A model for knowledge representation in distributed systems

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Introduction

Based on attempts to formalize notions like information or knowledge, the techniques which are now known as information flow analysis have helped to understand the way communication and interaction operate as an exchange of information. They have successfully been applied to different domains such as the study of multi-agent systems, communication protocols and enabled to make connections with philosophical issues such as knowledge and belief.

In the present paper, we analyse the problem of information representation in distributed systems and use it to set up a generic framework for the study of properties such as knowledge preservation or cancellation between agents. The starting point for this work is the *First Principle of Information Flow* given in [1] :

Information flow results from regularities in a distributed system

This way, information flow can be described as a structural property of the system, permitting a statical approach to the problem. The main idea used for designing this framework expresses the fact that a complex system can be seen as a "black-box" whose inner structure is known but which can only be observed in approximate ways. For instance, in a multi-agent system, if an agent can only have access to its own state, this can be seen as a partial observation of the state of the whole system.

It follows that the notion of approximation (here, an agent knows only an approximation of the state of the system) has to be present in a general theory of information representation in distributed system.

Following this, we set up a framework based on partial orders (which embody the notion of approximation) and a class of operators which represent the way observations keep only a limited amount of information. Similarities between those operators and some axioms of modal logic (namely, the axioms **T** and **4**) lead us to the study of the relationship between such systems and modal logic, and we show that the description systems defined in our framework act as a model for a variant of the modal logic **IS4**.

In a first part of this article, we present a simplified formalism of distributed systems in order to show some concepts which we think are important in the study of the flow of information between different parts or agents of a system. Then, we use those basic concepts and generalize them to define an algebraic framework for formalizing distributed systems. Finally, we study the logical structure of this framework and show that in this formalism, the distributed systems form a model for the intuitionistic modal logic **IS4+KV**.

1 Partial Knowledge in Distributed Systems

In the most basic approach, we consider that a "system" (in an extremely general way) can be defined by a set of states $\mathbf{S}$ which represents its structure, and a particular state $s \in \mathbf{S}$, its actual state. We suppose that the structure $\mathbf{S}$ is common knowledge, but not its state s.

Now, let us consider that there exists a set of possible observations $\{\mathcal{O}_i\}_{i\in\mathcal{I}}$ of the system. Since its structure $\mathbf{S}$ is known, the observations only provide information about the actual state s of the system. But different observations shall give different results, so that one actually gets an approximation of the actual state, that is a collection of possible candidates for it. In this approach, each observation $\mathcal{O}_i$ can be formalized by a function $f_i : \mathbf{S} \to \wp(\mathbf{S})$, so that if the actual state is s, the possible candidates given by observation $\mathcal{O}_i$ are the elements of $f_i(s)$. The first property we want those functions to verify is that they are consistent, by always letting the actual state be a possible candidate :

$$\forall\, i \in \mathcal{I}, \forall\, s \in \mathbf{S}, s \in f_i(s)$$

The f_i functions can be seen as a possibility function : if x is possible (or said another way, is a candidate for the actual state), and y is in $f_i(x)$, then y is also possible. It acts as a binary relation $\to_i$ such that $x \to_i y \Leftrightarrow y \in f_i(x)$. The previous property is then equivalent to saying that $\to_i$ is reflexive. In the following, we want the further assumption that the relations be also transitive, which in terms of f_i rewrites as $\forall\, y \in f_i(x), f_i(y) \subseteq f_i(x)$, so that all the possible states are found in only one step. Such relations are pre-orders, announcing the poset structure we will introduce in the next section.

Partial observations can be used to formalize knowledge issues in a multi-agent system, since the "knowledge" of an agent (which generally amounts to its internal state) is an approximation of the state of the whole system, and thus can be obtained by using a partial observation which selects the information available to the agent. This motivates the definition of a partial observation based framework for modelling multi-agent systems.

2 Description Systems

In the previous section, we have presented a simple way of obtaining partial information from the state of a system, by the means of observations which would return a description of the system in the form of a set of states containing the actual one. Thus, the set of possible descriptions of the actual state is a set of subsets, which verifies the property of being ordered (using the inclusion relation). This is an expected property, since it is natural to compare descriptions depending on the amount of information they carry.

Such orderedness is the only property we want to have in our framework so as to keep it as general as possible. This gives the set of possible descriptions of the system a structure of poset. We will use the convention that given two description x and y, $x \leq y$ means that x is more accurate than y.

We now want to generalize the notion of partial observations introduced previously. They were presented as filters which would only keep available information with regards to a particular "point of view". In the general case, this loss of information corresponds to returning a description which is less accurate that the initial one. Mathematically, if a function ρ_i embodies the loss of information done by observation $\mathcal{O}_i$, then one can write : $\forall\, x, x \leq \rho_i(x)$. Another important property that ρ_i verifies is that it is monotonous, since if one has two descriptions $x \leq y$, any piece of information in y which is not lost through observation $\mathcal{O}_i$ remains in description x too. We want a last property to be verified, that of idempotence ($\rho_i \circ \rho_i(x) = \rho_i(x)$). It corresponds to the transitivity assumption for $\rightarrow_i$ in the previous section, which meaning is that all the unwanted information is lost at once, or equivalently, that all the possible states are found in one step in the case of set of states as seen previously.

Functions verifying these three properties are called upper closure operators (uco) in lattice literature [5] where they are commonplace.

We sum up the definition of our framework by introducing the following notion :

Definition 1 (Description system)
A description system *over a set $\mathcal{I}$ of agents (or more generally, of indexes) is a tuple $\langle \mathbf{P}, \leq, \{\rho_i\}_{\mathcal{I}} \rangle$ where $\langle \mathbf{P}, \leq \rangle$ is a poset and the ρ_i's are upper closure operators on $\mathbf{P}$.*

3 The Logic of Knowledge

With the structure defined in the previous section, we can now set up a logical language for expressing propositions on our system. Aside from this language, we introduce the modal logic $\mathbf{IS4} + \mathbf{KV}$ and show that this logic exactly reflects the behaviour of description systems, as we will show in a determination theorem.

3.1 The logical language $\mathcal{L}(\Psi, \mathcal{I})$

To study the properties of the behaviour of information in a distributed system, we need a logical language for expressing propositions about the knowledge of the different agents. For this purpose, we define $\mathcal{L}(\Psi, \mathcal{I})$ as the least language containing some primitive propositions (the elements of a set Ψ), closed under the classical operations – disjunction $\vee$, conjunction $\wedge$, implication $\rightarrow$ and falsehood $\perp$ – and containing a collection of unary operators $\{K_i\}_{i \in \mathcal{I}}$ where $K_i\, \varphi$ means that agent $\mathcal{A}_i$ knows that proposition φ holds. Following usual logical convention, we define equivalence $\varphi \leftrightarrow \psi$ as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and negation $\neg\varphi$ as $\varphi \rightarrow \perp$.

We now want to define the semantics of this logical language in terms of description systems. Thus, given such a system $\langle \mathbf{P}, \leq, \{\rho_i\} \rangle$, we want to relate a proposition φ to a subset $[\![\varphi]\!]$ of those elements of $\mathbf{P}$ where the property holds. Such a subset $[\![\varphi]\!]$ has to verify the fact that if φ holds at a description x, φ must also hold at any description y which is more informative than x (i.e. such that $y \leq x$), since the truthness of a proposition is not affected by the addition of consistent information. It follows that $[\![\varphi]\!]$ is an ideal (i.e. a downward closed subset) of $\langle \mathbf{P}, \leq \rangle$. Let $\mathbf{Id}(\mathbf{P})$ denote the set of ideals of $\mathbf{P}$.

We can now define an interpretation function $[\![\cdot]\!]$ from $\mathcal{L}(\Psi, \mathcal{I})$ to $\mathbf{Id}(\mathbf{P})$. We first have to define an interpretation of the primitive propositions and then to give a structural definition of $[\![\cdot]\!]$. The meaning of the primitive propositions is provided by a function $\nu : \Psi \to \mathbf{Id}(\mathbf{P})$. We give the interpretation of a binary connective $\mathbf{c}$, by defining $[\![\varphi \; \mathbf{c} \; \psi]\!] = \{ x \mid \forall y \leq x, (y \in [\![\varphi]\!]) \; \mathbf{c} \; (y \in [\![\psi]\!]) \}$. This ensures that the computed subset is indeed an ideal. The last rule, that of K_i, is based on the fact that $x \in [\![K_i \, \varphi]\!]$ is equivalent to $\rho_i(x) \in [\![\varphi]\!]$. The translation rules are summed up in Table 1.

$$
\begin{aligned}
\forall \varphi \in \Psi, [\![\varphi]\!] &= \nu(\varphi) \\
[\![\bot]\!] &= \emptyset \\
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \to \psi]\!] &= \{ x \mid \forall y \leq x, y \in [\![\varphi]\!] \Rightarrow y \in [\![\psi]\!] \} \\
[\![K_i \, \varphi]\!] &= \{ x \mid \rho_i(x) \in [\![\varphi]\!] \}
\end{aligned}
$$

Table 1: Translation rules from $\mathcal{L}(\Psi)$ to $\mathbf{Id}(\mathbf{P})$

The interpretation function $[\![\cdot]\!]$ can then be defined given a description structure and an interpretation for primitive propositions. Thus, we introduce the definition of a description model which contains all the information necessary to define an interpretation of the language $\mathcal{L}(\Psi, \mathcal{I})$.

Definition 2 (Description Model)
A description model of $\mathcal{L}(\Psi, \mathcal{I})$ is a tuple $\mathcal{D} = \langle \mathbf{P}, \leq, \{\rho_i\}_{\mathcal{I}}, \nu \rangle$ where $\langle \mathbf{P}, \leq, \{\rho_i\}_{\mathcal{I}} \rangle$ is a description system over $\mathcal{I}$ and $\nu : \Psi \to \mathbf{Id}(\mathbf{P})$ is the interpretation function for the primitive propositions.

In the following, we might write $[\![\cdot]\!]_{\mathcal{D}}$ to emphasize that the interpretation function is the one defined by the description model $\mathcal{D}$.

3.2 The logics IS4+KV and IS4+KV$_{\mathcal{I}}$

Intuitionistic logic was developed by Brouwer and formalized by Heyting in the thirties as a constructivist approach to logic [10], and is based on the idea that if an object can be proved to exist, it can be constructed. One of the most striking features of intuitionistic logic is that the rule of excluded middle ($\varphi \vee \neg\varphi$) is no longer valid.

Apart from its philosophical interest, intuitionistic logic proved to be extremely useful in computer science (illustrated by the Curry-Howard correspondence [2]). We define the modal logic **IS4** as an extension of the propositional intuitionistic logic, to which a modal operator $\Box$ (usually called *necessity* operator) is added. This operator verifies the following axioms and rules :

$\vdash \varphi$ entails $\vdash \Box\varphi$	Knowledge Generalization Rule
$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Deduction axiom $\mathbf{K}$
$\vdash \Box\varphi \to \varphi$	Knowledge axiom $\mathbf{T}$
$\vdash \Box\varphi \to \Box\Box\varphi$	Positive introspection axiom $\mathbf{4}$

We also introduce an axiom which permits to distribute the disjunction over the $\Box$ operator :

$$\vdash \Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi \qquad \text{Distribution axiom } \mathbf{KV}$$

Considering this axiom in addition to **IS4**, one gets **IS4+KV**. We define the logic **IS4+KV**$_{\mathcal{I}}$ similarly as **IS4+KV**, but in which the operator $\Box$ is replaced by modal operators K_i where i is an element of a set of indexes $\mathcal{I}$ and where each K_i behaves like $\Box$. An axiomatic formulation of this logic is given in Appendix A.

In the following, we will only deal with $\mathbf{IS4} + \mathbf{KV}_{\mathcal{I}}$, even though we might sometimes write **IS4+KV**. This is just for a matter of readability, since a more complete denotation would be $\mathbf{IS4} + \mathbf{KV}_{\Psi,\mathcal{I}}$.

Much work exist on different models for intuitionistic modal logic. The classical approach is based on Kripke's "possible worlds" models, where both modal logic and intuitionistic logic have a natural translation. Such semantics are described in [13]. Other approach include categorical ones [3, 4], computational ones [3, 11, 9] and others [8].

3.3 Relating IS4+KV and description models

We now show that there is a close relation between the logic $\mathbf{IS4} + \mathbf{KV}_{\Psi,\mathcal{I}}$ and the description models of $\mathcal{L}(\Psi, \mathcal{I})$. For this, we define two notions of "truth" and prove that they are equivalent.

First, we define $\vdash_{\mathbf{IS4+KV}} \varphi$ as the fact that there exists a finite proof of φ using the axioms of **IS4+KV**. We then define $\models_{\Psi,\mathcal{I}} \varphi$ to express that φ holds everywhere in any description model of $\mathcal{L}(\Psi, \mathcal{I})$. More formally, if $\mathbf{D}_{\Psi,\mathcal{I}}$ denotes the set of description models of $\mathcal{L}(\Psi, \mathcal{I})$, then

$$\models_{\Psi,\mathcal{I}} \varphi \;\Leftrightarrow \forall \mathcal{D} \in \mathbf{D}_{\Psi,\mathcal{I}}, \mathcal{D} \models \varphi$$

where if $\mathcal{D} = \langle \mathbf{P}, \{\rho_i\}_{\mathcal{I}}, \nu \rangle$, $\mathcal{D} \models \varphi$ means that $[\![\varphi]\!]_{\mathcal{D}} = \mathbf{P}$. With those notions, we can now give the following theorem which expresses that they are equivalent :

Theorem 1 (Determination)
The class of description models determines the logic **IS4+KV**, *that is :*

$$\forall\,\varphi \in \mathcal{L}(\Psi,\mathcal{I}),\ \vdash_{\mathbf{IS4+KV}} \varphi \ \Leftrightarrow\ \models_{\Psi,\mathcal{I}} \varphi$$

Sketch of Proof The proof of this theorem can be divided in two parts. The first one, the soundness part, states that given a correct sequent $\Gamma_1, \ldots, \Gamma_n \vdash \varphi$ (i.e. such that φ can be proved by the axioms of **IS4+KV** using propositions Γ_i), then for any $\mathcal{D} \in \mathbf{D}_{\Psi,\mathcal{I}}$, we have $[\![\varphi]\!]_{\mathcal{D}} \subseteq \bigcap_i [\![\Gamma_i]\!]_{\mathcal{D}}$.

The second part, that of completeness, states that in a special model $\mathcal{C}$, called the canonical model, the formulas φ such that $\mathcal{C} \models \varphi$ are exactly those provable in **IS4+KV** (that is those which verify $\vdash_{\mathbf{IS4+KV}}\ \varphi$). The proof of this theorem is given more precisely in Appendix B.

$\square$

This theorem provides a simple and general class of model for the modal logic **IS4+KV**. While many classes of model exist, either based on Kripke structures [13], on categories [3, 4] or on adaptations of λ-calculus [9, 11], the present model originates from approximation techniques and its application to information flow formalisms [6], offering new possibilities in the logical study of complex systems and knowledge representation.

4 Conclusion

In this paper, we have introduced description systems, a general framework for formalizing complex systems. It is based on the notion of partial observation, which appears as a central feature for reasoning about a system when one has not got a total description of it, which is the case in multi-agent systems.

The study of the logical behavior of those systems has shown us that in such case, the properties concerning the knowledge given from the different observations have to behave in an intuitionistic way. This is not completely surprising though, since by observing a system, if one is not sure whether a property φ holds – said another way, if one has not got enough information to ensure that φ holds – it does not entail that he is sure that $\neg\varphi$ holds too. The need for a proof in intuitionistic logic for truthness is replaced by the need for information.

This approach is to be further developped, by adding functional and dynamical aspects, but we feel that it constitutes an interesting foundation for the study of general knowledge-related issues, and would in the long term be a basis for a semantical (as opposed to probabilistic) theory of information.

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A The Axioms and Rules for IS4+KV

$$\begin{aligned}
\varphi &\to \varphi \wedge \varphi \\
\varphi \wedge \psi &\to \psi \wedge \varphi \\
(\varphi \to \psi) &\to ((\varphi \wedge \vartheta) \to (\psi \wedge \vartheta)) \\
\varphi &\to (\psi \to \varphi) \\
\varphi \wedge (\varphi \to \psi) &\to \psi \\
\varphi &\to \varphi \vee \psi \\
\varphi \vee \psi &\to \psi \vee \varphi \\
(\varphi \to \vartheta) \wedge (\psi \to \vartheta) &\to ((\varphi \vee \psi) \to \vartheta) \\
\bot &\to \varphi \\
K_i\,(\varphi \to \psi) &\to (K_i\,\varphi \to K_i\,\psi) \\
K_i\,\varphi &\to \varphi \\
K_i\,\varphi &\to K_i\,K_i\,\varphi \\
K_i\,(\varphi \vee \psi) &\to K_i\,\varphi \vee K_i\,\psi \\
\vdash \varphi &\text{ entails } \vdash K_i\,\varphi
\end{aligned}$$

B Proof the Determination Theorem

B.1 Proof of Soundness

The soundness part of the determination theorem is stated as follows :

Theorem 2 (Soundness)
Given a sequent $\Gamma_1, \ldots, \Gamma_n \vdash \varphi$*, if it can be proved in* **IS4+KV***, then for any description model* $\mathcal{D}$*, we have* $\bigcap_i [\![\Gamma_i]\!]_{\mathcal{D}} \subseteq [\![\varphi]\!]_{\mathcal{D}}$.

Proof We prove this theorem by induction on the length of the proof, by case depending on the last rule used in this proof.

The proofs for the rules corresponding to non-modal connectives are routine. We shall only focus on the rules K_iL, K_iR and K_iV, which define the behavior of K_i operators in the sequent calculus formulation of **IS4+KV**. Those rules are defined as follows :

$$K_iL\,\frac{\Gamma, \varphi \vdash \psi}{\Gamma, K_i\,\varphi \vdash \psi} \qquad K_iR\,\frac{K_i\,\Gamma \vdash \varphi}{K_i\,\Gamma \vdash K_i\,\varphi} \qquad K_iV\,\frac{\Gamma \vdash K_i\,(\varphi \vee \psi)}{\Gamma \vdash K_i\,\varphi \vee K_i\,\psi}$$

In the first case (K_iL), it suffices to see that $[\![K_i\,\varphi]\!] \subseteq [\![\varphi]\!]$. Similarly, in the third case, one has $[\![K_i\,(\varphi \vee \psi)]\!] = [\![K_i\,\varphi \vee K_i\,\psi]\!]$.

In the second case ($K_i R$), suppose that the sequent to prove is $K_i\,\Gamma \vdash K_i\,\varphi$ where $K_i\,\Gamma$ is a short-cut for $K_i\,\psi_1, \ldots, K_i\,\psi_n$. Then we have $K_i\,\Gamma \vdash \varphi$, which by induction implies :

$$\bigcap_j [\![K_i\,\psi_j]\!] \subseteq [\![\varphi]\!]$$

But since $[\![K_i\,\psi]\!] = \{\, x \mid \rho_i(x) \in [\![\psi]\!] \}$, the previous inclusion rewrites

$$\forall\, x,\ (\forall\, j, \rho_i(x) \in [\![\psi_j]\!]) \Rightarrow x \in [\![\varphi]\!]$$

so that

$$\forall\, y,\ \left(\forall\, j, \rho_i^2(y) \in [\![\psi_j]\!]\right) \Rightarrow \rho_i(y) \in [\![\varphi]\!]$$

which, since $\rho_i^2 = \rho_i$, is equivalent to

$$\bigcap_j [\![K_i\,\psi_j]\!] \subseteq [\![K_i\,\varphi]\!]$$

$\square$

B.2 Proof of Completeness

To prove that $\mathbf{D}_{\Psi,\mathcal{I}}$ is complete for $\mathbf{IS4} + \mathbf{KV}_{\mathcal{I}}$, we develop an appropriate model $\mathcal{C}$ called the canonical model, and show that any φ valid in this model is provable in **IS4+KV**, following the usual method for completeness [7, 12].

Definition 3 (Prime Sets)
A prime set *is a subset* Γ *of* $\mathcal{L}(\Psi)$ *which is consistent (*$\perp \notin \Gamma$*), closed under deduction (*$\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$*).*

We introduce two notations for selecting the knowledge of a given agent. Let $\Gamma/_i = \{\, \varphi \mid K_i\,\varphi \in \Gamma \}$ and $K\Gamma/_i = \{\, K_i\,\varphi \mid K_i\,\varphi \in \Gamma \}$. One easily proves that if $\Gamma \subseteq \mathcal{L}(\Psi)$ is prime, then so is $\Gamma/_i$.

We can now define our canonical model :

Definition 4 (Canonical Model)
We define our canonical model $\mathcal{C}$ *as* $\langle \mathbf{P}_\mathcal{C}, \leq_\mathcal{C}, \{\rho_{i,\mathcal{C}}\}, \nu_\mathcal{C} \rangle$ *where :*

$$\mathbf{P}_\mathcal{C} = \{\, \Gamma \subseteq \mathcal{L}(\Psi) \mid \Gamma \textit{ is prime} \} \quad \Gamma \leq_\mathcal{C} \Delta \Leftrightarrow \Delta \subseteq \Gamma$$
$$\rho_{i,\mathcal{C}}(\Gamma) = \{\, \varphi \mid K\Gamma/_i \vdash \varphi \} \quad \nu_\mathcal{C}(\alpha) = \{\, \Gamma \mid \alpha \in \Gamma \}$$

In this definition, the poset $\langle \mathbf{P}_\mathcal{C}, \leq_\mathcal{C} \rangle$ has a greatest element, which is precisely $\{\, \varphi \mid \vdash_{\mathbf{IS4+KV}} \varphi \}$. $\rho_{i,\mathcal{C}}(\Gamma)$ is the deductive closure of the assertions of Γ known by agent i. The following proposition shows that it is indeed prime.

Proposition 3
Given a prime set Γ*, one has* $\rho_{i,\mathcal{C}}(\Gamma) = \Gamma/_i$*.*

Proof Because of axiom **T** ($K_i\,\varphi \to \varphi$), we have $\Gamma/_i \subseteq \rho_{i,\mathcal{C}}(\Gamma)$. Conversely, using axiom **4** ($K_i\,\varphi \to K_i\,K_i\,\varphi$), if $K_i\,\varphi \in K\Gamma/_i$, then $K_i\,K_i\,\varphi \in K\Gamma/_i$, so that $K_i\,\varphi \in \Gamma/_i$. Thus, we have shown that $K\Gamma/_i \subseteq \Gamma/_i$. As a consequence, $\varphi \in \rho_{i,\mathcal{C}}(\Gamma) \Leftrightarrow K\Gamma/_i \vdash \varphi \Rightarrow \Gamma/_i \vdash \varphi \Leftrightarrow \varphi \in \Gamma/_i$.

□

Proposition 4
For all Γ prime and φ in $\mathcal{L}(\Psi)$, we have :

$$K\Gamma/_i \vdash \varphi \Leftrightarrow K\Gamma/_i \vdash K_i\,\varphi \Leftrightarrow \Gamma \vdash K_i\,\varphi$$

Proof The first equivalence comes directly from axioms **4** and **T**. For the second equivalence, as $K\Gamma/_i \subseteq \Gamma$, if $K\Gamma/_i \vdash K_i\,\varphi$, then $\Gamma \vdash K_i\,\varphi$. Conversely, if $\Gamma \vdash K_i\,\varphi$, as Γ is prime, it implies that $K_i\,\varphi \in \Gamma$, so that $K_i\,\varphi \in K\Gamma/_i$.

□

We can now prove the following theorem :

Theorem 5
For all $\varphi \in \mathcal{L}(\Psi)$,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \in [\![\varphi]\!]_{\mathcal{C}}$$

Proof This proof in done by induction on the size of φ. In the case of a primitive proposition, one has : $\Gamma \in [\![\alpha]\!] \Leftrightarrow \Gamma \in \nu_{\mathcal{C}}(\alpha) \Leftrightarrow \alpha \in \Gamma \Leftrightarrow \Gamma \vdash \alpha$.

In the case of a disjunction, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$, so that by induction, $\Gamma \in [\![\varphi \vee \psi]\!]$ is equivalent to $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$. We conclude by considering the definition of a prime set.

The case of the conjunction is straightforward.

By definition of $[\![\varphi \to \psi]\!]$ and induction, $\Gamma \in [\![\varphi \to \psi]\!]$ is equivalent to the implication $\forall\,\Delta \leq \Gamma$, $\Delta \vdash \varphi \Rightarrow \Delta \vdash \psi$. Now, with $\Delta = \Gamma \cup \{\varphi\}$, this implies $\Gamma, \varphi \vdash \psi$, and then $\Gamma \vdash \varphi \to \psi$ or equivalently $\varphi \to \psi \in \Gamma$ as Γ is prime. Conversely, if $\varphi \to \psi \in \Gamma$, then for any $\Delta \subseteq \Gamma$, $\Delta \vdash \varphi$ implies that $\Delta \vdash \psi$ since it is deductively closed.

The last case to prove is that of $K_i\,\varphi$. By definition of $[\![K_i\,\varphi]\!]$, we have $\Gamma \in [\![K_i\,\varphi]\!]_{\mathcal{C}} \Leftrightarrow \rho_{i,\mathcal{C}}(\Gamma) \in [\![\varphi]\!]$. But by induction, this is equivalent to $\rho_{i,\mathcal{C}}(\Gamma) \vdash \varphi$, or $K\Gamma/_i \vdash \varphi$ using the definition of $\rho_{i,\mathcal{C}}$. We conclude by using Proposition 2.

□

B.3 Proof of Determination

What remains to do now is to prove the determination theorem itself, that is :

$$\forall\,\varphi \in \mathcal{L}(\Psi, \mathcal{I}),\ \vdash_{\mathbf{IS4}_{\Psi,\mathcal{I}}} \varphi \Leftrightarrow \models_{\Psi,\mathcal{I}} \varphi$$

Suppose that $\vdash_{\mathbf{IS4}_{\Psi,\mathcal{I}}} \varphi$. Then, the soundness theorem tells us that for all description model $\mathcal{D}$, we have $[\![\varphi]\!]_{\mathcal{D}} = \mathbf{P}_{\mathcal{D}}$, implying that $\models_{\Psi,\mathcal{I}} \varphi$. Conversely, suppose that

$\models_{\Psi,\mathcal{I}} \varphi$. It implies that in particular, $\mathcal{C} \models \varphi$, and considering the greatest element of $\mathbf{P}_{\mathcal{C}}$, it follows that $\vdash_{\mathbf{IS4}} \varphi$.