdot 2^{n - \frac{n_3}{2}}$ .*

*Proof.* If  $G$  is a cycle, then the result follows from Proposition 1. Hence we may assume that  $G$  is not a cycle and thus has at least two vertices of degree 3, say  $s$  and  $t$ . By Lemma 3, there exists an  $(s, t)$ -ordering  $v_1, v_2, \dots, v_n$  of  $G$ . Orient each edge of  $G$  according to this order, that is from the lower-indexed end-vertex towards its higher-indexed one. Let us denote by  $D$  the obtained digraph. Observe that  $d^+(v_1) = 3 = d^-(v_n)$  and  $d^-(v_1) = 0 = d^+(v_n)$ . Let  $A^+$  (resp.  $A^-$ ) be the set of vertices with outdegree 2 (resp. indegree 2) in  $D$  and  $A_2$  be the set of vertices with degree 2 in  $G$  (and thus with indegree 1 and outdegree 1 in  $D$ ). Clearly,  $(A_2, A^-, A^+)$  is a partition of  $V(D) \setminus \{v_1, v_n\}$ . Observe that  $|A_2| = n - n_3$ . Since  $\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v)$ , we have  $|A^+| = |A^-|$ , and so  $|A^+| = (n_3 - 2)/2$ .

Now for  $i = 1$  to  $n - 1$ , we enumerate the  $p_i$  partial 3-edge-colourings of the arcs whose tail is in  $\{v_1, \dots, v_i\}$ . For  $i = 1$ , there are 6 such colourings, since  $d^+(v_1) = 3$ .

Now, for each  $i$ , when we want to extend the partial colourings, two cases may arise.

- If  $d_D^-(v_i) = 1$ , then we need to colour one or two arcs, and one colour (the one of the arc entering  $v_i$ ) is forbidden, so there are at most 2 possibilities. Hence  $p_i \leq 2p_{i-1}$ .
- If  $d_D^-(v_i) = 2$ , then we need to colour one arc, and at least two colours (the ones of the arcs entering  $v_i$ ) are forbidden, so there is at most one possibility. Hence  $p_i \leq p_{i-1}$ .

At the end, all the edges of  $G$  are coloured, and a simple induction shows that  $c_3(G) = p_{n-1} \leq 6 \cdot 2^{|A_2|+|A^+|} = 3 \cdot 2^{n-\frac{n_3}{2}}$ .  $\square$

In particular, for a connected cubic graph  $G$ , we obtain  $c_3(G) \leq 3 \cdot 2^{n/2}$ . We now extend this result to  $k$ -regular graphs.

First, remark that every  $k$ -edge-colouring of a  $k$ -regular graph  $G$  is precisely the union of  $k$  (pairwise disjoint) perfect matchings of  $G$ . So, a bound on the number of perfect matchings in a  $k$ -regular graph yields to a bound on the number of  $k$ -edge-colourings of this graph. Computing the maximal number of perfect matchings in a graph is a classical problem in Graph Theory [7]. The best bound on graphs with degree constraints is due to Alon and Friedland [1]. They proved that a  $k$ -regular graph admits at most  $(k!)^{\frac{n}{2k}}$  distinct perfect matchings. So, as a  $k$ -edge-colouring is made of  $k$  perfect matchings, a  $k$ -regular graph admits at most  $(k!)^{\frac{n}{2}}$   $k$ -edge-colourings. Under the assumption that the graph is connected, we improve this bound.

**Theorem 5.** *Let  $G$  be a connected  $k$ -regular graph, with  $k \geq 3$ . Then  $c_k(G) \leq k \cdot ((k-1)!)^{n/2}$ .*

*Proof.* First, observe that if a connected  $k$ -regular graph  $G$  admits a  $k$ -edge-colouring, then every colour induces a perfect matching of  $G$ , and then  $n$  is even. Furthermore, observe that  $G$  is 2-connected. Indeed, assume that  $G$  has a cutvertex  $x$  and admits a  $k$ -edge-colouring  $c$ . As  $G$  has an even number of vertices, one of the connected components, say  $C$ , of  $G - x$  has odd cardinality. A colour appearing on an edge between  $x$  and a connected component of  $G - x$  different from  $C$  must form a perfect matching on  $C$  which is impossible. So,  $G$  is 2-connected.

Hence we can use the method of the proof of Theorem 4 and consider an  $(s, t)$ -ordering  $v_1, \dots, v_n$  of  $G$  and  $D$  the orientation of  $G$  obtained from this ordering (i.e.  $v_i v_j \in A(D)$  if and only if  $v_i v_j \in E(G)$  and  $i < j$ ). The analysis made in the proof of Theorem 4 yields  $c_k(G) \leq \prod_{x \in V(G)} (d^+(x)!)^k$ . For  $i = 1, \dots, k-1$ , we define  $A_i = \{x \in V(G) \setminus \{v_1, v_n\} : d^+(x) = i\}$ . It is clear that  $(A_i)_{1 \leq i \leq k-1}$  form a partition of  $V(G) \setminus \{v_1, v_n\}$ . If we denote  $|A_i|$  by  $a_i$ , then  $c_k(G) \leq P := k! \prod_{i=1}^{k-1} (i!)^{a_i}$ . Moreover  $S_1 := \sum_{i=1}^{k-1} a_i = n-2$  (by counting the number of vertices of  $G$ ) and  $S_2 := \sum_{i=1}^{k-1} i \cdot a_i = k(n-2)/2$  (by counting the number of arcs of  $D - v_1$ ).

Let us now find the maximum value of  $P$  under the conditions  $S_1 = n-2$  and  $S_2 = k(n-2)/2$ . If we can find  $1 < p \leq q < k-1$  with  $a_p \neq 0$  and  $a_q \neq 0$  (or  $a_p \geq 2$  if  $p = q$ ), then we decrease  $a_p$  and  $a_q$  by one and increase  $a_{p-1}$  and  $a_{q+1}$  by one. Doing this,  $S_1$  and  $S_2$  are unchanged and  $P$  is multiplied by  $\frac{q+1}{p} > 1$ . We repeat this operation as many times as possible and stop when (a) for every  $i = 2, \dots, k-2$ ,  $a_i = 0$  or (b) there exists  $j \in \{2, \dots, k-2\}$  such that for every  $i = 2, \dots, k-2$  and  $i \neq j$ ,  $a_i = 0$  and  $a_j = 1$ . In case (b),  $S_1$  gives  $a_1 + 1 + a_{k-1} = n-2$  and  $S_2$  is  $a_1 + j + (k-1)a_{k-1} = k(n-2)/2$ . Combining  $S_1$  and  $S_2$ , we obtain  $2(k-2)a_{k-1} + 2(j-1) = (k-2)(n-2)$  and we conclude that  $k-2$  divides  $2(j-1)$  and so that  $j = k/2$ . Solving  $S_1$  and  $S_2$  we have in particular that  $a_1 = a_{k-1}$  and  $2a_1 = n-3$  which is impossible, as  $n$  is even. Hence, we are in case (a), and solving  $S_1$  and  $S_2$  yields to  $a_1 = a_k = (n-2)/2$ . We conclude that  $P \leq k!((k-1)!)^{n/2-1}$ .  $\square$

We turn now the proof of Theorem 5 into an algorithm to enumerate all the  $k$ -edge-colourings of a connected  $k$ -regular graph.

**Corollary 6.** *There is an algorithm to enumerate all the  $k$ -edge-colourings of a connected  $k$ -regular graph on  $n$  vertices in time  $O^*((k-1)!)^{n/2}$  and polynomial space.*

*Proof.* Let  $G$  be a connected  $k$ -regular graph. We first check the 2-connectivity of  $G$ . If it is not 2-connected, then we return ‘The graph is not  $k$ -edge-colourable’.

If it is 2-connected, then we proceed as follows. We compute an  $(s, t)$ -ordering  $v_1, \dots, v_n$  of  $G$ , which can be done in polynomial time (see [8] and [9] for instance), and orient  $G$  accordingly to this ordering. Now, it is classical to enumerate all the permutations of a set of size  $p$  in time  $O(p!)$  and linear space, in such way that, being given a permutation we compute in average constant time the next permutation in the enumeration (with the Steinhaus-Johnson-Trotter algorithm for instance, see [15]).

Using this and the odometer principle, it is now easy to enumerate all the edge colourings we want. In the enumeration of all the permutations of  $\{1, \dots, k\}$ , we take the first one and assign the corresponding colours to the arc with tail  $x_1$ . For any index  $i$  with  $2 \leq i \leq k$ , we assign to the arcs with tail  $x_i$  the first permutation in the enumeration of the permutations of the possible colours for these arcs (i.e. all the colours of  $\{1, \dots, k\}$  minus the one of the arcs entering in  $x_i$ ). Then, we have the first colouring, and we check if it is a proper edge colouring of  $G$  (in polynomial time). To obtain the next colouring, we take the next permutation on the colours possible on the arcs with tail  $x_{n-1}$ , and so on. Once all the possible permutations have been enumerated for these arcs, we take the next permutation on the colours possible on the arcs with tail  $x_{n-2}$  and re-enumerate the permutation of possible colours for the arcs with tail  $x_{n-1}$ , and so on, following the odometer principle.  $\square$

The bound given by Theorem 5 is optimal on the class of connected  $k$ -regular graphs. For all  $k \geq 3$ , and  $n \geq 2$ ,  $n = 2p$  even, the  $k$ -noodle necklace  $N_n^k$  is the  $k$ -regular graph obtained from a cycle on  $2p$  vertices  $(v_1, v_2, \dots, v_{2p}, v_1)$  by replacing all the edges  $v_{2i-1}v_{2i}$ ,  $1 \leq i \leq p$  by  $k - 1$  parallel edges.

**Proposition 7.** *Let  $k \geq 3$  and  $n \geq 2$ ,*

$$c_k(N_n^k) = k \cdot ((k - 1)!)^{n/2}.$$

*Proof.* Observe that in every  $k$ -edge-colouring of  $N_n^k$  the edges which are not multiplied (i.e.  $v_{2i}v_{2i+1}$ ) are coloured the same. There are  $k$  choices for such a colour. Once this colour is fixed, there are  $(k - 1)!$  choices for each set of  $k - 1$  parallel edges. Hence  $c_k(N_n^k) = k \cdot ((k - 1)!)^{n/2}$ .  $\square$

## 2.2 A more precise bound for cubic graphs

For simple cubic graphs, we lower the bound on the number of 3-edge-colourings from  $3 \cdot 2^{n/2}$  to  $\frac{9}{4} \cdot 2^{n/2}$ .

**Lemma 8.** *If  $G$  is a connected cubic simple graph, then  $c_3(G) \leq \frac{9}{4} \cdot 2^{n/2}$ .*

*Proof.* As in Theorem 4, let us consider an  $(s, t)$ -ordering  $v_1, v_2, \dots, v_n$  of the vertices and the acyclic digraph  $D$  obtained by orienting all the edges of  $G$  according to this ordering.

Let  $i$  be the smallest integer such that  $d^-(v_i) = 2$ . Since every vertex (except  $v_1$ ) has an inneighbour, there exists  $j$  such that there are two internally-disjoint directed paths from  $v_j$  to  $v_i$  in  $D$ . In  $G$ , the union of these two paths forms a cycle  $C$ . By definition of  $i$ , all vertices of  $C$  but  $v_i$  have outdegree 2. So, if there is  $k$  such that  $j < k < i$  and  $v_k \notin V(C)$ , then  $v_k$  has no outneighbour in  $C$  and the ordering  $v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_i, v_k, v_{i+1}, \dots, v_n$  is also an  $(s, t)$ -ordering. Repeating

this operation as many times as necessary, we may obtain that all the vertices of  $C$  are consecutive in the ordering, that is  $C = (v_j, v_{j+1}, \dots, v_i, v_j)$ .

We enumerate the 3-edge-colourings of  $G$  in a similar way to the proof of Theorem 4, except that instead of examining the colour of arcs with tail in  $\{v_j, \dots, v_i\}$  one after another, we look at  $C$  globally. If  $j = 1$ , then there are exactly  $c_3(C)$  3-edge-colourings of  $C$ , because no arcs has head in  $C$ . If  $j > 1$ , then there are  $c_3(C)/3$  3-edge-colourings of  $C$ , because one arc has head  $v_j$  and we need the colours of the two arcs with tail  $v_j$  to have a colour distinct from it.

Recall that in  $D$ ,  $v_1$  has indegree 0,  $v_p$  has indegree 3,  $\frac{n-2}{2}$  vertices have indegree 2 and  $\frac{n-2}{2}$  vertices have indegree 1. If  $j > 1$  (resp.  $j = 1$ ), then there are  $i - j$  (resp.  $i - 2$ ) vertices of indegree 1 in  $C$ , so there are  $\frac{n-2i+2j-2}{2}$  (resp.  $\frac{n-2i+2}{2}$ ) vertices of indegree 1 in  $V(G) \setminus V(C)$ .

If  $j = 1$ , then we start by colouring  $C$  and then extend the colouring to  $G$ . Once  $C$  is coloured, there is at most one possibility to colour each arc with tail in  $C$ , so  $c_3(G) \leq c_3(C) \cdot 2^{\frac{n-2i+2}{2}} = (c_3(C)/2^{i-1}) \cdot 2^{n/2}$ . If  $j > 1$ , we colour the arcs with tail in  $\{v_1, \dots, v_{j-1}\}$  as usual. Remark that, by the choice of  $C$ , there is exactly one of these arcs, denoted by  $e$ , which has head in  $C$  and more precisely  $e$  has tail  $v_j$ . Then, we consider all the 3-edge-colourings of  $C$  that agree with the colour of  $e$  (i.e.  $c(v_j v_{j+1}) \neq c(e)$  and  $c(v_j v_i) \neq c(e)$ ). There are exactly  $c_3(C)/3$  such colourings. Finally, we extend the edge colourings in all possible ways to  $G$  using the usual method. So, in this case, we obtain  $c_3(G) \leq 6 \cdot c_3(C)/3 \cdot 2^{\frac{n-2i+2j-2}{2}} = (c_3(C)/2^{i-j}) \cdot 2^{n/2}$ .

In all cases, we have to bound the value  $\frac{c_3(C)}{2^{i-j}}$  for  $1 \leq i < j$ . Since  $G$  has no 2-cycles,  $C$  has length at least 3, and so  $i - j \geq 2$ . By Proposition 1,  $c_3(G) = 2^{i-j+1} + 2$  if  $i - j$  is odd, and  $c_3(G) = 2^{i-j+1} - 2$  if  $i - j$  is even. Easily one sees that the value  $\frac{c_3(C)}{2^{i-j}}$  is maximized when  $i - j = 3$  (i.e.  $C$  has length four), and so  $\frac{c_3(C)}{2^{i-j}} \leq \frac{18}{8} = \frac{9}{4}$ . Thus  $c_3(G) \leq \frac{9}{4} \cdot 2^{n/2}$ .  $\square$

Theorem 5 for  $k = 3$  states that a connected cubic graph  $G$  has at most  $3 \cdot 2^{n/2}$  3-edge-colourings. We shall now describe all connected cubic graphs attaining this bound.

Let  $G$  be a cubic graph and  $C = uvu$  be a 2-cycle in  $G$ . Then  $G/C$  is the graph obtained from  $G - \{u, v\}$  by adding an edge between the neighbour of  $u$  distinct from  $v$  and the neighbour of  $v$  distinct from  $u$ .

Let  $\Theta$  be the graph with two vertices joined by three edges. And let  $\mathcal{L}$  be the family of graphs defined recursively as follows:

- $\Theta \in \mathcal{L}$ .
- if  $G$  has a 2-cycle  $C$  such that  $G/C$  is in  $\mathcal{L}$ , then  $G$  is in  $\mathcal{L}$ .

Remark that the 3-noodle necklaces  $N_n^3$  (with  $n$  even) belongs to the family  $\mathcal{L}$ .

**Theorem 9.** *Let  $G$  be a connected cubic graph. If  $G \in \mathcal{L}$ , then  $c_3(G) = 3 \cdot 2^{n/2}$ . Otherwise  $c_3(G) \leq \frac{9}{4} \cdot 2^{n/2}$ .*

*Proof.* By induction on  $n$ , the result holding for simple graphs by Lemma 8 and for  $\Theta$  because  $c_3(\Theta) = 6$ .

Assume that  $n \geq 4$  and that  $G$  has a 2-cycle  $C = uvu$ . In any 3-edge-colouring of  $G$ , the edges not in  $C$  incident to  $u$  and  $v$  are coloured the same. Hence to each 3-edge-colouring  $c$  of  $G/C$  corresponds the two 3-edge-colourings of  $G$  that agrees with  $c$  on  $G - \{u, v\}$ . Hence  $c_3(G) = 2c_3(G/C)$ .

If  $G/C$  is in  $\mathcal{L}$ , then  $G$  is also in  $\mathcal{L}$ . Moreover, by the induction hypothesis,  $c_3(G/C) = 3 \cdot 2^{(n-2)/2}$ . So  $c_3(G) = 3 \cdot 2^{n/2}$ .

If  $G/C$  is not in  $\mathcal{L}$ , then  $G$  is not in  $\mathcal{L}$ . Moreover, by the induction hypothesis,  $c_3(G/C) \leq \frac{9}{4} \cdot 2^{(n-2)/2}$ . So  $c_3(G) \leq \frac{9}{4} \cdot 2^{n/2}$ .  $\square$

We have no example of cubic simple graphs admitting exactly  $\frac{9}{4} \cdot 2^{n/2}$  3-edge-colourings, and we believe that  $\frac{9}{4}$  could be replaced by a lower constant in the statement of Theorem 9. In fact, we conjecture that the maximum number of 3-edge-colourings of cubic simple graphs of order  $n$  is attained by some special graphs that we now describe.

For all  $n \geq 2$ ,  $n = 2p$  even, the *hamster wheel*  $H_n$  is the cubic graph obtained from two cycles on  $p$  vertices  $C_v = (v_1, v_2, \dots, v_p, v_1)$  and  $C_w = (w_1, w_2, \dots, w_p, w_1)$  by adding the matching  $M = \{v_i w_i : 1 \leq i \leq p\}$ . This construction for a lower bound was proposed by Pyatkin as it is mentioned in [12].

**Proposition 10.**

$$c_3(H_n) = \begin{cases} 2^{n/2} + 8, & \text{if } n/2 \text{ is even,} \\ 2^{n/2} - 2, & \text{if } n/2 \text{ is odd.} \end{cases}$$

*Proof.* Let  $\phi$  is a 3-edge-colouring of  $C_v$ .

If the three colours appear on  $C_v$ , then there is a unique 3-edge-colouring of  $H_n$  extending  $\phi$ . Indeed, to extend  $\phi$ , the colours of the edges of  $M$  are forced. Since the three colours appear on  $C_v$ , there are two edges of  $M$  which are coloured differently. Without loss of generality, we may assume that these two edges are consecutive, that is there exists  $i$  such that they are  $v_j w_j$  and  $v_{j+1} w_{j+1}$ . But then the colour of  $w_j w_{j+1}$  must be equal to the one of  $v_j v_{j+1}$ . Then, from edge to edge along the cycle, one shows that for all  $i$ , the colour of  $w_i w_{i+1}$  is the one of  $v_i v_{i+1}$ .

If only two colours appear on  $C_v$ , then there are two 3-edge-colourings of  $H_n$  extending  $\phi$ . Indeed in this case,  $n$  is even and all the edges of  $M$  must be coloured by the colour not appearing on  $C_v$ . So, there are two possible 3-edge-colourings of  $C_w$  with the colours appearing on  $C_v$ .

Hence the number of 3-edge-colourings of  $G$  is equal to the number of 3-edge-colourings of  $C_v$  plus the number of 3-edge-colourings of  $C_v$  in which two colours appear. If  $n/2$  is odd, this last number is 0, and if  $n/2$  is even, this number is 6. So, by Proposition 1,  $c_3(H_n) = 2^{n/2} - 2$  if  $n/2$  is odd and  $c_3(H_n) = 2^{n/2} + 8$  if  $n/2$  is even.  $\square$

For all  $n \geq 2$ ,  $n = 2p$  even, the *Möbius ladder*  $M_n$  is the cubic graph obtained from a cycle on  $n$  vertices  $C = (v_1, v_2, \dots, v_n, v_1)$  by adding the matching  $M = \{v_i v_{i+p} : 1 \leq i \leq p\}$  (indices are modulo  $n$ ).

Two edges  $e$  and  $f$  of the cycle  $C$  are said to be *antipodal*, if there exists  $1 \leq i \leq p$  such that  $\{e, f\} = \{v_i v_{i+1}, v_{i+p} v_{i+p+1}\}$ . A 3-edge-colouring  $c$  of  $M_n$  is said to be *antipodal* if  $c(e) = c(f)$  for every pair  $(e, f)$  of antipodal edges.

**Proposition 11.** *Let  $c$  be a 3-edge-colouring of  $M_n$ . If  $c$  is not antipodal, then  $n/2$  is odd and all the arcs  $v_i v_{i+n/2+1}$  are coloured the same.*

*Proof.* Suppose that two antipodal edges are not coloured the same. Without loss of generality,  $c(v_n v_1) = 2$  and  $c(v_p v_{p+1}) = 3$ , where  $n = 2p$ . Hence, we have  $c(v_1 v_{p+1}) = 1$ ,  $c(v_1 v_2) = 3$  and  $c(v_{p+1} v_{p+2}) = 2$ . And so on by induction, for all  $1 \leq i \leq p$ ,  $c(v_i v_{i+p}) = 1$  and  $\{c(v_i v_{i+1}), c(v_{i+p} v_{i+p+1})\} = \{2, 3\}$ . Hence, the edges of  $C$  are coloured alternately with 2 and 3, Since  $c(v_{2p} v_1) = 2$  and  $c(v_p v_{p+1}) = 3$ ,  $p$  must be odd.  $\square$



**Proposition 12.**

$$c_3(M_n) = \begin{cases} 2^{n/2} + 2, & \text{if } n/2 \text{ is even,} \\ 2^{n/2} + 4, & \text{if } n/2 \text{ is odd.} \end{cases}$$

*Proof.* Clearly, there is a one-to-one mapping between the antipodal 3-edge-colourings of  $M_n$  and the 3-edge-colourings of  $C_{n/2}$ . Hence, by Proposition 11, if  $n/2$  is even, then  $c_3(M_n) = c_3(C_{n/2}) = 2^{n/2} + 2$  by Proposition 1.

If  $n/2$  is odd, non-antipodal 3-edge-colourings are those such that all arcs  $v_i v_{i+n/2+1}$  are coloured the same, by Proposition 11. There are 6 such edge colourings (three choices for the colour of the edges  $v_i v_{i+n/2+1}$  and for each of these choices, two possible edge colourings of  $C$ ). Hence  $c_3(M_n) = c_3(C_{n/2}) + 6 = 2^{n/2} + 4$  by Proposition 1.  $\square$

We think that  $H_n$  and  $M_n$  are the connected cubic graphs which admit the maximum number of 3-edge-colourings. Precisely, we raise the following conjecture.

**Conjecture 13.** Let  $G$  be a connected cubic simple graph on  $n$  vertices. If  $n/2$  is even, then  $c_3(G) \leq c_3(H_n)$  and if  $n/2$  is odd, then  $c_3(G) \leq c_3(M_n)$ .

### 3 Total colouring

A *total colouring* of a graph  $G$  into  $k$  colours is a colouring of its vertices and edges such that two adjacent vertices receive different colours, two adjacent edges receive different colours and a vertex and an edge incident to it receive different colours. A total colouring with  $k$  colours is a *k-total-colouring*. For every graph  $G$ , let  $c_k^T(G)$  be the number of  $k$ -total-colourings of  $G$ .

For each 4-edge-colouring  $c$  of a cubic graph  $G$ , there is at most one 4-total-colouring of  $G$  whose restriction to  $E(G)$  equals  $c$ . Indeed, the colours of the three edges incident to a vertex force the colour of this vertex. Hence if  $G$  is cubic, we have that  $c_4^T(G) \leq c_4(G)$ .

By the method described in the previous section, one can show that if  $G$  is 2-connected, then  $c_4(G) = O(2^{n/2} \cdot 6^{n/2})$ , and so  $c_4^T(G) = O(2^{n/2} \cdot 6^{n/2})$ . We now obtain better upper bounds for  $c_4^T$ .

**Theorem 14.** *Let  $G$  be a 2-connected subcubic graph. Then  $c_4^T(G) \leq 3 \cdot 2^{2n-n_3/2}$ .*

*Proof.* Assume first that  $G$  is a cycle  $(v_1, \dots, v_n, v_1)$ . Let us totally colour it greedily starting from  $v_1$ . There are 4 possible colours for  $v_1$ , and then 3 possible colours for  $v_1 v_2$ . Afterwards for every  $i \geq 2$ , there at most two possible colours for  $v_i$  (the ones distinct from the colours of  $v_{i-1}$  and  $v_{i-1} v_i$ ) and then at most two possible colours for  $v_i v_{i+1}$  (the ones distinct from the colours of  $v_{i-1} v_i$  and  $v_i$ ). Hence  $c_4^T(G) \leq 4 \cdot 3 \cdot 2^{2n-2} = 3 \cdot 2^{2n}$ .

Assume now that  $G$  is not a cycle. Let  $s$  and  $t$  be two distinct vertices of degree 3. Consider an  $(s, t)$ -ordering  $v_1, v_2, \dots, v_n$  of  $V(G)$ , which exists by Lemma 3, and the orientation  $D$  of  $G$  according to this ordering. Then  $d^+(v_1) = 3 = d^-(v_n)$  and  $d^-(v_1) = 0 = d^+(v_n)$ . Let  $A^+$  (resp.  $A^-$ ) be the set of vertices of outdegree 2 (resp. indegree 2) in  $D$  and  $A_2$  be the set of vertices of degree 2 in  $G$ . As in the proof of Theorem 4, we have  $|A_2| = n - n_3$ , and  $|A^+| = |A^-| = (n_3 - 2)/2$ .

Now for  $i = 1$  to  $n - 1$ , we enumerate the  $p_i$  partial 4-total-colourings of vertices in  $\{v_1, \dots, v_i\}$  and arcs with tail in  $\{v_1, \dots, v_i\}$ . For  $i = 1$ , there are  $4! = 24$  such colourings, since  $v_1$  and its three incident arcs must receive different colours.

For each  $1 < i < n$ , when we extend the partial total colourings. Two cases may arise.

- If  $d_D^-(v_i) = 1$ , then there are two choices to colour  $v_i$  and then two other choices to colour the (at most two) arcs leaving  $v_i$ . Hence  $p_i \leq 4p_{i-1}$ .
- If  $d_D^-(v_i) = 2$ , then there are at most two choices to colour  $v_i$  and then the colour of the arc leaving  $v_i$  is forced since three colours are forbidden by  $v_i$  and its two entering arcs. Hence  $p_i \leq 2p_{i-1}$ .

Finally, we need to colour  $v_n$ . Since its three entering arcs are coloured its colour is forced (or it is impossible to extend the colouring).

Hence an easy induction shows that  $c_4^T(G) = p_{n-1} \leq 24 \cdot 4^{|A_2|+|A^+|} \cdot 2^{|A^-|} = 3 \cdot 2^{2n-n_3/2}$ .  $\square$

A *leaf* of a tree is a degree one vertex. A vertex of a tree which is not a leaf is called a *node*. A tree is *binary* if all its nodes have degree 3.

**Proposition 15.** *If  $T$  is a binary tree of order  $n$ , then  $c_4^T(T) = 3 \cdot 2^{3n/2}$ .*

*Proof.* By induction on  $n$ , the results holding easily when  $n = 2$ , that is when  $T = K_2$ .

Suppose now that  $T$  has more than two vertices. There is a node  $x$  which is adjacent to two leaves  $y_1$  and  $y_2$ . Consider the tree  $T' = T - \{y_1, y_2\}$ . By the induction hypothesis,  $c_4^T(T') = 3 \cdot 2^{3(n-2)/2}$ . Now each 4-total-colouring of  $T'$  may be extended into exactly eight 4-total-colourings of  $T'$ . Indeed the two colours of  $x$  and its incident edge in  $T'$  are forbidden for  $xy_1$  and  $xy_2$ , so there are two possibilities to extend the colouring to these edges, and then for each  $y_i$ , there are two possible colours available. Hence  $c_4^T(T) = 8 \cdot c_4^T(T') = 3 \cdot 2^{3n/2}$ .  $\square$

**Theorem 16.** *Let  $G$  be a connected cubic graph. Then  $c_4^T(G) \leq 3 \cdot 2^{3n/2}$ .*

*Proof.* Let  $F$  be the subgraph induced by the cutedges of  $G$ . Then  $F$  is a forest. Consider a tree of  $F$ . It is binary, its leaves are in different non-trivial 2-connected components of  $G$ , and every node is a trivial 2-connected component of  $G$ .

A subgraph  $H$  of  $G$  is *full* if it is induced on  $G$ , connected and such that for every non-trivial 2-connected component  $C$ ,  $H \cap C$  is empty or is  $C$  itself and for every tree  $T$  of  $F$ ,  $H \cap T$  is empty, or is just one leaf of  $T$  or is  $T$  itself. Observe that a full subgraph has minimum degree at least 2.

We shall prove that for every full subgraph  $H$ ,  $c_4^T(H) \leq 3 \cdot 2^{2n(H)-n_3(H)/2}$ . We proceed by induction on the number of 2-connected components of  $H$ . If  $H$  is 2-connected, then the result holds by Theorem 14.

Suppose now that  $H$  is not 2-connected. Then  $H$  contains a tree  $T$  of  $F$ . Let  $v_1, \dots, v_p$  be the leaves of  $T$ ,  $e_i$ ,  $1 \leq i \leq p$  the edge incident to  $v_i$  in  $T$  and  $N$  the set of nodes of  $T$ . Then  $H - N$  has  $p$  connected components  $H_1, \dots, H_p$  such that  $v_i \in H_i$  for all  $1 \leq i \leq p$ . Furthermore, each  $H_i$  is a full subgraph of  $G$ .

Let  $c$  be a 4-total-colouring of  $T$ . It can be extended to  $H_i$  by any 4-total-colouring of  $H_i$  such that  $v_i$  is coloured  $c(v_i)$  and the two edges incident to  $v_i$  in  $H_i$  are coloured in  $\{1, 2, 3, 4\} \setminus \{c(v_i), c(e_i)\}$ . There are  $\frac{1}{12}c_4^T(H_i)$  such colourings because each of them correspond to exactly twelve 4-total colourings of  $H_i$  obtained by permuting the colour of  $v_i$  (there are 4 possibilities) and then the colour of  $e_i$  (there are 3 possibilities). Hence each 4-total-colouring of  $T$  can be extended into  $\prod_{i=1}^p \frac{1}{12}c_4^T(H_i)$  4-total-colourings of  $H$  and so

$$c_4^T(H) = c_4^T(T) \cdot \prod_{i=1}^p \frac{1}{12}c_4^T(H_i).$$

Now by Proposition 15,  $T$  has  $3 \cdot 2^{3n(T)/2}$  4-total-colourings, and since  $H_i$  is full  $c_4^T(H_i) \leq 3 \cdot 2^{2n(H_i) - n_3(H_i)/2}$  by the induction hypothesis. Moreover,  $n_3(H) = n(T) + \sum_{i=1}^p n_3(H_i)$  and  $n(H) = n(T) + \sum_{i=1}^p n(H_i) - p$ . Hence  $c_4^T(H) \leq 3 \cdot 2^{2n(H) - n_3(H)/2}$ .  $\square$

As previously for the edge colourings of graphs, we derive from Theorem 16 an algorithm to enumerate all the 4-total-colourings of a cubic graph. The proof is similar to the one of Corollary 6.

**Corollary 17.** *There is an algorithm to enumerate all the 4-total-colourings of a connected cubic graph on  $n$  vertices in time  $O^*(2^{3n/2})$  and polynomial space.*

The bound of Theorem 16 is seemingly not tight. Indeed, in Theorem 14, the equation  $p_i \leq 2p_{i-1}$  when  $d_D^-(v_i)$  often overestimates  $p_i$ , because there are two choices to colour  $v_i$  only if the two colours appearing on its two entering arcs are the same two as the ones assigned to the tails of these arcs. If not the colour of  $v_i$  is forced or  $v_i$  cannot be coloured.

**Problem 18.** What is  $c_4^T(n)$ , the maximum of  $c_4^T(G)$  over all connected graphs of order  $n$ ?

We shall now give a lower bound on  $c_4^T(n)$ . A binary tree is *nice* if its set of leaves may be partitionned into pairs of *twins*, i.e. leaves at distance 2. Clearly, every nice binary tree  $T$  has an even number of leaves and thus  $n(T) \equiv 2 \pmod{4}$ . Moreover if  $n(T) = 4p + 2$ , then  $T$  has  $2p$  nodes and  $p + 1$  pairs of twins. A *noodle tree* is a cubic graph obtained from a nice binary tree by adding two parallel edges between each pair of twins.

**Proposition 19.** *Let  $p$  be a positive integer and  $n = 4p + 2$ . If  $G$  is a noodle tree  $G$  of order  $n$ , then  $c_4^T(G) = \frac{3}{\sqrt{2}} \cdot 2^{5n/4}$ .*

*Proof.* Let  $X_1, \dots, X_{p+1}$  be the pairs of twins of  $G$ , and let  $T$  be the binary tree  $G - \bigcup_{i=1}^{p+1} X_i$ . Let us label the leaves of  $T$ ,  $y_1, \dots, y_{p+1}$  such that for all  $1 \leq i \leq p + 1$ ,  $y_i$  is adjacent to the two vertices of  $X_i$  in  $G$ .

Every 4-total-colouring of  $T$ , may extended in exactly 4 ways to each pair of twins  $X_i = \{x_i, x'_i\}$  and their incident edges. Indeed, without loss of generality we may assume that  $y_i$  is coloured 1 and its incident edge in  $T$  is coloured 2. Then the edges  $y_i x_i$  and  $y_i x'_i$  must be coloured in  $\{3, 4\}$ , which can be done in two possible ways. For each of these possibilities, the parallel edges between  $x_i$  and  $x'_i$  must be coloured in  $\{1, 2\}$ , which again can be done in two possible ways. Finally, we must colour  $x_i$  (resp.  $x'_i$ ) with the colour of  $y_i x'_i$  (resp.  $y_i x_i$ ).

Hence  $c_4(G) = 4^{p+1} \cdot c_4(T)$ , and so by Proposition 15,  $c_4(G) = 3 \cdot 2^{5p+2}$ .  $\square$

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